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# On a New Class of Hypergeometric Function

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**Abstract**—The major objective of this article is to establish a new function called ${}_pG_q^{\eta,\zeta,m,\xi}(z)$ which is generalization of the generalized hypergeometric function and Mittag-Leffler function and obtain its properties particular differential property, integral representation, derivative formula and some integral transform, Euler transform, Laplace transform, Whittaker transform. We also derived the relations that exist between ${}_pG_q^{\eta,\zeta,m,\xi}(z)$ function with well known special functions. This article also deals with some fractional integral properties of the ${}_pG_q^{\eta,\zeta,m,\xi}(z)$ function using the Riemann–Liouville fractional integrals and derivatives operators.

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## 1. INTRODUCTION AND PRELIMINARIES

In 2008, Sharma [1] has introduced $M$-series and discussed its properties including fractional integration and fractional differentiation as follows

$$ {}_pM_q^{\eta}(w) = {}_pM_q^{\eta}(u_1,...,u_p,v_1,...,v_q;w) = \sum_{k=0}^{\infty} \frac{(u_1)_k....(u_p)_k}{(v_1)_k...(v_q)_k} \frac{w^k}{\Gamma(\eta k+1)}, \tag{1} $$

where $\eta \in \mathbb{C}$ such that $\Re(\eta) > 0$ and $(u_i)_k$ and $(v_j)_k$, $i = 1,..p$, $j = 1,...q$ are the Pochammer symbols.

Later Sharma and Jain [2], continuation of their previous research [1], have further generalized the $M$-series and studied generalized $M$-series in connection with the Fox's H-function and generalized hypergeometric function as follows

$$ {}_pM_q^{\eta,\zeta}(w) = {}_pM_q^{\eta,\zeta}(u_1,...,u_p,v_1,...,v_q;w) = \sum_{k=0}^{\infty} \frac{(u_1)_k....(u_p)_k}{(v_1)_k...(v_q)_k} \frac{w^k}{\Gamma(\eta k+\zeta)}, \tag{2} $$

where $\eta, \zeta \in \mathbb{C}$ such that $\Re(\eta), \Re(\zeta) > 0$ and $(u_i)_k$ and $(v_j)_k$, $i = 1,..p$, $j = 1,...q$ are the Pochammer symbols.

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In the sequence, Sharma [3] has introduced a new function called as $K$-function and obtained the connections that exists between the $K$-function and Riemann–Liouville fractional integrals and derivatives operators as follows

$$ {}_pK_q^{\eta,\zeta,\xi}(u_1,...,u_p,v_1,...,v_q;w) = {}_pK_q^{\eta,\zeta,\xi}(w) = \sum_{k=0}^{\infty} \frac{(u_1)_k....(u_p)_k(\xi)_k}{(v_1)_k...(v_q)_k\Gamma(\eta k+\zeta)} \frac{w^k}{k!}, \quad (3) $$

where $\eta, \zeta, \xi \in \mathbb{C}$ such that $\Re(\eta), \Re(\zeta), \Re(\xi) > 0$ and $(u_i)_k$ and $(v_j)_k$, $i = 1, ..p$, $j = 1, ...q$ are the Pochammer symbols.

Motivated and inspired by the above work [1–3], here, we introduce following ${}_pG_q^{\eta,\zeta,m,\xi}(a, b; z)$ function and its associated function.

**Definition 1.** A new ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function with $w \in \mathbb{C}$ is defined by

$$ {}_pG_q^{\eta,\zeta,m,\xi}(w) = {}_pG_q^{\eta,\zeta,m,\xi}(u_1,...,u_p,v_1,...,v_q;w) $$

$$ = {}_pG_q^{\eta,\zeta,m,\xi}\left[w\left|\begin{matrix}u_1,...,u_p\\ v_1,...,v_q\end{matrix}\right.\right] = \sum_{k=0}^{\infty} \frac{(u_1)_k....(u_p)_k(\xi)_{mk}}{(v_1)_k...(v_q)_k\Gamma(\eta k+\zeta)} \frac{w^k}{k!}, \quad (4) $$

where $\eta, \zeta, \xi \in \mathbb{C}$ such that $\Re(\eta), \Re(\zeta), \Re(\xi) > 0$, $m \in (0, 1) \cup \mathbb{N}$ and $(u_i)_k$ and $(v_j)_k$, $i = 1, ..p$, $j = 1, ...q$ are the Pochammer symbols.

**Remark 1.** The above series (4), is valid when none of the denominator parameter $v_k$; $\forall k = 1, 2, .., q$, is a zero or nonnegative integer. If any numerator parameter $u_k$; $\forall k = 1, 2, .., p$, zero or nonnegative integer, then the series (4), reduces to polynomial in one variable $w$ with certain degree.

For series (4), the following below are the convergence conditions:

- If $p < q + 1$, the series (4) converges for $\forall$ finite $w$.
- If $p = q + 1$, the series (4) diverges for $|w| > 1$ and converges $\forall$ $|w| < 1$.
- If $p > q + 1$, the series (4) diverges for $w \neq 0$.
- When $p = q + 1$ and $|w| = 1$, the series (4) can divergent on conditions depending on parameters and absolutely convergent on the circle $|w| = 1$, if $\Re\left(\xi \sum_{k=1}^{q} v_k - \sum_{k=1}^{q} u_k\right) > 0$.

**Remark 2.** (i) If we substitute $\zeta = \xi = 1$ and $m = 1$, then series (4), reduced to $M$-series defined in (1) as follows: ${}_pG_q^{\eta,1,1,1}(w) = {}_pM_q^{\eta}(w)$;

(ii) If we put $\xi = 1$ and $m = 1$, then series (4), reduced to $M$-series defined in (2) as follows: ${}_pG_q^{\eta,\zeta,1,1}(w) = {}_pM_q^{\eta,\zeta}(w)$;

(iii) If we consider $m = 1$, then series (4), reduced to $K$-function given in (3) as follows: ${}_pG_q^{\eta,\zeta,1,\xi}(w) = {}_pK_q^{\eta,\zeta,\xi}(w)$.

## 2. SPECIAL CASES

Many well known special functions can be obtained from ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function with the above mentioned convergence conditions. Here, we derive relations that exist between ${}_pG_q^{\eta,\zeta,m,\xi}(z)$ function with well known special functions like generalised hypergeometric function, Mittag-Leffler functions and Wright hypergeometric function and Fox H-function.

### *2.1. Relation to Generalised Hypergeometric Function*

Assume $\eta = 1 = \zeta = \xi$ and $m = 1$, then series (4) reduced to generalized hypergeometric function defined in [4]

$$ {}_pG_q^{1,1,1,1}(w) = {}_pG_q^{\eta,\zeta,m,\xi}(u_1, ..., u_p, v_1, ..., v_q; w) $$
$$ = \sum_{k=0}^{\infty} \frac{(u_1)_k....(u_p)_k(1)_k}{(v_1)_k...(v_q)_k\Gamma(k+1)} \frac{w^k}{k!} = {}_pF_q \left[ w \,\middle|\, \begin{matrix} u_1, ..., u_p \\ v_1, ..., v_q \end{matrix} \right]. \tag{5} $$

From (5) we can say that ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function (4), is the extension of generalised hypergeometric function ${}_pF_q(w)$.

**2.1.1. Relation to Gauss hypergeometric function.** Consider $|w| < 1$, $p = 2$ and $q = 1$, then (5), reduced to Gauss hypergeometric function ${}_2F_1(w)$ given in [5]

$$ {}_2G_1^{1,1,1,1}(w) = {}_2G_1^{1,1,1,1}(u_1, u_2, v_1; w) = \sum_{k=0}^{\infty} \frac{(u_1)_k(u_2)_k(1)_k}{(v_1)_k\Gamma(k+1)} \frac{w^k}{k!} = {}_2F_1(u_1, u_2, v_1; w). $$

**2.1.2. Relation to confluent hypergeometric function.** Let $p = 1$ and $q = 1$, then (5), reduced to confluent hypergeometric function ${}_1\Phi_1(w)$ studied in [5]

$$ {}_1G_1^{1,1,1,1}(w) = {}_1G_1^{1,1,1,1}(u_1, v_1; w) = \sum_{k=0}^{\infty} \frac{(u_1)(1)_k}{(v_1)_k\Gamma(k+1)} \frac{w^k}{k!} = {}_1\Phi_1(u_1, v_1; w). $$

### *2.2. Relation to Mittag-Leffler Functions*

We can also noticed that ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function is also generalization of Mittag-Leffler functions studied in [6–9].

Consider $p = 0 = q$, then (4), reduce to four parameter Mittag-Leffler function $E_{\eta,\zeta}^{\xi,m}(w)$ defined and studied by Shukla and Prajapati in [9]

$$ {}_0G_0^{\eta,\zeta,m,\xi}(w) = {}_0G_0^{\eta,\zeta,m,\xi}(-,-; w) = \sum_{k=0}^{\infty} \frac{(\xi)_{mk}}{\Gamma(\eta k + \zeta)} \frac{w^k}{k!} = E_{\eta,\zeta}^{\xi,m}(w). $$

In addition with above conditions, if we put $m = 1$, then (4), becomes three parameter Mittag-Leffler function $E_{\eta,\zeta}^{\xi}(w)$ studied by Prabhakar [8]

$$ {}_0G_0^{\eta,\zeta,1,\xi}(w) = {}_0G_0^{\eta,\zeta,1,\xi}(-,-; w) = \sum_{k=0}^{\infty} \frac{(\xi)_k}{\Gamma(\eta k + \zeta)} \frac{w^k}{k!} = E_{\eta,\zeta}^{\xi}(w). $$

If we further substitute $\xi = 1$, we get Wiman's function $E_{\eta,\zeta}(w)$ defined in [7] as follows

$$ {}_0G_0^{\eta,\zeta,1,1}(w) = {}_0G_0^{\eta,\zeta,1,1}(-,-; w) = \sum_{k=0}^{\infty} \frac{(1)_k}{\Gamma(\eta k + \zeta)} \frac{w^k}{k!} = E_{\eta,\zeta}(w). $$

Likewise if put $\zeta = 1$, then (4), reduce to classical Mittag-Leffler function $E_\eta(w)$ given in [6]

$$ {}_0G_0^{\eta,1,1,1}(w) = {}_0G_0^{\eta,1,1,1}(-,-; w) \sum_{k=0}^{\infty} \frac{(1)_k}{\Gamma(\eta k + 1)} \frac{w^k}{k!} = E_\eta(w). $$

In the sequence if we taking $\eta = 1$, then (4) becomes exponential function $e^w$ as follows

$$ {}_0G_0^{1,1,1,1}(w) = {}_0G_0^{1,1,1,1}(-,-; w) \sum_{k=0}^{\infty} \frac{(1)_k}{\Gamma k + 1)} \frac{w^k}{k!} = e^w. $$

### 2.3. Relation to Wright Hypergeometric Function

From the definition of ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function with the convergence condition, we can observe that it is closely related to Wright hypergeometric function given [10] as follows

$$ {}_pG_q^{\eta,\zeta,m,\xi}(w) = {}_pG_q^{\eta,\zeta,m,\xi}(u_1,...,u_p,v_1,...,v_q;w) = \Omega . {}_{p+1}\Psi_{q+1}\left[ w \left| \begin{matrix} (u_1,1),\cdots,(u_p,1),(\xi,m) \\ (v_1,1),\cdots,(v_q,1),(\zeta,\eta) \end{matrix} \right. \right], $$

where $\Omega = \frac{\Gamma(v_1),\cdots,\Gamma(v_q)}{\Gamma(u_1),\cdots,\Gamma(u_p)}$

### 2.4. Relation to Fox H-Function

Consider ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function with the convergence condition, we can found that it is related to Fox $H$-function defined in [11] as follows

$$ {}_pG_q^{\eta,\zeta,m,\xi}(w) ={}_p G_q^{\eta,\zeta,m,\xi}(u_1,...,u_p,v_1,...,v_q;w) = \Omega, $$

$$ H^{1,p+1}_{p+1,q+2}\left[ -w \left| \begin{matrix} (1-u_1,1),...,(1-u_p,1),(1-\xi,m) \\ (0,1),(1-v_1,1),...,(1-v_q,1),(1-\zeta,\eta) \end{matrix} \right. \right], $$

where $\Omega = \frac{\Gamma(v_1),\cdots,\Gamma(v_q)}{\Gamma(u_1),\cdots,\Gamma(u_p)}$.

### 2.5. Some Properties of ${}_pG_q^{\eta,\zeta,m,\xi}(w)$

**Theorem 1.** *The following integral representation for ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function holds true*

$$ {}_pG_q^{\eta,\zeta,m,\xi}(u_1,...,u_p,v_1,...,v_q;w) = \frac{\Gamma(v_1)}{\Gamma(u_1)\Gamma(v_1-u_1)} $$

$$ \times \int_0^1 t^{u_1-1}(1-t)^{v_1-u_1-1}{}_pG_q^{\eta,\zeta,m,\xi}(u_2,...,u_p,v_2,...,v_q;wt)dt, $$

*where $\eta,\zeta,\xi \in \mathbb{C}$ such that $\Re(\eta),\Re(\zeta),\Re(\xi) > 0$ and $(u_i)_k$ and $(v_j)_k$, $i = 1,..p$, $j = 1,...q$ are the Pochammer symbols.*

**Proof.** From the properties of Pochammer symbol given in [4], we have

$$ \frac{(u_1)_n}{(v_1)_n} = \frac{\Gamma(u_1+n)\Gamma(v_1)}{\Gamma(v_1+n)\Gamma(u_1)} = \frac{\Gamma(u_1+n)\Gamma(v_1)\Gamma(v_1-u_1)}{\Gamma(v_1+n)\Gamma(u_1)\Gamma(v_1-u_1)}. $$

By using the relation between beta and gamma function, we get

$$ \frac{(u_1)_n}{(v_1)_n} = \frac{\Gamma(v_1)}{\Gamma(u_1)\Gamma(v_1-u_1)}B(u_1+n, v_1-u_1), $$

where $\Re(v_1) > \Re(u_1)$. Then, using the integral representation of beta function [4], we have

$$ \frac{(u_1)_n}{(v_1)_n} == \frac{\Gamma(v_1)}{\Gamma(u_1)\Gamma(v_1-u_1)} \int_0^1 t^{u_1+n-1}(1-t)^{v_1-u_1-1}dt. $$

Substituting the value of $\frac{(u_1)_n}{(v_1)_n}$ in definition of ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function (4), we get

$$ {}_pG_q^{\eta,\zeta,m,\xi}(u_1,...,u_p,v_1,...,v_q;w) = \sum_{n=0}^{\infty} \frac{(u_2)_n....(u_p)_n(\xi)_{mn}}{(v_2)_n...(v_q)_n\Gamma(\eta n+\zeta)} $$

$$\times\left\{\frac{\Gamma(v_1)}{\Gamma(u_1)\Gamma(v_1-u_1)}\int\limits_0^1 t^{u_1+n-1}(1-t)^{v_1-u_1-1}dt\right\}\frac{w^n}{n!}.$$

On interchanging order of summation and integration with some calculation, we have

$${}_pG_q^{\eta,\zeta,m,\xi}(u_1,...,u_p,v_1,...,v_q;w)=\frac{\Gamma(v_1)}{\Gamma(u_1)\Gamma(v_1-u_1)}\int\limits_0^1 t^{u_1-1}(1-t)^{v_1-u_1-1}$$

$$\times\left(\sum_{n=0}^{\infty}\frac{(u_2)_n....(u_p)_n(\xi)_{mn}}{(v_2)_n...(v_q)_n\Gamma(\eta n+\zeta)}\frac{(wt)^n}{n!}\right)dt.$$

By the definition of ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function (4), we get our desired result

$${}_pG_q^{\eta,\zeta,m,\xi}(u_1,...,u_p,v_1,...,v_q;w)=\frac{\Gamma(v_1)}{\Gamma(u_1)\Gamma(v_1-u_1)}$$

$$\times\int\limits_0^1 t^{u_1-1}(1-t)^{v_1-u_1-1}{}_pG_q^{\eta,\zeta,m,\xi}(u_2,...,u_p,v_2,...,v_q;wt)dt.$$

□

**Theorem 2.** *For ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function the following derivative formula holds true*

$$\left(\frac{d}{dw}\right)^m{}_pG_q^{\eta,\zeta,k,\xi}(w)=\left(\frac{d}{dw}\right)^m{}_pG_q^{\eta,\zeta,k,\xi}\left[w\left|\begin{matrix}u_1,...,u_p\\ v_1,...,v_q\end{matrix}\right.\right]=\Psi.{}_pG_q^{\eta,\zeta+\eta m,k,\xi+km},$$

$$\left[u\left|\begin{matrix}u_1+m,...,u_p+m\\ v_1+m,...,v_q+m\end{matrix}\right.\right],$$

*where* $\Psi=\frac{(u_1)_m....(u_p)_m(\xi)_{km}}{(v_1)_m....(v_q)_m}$.

**Proof.** From the definition of ${}_pG_q^{\eta,\zeta,k,\xi}(w)$, we have

$$\left(\frac{d}{dw}\right)^m{}_pG_q^{\eta,\zeta,k,\xi}(w)=\left(\frac{d}{dz}\right)^m\sum_{n=0}^{\infty}\frac{(u_1)_n....(u_p)_n(\xi)_{nk}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta)}\frac{w^n}{n!}$$

$$=\sum_{n=m}^{\infty}\frac{(u_1)_n....(u_p)_n(\xi)_{nk}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta)}\frac{w^{n-m}}{(n-m)!}.$$

Then, replace $n$ by $n+m$, we get

$$\left(\frac{d}{dw}\right)^m{}_pG_q^{\eta,\zeta,k,\xi}(w)=\sum_{n=0}^{\infty}\frac{(u_1)_{n+m}....(u_p)_{n+m}(\xi)_{nk+km}}{(v_1)_{n+m}...(v_q)_{n+m}\Gamma(\eta n+\eta m+\zeta)}\frac{u^n}{n!}.$$

By using the property of Pochammer symbol $(a)_{n+m}=(a)_m(a+m)_n$, we have

$$\left(\frac{d}{dw}\right)^m{}_pG_q^{\eta,\zeta,k,\xi}(w)=\frac{(u_1)_m....(u_p)_m(\xi)_{km}}{(v_1)_m....(v_q)_m}\times\sum_{n=0}^{\infty}\frac{(u_1+m)_n....(u_p+m)_n(\xi+km)_{kn}}{(v_1+m)_n...(v_q+m)_n\Gamma(\eta n+\zeta+\eta m)}\frac{w^n}{(n)!}.$$

By using the equation (4) and substitute $\Psi=\frac{(u_1)_m....(u_p)_m(\xi)_{km}}{(v_1)_m....(v_q)_m}$, we get our desired result

$$\left(\frac{d}{dw}\right)^m{}_pG_q^{\eta,\zeta,k,\xi}(w)=\left(\frac{d}{dz}\right)^m{}_pG_q^{\eta,\zeta,k,\xi}\left[w\left|u_1,...,u_pv_1,...,v_q\right.\right]$$

$$= \Psi._{p}G_{q}^{\eta,\zeta+\eta m,k,\xi+km}, \left[ u \middle| \begin{matrix} u_1+m,...,u_p+m \\ v_1+m,...,v_q+m \end{matrix} \right].$$

□

**Corollary 1.** *For* ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ *function the result holds true*

$$\frac{d}{dw}{}_pG_q^{\eta,\zeta,k,\xi}(w) = \frac{d}{dz}{}_pG_q^{\eta,\zeta,k,\xi}\left[ w \middle| \begin{matrix} u_1,...,u_p \\ v_1,...,v_q \end{matrix} \right] = \frac{u_1....u_p(\xi)_k}{v_1....v_q}{}_pG_q^{\eta,\zeta+\eta,k,\xi+k}\left[ w \middle| \begin{matrix} u_1+1,...,u_p+1 \\ v_1+1,...,v_q+1 \end{matrix} \right].$$

**Theorem 3.** *The following differential property holds true*

$${}_pG_q^{\eta,\zeta,k,\xi}(w) = \zeta{}_pG_q^{\eta,\zeta+1,k,\xi}(w) + \eta w\frac{d}{dw}{}_pG_q^{\eta,\zeta+1,k,\xi}(w).$$

**Proof.** From the above Theorem 2, we have

$$\eta w\frac{d}{dw}{}_pG_q^{\eta,\zeta+1,k,\xi}(w) = \sum_{n=0}^{\infty}\frac{\eta.n.(u_1)_n....(u_p)_n(\xi)_{kn}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta+1)}\frac{w^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{(\eta.n+\zeta-\zeta).(u_1)_n....(u_p)_n(\xi)_{kn}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta+1)}\frac{w^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{(\eta.n+\zeta).(u_1)_n....(u_p)_n(\xi)_{kn}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta+1)}\frac{w^n}{n!} - \sum_{n=0}^{\infty}\frac{\zeta.(u_1)_n....(u_p)_n(\xi)_{kn}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta+1)}\frac{w^n}{n!}.$$

Then, using the equation (4), we have

$$\eta z\frac{d}{dw}{}_pG_q^{\eta,\zeta+1,k,\xi}(w) = \sum_{n=0}^{\infty}\frac{(\eta.n+\zeta).(u_1)_n....(u_p)_n(\xi)_{kn}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta+1)}\frac{w^n}{n!} - \zeta._pG_q^{\eta,\zeta+1,k,\xi}(w).$$

On re-arranging the terms, we get

$$\eta z\frac{d}{dw}{}_pG_q^{\eta,\zeta+1,k,\xi}(w) + \zeta._pG_q^{\eta,\zeta+1,k,\xi}(w) = \sum_{n=0}^{\infty}\frac{(\eta.n+\zeta).(u_1)_n....(u_p)_n(\xi)_{kn}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta+1)}\frac{w^n}{n!}.$$

By using the definition of ${}_pG_q^{\eta,\zeta,k,\xi}(w)$ function, we get our desired result

$${}_pG_q^{\eta,\zeta,k,\xi}(w) = \zeta{}_pG_q^{\eta,\zeta+1,k,\xi}(w) + \eta w\frac{d}{dz}{}_pG_q^{\eta,\zeta+1,k,\xi}(w).$$

□

### *2.6. Integral Transforms of* ${}_pG_q^{\eta,\zeta,m,\xi}(w)$

**Theorem 4.** *The Euler beta transform for* ${}_pG_q^{\eta,\zeta,k,\xi}(w)$ *is given by*

$$\int_0^1 w^{a-1}(1-w)^{b-1}{}_pG_q^{\eta,\zeta,k,\xi}(a,b;xw^\sigma)dw$$

$$= \frac{\Gamma(b)\Gamma(v_1)...\Gamma(v_q)}{\Gamma(\xi)\Gamma(u_1)...\Gamma(u_p)}{}_{p+2}\Psi_{q+2}\left[ x \middle| \begin{matrix} (u_1,1),\cdots,(u_p,1),(\xi,k),(a,\sigma) \\ (v_1,1),\cdots,(v_q,1),(\zeta,\eta),(a+b,\sigma) \end{matrix} \right].$$

**Proof.** Apply the definition of Euler beta transform to ${}_pG_q^{\eta,\zeta,k,\xi}(w)$ defined in [12], we have

$$\int_0^1 w^{a-1}(1-w)^{b-1}{}_pG_q^{\eta,\zeta,k,\xi}(a,b;xw^\sigma)dw$$

$$= \int_0^1 w^{a-1}(1-w)^{b-1} \sum_{n=0}^{\infty} \frac{(u_1)_n....(u_p)_n(\xi)_{nk}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta)} \frac{(xw^\sigma)^n}{n!} dw.$$

On interchanging order of summation and integration, we have

$$\int_0^1 w^{a-1}(1-w)^{b-1}{}_pG_q^{\eta,\zeta,k,\xi}(a,b;xw^\sigma)dw$$

$$= \sum_{n=0}^{\infty} \frac{(u_1)_n....(u_p)_n(\xi)_{nk}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta)} \frac{(x)^n}{n!} \int_0^1 w^{a+\sigma.n-1}(1-w)^{b-1}dw.$$

Then, by definition of classical Euler beta function defined in [4], we get

$$\int_0^1 w^{a-1}(1-w)^{b-1}{}_pG_q^{\eta,\zeta,k,\xi}(a,b;xw^\sigma)dw = \sum_{n=0}^{\infty} \frac{(u_1)_n....(u_p)_n(\xi)_{nk}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta)} \frac{(x)^n}{n!} B(a+\sigma.n,b).$$

By using the results of Pochammer symbol and gamma function given in [4], we have

$$\int_0^1 w^{a-1}(1-w)^{b-1}{}_pG_q^{\eta,\zeta,k,\xi}(a,b;xw^\sigma)dw$$

$$= \frac{\Gamma(b)\Gamma(v_1)...\Gamma(v_q)}{\Gamma(\xi)\Gamma(u_1)...\Gamma(u_p)} \sum_{n=0}^{\infty} \frac{\Gamma(u_1+n)...\Gamma(u_p+n)\Gamma(\xi+kn)\Gamma(a+\sigma n)}{\Gamma(v_1+n)...\Gamma(v_q+n)\Gamma(\zeta+\eta n)\Gamma(a+b+\sigma n)} \frac{(x)^n}{n!}.$$

By using the definition of Wright hypergeometric function [10], we get our result of Theorem 4

$$\int_0^1 w^{a-1}(1-w)^{b-1}{}_pG_q^{\eta,\zeta,k,\xi}(a,b;xw^\sigma)dw$$

$$= \frac{\Gamma(b)\Gamma(v_1)...\Gamma(v_q)}{\Gamma(\xi)\Gamma(u_1)...\Gamma(u_p)} {}_{p+2}\Psi_{q+2}\left[x \middle| \begin{matrix} (u_1,1),\cdots,(u_p,1),(\xi,k),(a,\sigma) \\ (v_1,1),\cdots,(v_q,1),(\zeta,\eta),(a+b,\sigma) \end{matrix}\right].$$

□

**Theorem 5.** *The Laplace transform for ${}_pG_q^{\eta,\zeta,k,\xi}(w)$ is given by*

$$\int_0^\infty w^{a-1}e^{-sw}{}_pG_q^{\eta,\zeta,k,\xi}(a,b;xw^\sigma)dw = \frac{s^{-a}\Gamma(v_1)...\Gamma(v_q)}{\Gamma(\xi)\Gamma(u_1)...\Gamma(u_p)}$$

$$\times {}_{p+2}\Psi_{q+1}\left[\frac{x}{s^\sigma} \middle| \begin{matrix} (u_1,1),\cdots,(u_p,1),(\xi,k),(a,\sigma) \\ (v_1,1),\cdots,(v_q,1),(\zeta,\eta) \end{matrix}\right].$$

**Proof.** Apply the definition of Laplace transform to ${}_pG_q^{\eta,\zeta,k,\xi}(w)$, we have

$$\int_0^\infty w^{a-1}w^{a-1}e^{-sw}{}_pG_q^{\eta,\zeta,k,\xi}(a,b;xw^\sigma)dw = \int_0^1 w^{a-1}e^{-sw} \sum_{n=0}^{\infty} \frac{(u_1)_n....(u_p)_n(\xi)_{nk}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta)} \frac{(xw^\sigma)^n}{n!} dw.$$

On interchanging order of summation and integration, we have

$$\int_0^\infty w^{a-1}e^{-sw}{}_pG_q^{\eta,\zeta,k,\xi}(a,b;xw^\sigma)dw = \sum_{n=0}^{\infty} \frac{(u_1)_n....(u_p)_n(\xi)_{nk}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta)} \frac{(x)^n}{n!} \int_0^1 w^{a+\sigma n-1}e^{-sw}dw.$$

Then, by definition of Laplace transform given in [12], we get

$$\int_0^\infty w^{a-1}w^{a-1}e^{-sw}{}_pG_q^{\eta,\zeta,k,\xi}(a,b;xw^\sigma)dw=\sum_{n=0}^\infty \frac{s^{-a}(u_1)_n....(u_p)_n(\xi)_{nk}\Gamma(a+\sigma n)}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta)}\frac{(\frac{x}{s^\sigma})^n}{n!}.$$

By using the results of Pochammer symbol and gamma function given in [4] and definition of Wright hypergeometric function [10], we get our desired result

$$\int_0^\infty w^{a-1}e^{-sw}{}_pG_q^{\eta,\zeta,k,\xi}(a,b;xw^\sigma)dw$$

$$=\frac{s^{-a}\Gamma(v_1)...\Gamma(v_q)}{\Gamma(\xi)\Gamma(u_1)...\Gamma(u_p)}{}_{p+2}\Psi_{q+1}\left[\frac{x}{s^\sigma}\middle|\begin{matrix}(u_1,1),\cdots,(u_p,1),(\xi,k),(a,\sigma)\\(v_1,1),\cdots,(v_q,1),(\zeta,\eta)\end{matrix}\right].$$

□

**Theorem 6.** *The Whittaker transform for ${}_pG_q^{\eta,\zeta,k,\xi}(z)$ is given by*

$$\int_0^\infty z^{a-1}e^{\frac{-bz}{2}}W_{\lambda,\mu}(bz){}_pG_q^{\eta,\zeta,k,\xi}(a,b;wz^\delta)dz$$

$$=\frac{b^{-a}\Gamma(v_1)...\Gamma(v_q)}{\Gamma(\xi)\Gamma(u_1)...\Gamma(u_p)}{}_{p+3}\Psi_{q+2}\left[\frac{w}{b^\delta}\middle|\begin{matrix}(u_1,1),\cdots,(u_p,1),(\xi,k),(\frac{1}{2}\pm\mu+a,\delta)\\(v_1,1),\cdots,(v_q,1),(\zeta,\eta),(1-\lambda+a,\delta)\end{matrix}\right]. \quad (6)$$

**Proof.** Consider the L.H.S of the equation (6), and using (4), we have

$$\int_0^\infty z^{a-1}e^{\frac{-bz}{2}}W_{\lambda,\mu}(bz){}_pG_q^{\eta,\zeta,k,\xi}(a,b;wz^\delta)dz$$

$$=\int_0^\infty z^{a-1}e^{\frac{-bz}{2}}W_{\lambda,\mu}(bz)\sum_{n=0}^\infty\frac{(u_1)_n....(u_p)_n(\xi)_{nk}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta)}\frac{(wz^\delta)^n}{n!}dz.$$

Then, substituting $bz=v$ in the above equation and re-arranging the terms, we get

$$\int_0^\infty z^{a-1}e^{\frac{-bz}{2}}W_{\lambda,\mu}(bz){}_pG_q^{\eta,\zeta,k,\xi}(a,b;wz^\delta)dz$$

$$=\int_0^\infty v^{a-1}e^{\frac{-v}{2}}W_{\lambda,\mu}(v)b^{-a}\sum_{n=0}^\infty\frac{(u_1)_n....(u_p)_n(\xi)_{nk}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta)}\frac{(\frac{w}{b^\delta})^n v^{\delta n}}{n!}dv.$$

On interchanging order of summation and integration, we have

$$\int_0^\infty z^{a-1}e^{\frac{-bz}{2}}W_{\lambda,\mu}(bz){}_pG_q^{\eta,\zeta,k,\xi}(a,b;wz^\delta)dz$$

$$=b^{-a}\sum_{n=0}^\infty\frac{(u_1)_n....(u_p)_n(\xi)_{nk}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta)}\frac{(\frac{w}{b^\delta})^n}{n!}\int_0^\infty v^{a+\delta n-1}e^{\frac{-v}{2}}W_{\lambda,\mu}(v)dv. \quad (7)$$

To get our result, we use the following integral

$$\int_0^\infty v^{a+\delta n-1}e^{\frac{-v}{2}}W_{\lambda,\mu}(v)dv=\frac{\Gamma(\frac{1}{2}+\mu+a+\delta n)\Gamma(\frac{1}{2}-\mu+a+\delta n)}{\Gamma(1-\lambda+a+\delta n)}. \quad (8)$$

Then, by using above result (8) in the equation (7), we get

$$\int_0^\infty z^{a-1} e^{\frac{-bz}{2}} W_{\lambda,\mu}(bz)\,{}_pG_q^{\eta,\zeta,k,\xi}(a,b;wz^\delta)dz$$

$$= b^{-a} \sum_{n=0}^\infty \frac{(u_1)_n....(u_p)_n(\xi)_{nk}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta)} \frac{(\frac{w}{b^\delta})^n}{n!} \frac{\Gamma(\frac{1}{2}+\mu+a+\delta n)\Gamma(\frac{1}{2}-\mu+a+\delta n)}{\Gamma(1-\lambda+a+\delta n)}.$$

By using the results of Pochammer symbol and gamma function given in [4] and definition of Wright hypergeometric function [10], we get our desired result

$$\int_0^\infty z^{a-1} e^{\frac{-bz}{2}} W_{\lambda,\mu}(bz)\,{}_pG_q^{\eta,\zeta,k,\xi}(a,b;wz^\delta)dz$$

$$= \frac{b^{-a}\Gamma(v_1)...\Gamma(v_q)}{\Gamma(\xi)\Gamma(u_1)...\Gamma(u_p)}\,{}_{p+3}\Psi_{q+2}\left[\frac{w}{b^\delta}\middle|\begin{matrix}(u_1,1),\cdots,(u_p,1),(\xi,k),\left(\frac{1}{2}\pm\mu+a,\delta\right)\\(v_1,1),\cdots,(v_q,1),(\zeta,\eta),(1-\lambda+a,\delta)\end{matrix}\right].$$

□

### *2.7. Fractional Integral and Derivative of ${}_pG_q^{\eta,\zeta,m,\xi}(w)$*

**Theorem 7.** *Consider $u>0$ and $\eta,\zeta,\xi\in\mathbb{C}$ such that $\Re(\eta),\Re(\zeta),\Re(\xi)>0$, $k\in(0,1)\cup\mathbb{N}$, then following result holds true*

$$I_w^u\left\{{}_pG_q^{\eta,\zeta,k,\xi}\left[w\middle|\begin{matrix}u_1,...,u_p\\v_1,...,v_q\end{matrix}\right]\right\} = \frac{w^u}{\Gamma(u+1)}\,{}_{p+1}G_{q+1}^{\eta,\zeta,k,\xi}\left[w\middle|\begin{matrix}u_1,...,u_p,1\\v_1,...,v_q,u+1\end{matrix}\right], \tag{9}$$

*where $I_w^u$ be the Riemann fractional integral operator defined in [13].*

**Proof.** Consider the L.H.S of the equation (9) and using definition of ${}_pG_q^{\eta,\zeta,k,\xi}(w)$ (4), we get

$$I_w^u\left\{{}_pG_q^{\eta,\zeta,k,\xi}\left[w\middle|\begin{matrix}u_1,...,u_p\\v_1,...,v_q\end{matrix}\right]\right\} = I_w^u\left\{\sum_{n=0}^\infty \frac{(u_1)_n....(u_p)_n(\xi)_{nk}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta)}\frac{w^n}{n!}\right\}.$$

Interchanging order of integration and summation, we have

$$I_w^u\left\{{}_pG_q^{\eta,\zeta,k,\xi}\left[w\middle|\begin{matrix}u_1,...,u_p\\v_1,...,v_q\end{matrix}\right]\right\} = \sum_{n=0}^\infty \frac{(u_1)_n....(u_p)_n(\xi)_{nk}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta)}\frac{1}{n!}I_w^u\{w^n\}. \tag{10}$$

Then, using the following result, we have

$$I_w^u\{w^n\} = \frac{\Gamma(n+1)}{\Gamma(n+1+u)}w^{n+u}.$$

Then, using the above result (2.2.7), in the equation (10), we get

$$I_w^u\left\{{}_pG_q^{\eta,\zeta,k,\xi}\left[w\middle|\begin{matrix}u_1,...,u_p\\v_1,...,v_q\end{matrix}\right]\right\} = \sum_{n=0}^\infty \frac{(u_1)_n....(u_p)_n(\xi)_{nk}}{(v_1)_n...(v_q)_n\Gamma(\eta n+\zeta)}\frac{1}{n!}\frac{\Gamma(n+1)}{\Gamma(n+1+u)}w^{n+u}.$$

By using the results of Pochammer symbol and gamma function given in [4] and Definition 1, we get our desired result

$$I_w^u\left\{{}_pG_q^{\eta,\zeta,k,\xi}\left[w\middle|\begin{matrix}u_1,...,u_p\\v_1,...,v_q\end{matrix}\right]\right\} = \frac{w^u}{\Gamma(u+1)}\,{}_{p+1}G_{q+1}^{\eta,\zeta,k,\xi}\left[w\middle|\begin{matrix}u_1,...,u_p,1\\v_1,...,v_q,u+1\end{matrix}\right].$$

□

**Theorem 8.** *Consider $u > 0$ and $\eta, \zeta, \xi \in \mathbb{C}$ such that $\Re(\eta), \Re(\zeta), \Re(\xi) > 0$, $k \in (0,1) \cup \mathbb{N}$, then following result holds true*

$$D_w^u \left\{ {}_pG_q^{\eta,\zeta,k,\xi} \left[ w \middle| \begin{matrix} u_1, ..., u_p \\ v_1, ..., v_q \end{matrix} \right] \right\} = \frac{w^{-u}}{\Gamma(1-u)} {}_{p+1}G_{q+1}^{\eta,\zeta,k,\xi} \left[ w \middle| \begin{matrix} u_1, ..., u_p, 1 \\ v_1, ..., v_q, 1-u \end{matrix} \right]. \tag{11}$$

*where $D_w^u$ be the Riemann fractional derivative operator defined in [13].*

**Proof.** With same parallel line of proof of Theorem 7,we get our result of Theorem 8. □

**Remark 3.** From above Theorems 7 and 8, we notice that Riemann fractional integral and derivative of ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function is again the same function with indices $p+1$ and $q+1$.

## 3. CONCLUSIONS

In this paper, firstly we have defined a new class of hypergeometric function which is called ${}_pG_q^{\eta,\zeta,m,\xi}(z)$. Then, we have discussed several basic properties like differential property, integral representation, derivative formula and some integral transforms, Euler transforms, Laplace transforms, Whittaker transforms etc and also derived the relations that exist between ${}_pG_q^{\eta,\zeta,m,\xi}(z)$ function with classical functions. Finally we have obtained Riemann–Liouville fractional integrals and derivatives operators of the ${}_pG_q^{\eta,\zeta,m,\xi}(z)$ function. We can also use our results for further investigations of the following results given in the papers [14–16].

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