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On the Dynamics of Solitary Waves to a (3+1)-Dimensional Stochastic Boiti–Leon–Manna–Pempinelli Model in Incompressible Fluid

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Abstract: We take into account the stochastic Boiti–Leon–Manna–Pempinelli equation (SBLMPE), which is perturbed by a multiplicative Brownian motion. By applying He's semi-inverse method and the Riccati equation mapping method, we can acquire the solutions of the SBLMPE. Since the Boiti–Leon–Manna–Pempinelli equation is utilized to explain incompressible liquid in fluid mechanics, the acquired solutions may be applied to explain a lot of fascinating physical phenomena. To address how Brownian motion effects the exact solutions of the SBLMPE, we present some 2D and 3D diagrams.

Keywords: stochastic Boiti–Leon–Manna–Pempinelli equation; Riccati equation mapping method; analytical stochastic solutions

MSC: 60H10; 60H15; 35Q51; 83C15; 35A20

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1. Introduction

Nonlinear wave propagation is one of the most significant natural phenomena, and there is a developing interest in studying nonlinear waves in dynamical systems. Nonlinear evolution equations (NEEs) have many applications in engineering and the sciences, such as in plasma physics, astrophysics, electrochemistry, cosmology, fluid dynamics, acoustics, and electromagnetic theory. In soliton theory, the issue of solitary wave solutions for NEEs is addressed by a number of well-known techniques, including (G'/G)-expansion [1,2], the tanh–sech method [3,4], the $exp(-\phi(\varsigma))$-expansion method [5], the extended auxiliary function [6], ansatz method [7], Hirota's method [8], the perturbation method [9,10], and the Jacobi elliptic function [11], etc.

Moreover, real-world phenomena are not typically deterministic, as actual systems cannot be completely isolated from their surroundings. For this reason, stochastic effects must consider these differential equations. The equations that are impacted by random factors are called stochastic partial differential equations (SPDEs). There has been a lot of study on these types of models in recent decades. SPDEs are used more and more in materials sciences, finance, information systems, biophysics, mechanical and electrical engineering, condensed matter climate, and physics system modeling to develop mathematical models of complex phenomena [12–14]. Recently, the exact solutions for some SPDEs, for instance, [15–20], have been obtained.

For this reason, stochastic effects in PDEs must be taken into consideration. Here, we consider the (3+1)-dimensional stochastic Boiti–Leon–Manna–Pempinelli equation (SBLMPE), induced by multiplicative noise:

$$d(\mathcal{B}_y+\mathcal{B}_z)+[\mathcal{B}_{yxxx}+\mathcal{B}_{zxxx}-3(\mathcal{B}_x(\mathcal{B}_y+\mathcal{B}_z))_x]dt=\gamma(\mathcal{B}_y+\mathcal{B}_z)dW, \tag{1}$$

where $\mathcal{B}(x,y,z,t)$ is an analytic function; W is Brownian motion (BM); γ represents the noise intensity. The BM $\{W(\tau)\}_{\tau\geq 0}$ is a stochastic process and satisfies the following: (1) $W(\tau)$ is continuous for $\tau\geq 0$; (2) $W(\tau_2)-W(\tau_1)$ is independent for $\tau_2>\tau_1$; (3) $W(0)=0$; (4) $W(\tau_2)-W(\tau_1)$ has a Gaussian distribution with mean 0 and variance $\tau_2-\tau_1$ for $\tau_2>\tau_1$.

Boiti et al. [21] discovered Equation (1) with $\gamma = 0$ while investigating a Korteweg–de Vries equation using weak Lax pair relations. It is employed to explain incompressible liquid in fluid mechanics. As a result, a lot of authors have studied different analytical solutions to Equation (1) with $\gamma=0$, including the extended tanh function [22], the Bäcklund transformation method [23], the modified exponential function [24], the auxiliary equation method [25], Hirota's direct method [26], ansatz functions and bilinear form [27], (G'/G)-expansion [28], He's semi-inverse method [29], the general bilinear form [30], and the modified hyperbolic tangent function [31]. Moreover, the analytical solutions of the (2+1)-dimensional BLMP equation were acquired in works [32–35] and the integrability of the (2+1)-dimensional BLMP equation was proved in [36]. Meanwhile, the stochastic solutions to Equation (1) have not addressed until now.

Our novelty of this study is to acquire the analytical stochastic solutions of SBLMPE (1) forced by Brownian motion. To obtain these solutions, we utilize two methods, namely the generalizing Riccati equation mapping method and He's semi-inverse method. The stochastic term in Equation (1) makes the solutions very accurate and useful for characterizing various important physical processes, and physicists would do well to consider them. Moreover, we offer some of graphs in MATLAB to explore the influence of noise on the solution of SBLMPE (1). To the best of our knowledge, this equation has never been investigated with a random term.

This is a brief summary of this article: The wave equation of SBLMPE (1) is obtained in Section 2. Achieving exact solutions for the SBLMPE is the focus of Section 3. In Section 4, we examine how Brownian motion effects the solutions of SBLMPE. Finally, the paper's conclusions are laid out.

2. Wave Equation for SBLMPE

The next transformation is employed to derive the SBLMPE (1) wave equation:

$$\mathcal{B}(x,y,z,t)=\mathcal{V}(\zeta)e^{(\gamma W(t)-\frac{1}{2}\gamma^2 t)},\ \zeta=\zeta_1 x+\zeta_2 y+\zeta_3 z+\zeta_4 t, \tag{2}$$

where the function $\mathcal{V}$ is deterministic and continuous in its domain; ζ_1, ζ_2, ζ_3 and ζ_4 are undefined constants. We observe that

$$\begin{aligned}
\mathcal{B}_x &= \zeta_1\mathcal{V}'e^{(\gamma W(t)-\frac{1}{2}\gamma^2 t)},\ \mathcal{B}_z=\zeta_3\mathcal{V}'e^{(\gamma W(t)-\frac{1}{2}\gamma^2 t)},\\
\mathcal{B}_{zx} &= \zeta_1\zeta_3\mathcal{V}''e^{(\gamma W(t)-\frac{1}{2}\gamma^2 t)},\ \mathcal{B}_{xx}=\zeta_1^2\mathcal{V}''e^{(\gamma W(t)-\frac{1}{2}\gamma^2 t)},\\
\mathcal{B}_{yxxx} &= \zeta_2\zeta_1^3\mathcal{V}''''e^{(\gamma W(t)-\frac{1}{2}\gamma^2 t)},\ \mathcal{B}_{zxxx}=\zeta_3\zeta_1^3\mathcal{V}''''e^{(\gamma W(t)-\frac{1}{2}\gamma^2 t)},\\
(\mathcal{B}_y+\mathcal{B}_z)_t &= (\zeta_2+\zeta_3)[\zeta_4\mathcal{V}''+\gamma\mathcal{V}'W_t]e^{(\gamma W(t)-\frac{1}{2}\gamma^2 t)}.
\end{aligned} \tag{3}$$

Inserting Equation (3) into Equation (1) yields

$$\zeta_4\mathcal{V}''+\zeta_1^3\mathcal{V}''''-6\zeta_1^2\mathcal{V}'\mathcal{V}''e^{(\gamma W(t)-\frac{1}{2}\gamma^2 t)}=0. \tag{4}$$

When we consider the expectations on both sides, we arrive at

$$\zeta_4\mathcal{V}''+\zeta_1^3\mathcal{V}''''-6\zeta_1^2\mathcal{V}'\mathcal{V}''e^{(-\frac{1}{2}\gamma^2 t)}\mathbb{E}e^{\gamma W(t)}=0. \tag{5}$$

Since $W(t)$ is Brownian motion, then $\mathbb{E}e^{\gamma W(t)} = e^{(\frac{1}{2}\gamma^2 t)}$; Equation (5) turn into

$$\mathcal{V}'''' + \ell_1\mathcal{V}'' + 2\ell_2\mathcal{V}'\mathcal{V}'' = 0, \tag{6}$$

where

$$\ell_1 = \frac{\zeta_4}{\zeta_1^3} \text{ and } \ell_2 = \frac{-3}{\zeta_1}. \tag{7}$$

Integrating Equation (6) yields

$$\mathcal{V}''' + \ell_1\mathcal{V}' + \ell_2(\mathcal{V}')^2 = 0, \tag{8}$$

where we ignored the integral constant.

3. Exact Solutions of SBLMPE

He's semi-inverse method (HSI-method) and the generalizing Riccati equation mapping method (GREM-method) are used to find the solutions to Equation (8). Consequently, the solutions to the SBLMPE (1) are found.

3.1. HSI-Method

We derive the next variational formulations by using the HSI-method, which is described in [37–39]:

$$\mathbb{J}(\mathcal{V}) = \int_0^\infty \{\frac{1}{2}(\mathcal{V}'')^2 - \frac{1}{2}\ell_1(\mathcal{V}')^2 + \frac{1}{3}\ell_2(\mathcal{V}')^3\}d\zeta. \tag{9}$$

Following the form given by [40], we assume the solution to (6) as

$$\mathcal{V}(\zeta) = \mathcal{K}\text{sech}(\zeta), \tag{10}$$

where $\mathcal{K}$ is an unidentified constant. Plugging Equation (10) into Equation (9), we attain

$$\begin{aligned}
\mathbb{J} &= \frac{1}{2}\mathcal{K}^2\int_0^\infty [\text{sech}^2(\zeta)\tanh^4(\zeta) + \text{sech}^4(\zeta)\tanh^2(\zeta) + \text{sech}^6(\zeta) \\
&\quad -\ell_1\text{sech}^2(\zeta)\tanh^2(\zeta) + \frac{2}{3}\ell_2\mathcal{K}\text{sech}^3(\zeta)\tanh^3(\zeta)]d\zeta \\
&= \frac{1}{2}\mathcal{K}^2\int_0^\infty [(\text{sech}^2(\zeta) - \ell_1\text{sech}^2(\zeta)\tanh^2(\zeta) + \frac{2}{3}\ell_2\mathcal{K}\text{sech}^3(\zeta)\tanh^3(\zeta)]d\zeta \\
&= \frac{\mathcal{K}^2}{2} - \ell_1\frac{\mathcal{K}^2}{6} - \frac{2}{45}\ell_2\mathcal{K}^3.
\end{aligned}$$

Making $\mathbb{J}$ stationary related to $\mathcal{K}$, as follows:

$$\frac{\partial\mathbb{J}}{\partial\mathcal{K}} = (1 - \frac{1}{3}\ell_1)\mathcal{K} - \frac{2}{15}\ell_2\mathcal{K}^2 = 0. \tag{11}$$

Solving Equation (11) yields

$$\mathcal{K} = \frac{15 - 5\ell_1}{2\ell_2}.$$

Therefore, the solution of Equation (6) is

$$\mathcal{V}(\zeta) = \frac{15 - 5\ell_1}{6\ell_2}\text{sech}(\zeta).$$

Now, the solution of SBLMPE (1) is

$$\mathcal{B}(x,y,z,t) = \frac{5\zeta_4 - 15\zeta_1^3}{18\zeta_1^2}\operatorname{sech}(\zeta_1 x + \zeta_2 y + \zeta_3 z + \zeta_4 t)e^{(\gamma W(t) - \frac{1}{2}\gamma^2 t)}. \quad (12)$$

Analogously, we can perform the same thing with the solution to Equation (6) as

$$\mathcal{V}(\zeta) = \mathcal{N}\operatorname{sech}(\zeta)\tanh^2(\zeta).$$

Repeating the previous procedures, we arrive at

$$\mathcal{N} = \frac{11(1199 - 213\ell_1)}{1456\ell_2}.$$

So, the solution of SBLMPE (1) is

$$\mathcal{B}(x,y,z,t) = \frac{11(1199 - 213\ell_1)}{1456\ell_2}\operatorname{sech}(\zeta)\tanh^2(\zeta)e^{(\gamma W(t) - \frac{1}{2}\gamma^2 t)}, \quad (13)$$

where $\zeta = \zeta_1 x + \zeta_2 y + \zeta_3 z + \zeta_4 t$.

3.2. GREM Method

In this subsection, we use the GREM method [35]. Let $\mathcal{V} = \mathcal{V}(\zeta)$ be the solution of the next generalized Riccati equation:

$$\mathcal{V}' = s\mathcal{V}^2 + r\mathcal{V} + p, \quad (14)$$

where s, r, p are constants. Utilizing Equation (14), we have

$$\mathcal{V}''' = 6s^3\mathcal{V}^4 + 12rs^2\mathcal{V}^3 + (8ps^2 + 7sr^2)\mathcal{V}^2 + (r^3 + 8rsp)\mathcal{V} + (r^2 + 2sp^2). \quad (15)$$

Plugging Equations (14) and (15) into (8), we obtain

$$\begin{array}{c}(6s^3 + s^2\ell_2)\mathcal{V}^4 + (12rs^2 + 2rs\ell_2)\mathcal{V}^3 + (8ps^2 + 7sr^2 + s\ell_1 + 2ps\ell_2 + r^2\ell_2)\mathcal{V}^2 \\ +(r^3 + 8rsp + r\ell_1 + 2pr\ell_2)\mathcal{V} + (r^2 + 2sp^2 + p\ell_1 + p^2\ell_2) = 0\end{array}.$$

By setting each coefficient of $\mathcal{V}^k$ to zero, we attain

$$6s^3 + s^2\ell_2 = 0,$$

$$12rs^2 + 2rs\ell_2 = 0,$$

$$8ps^2 + 7sr^2 + s\ell_1 + 2ps\ell_2 + r^2\ell_2 = 0,$$

$$r^3 + 8rsp + r\ell_1 + 2pr\ell_2,$$

and

$$r^2 + 2sp^2 + p\ell_1 + p^2\ell_2 = 0.$$

After working out these equations, we obtain

$$s = \frac{-\ell_2}{6},\ r = 0,\ \text{and}\ \ p = \frac{-3\ell_1}{2\ell_2}. \quad (16)$$

$$\frac{d\mathcal{V}}{\mathcal{V}^2 + (\frac{p}{s})} = sd\zeta. \quad (17)$$

There are various families depending on p and s as follows:

Family I: If $ps > 0$, then we have five different solutions to Equation (14):

$$\mathcal{V}_1(\zeta) = \sqrt{\frac{p}{s}}\tan\left(\sqrt{ps}\zeta\right), \text{ for } \frac{-\pi}{2} < \zeta < \frac{\pi}{2},$$

$$\mathcal{V}_2(\zeta) = -\sqrt{\frac{p}{s}}\cot\left(\sqrt{ps}\zeta\right), \text{ for } 0 < \zeta < \pi,$$

$$\mathcal{V}_3(\zeta) = \sqrt{\frac{p}{s}}\Big(\tan(\sqrt{4ps}\zeta) \pm \sec(\sqrt{4ps}\zeta)\Big), \text{ for } 0 < \zeta < \frac{\pi}{2},$$

$$\mathcal{V}_4(\zeta) = -\sqrt{\frac{p}{s}}\Big(\cot(\sqrt{4ps}\zeta) \pm \csc(\sqrt{4ps}\zeta)\Big), \text{ for } 0 < \zeta < \frac{\pi}{2},$$

$$\mathcal{V}_5(\zeta) = \frac{1}{2}\sqrt{\frac{p}{s}}\Big(\tan(\frac{1}{2}\sqrt{ps}\zeta) - \cot(\frac{1}{2}\sqrt{ps}\zeta)\Big), \text{ for } 0 < \zeta < \frac{\pi}{2}.$$

Thus, by using Equation (2), SBLMPE (1) has the trigonometric function solution:

$$\mathcal{B}_1(x,y,z,t) = \sqrt{\frac{p}{s}}\tan\left(\sqrt{ps}\zeta\right)e^{(\gamma W(t) - \frac{1}{2}\gamma^2 t)}, \text{ for } \frac{-\pi}{2} < \zeta < \frac{\pi}{2}, \tag{18}$$

$$\mathcal{B}_2(x,y,z,t) = -\sqrt{\frac{p}{s}}\cot\left(\sqrt{ps}\zeta\right)e^{(\gamma W(t) - \frac{1}{2}\gamma^2 t)}, \text{ for } 0 < \zeta < \pi, \tag{19}$$

$$\mathcal{B}_3(x,y,z,t) = \sqrt{\frac{p}{s}}\Big(\tan(\sqrt{4ps}\zeta) \pm \sec(\sqrt{4ps}\zeta)\Big)e^{(\gamma W(t) - \frac{1}{2}\gamma^2 t)}, \text{ for } 0 < \zeta < \frac{\pi}{2}, \tag{20}$$

$$\mathcal{B}_4(x,y,z,t) = -\sqrt{\frac{p}{s}}\Big(\cot(\sqrt{4ps}\zeta) \pm \csc(\sqrt{4ps}\zeta)\Big)e^{(\gamma W(t) - \frac{1}{2}\gamma^2 t)}, \text{ for } 0 < \zeta < \frac{\pi}{2}, \tag{21}$$

$$\mathcal{B}_5(x,y,z,t) = \frac{1}{2}\sqrt{\frac{p}{s}}\Big(\tan(\frac{1}{2}\sqrt{ps}\zeta) - \cot(\frac{1}{2}\sqrt{ps}\zeta)\Big)e^{(\gamma W(t) - \frac{1}{2}\gamma^2 t)}, \text{ for } 0 < \zeta < \frac{\pi}{2}, \tag{22}$$

where $\zeta = \zeta_1 x + \zeta_2 y + \zeta_3 z + \zeta_4 t$, respectively.
Family II: If $ps < 0$, then we have five different solutions to Equation (14):

$$\mathcal{V}_6(\zeta) = -\sqrt{\frac{-p}{s}}\tanh\left(\sqrt{-ps}\zeta\right), \text{ for } \zeta \in \mathbb{R},$$

$$\mathcal{V}_7(\zeta) = -\sqrt{\frac{-p}{s}}\coth\left(\sqrt{-ps}\zeta\right), \text{ for } \zeta \in \mathbb{R} - \{0\},$$

$$\mathcal{V}_8(\zeta) = -\sqrt{\frac{-p}{s}}\Big(\tanh(\sqrt{-4ps}\zeta + c) \pm i\,\mathrm{sech}(\sqrt{-4ps}\zeta)\Big), \text{ for } \zeta \in [0, \infty),$$

$$\mathcal{V}_9(\zeta) = -\sqrt{\frac{-p}{s}}\Big(\coth(\sqrt{-4ps}\zeta) \pm \mathrm{csch}(\sqrt{-4ps}\zeta)\Big), \text{ for } \zeta \in \mathbb{R} - \{0\},$$

$$\mathcal{V}_{10}(\zeta) = \frac{-1}{2}\sqrt{\frac{-p}{s}}\Big(\tanh(\frac{1}{2}\sqrt{-ps}\zeta + c) + \coth(\frac{1}{2}\sqrt{-ps}\zeta)\Big), \text{ for } \zeta \in \mathbb{R} - \{0\}.$$

Thus, by using Equation (2), SBLMPE (1) has the hyperbolic function solution:

$$\mathcal{B}_6(x,y,z,t) = -\sqrt{\frac{-p}{s}}\tanh\left(\sqrt{-ps}\zeta\right)e^{(\gamma W(t) - \frac{1}{2}\gamma^2 t)}, \text{ for } \zeta \in \mathbb{R}, \tag{23}$$

$$\mathcal{B}_7(x,y,z,t) = -\sqrt{\frac{-p}{s}}\coth\left(\sqrt{-ps}\zeta\right)e^{(\gamma W(t)-\frac{1}{2}\gamma^2 t)}, \text{ for } \zeta \in \mathbb{R}-\{0\}, \tag{24}$$

$$\mathcal{B}_8(x,y,z,t) = -\sqrt{\frac{-p}{s}}\Big(\tanh(\sqrt{-4ps}\zeta) \pm i\operatorname{sech}(\sqrt{-4ps}\zeta)\Big)e^{(\gamma W(t)-\frac{1}{2}\gamma^2 t)}, \tag{25}$$

for $\zeta \in [0,\infty)$,

$$\mathcal{B}_9(x,y,z,t) = -\sqrt{\frac{-p}{s}}\Big(\coth(\sqrt{-4ps}\zeta) \pm \operatorname{csch}(\sqrt{-4ps}\zeta)\Big)e^{(\gamma W(t)-\frac{1}{2}\gamma^2 t)}, \tag{26}$$

for $\zeta \in \mathbb{R}-\{0\}$,

$$\mathcal{B}_{10}(x,y,z,t) = \frac{-1}{2}\sqrt{\frac{-p}{s}}\Big(\tanh(\frac{1}{2}\sqrt{-ps}\zeta) + \coth(\frac{1}{2}\sqrt{-ps}\zeta)\Big)e^{(\gamma W(t)-\frac{1}{2}\gamma^2 t)}, \tag{27}$$

for $\zeta \in \mathbb{R}-\{0\}$. Where $\zeta = \zeta_1 x + \zeta_2 y + \zeta_3 z + \zeta_4 t$, respectively.
Family III: If $p = 0,\ s \neq 0$, then the solution of Equation (14) is

$$\mathcal{V}_{15}(\zeta) = \frac{-1}{s\zeta},$$

Then, we obtain the rational function solution of SBLMPE (1) as

$$\mathcal{B}_{15}(x,y,z,t) = \Big(\frac{-1}{s(\zeta_1 x + \zeta_2 y + \zeta_3 z + \zeta_4 t)}\Big)e^{(\gamma W(t)-\frac{1}{2}\gamma^2 t)}. \tag{28}$$

4. Impacts of Brownian Motion

We now investigate the impact of BM on the obtained solution of the SBLMPE (1). Many graphs illustrating the performance of different solutions are given. Let us fix the parameters $\zeta_1 = 1,\ \zeta_2 = -\zeta_3 = 1,\ \zeta_4 = -2,\ y = z = 1,\ t \in [0,3]$, and $x \in [0,4]$ for some solutions that have been found, such as (12), (13), and (23), so that we can study them further. In Figures 1–3, we can see the impact of noise on the solutions:

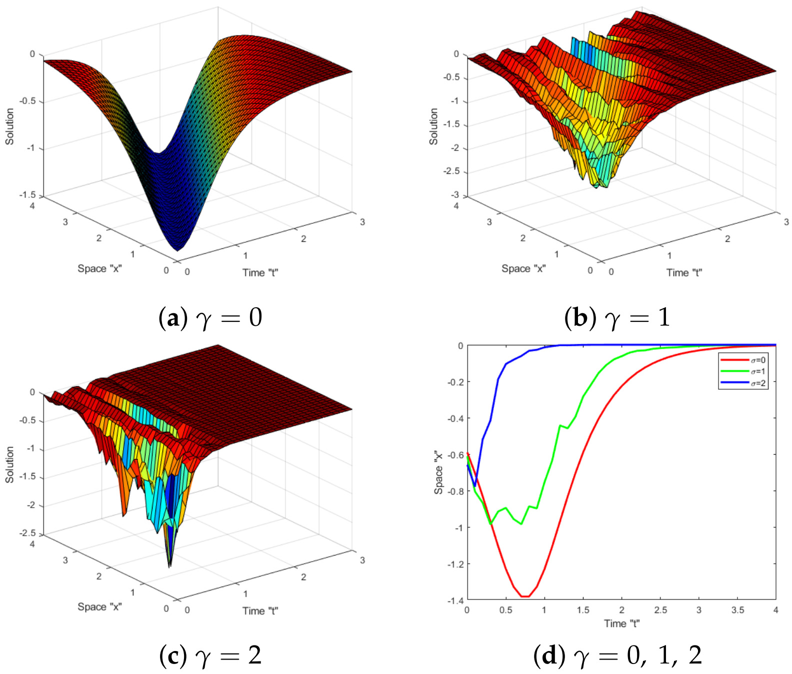
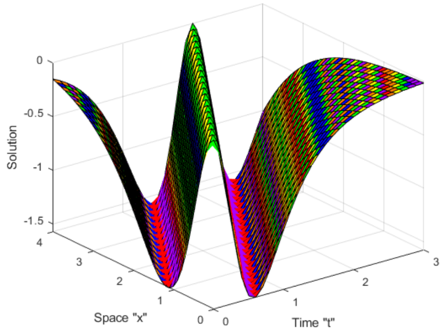
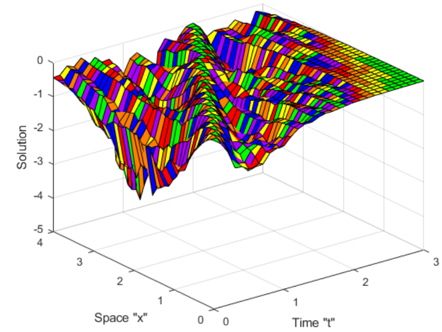


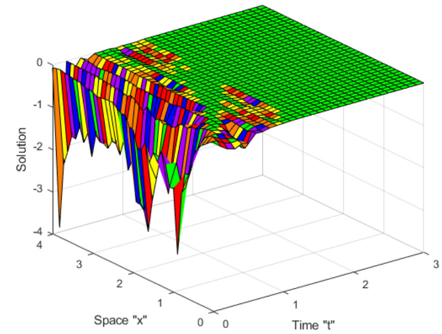
Figure 1. (**a**–**c**) introduce 3D shape of solution given in Equation (12) for several $\gamma = 0,\ 1,\ 2$; (**d**) shows 2D shape for these values of γ.



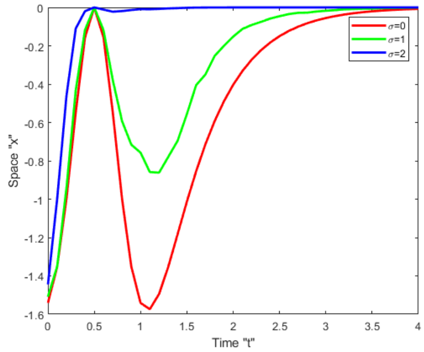
(**a**) $\gamma = 0$



(**b**) $\gamma = 1$

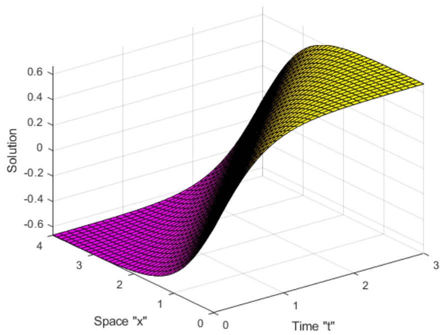


(**c**) $\gamma = 2$

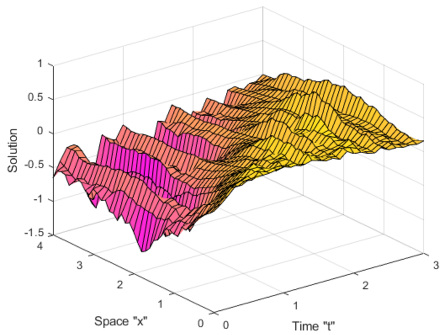


(**d**)$\gamma = 0,\ 1,\ 2$

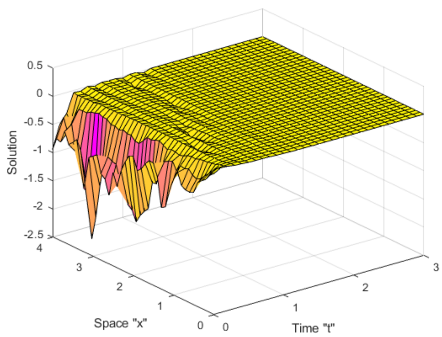
Figure 2. (**a**–**c**) introduce 3D shape of solution given in Equation (13) for various $\gamma = 0,\ 1,\ 2$; (**d**) shows 2D shape for these values of γ .



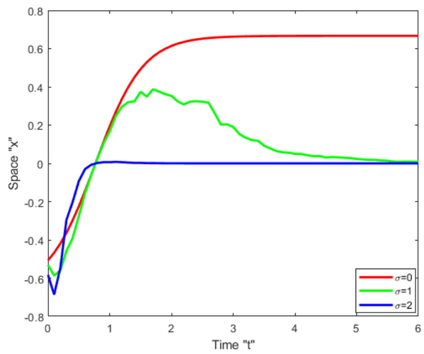
(**a**) $\gamma = 0$



(**b**) $\gamma = 1$



(**c**) $\gamma = 2$



(**d**)$\gamma = 0,\ 1,\ 2$

Figure 3. (**a**–**c**) introduce 3D shape of solution given in Equation (23) for various $\gamma = 0,\ 1,\ 2$; (**d**) shows 2D shape for these values of γ.

It can be seen from Figures 1–3 that there exist several solutions, such as bright, periodic, kink, and so on, when the noise has disappeared (i.e., at $\gamma = 0$). After a few modest transit patterns, the surface gets much flatter when noise is presented and the intensity is raised. This was confirmed using a 2D graph. This implies that the SBLMPE solutions are affected by Brownian motion and are stabilized at zero.

5. Conclusions

Real-world phenomena are not typically deterministic because actual systems cannot be completely isolated from their surroundings. For this reason, it is essential to consider stochastic effects when solving these differential equations. Therefore, we considered here a stochastic Boiti–Leon–Manna–Pempinelli equation (SBLMPE) that is perturbed by a multiplicative noise. The solution of the SBLMPE was achieved by using the Riccati equation mapping and He's semi-inverse methods. Since the Boiti–Leon–Manna–Pempinelli equation is utilized to explain incompressible liquid in fluid mechanics, the discovered solutions can be employed to explain a broad range of fascinating physical phenomena. We constructed some 2D and 3D diagrams using Matlab to demonstrate the impact of Brownian motion on the analytical solutions of the SBLMPE. Finally, we deduced that Brownian motion stabilized the solutions of SBLMPE at around zero.

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