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Generalizations of Hardy's Type Inequalities via Conformable Calculus

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Abstract: In this paper, we derive some new fractional extensions of Hardy's type inequalities. The corresponding reverse relations are also obtained by using the conformable fractional calculus from which the classical integral inequalities are deduced as special cases at $\alpha = 1$.

Keywords: Hardy's inequality; conformable fractional derivative; conformable fractional integral; Hölder's inequality

AMS Subject Class: 26A33; 26D10

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Citation: AlNemer, G.; Kenawy, M.; Zakarya, M.; Cesarano, C.; Rezk, H.M. Generalizations of Hardy's Type Inequalities via Conformable Calculus. *Symmetry* **2021**, *13*, 242. https://doi.org/10.3390/sym13020242

Academic Editor: Charles F. Dunkl

Received: 12 December 2020
Accepted: 21 January 2021
Published: 1 February 2021

1. Introduction

In 1925 Hardy [1] employed the calculus of variations and proved the inequality

$$\int_0^\infty \left(\frac{1}{\vartheta}\int_0^\vartheta h(s)ds\right)^\mu d\vartheta \le \left(\frac{\mu}{\mu-1}\right)^\mu \int_0^\infty h^\mu(\vartheta)d\vartheta, \tag{1}$$

where $h \ge 0$ and integrable over any finite interval $(0,\vartheta)$ and h^μ is integrable and convergent over $(0,\infty)$ and $\mu > 1$. The constant $(\mu/(\mu-1))^\mu$ is the best possible.

In 1928 Hardy [2] generalized the inequality (1) and proved that if $\mu > 1$ and h is non-negative for $\vartheta \ge 0$, then:

$$\int_0^\infty \vartheta^{-r}\left(\int_0^\vartheta h(s)ds\right)^\mu d\vartheta \le \left(\frac{\mu}{r-1}\right)^\mu \int_0^\infty \vartheta^{\mu-r}h^\mu(\vartheta)d\vartheta, \text{ for } r>1. \tag{2}$$

Moreover,

$$\int_0^\infty \vartheta^{-r}\left(\int_0^\infty h(s)ds\right)^\mu d\vartheta \le \left(\frac{\mu}{1-r}\right)^\mu \int_0^\infty \vartheta^{\mu-r}h^\mu(\vartheta)d\vartheta, \text{ for } r<1. \tag{3}$$

The constants $(\mu/(r-1))^\mu$ and $(\mu/(1-r))^\mu$ are the best possible.

In recent years, several scholars have examined fractional inequalities by using the fractional derivative of Caputo and Riemann–Liouville; we refer to the papers [3–7] for these results.

In [8,9], the authors expanded the calculus of fractional order to conformable calculus. Lately, some scholars have expanded classical inequalities by applying conformable fractional formulas such as Opial's inequality [10–12], Hermite–Hadamard's inequality [13–15], Chebyshev's inequality [16] and Steffensen's inequality [17].

The original motivation for this paper is obtaining the fractional forms of some extensions of Hardy's type inequalities and their reverses using conformable fractional calculus, and as a special case, we put $\alpha = 1$ to get the generalized ones.

The paper is structured as follows: In Section 2, we will present some concepts for the conformable fractional calculus and also the Hölder's inequality for α-fractional differentiable functions that will represent our key outcomes. In Section 3, we shall set out generalizations of Hardy's type inequalities and revers relations in each case for α-fractional differentiable functions.

2. Key Concepts and Lemmas

In this section, we present some basic definitions concerning the conformable fractional calculus that will be used throughout the paper. For the latest findings on conformable derivatives and integrals, we refer to [8,9].

Definition 1. *Let $h : [0, \infty) \to \mathbb{R}$. Then the conformable fractional derivative of order α of h is defined as:*

$$D_\alpha h(s) = \lim_{\in \to 0} \frac{h(s + \in s^{1-\alpha}) - h(s)}{\in}$$

for all $s > 0$ and $0 < \alpha \leq 1$.

For $0 < \alpha \leq 1$ and h, k be α-differentiable at a point s, then:

$$D_\alpha(hk) = hD_\alpha k + kD_\alpha h$$

Further, for $0 < \alpha \leq 1$ and h, k be α-differentiable at a point s with $k(s) \neq 0$, then:

$$D_\alpha\left(\frac{h}{k}\right) = \frac{kD_\alpha h - hD_\alpha k}{k^2} \tag{4}$$

Remark 1. *If h is a differentiable function, then:*

$$D_\alpha h(s) = s^{1-\alpha}\frac{dh(s)}{ds}$$

Definition 2. *Let $h : [0, s] \to \mathbb{R}$. Then the conformable fractional integral order α of h is defined as*

$$I_\alpha h(s) = \int_0^s h(\vartheta) d_\alpha \vartheta = \int_0^s \vartheta^{\alpha-1} h(\vartheta) d\vartheta, \tag{5}$$

for all s and $0 < \alpha \leq 1$.

Now, we state some lemmas which play important roles in our proofs of the main results. First, the integration by parts formula is given in the following lemma:

Lemma 1. *Suppose that ϕ, ψ: $[0, s] \to \mathbb{R}$ be two functions such that $\phi\psi$ is α-differentiable and $0 < \alpha \leq 1$ then:*

$$\int_0^s \phi(\vartheta) D_\alpha \psi(\vartheta) d_\alpha \vartheta = \phi(\vartheta)\psi(\vartheta)|_0^s - \int_0^s \psi(\vartheta) D_\alpha \phi(\vartheta) d_\alpha \vartheta, \tag{6}$$

Next, we state Hölder's and reversed Hölder's inequality for α-conformable functions, which will mainly be used to prove the results of this paper.

Lemma 2. *Let $h, k : [0, s] \to \mathbb{R}$ be a continuous function and $0 < \alpha \leq 1$. Then:*

$$\int_0^s |h(\vartheta)k(\vartheta)| d_\alpha \vartheta \leq \left(\int_0^s |h(\vartheta)|^\mu d_\alpha \vartheta \right)^{\frac{1}{\mu}} \left(\int_0^s |k(\vartheta)|^m d_\alpha \vartheta \right)^{\frac{1}{m}}, \tag{7}$$

where $\mu > 1$ and $1/\mu + 1/m = 1$. This inequality is reversed if $0 < \mu < 1$ and if $\mu < 0$ or $m < 0$.

3. Hardy's Type Inequalities of a Fractional Order

In this section, we state and prove the main outcomes of this paper and we begin with the following theorem:

Theorem 1. *Let $h \geq 0$ be a non-decreasing function and $\Lambda(\vartheta) = \int_0^\vartheta s^{1-\alpha} h(s) d_\alpha s$. If $\Psi \geq 0$ is non-decreasing and $0 < b \leq \infty$, then:*

$$\int_0^b \Psi\left(\frac{\Lambda(\vartheta)}{\vartheta} \right) d_\alpha \vartheta \leq \int_0^b \Psi(h(\vartheta)) d_\alpha \vartheta \tag{8}$$

When $\Psi(\vartheta) = \vartheta^\mu,\ \mu \geq 1$, we have

$$\int_0^b \left(\frac{\Lambda(\vartheta)}{\vartheta} \right)^\mu d_\alpha \vartheta \leq \int_0^b h^\mu(\vartheta) d_\alpha \vartheta. \tag{9}$$

Proof. We start with the following identity:

$$\int_0^b \Psi\left(\frac{\Lambda(\vartheta)}{\vartheta} \right) d_\alpha \vartheta = \int_0^b \Psi\left(\vartheta^{-1} \Lambda(\vartheta) \right) d_\alpha \vartheta = \int_0^b \Psi\left(\vartheta^{-1} \int_0^\vartheta s^{1-\alpha} h(s) d_\alpha s \right) d_\alpha \vartheta.$$

Since h is non-decreasing, then we have:

$$\begin{aligned} \int_0^b \Psi\left(\tfrac{\Lambda(\vartheta)}{\vartheta} \right) d_\alpha \vartheta &\leq \int_0^b \Psi\left(\vartheta^{-1} h(\vartheta) \int_0^\vartheta s^{1-\alpha} d_\alpha s \right) d_\alpha \vartheta \\ &= \int_0^b \Psi\left(\vartheta^{-1} h(\vartheta) \vartheta \right) d_\alpha \vartheta = \int_0^b \Psi(h(\vartheta)) d_\alpha \vartheta, \end{aligned}$$

which is (8). For $\Psi(\vartheta) = \vartheta^\mu, \mu \geq 1$, we have (9). The proof is complete. □

The theorem below is the generalization of Hardy's inequality (1) on conformable calculus.

Theorem 2. *Let h be a non-negative α-integrable function on $(0, \infty), k > 0,\ \vartheta/k(\vartheta)$ is non-increasing and*

$$\Lambda(\vartheta) = \int_0^\vartheta s^{1-\alpha} h(s) d_\alpha s,$$

where $0 < \alpha \leq 1,\ 0 < a < 1$ and $\mu > a/(1 + a - \alpha)$. Then

$$\int_0^\infty \left(\frac{\Lambda(\vartheta)}{k(\vartheta)} \right)^\mu d_\alpha \vartheta \leq \frac{1}{(\alpha - a)^{\mu-1}((\alpha - a)(1 - \mu) + \mu - \alpha)} \int_0^\infty \left(\frac{\vartheta h(\vartheta)}{k(\vartheta)} \right)^\mu d_\alpha \vartheta. \tag{10}$$

Proof. Since

$$\int_0^\infty \left(\frac{\Lambda(\vartheta)}{k(\vartheta)} \right)^\mu d_\alpha \vartheta = \int_0^\infty k^{-\mu}(\vartheta) \left(\int_0^\vartheta s^{1-\alpha} s^{a(\frac{\mu-1}{\mu})} s^{-a(\frac{\mu-1}{\mu})} h(s) d_\alpha s \right)^\mu d_\alpha \vartheta,$$

applying Hölder's inequality with index μ and $\mu/(\mu-1)$, we get

$$\begin{aligned}
\int_0^\infty \left(\frac{\Lambda(\vartheta)}{k(\vartheta)}\right)^\mu d_\alpha\vartheta &\le \int_0^\infty k^{-\mu}(\vartheta)\left(\begin{array}{c}\left(\int_0^\vartheta s^{\mu(1-\alpha)+a(\mu-1)}h^\mu(s)d_\alpha s\right)^{\frac{1}{\mu}} \\ \times\left(\int_0^\vartheta s^{-a}d_\alpha s\right)^{\frac{\mu-1}{\mu}}\end{array}\right)^\mu d_\alpha\vartheta \\
&= \int_0^\infty k^{-\mu}(\vartheta)\left(\begin{array}{c}\left(\int_0^\vartheta s^{\mu(1-\alpha)+a(\mu-1)}h^\mu(s)d_\alpha s\right) \\ \times\left(\frac{\vartheta^{\alpha-a}}{\alpha-a}\right)^{\mu-1}\end{array}\right)d_\alpha\vartheta \\
&= \frac{1}{(\alpha-a)^{\mu-1}}\left(\begin{array}{c}\left(\int_0^\vartheta s^{\mu(1-\alpha)+a(\mu-1)}h^\mu(s)d_\alpha s\right) \\ \times\left(\int_s^\vartheta \vartheta^{(\alpha-a)+(\mu-1)}k^{-\mu}(\vartheta)d_\alpha\vartheta\right)\end{array}\right).
\end{aligned}$$

Now, since $(\vartheta/k(\vartheta))^\mu$ is non-increasing, we have:

$$\begin{aligned}
\int_0^\infty \left(\frac{\Lambda(\vartheta)}{k(\vartheta)}\right)^\mu d_\alpha\vartheta &\le \frac{1}{(\alpha-a)^{\mu-1}}\left(\begin{array}{c}\int_0^\infty s^{\mu(1-\alpha)+a(\mu-1)}h^\mu(s)\left(\frac{s}{k(s)}\right)^\mu \\ \times\left(\int_s^\infty \vartheta^{(\alpha-a)(\mu-1)-\mu}d_\alpha\vartheta\right)d_\alpha s\end{array}\right) \\
&\le \frac{1}{(\alpha-a)^{\mu-1}}\left(\begin{array}{c}\int_0^\infty s^{\mu(1-\alpha)+a(\mu-1)}h^\mu(s)\left(\frac{s}{k(s)}\right)^\mu \\ \times\frac{-s^{(\alpha-a)(\mu-1)-\mu+\alpha}}{(\alpha-a)(\mu-1)-\mu+\alpha}d_\alpha s\end{array}\right) \\
&= \frac{1}{(\alpha-a)^{\mu-1}((\alpha-a)(1-\mu)+\mu-\alpha)}\left(\begin{array}{c}\int_0^\infty s^{\mu(1-\alpha)+a(\mu-1)+(\alpha-a)(\mu-1)-\mu+\alpha} \\ \times\left(\frac{sh(s)}{k(s)}\right)^\mu d_\alpha s\end{array}\right),
\end{aligned}$$

Then

$$\int_0^\infty \left(\frac{\Lambda(\vartheta)}{k(\vartheta)}\right)^\mu d_\alpha\vartheta \le \frac{1}{(\alpha-a)^{\mu-1}((\alpha-a)(1-\mu)+\mu-\alpha)}\int_0^\infty \left(\frac{\vartheta h(\vartheta)}{k(\vartheta)}\right)^\mu d_\alpha\vartheta.$$

The proof is complete. □

Corollary 1. *[In Theorem 2] For $\alpha=1$, then we have*

$$\int_0^\infty \left(\frac{\Lambda(\vartheta)}{k(\vartheta)}\right)^\mu d\vartheta \le \frac{1}{a(1-a)^{\mu-1}(\mu-1)}\int_0^\infty \left(\frac{\vartheta h(\vartheta)}{k(\vartheta)}\right)^\mu d\vartheta,$$

which is [[18], Theorem 2.2].

Proof. The proof follows from Theorem 2 for $\alpha=1$. □

Corollary 2. *[In Theorem 2] For $\alpha=1$, $a=\frac{1}{\mu}$ and $k(\vartheta)=\vartheta$, then we have the classical Hardy inequality:*

$$\int_0^\infty \left(\frac{\int_0^\vartheta h(s)ds}{\vartheta}\right)^\mu d\vartheta \le \left(\frac{\mu}{\mu-1}\right)^\mu \int_0^\infty h^\mu(\vartheta)d\vartheta.$$

Proof. The proof follows from Theorem 2 for $\alpha=1$, $a=\frac{1}{\mu}$ and $k(\vartheta)=\vartheta$. □

The following finding concerns the converse of Hardy's inequality.

Theorem 3. *Let h be a non-negative α-integrable function on $(0,\infty), k>0$, $\vartheta/k(\vartheta)$ is non-decreasing and*

$$\Lambda(\vartheta)=\int_0^\vartheta s^{1-\alpha}h(s)d_\alpha s,$$

where $0 < \alpha \leq 1$, $a > 0$ *and* $0 < \mu < a/(a + \alpha - 1)$. *Then*

$$\int_0^{\vartheta} \left(\frac{\Lambda(\vartheta)}{k(\vartheta)} \right)^{\mu} d_\alpha \vartheta \geq \frac{1}{(\alpha + a)^{\mu - 1}((\alpha + a)(1 - \mu) + \mu - \alpha)} \int_0^{\vartheta} \left(\frac{\vartheta h(\vartheta)}{k(\vartheta)} \right)^{\mu} d_\alpha \vartheta. \tag{11}$$

Proof. Since

$$\int_0^{\infty} \left(\frac{\Lambda(\vartheta)}{k(\vartheta)} \right)^{\mu} d_\alpha \vartheta = \int_0^{\infty} (k(\vartheta))^{-\mu} \left(\int_0^{\vartheta} s^{1-\alpha} s^{a(\frac{\mu-1}{\mu})} s^{-a(\frac{\mu-1}{\mu})} h(s) d_\alpha s \right)^{\mu} d_\alpha \vartheta,$$

applying reverse Hölder's inequality with index μ and $\mu/(\mu - 1)$, we get

$$\begin{aligned} \int_0^{\infty} \left(\frac{\Lambda(\vartheta)}{k(\vartheta)} \right)^{\mu} d_\alpha \vartheta &\geq \int_0^{\infty} k^{-\mu}(\vartheta) \begin{pmatrix} \left(\int_0^{\vartheta} s^{\mu(1-\alpha)+a(1-\mu)} h^{\mu}(s) d_\alpha s \right)^{\frac{1}{\mu}} \\ \times \left(\int_0^{\vartheta} s^{a} d_\alpha s \right)^{\frac{\mu-1}{\mu}} \end{pmatrix}^{\mu} d_\alpha \vartheta \\ &= \int_0^{\infty} k^{-\mu}(\vartheta) \begin{pmatrix} \left(\int_0^{\vartheta} s^{\mu(1-\alpha)+a(1-\mu)} h^{\mu}(s) d_\alpha s \right) \\ \times \left(\frac{\vartheta^{\alpha+a}}{\alpha+a} \right)^{\mu-1} \end{pmatrix} d_\alpha \vartheta \\ &= \frac{1}{(\alpha+a)^{\mu-1}} \begin{pmatrix} \int_0^{\infty} s^{\mu(1-\alpha)+a(1-\mu)} h^{\mu}(s) d_\alpha s \\ \times \int_s^{\infty} \vartheta^{(\alpha+a)(\mu-1)} k^{-\mu}(\vartheta) d_\alpha \vartheta \end{pmatrix}. \end{aligned}$$

Now, since $(\vartheta/k(\vartheta))^{\mu}$ is non-decreasing, we have:

$$\begin{aligned} \int_0^{\infty} \left(\frac{\Lambda(\vartheta)}{k(\vartheta)} \right)^{\mu} d_\alpha \vartheta &\geq \frac{1}{(\alpha+a)^{\mu-1}} \begin{pmatrix} \int_0^{\infty} s^{\mu(1-\alpha)+a(1-\mu)} h^{\mu}(s) \left(\frac{s}{k(s)} \right)^{\mu} \\ \times \left(\int_s^{\infty} \vartheta^{(\alpha+a)(\mu-1)-\mu} \, d_\alpha \vartheta \right) d_\alpha s \end{pmatrix} \\ &\geq \frac{1}{(\alpha+a)^{\mu-1}} \begin{pmatrix} \int_0^{\infty} s^{\mu(1-\alpha)+a(1-\mu)} h^{\mu}(s) \left(\frac{s}{k(s)} \right)^{\mu} \\ \times \frac{-s^{(\alpha+a)(\mu-1)+\alpha-\mu}}{(\alpha+a)(\mu-1)-\mu+\alpha} d_\alpha s \end{pmatrix} \\ &\geq \frac{1}{(\alpha+a)^{\mu-1}((\alpha+a)(1-\mu)+\mu-\alpha)} \begin{pmatrix} \int_0^{\infty} s^{\mu(1-\alpha)+a(1-\mu)+(\alpha+a)(\mu-1)+\alpha-\mu} \\ \times \left(\frac{sh(s)}{k(s)} \right)^{\mu} d_\alpha s \end{pmatrix}, \end{aligned}$$

then:

$$\int_0^{\infty} \left(\frac{\Lambda(\vartheta)}{k(\vartheta)} \right)^{\mu} d_\alpha \vartheta \geq \frac{1}{(\alpha + a)^{\mu - 1}((\alpha + a)(1 - \mu) + \mu - \alpha)} \int_0^{\infty} \left(\frac{\vartheta h(\vartheta)}{k(\vartheta)} \right)^{\mu} d_\alpha \vartheta.$$

The proof is complete. □

Corollary 3. *[In Theorem 3] For* $\alpha = 1$, *then we have*

$$\int_0^{\infty} \left(\frac{\Lambda(\vartheta)}{k(\vartheta)} \right)^{\mu} d\vartheta \geq \frac{1}{a(1 + a)^{\mu - 1}(1 - \mu)} \int_0^{\infty} \left(\frac{\vartheta h(\vartheta)}{k(\vartheta)} \right)^{\mu} d\vartheta,$$

which is [[18], Theorem 2.3].

Proof. The proof follows from Theorem 3 for $\alpha = 1$. □

Theorem 4. *Let* h *be a non-negative* α*-integrable function on* $(0, \infty), k > 0$, $\vartheta/k(\vartheta)$ *is non-increasing and*

$$\Lambda(\vartheta) = \int_0^{\vartheta} s^{1-\alpha} h(s) d_\alpha s,$$

where $0 < \alpha \leq 1$, $0 < a < 1$, $m > \alpha\mu - \alpha(\mu-1)$ *and* $\mu > 1$. *Then*

$$\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{k^m(\vartheta)} d_\alpha\vartheta \leq \frac{1}{(\alpha-a)^{\mu-1}((a-\alpha)(\mu-1)+m-\alpha)} \tag{12}$$

Proof. Since

$$\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{k^m(\vartheta)} d_\alpha\vartheta = \int_0^\infty k^{-m}(\vartheta)\left(\int_0^\vartheta s^{1-\alpha} s^{a(\frac{\mu-1}{\mu})} s^{-a(\frac{\mu-1}{\mu})} h(s) d_\alpha s\right)^\mu d_\alpha\vartheta,$$

applying Hölder's inequality with index μ and $\mu/(\mu-1)$, we get

$$\begin{aligned}
\int_0^\infty \tfrac{\Lambda^\mu(\vartheta)}{k^m(\vartheta)} d_\alpha\vartheta &\leq \int_0^\infty k^{-m}(\vartheta)\left(\begin{array}{c}\left(\int_0^\vartheta s^{\mu(1-\alpha)+a(\mu-1)} h^\mu(s) d_\alpha s\right)^{\frac{1}{\mu}} \\ \times\left(\int_0^\vartheta s^{-a} d_\alpha s\right)^{\frac{\mu-1}{\mu}}\end{array}\right)^\mu d_\alpha\vartheta \\
&= \int_0^\infty k^{-m}(\vartheta)\left(\begin{array}{c}\left(\int_0^\vartheta s^{\mu(1-\alpha)+a(\mu-1)} h^\mu(s) d_\alpha s\right) \\ \times\left(\frac{\vartheta^{\alpha-a}}{\alpha-a}\right)^{\mu-1}\end{array}\right) d_\alpha\vartheta \\
&= \tfrac{1}{(\alpha-a)^{\mu-1}}\left(\begin{array}{c}\int_0^\infty s^{\mu(1-\alpha)+a(\mu-1)} h^\mu(s) d_\alpha s \\ \times\int_s^\infty \vartheta^{(\alpha-a)(\mu-1)} k^{-m}(\vartheta) d_\alpha\vartheta\end{array}\right).
\end{aligned}$$

Now, since $(\vartheta/k(\vartheta))^m$ is non-decreasing, we have:

$$\begin{aligned}
\int_0^\infty \tfrac{\Lambda^\mu(\vartheta)}{k^m(\vartheta)} d_\alpha\vartheta &\leq \tfrac{1}{(\alpha-a)^{\mu-1}}\left(\begin{array}{c}\int_0^\infty s^{\mu(1-\alpha)+a(\mu-1)} h^\mu(s)\left(\frac{s}{k(s)}\right)^m \\ \times\left(\int_s^\infty \vartheta^{(\alpha-a)(\mu-1)-m} d_\alpha\vartheta\right) d_\alpha s\end{array}\right) \\
&= \tfrac{1}{(\alpha-a)^{\mu-1}}\left(\begin{array}{c}\int_0^\infty s^{\mu(1-\alpha)+a(\mu-1)} h^\mu(s)\left(\frac{s}{k(s)}\right)^m \\ \times\frac{-s^{(\alpha-a)(\mu-1)-m+\alpha}}{(\alpha-a)(\mu-1)-m+\alpha} d_\alpha s\end{array}\right) \\
&\geq \tfrac{1}{(\alpha-a)^{\mu-1}((a-\alpha)(\mu-1)+m-\alpha)}\left(\begin{array}{c}\int_0^\infty s^{\mu(1-\alpha)+a(\mu-1)+(\alpha-a)(\mu-1)-\mu+\alpha} \\ \times\frac{(sh(s))^\mu}{k^m(s)} d_\alpha s\end{array}\right)
\end{aligned}$$

Then

$$\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{k^m(\vartheta)} d_\alpha\vartheta \leq \frac{1}{(\alpha-a)^{\mu-1}((a-\alpha)(\mu-1)+m-\alpha)} \int_0^\infty \frac{(\vartheta h(\vartheta))^\mu}{k^m(\vartheta)} d_\alpha\vartheta.$$

The proof is complete. □

Corollary 4. *[In Theorem 4] For* $\alpha = 1$, *then we have*

$$\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{k^m(\vartheta)} d\vartheta \leq \frac{1}{(1-a)^{\mu-1}((a-1)(\mu-1)+m-1)} \int_0^\infty \frac{(\vartheta h(\vartheta))^\mu}{k^m(\vartheta)} d\vartheta,$$

which is [[19], Theorem 2.1].

Proof. The proof follows from Theorem 4 for $\alpha = 1$. □

Corollary 5. *[In Theorem 4] For* $\alpha = 1$, $a = \frac{1}{\mu}$, $\mu = m$ *and* $k(\vartheta) = \vartheta$, *then we have the classical Hardy inequality.*

$$\int_0^\infty \left(\frac{\int_0^\vartheta h(s) ds}{\vartheta}\right)^\mu d\vartheta \leq \left(\frac{\mu}{\mu-1}\right)^\mu \int_0^\infty h^\mu(\vartheta) d\vartheta.$$

Proof. The proof follows from Theorem 4 for $\alpha = 1,\ a = \frac{1}{\mu},\ \mu = m$ and $k(\vartheta) = \vartheta$. □

Theorem 5. *Let h be a non-negative α-integrable function on $(0,\infty)$, $k > 0$, $\vartheta/k(\vartheta)$ is non-decreasing and:*

$$\Lambda(\vartheta) = \int_0^\vartheta s^{1-\alpha} h(s) d_\alpha s,$$

where $0 < \alpha \le 1,\ a > 0,\ \ m > \alpha\mu + a(\mu - 1)$ *and* $0 < \mu < 1$. *Then:*

$$\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{k^m(\vartheta)} d_\alpha\vartheta \ge \frac{1}{(\alpha + a)^{\mu-1}((\alpha + a)(1-\mu) + m - \alpha)} \int_0^\infty \frac{(\vartheta h(\vartheta))^\mu}{k^m(\vartheta)} d_\alpha\vartheta. \tag{13}$$

Proof. Since

$$\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{k^m(\vartheta)} d_\alpha\vartheta = \int_0^\infty k^{-m}(\vartheta) \left(\int_0^\vartheta s^{1-\alpha} s^{a(\frac{\mu-1}{\mu})} s^{-a(\frac{\mu-1}{\mu})} h(s) d_\alpha s \right)^\mu d_\alpha\vartheta,$$

applying reverse Hölder's inequality with index μ and $\mu/(\mu-1)$, we get:

$$\begin{aligned}
\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{k^m(\vartheta)} d_\alpha\vartheta &\ge \int_0^\infty k^{-m}(\vartheta) \begin{pmatrix} \left(\int_0^\vartheta s^{\mu(1-\alpha)+a(1-\mu)} h^\mu(s) d_\alpha s \right)^{\frac{1}{\mu}} \\ \times \left(\int_0^\vartheta s^a d_\alpha s \right)^{\frac{\mu-1}{\mu}} \end{pmatrix} d_\alpha\vartheta \\
&= \int_0^\infty k^{-m}(\vartheta) \begin{pmatrix} \left(\int_0^\vartheta s^{\mu(1-\alpha)+a(1-\mu)} h^\mu(s) d_\alpha s \right) \\ \times \left(\frac{\vartheta^{\alpha+a}}{\alpha+a} \right)^{\mu-1} \end{pmatrix} d_\alpha\vartheta \\
&= \frac{1}{(\alpha+a)^{\mu-1}} \begin{pmatrix} \int_0^\infty s^{\mu(1-\alpha)+a(1-\mu)} h^\mu(s) d_\alpha s \\ \times \int_s^\infty \vartheta^{(\alpha+a)(\mu-1)}\, k^{-m}(\vartheta)\, d_\alpha\vartheta \end{pmatrix}.
\end{aligned}$$

Now, since $(\vartheta/k(\vartheta))^m$ is non-decreasing, we have:

$$\begin{aligned}
\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{k^m(\vartheta)} d_\alpha\vartheta &\ge \frac{1}{(\alpha+a)^{\mu-1}} \begin{pmatrix} \int_0^\infty s^{\mu(1-\alpha)+a(1-\mu)} h^\mu(s) \left(\frac{s}{k(s)} \right)^m \\ \times \left(\int_s^\infty \vartheta^{(\alpha+a)(\mu-1)-m} d_\alpha\vartheta \right) d_\alpha s \end{pmatrix} \\
&= \frac{1}{(\alpha+a)^{\mu-1}} \begin{pmatrix} \int_0^\infty s^{\mu(1-\alpha)+a(1-\mu)} h^\mu(s) \left(\frac{s}{k(s)} \right)^m \\ \times \frac{-s^{(\alpha+a)(\mu-1)+\alpha-m}}{(\alpha+a)(\mu-1)-m+\alpha} d_\alpha s \end{pmatrix} \\
&= \frac{1}{(\alpha+a)^{\mu-1}((\alpha+a)(1-\mu)+m-\alpha)} \begin{pmatrix} \int_0^\infty s^{\mu(1-\alpha)+a(1-\mu)+(\alpha+a)(\mu-1)+\alpha-\mu} \\ \times \frac{(sh(s))^\mu}{k^m(s)} d_\alpha s \end{pmatrix}
\end{aligned}$$

Then:

$$\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{k^m(\vartheta)} d_\alpha\vartheta \ge \frac{1}{(\alpha + a)^{\mu-1}((\alpha + a)(1-\mu) + m - \alpha)} \int_0^\infty \frac{(\vartheta h(\vartheta))^\mu}{k^m(\vartheta)} d_\alpha\vartheta.$$

The proof is complete. □

Corollary 6. *[In Theorem 5] For $\alpha = 1$, then we have:*

$$\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{k^m(\vartheta)} d\vartheta \ge \frac{1}{(1 + a)^{\mu-1}((1 + \alpha)(1-\mu) + m - 1)} \int_0^\infty \frac{(\vartheta h(\vartheta))^\mu}{k^m(\vartheta)} d\vartheta,$$

which is [[19], Theorem 2.2].

Proof. The proof follows from Theorem 5 for $\alpha = 1$. □

Theorem 6. *Let h be a non-negative α-integrable function on $(0,\infty)$, $k>0$, $\vartheta/k(\vartheta)$ is non-decreasing and*

$$\Omega(\vartheta)=\int_0^\vartheta s^{1-\alpha}k(s)d_\alpha s,\ \Lambda(\vartheta)=\int_0^\vartheta s^{1-\alpha}h(s)d_\alpha s,$$

where $0<\alpha\le 1$, $0<a<1$, $m>(\alpha\mu-a(\mu-1)/2)$ *and* $\mu>1$. *Then*

$$\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{\Omega^m(\vartheta)}d_\alpha\vartheta \le \frac{1}{(\alpha-a)^{\mu-1}((\alpha-a)(1-\mu)+2m-\alpha)}\int_0^\infty \frac{(\vartheta h(\vartheta))^\mu}{\Omega^m(\vartheta)}d_\alpha\vartheta. \tag{14}$$

Proof. Since

$$\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{\Omega^m(\vartheta)}d_\alpha\vartheta=\int_0^\infty \Omega^{-m}(\vartheta)\left(\int_0^\vartheta s^{1-\alpha}s^{a(\frac{\mu-1}{\mu})}s^{-a(\frac{\mu-1}{\mu})}h(s)d_\alpha s\right)^\mu d_\alpha\vartheta,$$

applying Hölder's inequality with index μ and $\mu/(\mu-1)$, we get

$$\begin{aligned}\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{\Omega^m(\vartheta)}d_\alpha\vartheta &\le \int_0^\infty \Omega^{-m}(\vartheta)\begin{pmatrix}\left(\int_0^\vartheta s^{\mu(1-\alpha)+a(\mu-1)}h^\mu(s)d_\alpha s\right)^{\frac{1}{\mu}}\\ \times\left(\int_0^\vartheta s^{-a}d_\alpha s\right)^{\frac{\mu-1}{\mu}}\end{pmatrix}d_\alpha\vartheta\\ &=\int_0^\infty \Omega^{-m}(\vartheta)\begin{pmatrix}\left(\int_0^\vartheta s^{\mu(1-\alpha)+a(\mu-1)}h^\mu(s)d_\alpha s\right)\\ \times\left(\frac{\vartheta^{\alpha-a}}{\alpha-a}\right)^{\mu-1}\end{pmatrix}d_\alpha\vartheta\\ &=\frac{1}{(\alpha-a)^{\mu-1}}\begin{pmatrix}\int_0^\infty s^{\mu(1-\alpha)+a(\mu-1)}h^\mu(s)d_\alpha s\\ \times\int_s^\infty \vartheta^{(\alpha-a)(\mu-1)}\,\Omega^{-m}(\vartheta)\,d_\alpha\vartheta\end{pmatrix}.\end{aligned}$$

Now, since $\left(\vartheta^2/\Omega(\vartheta)\right)^m$ is non-decreasing, we have:

$$\begin{aligned}\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{\Omega^m(\vartheta)}d_\alpha\vartheta &\le \frac{1}{(\alpha-a)^{\mu-1}}\begin{pmatrix}\int_0^\infty s^{\mu(1-\alpha)+a(\mu-1)}h^\mu(s)\left(\frac{s^2}{\Omega(s)}\right)^m\\ \times\left(\int_s^\infty \vartheta^{(\alpha-a)(\mu-1)-2m}\,d_\alpha\vartheta\right)d_\alpha s\end{pmatrix}\\ &=\frac{1}{(\alpha-a)^{\mu-1}}\begin{pmatrix}\int_0^\infty s^{\mu(1-\alpha)+a(\mu-1)}h^\mu(s)\left(\frac{s^2}{\Omega(s)}\right)^m\\ \times\frac{-s^{(\alpha-a)(\mu-1)-2m+\alpha}}{(\alpha-a)(\mu-1)-2m+\alpha}d_\alpha s\end{pmatrix}\\ &=\frac{1}{(\alpha-a)^{\mu-1}((a-\alpha)(\mu-1)+2m-\alpha)}\begin{pmatrix}\int_0^\infty s^{\mu(1-\alpha)+a(\mu-1)+(\alpha-a)(\mu-1)-\mu+\alpha}\\ \times\frac{(sh(s))^\mu}{\Omega^m(s)}d_\alpha s\end{pmatrix},\end{aligned}$$

Then

$$\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{\Omega^m(\vartheta)}d_\alpha\vartheta \le \frac{1}{(\alpha-a)^{\mu-1}((a-\alpha)(\mu-1)+2m-\alpha)}\int_0^\infty \frac{(\vartheta h(\vartheta))^\mu}{\Omega^m(\vartheta)}d_\alpha\vartheta.$$

The proof is complete. □

Corollary 7. *[In Theorem 6] For $\alpha=1$, we have*

$$\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{\Omega^m(\vartheta)}d\vartheta \le \frac{1}{(1-a)^{\mu-1}((a-1)(\mu-1)+2m-1)}\int_0^\infty \frac{(\vartheta h(\vartheta))^\mu}{\Omega^m(\vartheta)}d\vartheta,$$

which is [[20], Theorem 1].

Proof. The proof follows from Theorem 6 for $\alpha=1$. □

Corollary 8. *[In Theorem 6] For $\alpha = 1, a = \frac{1}{\mu},\ m = \frac{\mu}{2}$ and $\Omega(\vartheta) = \vartheta^2$, then we have the classical Hardy inequality*

$$\int_0^\infty \left(\frac{\int_0^\vartheta h(s)ds}{\vartheta}\right)^\mu d\vartheta \leq \left(\frac{\mu}{\mu-1}\right)^\mu \int_0^\infty h^\mu(\vartheta)d\vartheta.$$

Proof. The proof follows from Theorem 6 for $\alpha = 1, a = \frac{1}{\mu},\ m = \frac{\mu}{2}$ and $\Omega(\vartheta) = \vartheta^2$. □

Theorem 7. *Let h be a non-negative α-integrable function on $(0,\infty)$, $k > 0, \vartheta/k(\vartheta)$ is non-decreasing and:*

$$\Omega(\vartheta) = \int_0^\vartheta s^{1-\alpha}k(s)d_\alpha s,\ \Lambda(\vartheta) = \int_0^\vartheta s^{1-\alpha}h(s)d_\alpha s,$$

where $0< \alpha \leq 1$, $a >0$, $m > (\alpha\mu + a(\mu-1)/2)$ and $0 < \mu < 1$. Then:

$$\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{\Omega^m(\vartheta)}d_\alpha\vartheta \geq \frac{1}{(\alpha+a)^{\mu-1}((\alpha+a)(1-\mu)+2m-\alpha)}\int_0^\infty \frac{(\vartheta h(\vartheta))^\mu}{\Omega^m(\vartheta)}d_\alpha\vartheta. \tag{15}$$

Proof. Since

$$\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{\Omega^m(\vartheta)}d_\alpha\vartheta = \int_0^\infty \Omega^{-m}(\vartheta)\left(\int_0^\vartheta s^{1-\alpha}s^{a(\frac{\mu-1}{\mu})}s^{-a(\frac{\mu-1}{\mu})}h(s)d_\alpha s\right)^\mu d_\alpha\vartheta,$$

applying reverse Hölder's inequality with index μ and $\mu/(\mu-1)$ we get:

$$\begin{aligned}\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{\Omega^m(\vartheta)}d_\alpha\vartheta &\geq \int_0^\infty \Omega^{-m}(\vartheta)\left(\begin{array}{c}\left(\int_0^\vartheta s^{\mu(1-\alpha)+a(1-\mu)}h^\mu(s)d_\alpha s\right)^{\frac{1}{\mu}}\\ \times\left(\int_0^\vartheta s^a d_\alpha s\right)^{\frac{\mu-1}{\mu}}\end{array}\right)^\mu d_\alpha\vartheta\\ &= \int_0^\infty \Omega^{-m}(\vartheta)\left(\begin{array}{c}\left(\int_0^\vartheta s^{\mu(1-\alpha)+a(1-\mu)}h^\mu(s)d_\alpha s\right)\\ \times\left(\frac{\vartheta^{\alpha+a}}{\alpha+a}\right)^{\mu-1}\end{array}\right)d_\alpha\vartheta\\ &= \frac{1}{(\alpha+a)^{\mu-1}}\left(\begin{array}{c}\int_0^\infty s^{\mu(1-\alpha)+a(1-\mu)}h^\mu(s)d_\alpha s\\ \times\left(\int_s^\infty \vartheta^{(\alpha+a)(\mu-1)}\ \Omega^{-m}(\vartheta)\ \right)d_\alpha\vartheta\end{array}\right).\end{aligned}$$

Now, since $\left(\vartheta^2/\Omega(\vartheta)\right)^m$ is non-decreasing, we have:

$$\begin{aligned}\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{\Omega^m(\vartheta)}d_\alpha\vartheta &\geq \frac{1}{(\alpha+a)^{\mu-1}}\left(\begin{array}{c}\int_0^\infty s^{\mu(1-\alpha)+a(1-\mu)}h^\mu(s)\left(\frac{s^2}{\Omega(s)}\right)^m\\ \times\left(\int_s^\infty \vartheta^{(\alpha+a)(\mu-1)-2m}\ d_\alpha\vartheta\right)\ d_\alpha s\end{array}\right)\\ &= \frac{1}{(\alpha+a)^{\mu-1}}\left(\begin{array}{c}\int_0^\infty s^{\mu(1-\alpha)+a(1-\mu)}h^\mu(s)\left(\frac{s^2}{\Omega(s)}\right)^m\\ \times\frac{-s^{(\alpha+a)(\mu-1)+\alpha-2m}}{(\alpha+a)(\mu-1)-2m+\alpha}d_\alpha s\end{array}\right)\\ &= \frac{1}{(\alpha+a)^{\mu-1}((\alpha+a)(1-\mu)+2m-\alpha)}\left(\begin{array}{c}\int_0^\infty s^{\mu(1-\alpha)+a(1-\mu)+(\alpha+a)(\mu-1)+\alpha-\mu}\\ \times\frac{(sh(s))^\mu}{\Omega^m(s)}d_\alpha s\end{array}\right),\end{aligned}$$

Then

$$\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{\Omega^m(\vartheta)}d_\alpha\vartheta \geq \frac{1}{(\alpha+a)^{\mu-1}((\alpha+a)(1-\mu)+2m-\alpha)}\int_0^\infty \frac{(\vartheta h(\vartheta))^\mu}{\Omega^m(\vartheta)}d_\alpha\vartheta.$$

The proof is complete. □

Corollary 9. *[In Theorem 7] For $\alpha = 1$, then we have*

$$\int_0^\infty \frac{\Lambda^\mu(\vartheta)}{\Omega^m(\vartheta)} d\vartheta \geq \frac{1}{(1+a)^{\mu-1}((1+a)(1-\mu)+2m-1)} \int_0^\infty \frac{(\vartheta h(\vartheta))^\mu}{\Omega^m(\vartheta)} d\vartheta,$$

which is [[20], Theorem 2].

Proof. The proof follows from Theorem 7 for $\alpha = 1$. □

4. Applications

Lyapunov's inequality is an important result in mathematics with many different applications see ([21,22] and the reference therein). The result, as proved by Lyapunov in 1907 [23], asserts that if $q(\vartheta)$ is real and continuous functions on $[a, b]$, then a necessary condition for the boundary value problem:

$$\begin{gathered} x''(\vartheta) + q(\vartheta)x(\vartheta) = 0,\ a < \vartheta < b, \\ x(a) = x(b) = 0, \end{gathered}$$

to have nontrivial solutions is given by:

$$\int_a^b q(\vartheta) d\vartheta > \frac{4}{b-a}$$

In this section, as the application of conformable fractional calculus, we obtain a Lyapunov-type inequality for a conformable fractional Sturm-Liouville equation subject to Dirichlet-type boundary conditions. To this aim, we must prove the following lemma.

Lemma 3. *Let $\mu \geq 1$ be a given real number and Let $q(\vartheta)$ be a non-negative and continuous function on $[a, b]$. Further, Let $x(\vartheta)$ be an absolutely continuous function on $[a, b]$, with $x(a) = x(b) = 0$. Then, the following inequality holds:*

$$\int_a^b q(\vartheta)|x(\vartheta)|^\mu d_\alpha\vartheta \leq \frac{1}{2}\left(\frac{b^\alpha - a^\alpha}{2\alpha}\right)^{\mu-1} \left(\int_a^b q(\vartheta) d_\alpha\vartheta\right) \int_a^b |D_\alpha x(\vartheta)|^\mu d_\alpha\vartheta. \tag{16}$$

Proof. Since $x(\vartheta) = \int_a^b D_\alpha x(s) d_\alpha s$, then $|x(\vartheta)| \leq \int_a^\vartheta |D_\alpha x(s)| d_\alpha s$ and also $|x(\vartheta)| \leq \int_\vartheta^b |D_\alpha x(s)| d_\alpha s$ and so

$$\begin{gathered} |x(\vartheta)| \leq \frac{1}{2}\left(\int_a^\vartheta |D_\alpha x(s)| d_\alpha s + \int_\vartheta^b |D_\alpha x(s)| d_\alpha s\right) \\ |x(\vartheta)|^\mu \leq \frac{1}{2^\mu}\left(\int_a^b |D_\alpha x(s)|\, d_\alpha\vartheta\right)^\mu \end{gathered}$$

Applying the Hölder's inequality with index μ and $\frac{\mu}{\mu-1}$, it follows that:

$$\begin{gathered} |x(\vartheta)|^\mu \leq \frac{1}{2^\mu}\left(\left(\int_a^b |D_\alpha x(\vartheta)|^\mu d_\alpha\vartheta\right)^{\frac{1}{\mu}} \cdot \left(\int_a^b d_\alpha\vartheta\right)^{\frac{\mu-1}{\mu}}\right)^\mu \\ |x(\vartheta)|^\mu \leq \frac{1}{2^\mu}\left(\int_a^b |D_\alpha x(\vartheta)|^\mu d_\alpha\vartheta\right) \cdot \left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{\mu-1} \\ |x(\vartheta)|^\mu \leq \frac{1}{2}\left(\frac{b^\alpha - a^\alpha}{2\alpha}\right)^{\mu-1}\left(\int_a^b |D_\alpha x(\vartheta)|^\mu d_\alpha\vartheta\right). \end{gathered}$$

Now, multiplying both sides of the above inequality by $q(\vartheta)$ and integrating the resulting inequality from a to b, we obtain the inequality (16). □

Corollary 10. *In (Lemma 3) at $\mu = 2$, then we have the inequality:*

$$\int_0^h q(\vartheta)(x(\vartheta))^2 d_\alpha \vartheta \leq \left(\frac{b^\alpha - a^\alpha}{4\alpha}\right)\left(\int_a^b q(\vartheta) d_\alpha \vartheta\right)\int_a^b (D_\alpha x(\vartheta))^2 d_\alpha \vartheta. \tag{17}$$

Theorem 8. *If the following fractional boundary value problem*

$$\begin{gathered} D_\alpha(D_\alpha x(\vartheta)) + q(\vartheta)x(\vartheta) = 0,\ a < \vartheta < b, \\ x(a) = x(b) = 0, \end{gathered} \tag{18}$$

has a nontrivial solution $x(\vartheta)$, where $q(\vartheta)$ is real and continuous functions, then:

$$\int_a^b q(\vartheta) d_\alpha \vartheta \ \geq \ \frac{4\alpha}{b^\alpha - a^\alpha}$$

Proof. Multiplying (18) by $x(\vartheta)$ and integrating by parts from a to b, where:

$$\begin{gathered} U(x) = x(\vartheta), D_\alpha V(x) = D_\alpha(D_\alpha x(\vartheta)) \\ D_\alpha U(x) = D_\alpha x(\vartheta),\ V(x) = D_\alpha x(\vartheta) \end{gathered}$$

then:

$$\begin{gathered} \int_a^b x(\vartheta) D_\alpha(D_\alpha x(\vartheta)) d_\alpha \vartheta + \int_a^b q(\vartheta)(x(\vartheta))^2 d_\alpha \vartheta = 0 \\ (x(\vartheta)(D_\alpha x(\vartheta)))|_a^b - \int_a^b (D_\alpha x(\vartheta) \cdot (D_\alpha x(\vartheta)\, d_\alpha \vartheta = -\int_a^b q(\vartheta)(x(\vartheta))^2 d_\alpha \vartheta \\ \int_a^b \left(D_\alpha x(\vartheta)\right)^2 d_\alpha \vartheta = \int_a^b q(\vartheta)(x(\vartheta))^2 d_\alpha \vartheta \end{gathered}$$

Applying the inequality (17)

$$\int_a^b \left(D_\alpha x(\vartheta)\right)^2 d_\alpha \vartheta \leq \left(\frac{b^\alpha - a^\alpha}{4\alpha}\right)\left(\int_a^b q(\vartheta) d_\alpha \vartheta\right)\int_a^b (D_\alpha x(\vartheta))^2 d_\alpha \vartheta,$$

then we have

$$\int_a^b q(\vartheta) d_\alpha \vartheta \geq \frac{4\alpha}{b^\alpha - a^\alpha}.$$

□

Corollary 11. *In (Theorem 8) at $\alpha = 1$, then we obtain the classical Lyapunov-type inequality for differential equation subject to Dirichlet-type boundary conditions.*

Author Contributions: Formal analysis, G.A., M.K., M.Z. and C.C.; conceptualization, H.M.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Deanship of Scientific Research at Princess Nourah Bint Abdulrahman University through the Fast-track Research Funding Program. This paper is supported financially by the Academy of Scientific Research and Technology (ASRT), Egypt, under initiatives of Science Up Faculty of Science (Grant No. 6695).

Data Availability Statement: No data were used to support this study.

Conflicts of Interest: The authors declare that they have no conflict of interest.

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