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The Influence of White Noise and the Beta Derivative on the Solutions of the BBM Equation

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Abstract: In the current study, we investigate the stochastic Benjamin–Bona–Mahony equation with beta derivative (SBBME-BD). The considered stochastic term is the multiplicative noise in the Itô sense. By combining the $\mathcal{F}$-expansion approach with two separate equations, such as the Riccati and elliptic equations, new hyperbolic, trigonometric, rational, and Jacobi elliptic solutions for SBBME-BD can be generated. The solutions to the Benjamin–Bona–Mahony equation are useful in understanding various scientific phenomena, including Rossby waves in spinning fluids and drift waves in plasma. Our results are presented using MATLAB, with numerous 3D and 2D figures illustrating the impacts of white noise and the beta derivative on the obtained solutions of SBBME-BD.

Keywords: stability by noise; exact solutions; $\mathcal{F}$-expansion method; beta-derivative; stochastic BBM

MSC: 35C08; 35C07; 35C05; 83C15; 35A20

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1. Introduction

Nonlinear evolution equations (NEEs) are utilized to explain complex phenomena in many disciplines, including optical fiber communication, chemical kinetics, population dynamics, chaotic systems, photonic, plasma physics, electromagnetism, ocean wave, wave propagation, nuclear physics, fluid mechanics, and solid-state physics. Obtaining traveling wave solutions for NEEs is the most significant physical challenge. There are several effective methods for solving NEEs, including the generalized Kudryashov approach [1], modified decomposition approach [2], Riccati equation expansion [3], sine-Gordon expansion [4], sine-cosine method [5], Exp-function [6], improved $\tan(\varphi/2)$-expansion [7], Lie symmetry [8], Jacobi elliptic function [9], and the tanh–sech method [10] .

Recently, numerous mathematicians have introduced several fractional derivatives. Some of the most well-known are those presented by Caputo, Riemann–Liouville, Grunwald–Letnikov, Kober, Erdelyi, Marchaud, Hadamard, and Riesz [11–14]. Most kinds of fractional derivatives do not follow the chain rule, quotient rule, or product rule. In recent years, Atangana et al. [15] produced a new operator derivative called the beta-derivative (BD), which extends the classical derivative. If $f:(0,\infty)\to\mathbb{R}$ then its beta derivative [15] is defined as:

$$\mathcal{D}_x^{\beta}\phi(x)=\frac{d^{\beta}\phi}{dx^{\beta}}=\lim_{h\to 0}\frac{\phi(x+h(x+\frac{1}{\Gamma(\beta)})^{1-\beta})-\phi(x)}{h},\ 0<\beta\leq 1.$$

Moreover, the BD possesses the following properties [15] for all real numbers a and b:
(1) $\mathcal{D}_x^{\beta}f(y)=(x+\frac{1}{\Gamma(\beta)})^{1-\beta}\frac{df}{dx}$,

(2) $\mathcal{D}_x^\beta(af+bg) = a\mathcal{D}_x^\beta(f) + b\mathcal{D}_x^\beta(g)$,
(3) $\mathcal{D}_x^\beta(f \circ g(x)) = (x + \frac{1}{\Gamma(\beta)})^{1-\beta} g'(x) f'(g(x))$, (4) $\mathcal{D}_x^\beta(a) = 0$.

Moreover, stochastic partial differential equations (SDEs) have a wide range of applications in physics, including molecular dynamics, neurodynamics, climate dynamics, geophysics, biology, physics, chemistry, and other scientific disciplines [16–18]. More precisely, SDEs characterize all dynamical equations in which quantum influences are either insignificant or can be accounted for as perturbations. SDEs are an extension of the theory of dynamical systems to models with noise. This is a significant generalization because actual systems cannot be entirely isolated from their surroundings and, as a result, are always subject to external stochastic influence.

As a consequence, obtaining exact solutions to fractional or stochastic differential equations is critical. Many analytical and numerical methods, including the (G'/G)-expansion method [19], the mapping method [20], the Jacobi elliptic function technique [21], the extended tanh-coth method [22], bifurcation analysis [23,24], and more.

Therefore, it is critical here to look at the stochastic Benjamin–Bona–Mahony equation with beta derivative (SBBME-BD) as follows:

$$\mathcal{Q}_t + 6\mathcal{Q}\mathcal{D}_x^\beta \mathcal{Q} + \mathcal{D}_{xxx}^\beta \mathcal{Q} - \alpha \mathcal{D}_{xx}^\beta \mathcal{Q}_t = \sigma(\mathcal{Q} - \alpha \mathcal{D}_{xx}^\beta \mathcal{Q})\mathcal{B}_t, \tag{1}$$

where the function $\mathcal{Q} = \mathcal{Q}(x,t)$ is real, σ is the strength of the noise, $\mathcal{B} = \mathcal{B}(t)$ is a white noise that satisfies the following properties: (i) $\mathcal{B}$ has continuous trajectories, (ii) $\mathcal{B}(0) = 0$, and (iii) $\mathcal{B}(t_{i+1}) - \mathcal{B}(t_i)$ has standard normal distribution. If we put $\sigma = 0$, and $\beta = 1$, then we obtain the Benjamin–Bona–Mahony equation as follows:

$$\mathcal{Q}_t + 6\mathcal{Q}\mathcal{Q}_x + \mathcal{Q}_{xxx} - \alpha \mathcal{Q}_{xxt} = 0. \tag{2}$$

Benjamin, Bona, and Mahony [25] investigated Equation (2) as a modification of the KdV equation. The modified equation was proposed to simulate long surface gravity waves with small amplitudes propagating in a 1 + 1 dimension. Many researchers have acquired the exact solutions of Equation (2) by applying many various methods, such as the generalized (G'/G)-expansion method [26], (G'/G)-expansion method [27], Hirota's bilinear method [28], the Lie group method [29], the exp-function method [30], the tanh–coth method, and the sn–ns method [31]. The stochastic Benjamin–Bona–Mahony equation with beta derivative has not been considered until now.

The motivation behind this study is to obtain exact stochastic solutions of SBBME-BD (1) using the $\mathcal{F}$-expansion approach combined with two distinct equations, namely the Riccati and elliptic equations. The presence of a stochastic term in the equation makes these solutions particularly useful for physicists in understanding important physical phenomena. Moreover, we present various 2D and 3D graphical representations using the MATLAB program to explore the impact of the Beta derivative and noise on the exact solution of SBBME-BD (1).

The sequence of the paper is as follows: In Section 2, we derive the wave equation for the SBBME-BD (1). In Section 3, the solution of the SBBME-BD (1) may be obtained by using $\mathcal{F}$white noise and the BD on the obtained solutions of SBBME-BD (1). In the end, the conclusions of this paper are introduced.

2. Traveling Wave Equation for SBBME-BD

The wave equation for SBBME-BD (1) is achieved by applying:

$$\mathcal{Q}(x,t) = \mathcal{G}(\zeta) e^{[\sigma \mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}, \quad \zeta = \frac{\zeta_1}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta + \zeta_2 t, \tag{3}$$

where $\mathcal{G}$ is a deterministic function, and $\zeta_1,\ \zeta_2$ are unknown constants. We note that

$$\begin{aligned}\mathcal{Q}_t &= [\zeta_2\mathcal{G}' + \sigma\mathcal{G}\mathcal{B}_t + \frac{1}{2}\sigma^2\mathcal{G} - \frac{1}{2}\sigma^2\mathcal{G}]e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]} \\ &= [\zeta_2\mathcal{G}' + \sigma\mathcal{G}\mathcal{B}_t]e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]},\end{aligned} \tag{4}$$

and

$$\mathcal{D}^{\beta}_{xx}\mathcal{Q}_t = [\zeta_1^2\zeta_2\mathcal{G}''' + \sigma\zeta_1^2\mathcal{G}''\mathcal{B}_t]e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]} \tag{5}$$

$$\mathcal{D}^{\beta}_{x}\mathcal{Q} = \zeta_1\mathcal{G}'e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]},\ \mathcal{D}^{\beta}_{xxx}\mathcal{Q} = \zeta_1^3\mathcal{G}'''e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}. \tag{6}$$

Inserting Equation (3) into Equation (1) and utilizing (4)–(6), we obtain

$$\zeta_2\mathcal{G}' + (\zeta_1^3 - \alpha\zeta_1^2\zeta_2)\mathcal{G}''' + 6\zeta_1\mathcal{G}\mathcal{G}'e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]} = 0. \tag{7}$$

Taking the expectations on both sides, we have

$$\zeta_2\mathcal{G}' + (\zeta_1^3 - \alpha\zeta_1^2\zeta_2)\mathcal{G}''' + 6\zeta_1\mathcal{G}\mathcal{G}'e^{-\frac{1}{2}\sigma^2 t}\mathbb{E}e^{[\sigma\mathcal{B}(t)]} = 0. \tag{8}$$

Since $\mathcal{B}(t)$ is a Gaussian process, then $\mathbb{E}(e^{\sigma\mathcal{B}(t)}) = e^{\frac{1}{2}\sigma^2 t}$. Thus, Equation (8) becomes

$$\zeta_2\mathcal{G}' + (\zeta_1^3 - \alpha\zeta_1^2\zeta_2)\mathcal{G}''' + 6\zeta_1\mathcal{G}\mathcal{G}' = 0. \tag{9}$$

Integrating Equation (9) once with a zero integration constant yields

$$\mathcal{G}'' + \gamma_1\mathcal{G} + \gamma_2\mathcal{G}^2 = 0, \tag{10}$$

where

$$\gamma_1 = \frac{\zeta_2}{(\zeta_1^3 - \alpha\zeta_1^2\zeta_2)} \text{ and } \gamma_2 = \frac{3}{(\zeta_1^2 - \alpha\zeta_1\zeta_2)}.$$

3. Exact Solutions of SBBME-BD

Utilizing the $\mathcal{F}$-expansion method ($\mathcal{F}$-EM) with two different equations, such as the Riccati equation and elliptic equation, the solutions to Equation (10) are discovered. Afterward, the SBBME-BD solutions (1) can be obtained.

3.1. $\mathcal{F}$-EM with Riccati Equation

Assuming that the solution $\mathcal{G}$ of Equation (10) has the form:

$$\mathcal{G}(\zeta) = \hbar_0 + \sum_{k=1}^{J}\hbar_k\mathcal{F}^k, \tag{11}$$

where $\mathcal{F}$ is the solution of the Riccati equation:

$$\mathcal{F}' = \mathcal{F}^2 + \phi, \tag{12}$$

Determining J needs balancing $\mathcal{G}''$ with $\mathcal{G}^2$ in Equation (10) as

$$J + 2 = 2J \ \Rightarrow\ J = 2.$$

Equation (11) becomes

$$\mathcal{G}(\zeta) = \hbar_0 + \hbar_1\mathcal{F} + \hbar_2\mathcal{F}^2. \tag{13}$$

Equation (12) has the following solution:

$$\mathcal{F}(\zeta) = \sqrt{\phi}\tan(\sqrt{\phi}\zeta) \;\text{ or } \mathcal{F}(\zeta) = -\sqrt{\phi}\cot(\sqrt{\phi}\zeta), \tag{14}$$

If $\phi > 0$, or

$$\mathcal{F}(\zeta) = -\sqrt{-\phi}\tanh(\sqrt{-\phi}\zeta) \;\text{ or } \mathcal{F}(\zeta) = -\sqrt{-\phi}\coth(\sqrt{-\phi}\zeta), \tag{15}$$

If $\phi < 0$, or

$$\varphi(\zeta) = \frac{-1}{\zeta}, \tag{16}$$

If $\phi = 0$.

Now, putting Equation (13) into Equation (10), we have

$$(6\hbar_2 + \gamma_2\hbar_2^2)\mathcal{F}^4 + (2\hbar_1 + 2\gamma_2\hbar_1\hbar_2)\mathcal{F}^3 + (8\phi\hbar_2 + 2\hbar_0\hbar_2\gamma_2 + \hbar_1^2\gamma_2 + \gamma_1\hbar_2)\mathcal{F}^2$$
$$(2\phi\hbar_1 + \gamma_1\hbar_1 + 2\gamma_2\hbar_0\hbar_1)\mathcal{F} + (2\phi^2\hbar_2 + \gamma_1\hbar_0 + \gamma_2\hbar_0^2) = 0$$

Putting the coefficients of $\mathcal{F}$ to zero:

$$6\hbar_2 + \gamma_2\hbar_2^2 = 0,$$

$$2\hbar_1 + 2\gamma_2\hbar_1\hbar_2 = 0,$$

$$8\phi\hbar_2 + 2\hbar_0\hbar_2\gamma_2 + \hbar_1^2\gamma_2 + \gamma_1\hbar_2 = 0,$$

$$2\phi\hbar_1 + \gamma_1\hbar_1 + 2\gamma_2\hbar_0\hbar_1 = 0,$$

and

$$2\phi^2\hbar_2 + \gamma_1\hbar_0 + \gamma_2\hbar_0^2 = 0.$$

By solving these equations, we obtain the two families of solutions:

First family:

$$\hbar_0 = \frac{-6\phi}{\gamma_2},\; \hbar_1 = 0,\; \hbar_2 = \frac{-6}{\gamma_2},\; \zeta_2 = \frac{4\phi\zeta_1^3}{1 + 4\alpha\phi\zeta_1^2}, \tag{17}$$

Second family:

$$\hbar_0 = \frac{-2\phi}{\gamma_2},\; \hbar_1 = 0,\; \hbar_2 = \frac{-6}{\gamma_2},\; \zeta_2 = \frac{-4\phi\zeta_1^3}{1 - 4\alpha\phi\zeta_1^2}, \tag{18}$$

First family: The solution to Equation (10) is as follows:

$$\mathcal{G}(\zeta) = \frac{-6\phi}{\gamma_2} - \frac{6}{\gamma_2}\mathcal{F}^2(\zeta).$$

There are three distinct cases for $\mathcal{F}(\zeta)$:

Case 1: If $\phi > 0$, then with (14), we have

$$\mathcal{G}(\zeta) = \frac{-6\phi}{\gamma_2} - \frac{6\phi}{\gamma_2}\tan^2(\sqrt{\phi}\zeta) = -\frac{6\phi}{\gamma_2}\sec^2(\sqrt{\phi}\zeta),$$

and

$$\mathcal{G}(\zeta) = \frac{-6\phi}{\gamma_2} - \frac{6\phi}{\gamma_2}\cot^2(\sqrt{\phi}\zeta) = \frac{-6\phi}{\gamma_2}\csc^2(\sqrt{\phi}\zeta).$$

Consequently, the solution of SBBME-BD (1) is

$$\mathcal{Q}(x,t) = -\frac{6\phi}{\gamma_2}\sec^2(\sqrt{\phi}\zeta)e^{(\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t)}, \tag{19}$$

and

$$\mathcal{Q}(x,t) = \frac{-6\phi}{\gamma_2}\csc^2(\sqrt{\phi}\zeta)e^{(\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t)}, \tag{20}$$

where $\zeta = \frac{\zeta_1}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{4\phi\zeta_1^3}{1+4\alpha\phi\zeta_1^2}t$.

Case 2: If $\phi < 0$, then by using (15), we have

$$\mathcal{G}(\zeta) = \frac{-6\phi}{\gamma_2} + \frac{6\phi}{\gamma_2}\tanh^2(\sqrt{-\phi}\zeta) = \frac{-6\phi}{\gamma_2}\mathrm{sech}^2(\sqrt{-\phi}\zeta),$$

and

$$\mathcal{G}(\zeta) = \frac{-6\phi}{\gamma_2} + \frac{6\phi}{\gamma_2}\coth^2(\sqrt{-\phi}\zeta) = \frac{6\phi}{\gamma_2}\mathrm{csch}^2(\sqrt{-\phi}\zeta).$$

Consequently, the solution of SBBME-BD (1) is

$$\mathcal{Q}(x,t) = \frac{-6\phi}{\gamma_2}\mathrm{sech}^2(\sqrt{-\phi}\zeta)e^{(\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t)}, \tag{21}$$

and

$$\mathcal{Q}(x,t) = \frac{6\phi}{\gamma_2}\mathrm{csch}^2(\sqrt{-\phi}\zeta)e^{(\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t)}. \tag{22}$$

Case 3: If $\phi = 0$, then by using (16), we have

$$\mathcal{G}(\zeta) = \frac{6}{\gamma_2}\frac{1}{\zeta^2}.$$

Consequently, the solution of SBBME-BD (1) is

$$\mathcal{Q}(x,t) = [-\frac{6}{\gamma_2}\frac{1}{\zeta^2}]e^{(\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t)}, \tag{23}$$

where $\zeta = \frac{\zeta_1}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{4\phi\zeta_1^3}{1+4\alpha\phi\zeta_1^2}t$.

Second family: Equation (10) has the solution

$$\mathcal{G}(\zeta) = \frac{-2\phi}{\gamma_2} - \frac{6}{\gamma_2}\mathcal{F}^2(\zeta)$$

There are three distinct cases for $\mathcal{F}(\zeta)$:

Case 1: If $\phi > 0$, then by using (14), we have

$$\mathcal{G}(\zeta) = \frac{-2\phi}{\gamma_2} - \frac{6\phi}{\gamma_2}\tan^2(\sqrt{\phi}\zeta),$$

and

$$\mathcal{G}(\zeta) = \frac{-2\phi}{\gamma_2} - \frac{6\phi}{\gamma_2}\cot^2(\sqrt{\phi}\zeta).$$

Consequently, the solution of SBBME-BD (1) is

$$\mathcal{Q}(x,t) = [\frac{-2\phi}{\gamma_2} - \frac{6\phi}{\gamma_2}\tan^2(\sqrt{\phi}\zeta)]e^{(\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t)}, \tag{24}$$

and

$$\mathcal{Q}(x,t) = [\frac{-2\phi}{\gamma_2} - \frac{6\phi}{\gamma_2}\cot^2(\sqrt{\phi}\zeta)]e^{(\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t)}, \tag{25}$$

where $\zeta = \frac{\zeta_1}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{4\phi\zeta_1^3}{1-4\alpha\phi\zeta_1^2}t$.

Case 2: If $\phi < 0$, then by using (15), we have

$$\mathcal{G}(\zeta) = \frac{-2\phi}{\gamma_2} + \frac{6\phi}{\gamma_2}\tanh^2(\sqrt{-\phi}\zeta),$$

and

$$\mathcal{G}(\zeta) = \frac{-2\phi}{\gamma_2} + \frac{6\phi}{\gamma_2}\coth^2(\sqrt{-\phi}\zeta).$$

Consequently, the solution of SBBME-BD (1) is

$$\mathcal{Q}(x,t) = [\frac{-2\phi}{\gamma_2} + \frac{6\phi}{\gamma_2}\tanh^2(\sqrt{-\phi}\zeta)]e^{(\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t)}, \tag{26}$$

and

$$\mathcal{Q}(x,t) = [\frac{-2\phi}{\gamma_2} + \frac{6\phi}{\gamma_2}\coth^2(\sqrt{-\phi}\zeta)]e^{(\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t)}, \tag{27}$$

where $\zeta = \frac{\zeta_1}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{4\phi\zeta_1^3}{1-4\alpha\phi\zeta_1^2}t$.

Case 3: If $\phi = 0$, then by using (16), we have

$$\mathcal{G}(\zeta) = \frac{6}{\gamma_2}\frac{1}{\zeta^2}.$$

Consequently, the solution of SBBME-BD (1) is

$$\mathcal{Q}(x,t) = \frac{6}{\gamma_2}\frac{1}{\zeta^2}e^{(\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t)}, \tag{28}$$

where $\zeta = \frac{\zeta_1}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{4\phi\zeta_1^3}{1-4\alpha\phi\zeta_1^2}t$.

3.2. $\mathcal{F}$-EM with Elliptic Equation

Suppose that the solution of Equation (10) has the form (13). However, at this time, $\mathcal{F}$ solves the following elliptic equation:

$$\mathcal{F}' = \sqrt{R + K\mathcal{F}^2 + P\mathcal{F}^4}, \tag{29}$$

where R, K, and P are constants. Differentiating Equation (13) twice and using (29), we have

$$\mathcal{G}'' = \hbar_1(K\mathcal{F} + 2P\mathcal{F}^3) + 2\hbar_2(R + 2K\mathcal{F}^2 + 3P\mathcal{F}^4). \tag{30}$$

Setting Equation (13) and Equation (30) into Equation (10), we have

$$(6\hbar_2 P + \gamma_2\hbar_2^2)\mathcal{F}^4 + (2P\hbar_1 + 2\hbar_1\hbar_2\gamma_2)\mathcal{F}^3 + (4\hbar_2 K + 2\gamma_2\hbar_0\hbar_2 + \hbar_1^2$$
$$+\hbar_2\gamma_1)\mathcal{F}^2 + (\hbar_1 K + 2\gamma_2\hbar_0\hbar_1 + \gamma_1\hbar_1)\mathcal{F} + (2R\hbar_2 + \gamma_1\hbar_0 + \gamma_2\hbar_0^2) = 0.$$

If we assign each coefficient of $\mathcal{F}^k$ to 0, we will have a system of equations. Here are the two families we obtain when we solve this system for $K^2 - 3RP > 0$:

First family:

$$\hbar_0 = -2(\frac{K + \sqrt{(K^2 - 3RP)}}{\gamma_2}),\ \hbar_1 = 0,\ \hbar_2 = \frac{-6P}{\gamma_2},\ \zeta_2 = \frac{4\sqrt{(K^2 - 3RP)}\zeta_1^3}{1 + 4\alpha\sqrt{(K^2 - 3RP)}\zeta_1^2}.$$

Second family:

$$\hbar_0 = -2(\frac{K - \sqrt{(K^2 - 3RP)}}{\gamma_2}),\ \hbar_1 = 0,\ \hbar_2 = \frac{-6P}{\gamma_2},\ \zeta_2 = \frac{-4\sqrt{(K^2 - 3RP)}\zeta_1^3}{1 - 4\alpha\sqrt{(K^2 - 3RP)}\zeta_1^2}.$$

In both families, the solution of Equation (10) takes the form:

$$\mathcal{G}(\zeta) = \hbar_0 + \hbar_2\mathcal{F}^2(\zeta). \tag{31}$$

There are many cases for $\mathcal{F}$ depending on $P,\ K$ and R such that $K^2 - 3RP > 0$ as follows :

Case	P	K	R	$F(\zeta)$
1	ρ^2	$-(1+\rho^2)$	1	$sn(\zeta)$
2	1	$2\rho^2 - 1$	$-\rho^2(1-\rho^2)$	$ds(\zeta)$
3	1	$2 - \rho^2$	$(1-\rho^2)$	$cs(\zeta)$
4	$-\rho^2$	$2\rho^2 - 1$	$(1-\rho^2)$	$cn(\zeta)$
5	-1	$2 - \rho^2$	$(\rho^2 - 1)$	$dn(\zeta)$
6	$\frac{\rho^2}{4}$	$\frac{(\rho^2-2)}{2}$	$\frac{1}{4}$(or $\frac{\rho^2}{4}$)	$\frac{sn(\zeta)}{1\pm dn(\zeta)}$
7	$\frac{-1}{4}$	$\frac{(\rho^2+1)}{2}$	$\frac{-(1-\rho^2)^2}{4}$	$\rho cn(\zeta) \pm dn(\zeta)$
8	$\frac{\rho^2-1}{4}$	$\frac{(\rho^2+1)}{2}$	$\frac{(\rho^2-1)}{4}$	$\frac{dn(\zeta)}{1\pm sn(\zeta)}$
9	$\frac{1-\rho^2}{4}$	$\frac{(1-\rho^2)}{2}$	$\frac{(1-\rho^2)}{4}$	$\frac{cn(\zeta)}{1\pm sn(\zeta)}$

For the first family: the solutions of SBBME-BD (1) are

$$\mathcal{Q}_1(x,t) = [\frac{2(1+\rho^2) - 2\sqrt{\rho^4 - \rho^2 + 1}}{\gamma_2} - \frac{6\rho^2}{\gamma_2}sn^2(\zeta)]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}, \tag{32}$$

$$\mathcal{Q}_2(x,t) = [\frac{(2 - 4\rho^2) - 2\sqrt{\rho^4 - \rho^2 + 1}}{\gamma_2} - \frac{6}{\gamma_2}ds^2(\zeta)]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}, \tag{33}$$

$$\mathcal{Q}_3(x,t) = [\frac{(2\rho^2 - 4) - 2\sqrt{\rho^4 + \rho^2 + 1}}{\gamma_2} - \frac{6}{\gamma_2}cs^2(\zeta)]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}, \tag{34}$$

$$\mathcal{Q}_4(x,t) = [\frac{(2 - 4\rho^2) - 2\sqrt{\rho^4 - \rho^2 + 1}}{\gamma_2} + \frac{6\rho^2}{\gamma_2}ds^2(\zeta)]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}. \tag{35}$$

$$\mathcal{Q}_5(x,t) = [\frac{(2\rho^2 - 4) - 2\sqrt{\rho^4 - \rho^2 + 1}}{\gamma_2} + \frac{6}{\gamma_2}dn^2(\zeta)]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}. \tag{36}$$

$$\mathcal{Q}_6(x,t) = [\frac{(4 - 2\rho^2) - \sqrt{4\rho^4 - 19\rho^2 + 16}}{2\gamma_2} - \frac{3\rho^2}{2\gamma_2}\frac{sn^2(\zeta)}{(1 \pm dn(\zeta))^2}]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}. \tag{37}$$

$$\mathcal{Q}_7(x,t) = [\frac{-(2\rho^2 + 2) - \sqrt{\rho^4 + 14\rho^2 + 1}}{2\gamma_2} + \frac{3}{2\gamma_2}(\rho cn(\zeta) \pm dn(\zeta))^2]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}. \tag{38}$$

$$\mathcal{Q}_8(x,t) = [\frac{-(2\rho^2 + 2) - \sqrt{\rho^4 + 14\rho^2 + 1}}{2\gamma_2} - \frac{3(\rho^2 - 1)}{2\gamma_2}\frac{dn^2(\zeta)}{[1 \pm sn(\zeta)]^2}]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}. \tag{39}$$

$$\mathcal{Q}_9(x,t) = [\frac{(2\rho^2-2) - \sqrt{\rho^4 - 2\rho^2 + 1}}{2\gamma_2} - \frac{3(1-\rho^2)}{2\gamma_2}\frac{cn^2(\zeta)}{[1 \pm sn(\zeta)]^2}]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}. \quad (40)$$

If $\rho \to 1$ in Equations (32)–(40), then we attain the soliton solutions for SBBME-BD (1) as:

$$\mathcal{Q}(x,t) = [\frac{2}{\gamma_2} - \frac{6}{\gamma_2}\tanh^2(\zeta)]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}. \quad (41)$$

$$\mathcal{Q}(x,t) = [\frac{-4}{\gamma_2} - \frac{6}{\gamma_2}\text{csch}^2(\zeta)]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}. \quad (42)$$

$$\mathcal{Q}(x,t) = [\frac{-4}{\gamma_2} + \frac{6}{\gamma_2}\text{sech}^2(\zeta)]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}. \quad (43)$$

$$\mathcal{Q}(x,t) = [\frac{1}{2\gamma_2} - \frac{3}{2\gamma_2}\frac{\tanh^2(\zeta)}{(1 \pm \text{sech}(\zeta))^2}]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}. \quad (44)$$

$$\mathcal{Q}(x,t) = [\frac{1}{2\gamma_2} - \frac{3}{2\gamma_2}(\coth(\zeta) \mp \text{csch}(\zeta))^2]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}. \quad (45)$$

If $\rho \to 0$ in Equations (32)–(40), then we acquire the triangular periodic solutions for SBBME-BD (1) as:

$$\mathcal{Q}(x,t) = -\frac{6}{\gamma_2}\csc^2(\zeta)]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}. \quad (46)$$

$$\mathcal{Q}(x,t) = [\frac{-6}{\gamma_2} - \frac{6}{\gamma_2}\cot^2(\zeta)]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]} = -\frac{6}{\gamma_2}\csc^2(\zeta)e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}. \quad (47)$$

$$\mathcal{Q}(x,t) = \frac{-3}{2\gamma_2}[1 - \frac{1}{[1 \pm \sin(\zeta)]^2}]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}. \quad (48)$$

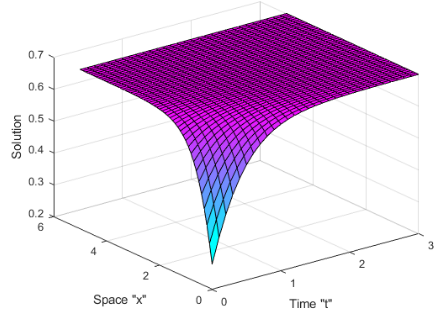
$$\mathcal{Q}(x,t) = \frac{-3}{2\gamma_2}[1 + \frac{\cos^2(\zeta)}{[1 \pm \sin(\zeta)]^2}]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}. \quad (49)$$

Second Family: By following the same steps as the first family, the same solutions may be found with various coefficients.

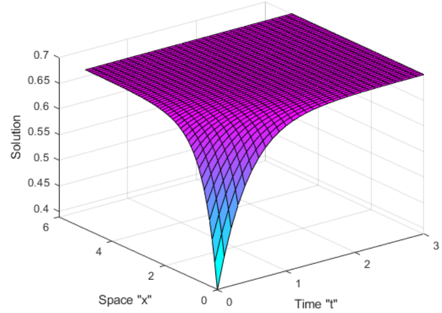
4. Impacts of the Beta Derivative and Noise on SBBME-BD Solutions

We discuss the impact of the BD and white noise on the exact solutions of the SBBME-BD (1). To demonstrate the behavior of these solutions, we provide various graphs. For a different σ (noise intensity), we run some simulations for acquired solutions, including Equations (26) and (32). Let us first fix the parameters $\zeta_1 = 1$, $\phi = -1$, $\alpha = \frac{1}{2}$ and $\rho = 0.5$. Moreover, let $x \in [0,6]$ and $t \in [0,3]$.

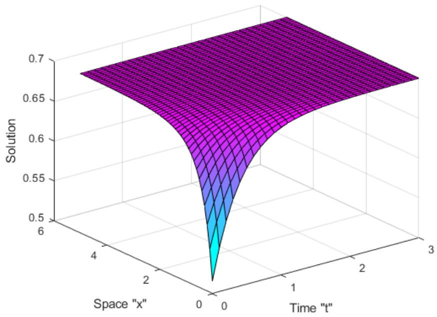
Effects of the beta derivative: When β decreases, we can observe in Figures 1 and 2 that the form of the graph is compressed:



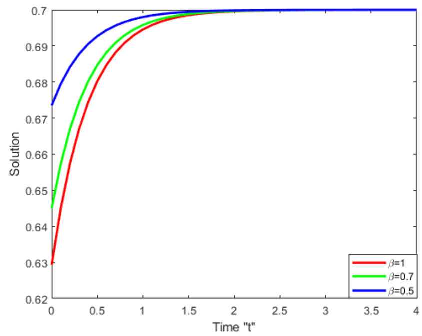
(**a**) $\sigma = 0$, and $\beta = 1$



(**b**) $\sigma = 0$, and $\beta = 0.7$

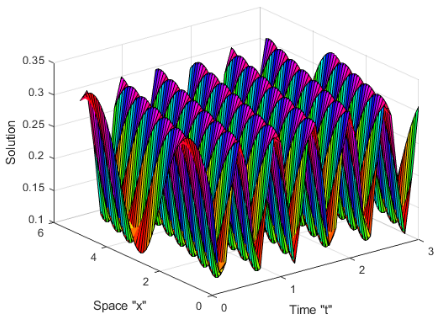


(**c**) $\sigma = 0$, and $\beta = 0.5$

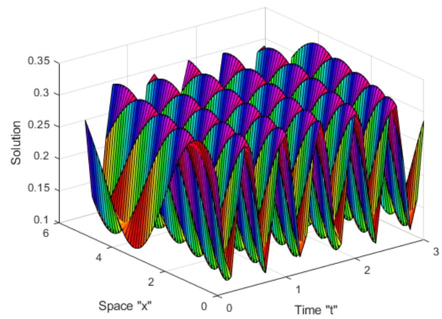


(**d**) $\sigma = 0$, and $\beta = 1,\ 0.7,\ 0.5$

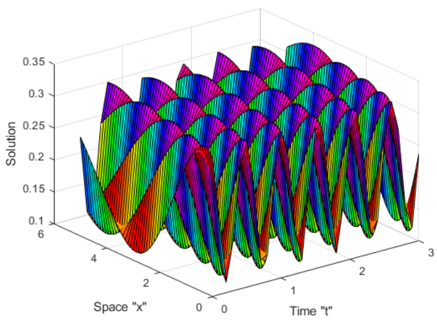
Figure 1. (**a**–**c**) show the 3D shapes of Equation (26) with $\sigma = 0$ and different values of $\beta = 1,\ 0.7,\ 0.5$ (**d**) Depicts a graph in two dimensions for these values of β.



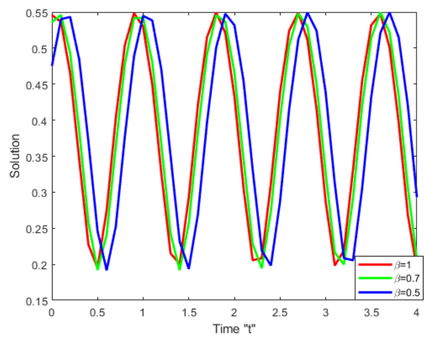
(**a**) $\sigma = 0$, and $\beta = 1$



(**b**) $\sigma = 0$, and $\beta = 0.7$



(**c**) $\sigma = 0$, and $\beta = 0.5$

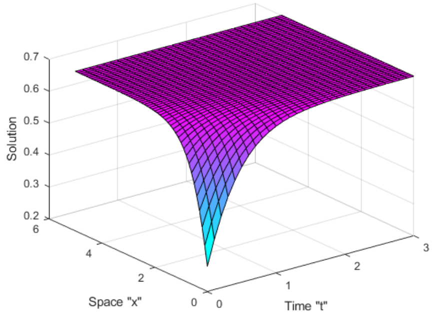


(**d**) $\sigma = 0$, and $\beta = 1,\ 0.7,\ 0.5$

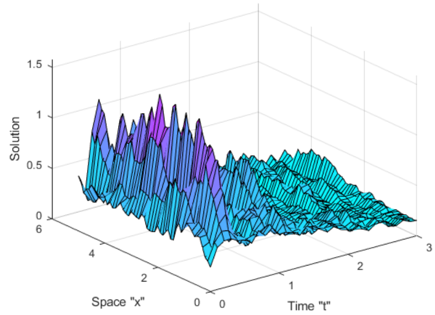
Figure 2. (**a**–**c**) show the 3D shapes of Equation (32) with $\sigma = 0$ and various values of $\beta = 1,\ 0.7,\ 0.5$ (**d**) Depicts a graph in two dimensions for these values of β.

As we can see in Figures 1 and 2, the solution curves do not intersect. Additionally, the curves shift to the right when the order of the beta derivative increases.

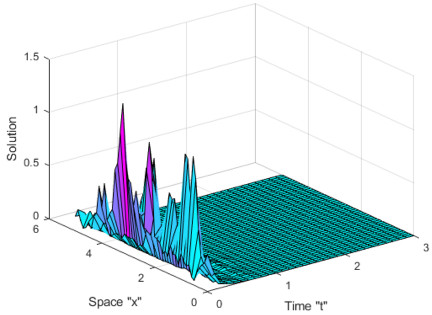
Impacts of white noise: The impact of noise on the solutions is seen in Figures 3 and 4 as follows:



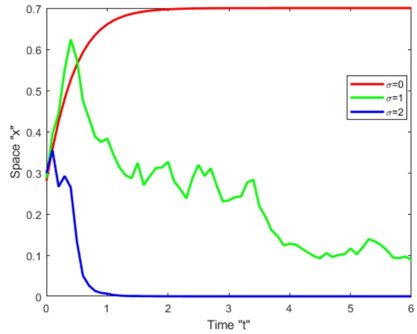
(**a**) $\sigma = 0$



(**b**) $\sigma = 1$

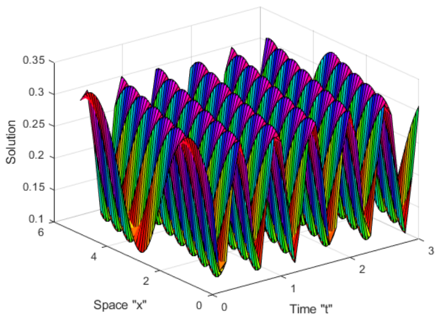


(**c**) $\sigma = 2$

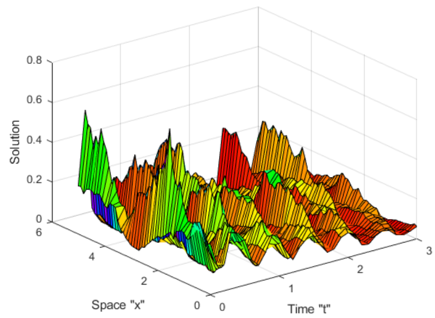


(**d**) $\sigma = 0,\ 1,\ 2$

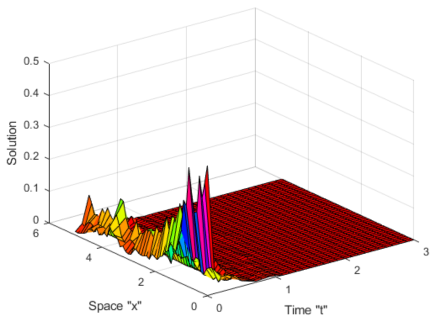
Figure 3. (**a**–**c**) show the 3D shapes of the solution $\mathcal{Q}(x,t)$ to Equation (26) for various values of $\sigma = 0,\ 1,\ 2$ (**d**) Depicts a graph in two dimensions for these values of σ.



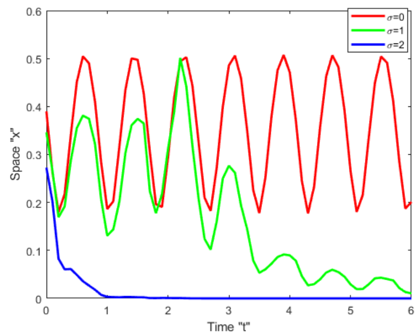
(**a**) $\sigma = 0$



(**b**) $\sigma = 1$



(**c**) $\sigma = 2$



(**d**) $\sigma = 0,\ 1,\ 2$

Figure 4. (**a**–**c**) show the 3D shapes of the solution $\mathcal{Q}(x,t)$ to Equation (32) for various values of $\sigma = 0,\ 1,\ 2$ (**d**) Depicts a graph in two dimensions for these values of σ.

From Figures 3 and 4, we can conclude that there are distinct types of solutions, such as hyperbolic, trigonometric, rational, and Jacobi elliptic solutions, when the noise is ignored (i.e., at $\sigma = 0$). Adding noise with a strength of $\sigma = 1, 2$ causes the surface to become much flatter following tiny transit patterns, as verified by the 2D graph. This demonstrates that the solutions of SBBME-BD (1) tend to converge around zero when white noise is present.

5. Conclusions

We looked at the stochastic Benjamin–Bona–Mahony Equation (1) with beta derivative (SBBME-BD). The solutions to the Benjamin–Bona–Mahony equation are helpful in understanding several exciting scientific phenomena, such as Rossby waves in rotating fluids and drift waves in plasma. New hyperbolic, trigonometric, rational, and Jacobi elliptic solutions for SBBME-BD were obtained by combining the $\mathcal{F}$-expansion approach with two separate equations, namely the Riccati and elliptic equations. Numerous fascinating and difficult physical occurrences may only be understood with these solutions. The MATLAB program was utilized to investigate the impact of the Gaussian process and beta derivative on the solutions of SBBME-BD (1). It was observed that the white noise component kept the solutions centered around zero. It was concluded that reducing the derivative order resulted in an enlargement of the surface. In future work, we can address Equation (1) with additive noise.

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