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NUMERICAL SOLUTION OF COUPLED FRACTIONAL BURGER'S EQUATION

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Abstract

In this paper, we use the homotopy analysis method to obtain an analytical solution for the fractional coupled Burgers' equation. This method yields an approximate solution that exhibits convergence across a range of values for both the convergence control parameter, denoted as $\hbar$, and the parameter α. Optimal $\hbar$ value, which minimizes the error determined by the residual, is identified. We subsequently conducted a comparative analysis of the results obtained through this method with solutions obtained from diverse numerical approach.

Keywords: Burgers' Equation; Numerical Solution; Homotopy Analysis Method.

1. INTRODUCTION

Differential equations have played an important role in science over time as interactions occur in real life such as medicine, epidemiology, flow, engineering, etc., and can be modeled into differential equations of any "ordinary, to partial derivatives, delay, stochastics, etc.". It is normal for many mathematical researchers and scientists have tried to find the best method to solve this type of equation, to better understand and master these phenomena, for the coupled fractional Burgers' equation we found the fractional natural decomposition method used by Prakasha *et al.*[1] Bahgat *et al.* used both of the Laplace–Adomian decomposition method, Laplace-variational iteration method and reduced differential transform method,[2] we have also the $(G'/G, 1/G)$ expansion method using by Inc *et al.*[3] and Nurul Islam and Ali Akbar,[4] the Hahn polynomials method by Heydari and Avazzadeh,[5] Safari and Chen used the semi-analytical method to solve the space-time fractional Burgers' equations,[6] Hussein used in Ref. 7 the weak Galerkin finite element method, and both of B-spline and shifted Jacobi spectral collocation techniques by Hadhoud *et al.*,[8] Ahmad *et al.* applied modified variational iteration algorithm[9] for the following coupled Burgers' equation:

$$\begin{cases} u_t - \alpha_1 \dfrac{\partial^2 u}{\partial x^2} + \beta_{1,1}\dfrac{\partial u}{\partial x} + \beta_{1,2}\dfrac{\partial uv}{\partial x} = 0 \\ \qquad t \in [0,T], \quad x \in [a,b], \\ v_t - \alpha_2 \dfrac{\partial^2 v}{\partial x^2} + \beta_{2,2}\dfrac{\partial v}{\partial x} + \beta_{2,1}\dfrac{\partial uv}{\partial x} = 0 \\ \qquad t \in [0,T], \quad x \in [a,b] \end{cases} \tag{1}$$

with I.C:

$$\begin{cases} u(x,0) = \phi_1(x) \quad x \in [a,b], \\ v(x,0) = \phi_2(x) \quad x \in [a,b] \end{cases} \tag{2}$$

and B.C:

$$\begin{cases} u(a,t) = \psi_1(t) \quad u(b,t) = \psi_2(t) \quad x \in [a,b], \\ v(a,t) = \psi_3(t) \quad v(b,t) = \psi_4(t) \quad x \in [a,b]. \end{cases} \tag{3}$$

The Homotopy Analysis Method (HAM) is a significant numerical technique for solving fractional differential equations, HAM is a semi-analytic method created by Liao[10–12] in 1992 in his Ph.D. The original method has known several improvements due to the same author. Sunil[13] and Alomari *et al.*[14] used the homotopy analysis method combined with the Laplace operator. This method is known as the "Laplace Homotopy Analysis Method" and is employed to solve fractional differential equations.

In this paper, we utilize the Laplace Homotopy Analysis Method (LHAM) to address the fractional coupled Burgers' equation and assess the method's effectiveness and accuracy[15]:

$$\begin{cases} D_t^{\alpha} u - \alpha_1 \dfrac{\partial^2 u}{\partial x^2} + \beta_{1,1}\dfrac{\partial u}{\partial x} + \beta_{1,2}\dfrac{\partial uv}{\partial x} = 0 \\ \qquad t \in [0,T], \quad x \in [a,b], \\ D_t^{\beta} v - \alpha_2 \dfrac{\partial^2 v}{\partial x^2} + \beta_{2,2}\dfrac{\partial v}{\partial x} + \beta_{2,1}\dfrac{\partial uv}{\partial x} = 0 \\ \qquad t \in [0,T], \quad x \in [a,b] \end{cases} \tag{4}$$

the I.C:

$$\begin{cases} u(x,0) = \phi_1(x) \quad x \in [a,b], \\ v(x,0) = \phi_2(x) \quad x \in [a,b] \end{cases} \tag{5}$$

and B.C:

$$\begin{cases} u(a,t) = \psi_1(t) \quad u(b,t) = \psi_2(t) \quad x \in [a,b], \\ v(a,t) = \psi_3(t) \quad v(b,t) = \psi_4(t) \quad x \in [a,b], \end{cases} \tag{6}$$

where D^{α} represents the fractional derivative operator of order α in the Caputo sens, and α_1 and α_2 are the positive viscosity parameters, and $\beta_{1,1}, \beta_{1,2}, \beta_{2,1}$, and $\beta_{2,2}$ are the constants of the Stokes velocity.

2. FRACTIONAL CALCULUS

In this section, we provide fundamental definitions from the theory of fractional calculus, which are utilized in this paper

Definition 1. The Riemann–Liouville fractional operator of order $\alpha \geq 0$, of a function $f \in C_{\alpha}, \alpha \geq -1$ is defined as[15]

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} \times f(s) ds, \quad \mu > 0, \; t > 0,$$

$$I^0 f(t) = f(t)$$

and for $\alpha, \beta \geq 0$, $n \in \mathbb{N}$:

$$I^{\alpha} I^{\beta} f(t) = I^{\alpha+\beta} f(t),$$

$$I^{\alpha} I^{\beta} f(t) = I^{\beta} I^{\alpha} f(t),$$

$$I^{\alpha} x^n = \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} x^{\alpha+n}.$$

Definition 2. The fractional Caputo derivative of $f, f \in C_{-1}^{m}$ is characterized as

$$D_C^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{(m-\alpha-1)} f^{(m)}(s) ds,$$

where $m-1 < \alpha < m$, $m \in \mathbb{N}$, and for $h \in C_{\mu}^{n}$, $\mu \geq -1$, $m-1 < \alpha \leq m$, $m \in \mathbb{N}$ we have

$$(I^{\alpha} D^{\alpha}) f(t) = f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} h^{(k)}(0^+).$$

Definition 3. The Laplace transform operator applies to the Caputo fractional derivative of $f(t)$[16]

$$L[D_t^{\alpha} f(t)] = s^{\alpha} L[f(t)] - \sum_{k=0}^{n-1} s^{(\alpha-k-1)} f^{(k)}(0), \quad n-1 < \alpha \leq n. \tag{7}$$

3. THE HOMOTOPY ANALYSIS METHOD

To describe some of basic ideas of HAM briefly, let's consider the following:

$$D_t^{\alpha} h(x,t) + \mathfrak{L} h(x,t) + \mathfrak{T} h(x,t) = f(x,t), \quad 0 < \alpha \leq 1, \tag{8}$$

where $D_t^\alpha h(x,t)$ denotes the Caputo derivative of $h(x,t)$, $\mathfrak{L}$ and $\mathfrak{T}$ represent the linear and nonlinear operators, respectively, and $f(x,t)$ stands as the source term. Initially, by subjecting Eq. (4) to the Laplace transform L, we get

$$\begin{aligned} s^\alpha L[h(x,t)] &- s^{\alpha-1}h(x,0) \\ &+ L[\mathfrak{L}h(x,t)] + L[\mathfrak{T}h(x,t)] \\ &= L[f(x,t)], \quad 0<\alpha\le 1. \end{aligned} \tag{9}$$

Upon simplification, we have the following:

$$\begin{aligned} L[h(x,t)] &= \frac{1}{s}h(x,0) \\ &\quad + \frac{1}{s^\alpha}(L[f(x,t)] - L[\mathfrak{L}h(x,t)] \\ &\quad - L[\mathfrak{T}h(x,t)]), \quad 0<\alpha\le 1 \end{aligned} \tag{10}$$

using HAM,[10–12] we define the nonlinear operator as follows:

$$\begin{aligned} N[\phi(x,t,q)] &= L[\phi(x,t,q)] - \frac{1}{s}h(x,0) \\ &\quad - \frac{1}{s^\alpha}(L[f(x,t)] + L[\mathfrak{R}h(x,t)] \\ &\quad + L[\mathfrak{N}h(x,t)]). \end{aligned} \tag{11}$$

Let $\phi(t,q)$ be a real-valued function of t with q ranging from 0 to 1. The zeroth-order deformation is

$$\begin{aligned} (1-q)\mathscr{L}&[\phi(x,t,q)-h_0(x,t)] \\ &= \hbar q N[\phi(x,t,q)], \end{aligned} \tag{12}$$

where $\hbar$ is a nonzero convergent control parameter, $h_0(x,t)$ represents the initial guess, N stands for the nonlinear operator, and $\mathscr{L}$ serves as an injective linear operator. In this paper, we select the linear operator as the Laplace operator, denoted by $\mathscr{L} = L$. It is evident that $\phi(x,t,0) = u_0(x,t)$ and $\phi(x,t,1) = h(x,t)$.

We proceed to expand $\phi(x,t,q)$ in a Taylor series with respect to q and t,

$$\phi(x,t,q) = \sum_{i=0}^{n} v_i(x,t)q^i,$$

where

$$v_i(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t,q)}{\partial q^m}|_{q=0}, \tag{13}$$

the mth-order deformation equation is

$$\begin{aligned} L[v_m(x,t) &- \ell_m v_{m-1}(x,t)] \\ &= \hbar H(x,t) R_m(v_{m-1}(x,t)). \end{aligned} \tag{14}$$

By upon applying the inverse operator L^{-1} to Eq. (14), we obtain

$$v_m(x,t) = \ell_m v_{m-1}(x,t) + \hbar L^{-1}[R_m(v_{m-1}(x,t))], \tag{15}$$

where

$$\ell_m = \begin{cases} 0, & m\le 1, \\ 1, & m>1. \end{cases}$$

For our problem (4), we define the system

$$\begin{cases} N_1[\phi(x,t,q),\psi(x,t,q)] \\ \quad = D_t^\alpha \phi(x,t,q) - \alpha_1 \dfrac{\partial^2\phi(x,t,q)}{\partial x^2} \\ \qquad + \beta_{1,1}\dfrac{\partial \phi(x,t,q)}{\partial x} + \beta_{1,2}\dfrac{\partial \phi(x,t,q)\psi(x,t,q)}{\partial x}, \\ N_1[\phi(x,t,q),\psi(x,t,q)] \\ \quad = D_t^\beta \psi(x,t,q) - \alpha_2 \dfrac{\partial^2\psi(x,t,q)}{\partial x^2} \\ \qquad + \beta_{2,2}\dfrac{\partial \psi(x,t,q)}{\partial x} + \beta_{2,1}\dfrac{\partial \phi(x,t,q)\psi(x,t,q)}{\partial x}, \\ \qquad\qquad t\in[0,T], \quad x\in[a,b] \ \text{and} \ q\in[0,1]. \end{cases} \tag{16}$$

For more details, see the following examples.

4. APPLICATION

Test Example 1:

First, let's consider the following problem:

$$\begin{cases} D_t^\alpha u - \dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial u}{\partial x} + (0.1)\dfrac{\partial uv}{\partial x} = 0 \\ \qquad t\ge 0, \quad x\in[-10,10], \\ D_t^\beta v - \dfrac{\partial^2 v}{\partial x^2} - 2\dfrac{\partial v}{\partial x} + (0.3)\dfrac{\partial uv}{\partial x} = 0 \\ \qquad t\ge 0, \quad x\in[-10,10] \end{cases} \tag{17}$$

with

$$\begin{cases} u(x,0) = K(1-\tanh(x)) \quad x\in[-10,10], \\ v(x,0) = K\left(\dfrac{2(0.3)-1}{2(0.1)-1} - \tanh(Bx)\right) \\ \qquad\qquad\qquad\qquad x\in[-10,10]. \end{cases}$$

The zeroth-order deformation is as follows:

$$\begin{cases} (1-q)\mathscr{L}_1[\phi(x,t,q) - h_0(x,t)] \\ \qquad = \hbar_1 q N_1[\phi(x,t,q),\psi(x,t,q)], \\ (1-q)\mathscr{L}_2[\psi(x,t,q) - h_0(x,t)] \\ \qquad = \hbar_2 q N_2[\phi(x,t,q),\psi(x,t,q)], \end{cases}$$

where $\mathscr{L}_1 = \mathscr{L}_2 = D_t^\alpha$, and

$$\begin{cases} N_1[\phi(x,t,q),\psi(x,t,q)] \\ \qquad = D_t^\alpha \phi - \dfrac{\partial^2 \phi}{\partial x^2} - 2\dfrac{\partial \phi(x,t,q)}{\partial x} + (0.1)\dfrac{\partial \phi\psi}{\partial x}, \\ N_2[\phi(x,t,q),\psi(x,t,q)] \\ \qquad = D_t^\beta \psi - \dfrac{\partial^2 \psi}{\partial x^2} - 2\dfrac{\partial \psi}{\partial x} + (0.3)\dfrac{\partial \phi\psi}{\partial x}. \end{cases}$$

The mth-order deformation equation is as follows:

$$\begin{cases} \mathscr{L}_1(u_m - \chi_m u_{m-1}) \\ \qquad = D_t^\alpha u_{m-1} - \dfrac{\partial^2 u_{m-1}}{\partial x^2} - 2\dfrac{\partial u_{m-1}}{\partial x} \\ \qquad\quad + (0.1)\displaystyle\sum_{i=0}^{m-1}\left(\dfrac{\partial u_i}{\partial x}v_{m-1-i} + \dfrac{\partial v_i}{\partial x}u_{m-1-i}\right), \\ \mathscr{L}_2(v_m - \chi_m v_{m-1}) \\ \qquad = D_t^\beta v_{m-1} - \dfrac{\partial^2 v_{m-1}}{\partial x^2} - 2\dfrac{\partial v_{m-1}}{\partial x} \\ \qquad\quad + (0.3)\displaystyle\sum_{i=0}^{m-1}\left(\dfrac{\partial u_i}{\partial x}v_{m-1-i} + \dfrac{\partial v_i}{\partial x}u_{m-1-i}\right), \end{cases}$$

where

$$\begin{aligned} u_0(x,0) &= K(1 - \tanh(Bx)), \\ v_0(x,0) &= K\left(\frac{2\beta_{2,1} - 1}{2\beta_{1,2} - 1} - \tanh(Bx)\right). \end{aligned}$$

By applying the inverse operator $\mathscr{L}_1^{-1} = \mathscr{L}_2^{-1} = I^\alpha$ and "The fractional Riemann operator", we obtain

$$\begin{aligned} u_m &= \chi_m u_{m-1} + I^\alpha \\ &\quad \times \Bigg(D_t^\alpha u_{m-1} - \frac{\partial^2 u_{m-1}}{\partial x^2} - 2\frac{\partial u_{m-1}}{\partial x} \\ &\quad + (0.1)\sum_{i=0}^{m-1}\left(\frac{\partial u_i}{\partial x}v_{m-1-i} + \frac{\partial v_i}{\partial x}u_{m-1-i}\right)\Bigg), \\ v_m &= \chi_m v_{m-1} + I^\beta \\ &\quad \times \Bigg(D_t^\beta v_{m-1} - \frac{\partial^2 v_{m-1}}{\partial x^2} - 2\frac{\partial v_{m-1}}{\partial x} \\ &\quad + (0.3)\sum_{i=0}^{m-1}\left(\frac{\partial u_i}{\partial x}v_{m-1-i} + \frac{\partial v_i}{\partial x}u_{m-1-i}\right)\Bigg), \end{aligned}$$

we have successfully obtained the first-term approximation of the approach solution for $\alpha_1 = \alpha_1 = 1$, $\beta_{1,1} = \beta_{2,2} = -2$, $\beta_{1,2} = 0.1, \beta_{2,1} = 0.3, K = 0.05$, and $B = \frac{K(4\beta_{2,1}-1)}{2(2\beta_{1,2}-1)}$:

$$\begin{aligned} u_1(x,t) &= \frac{1}{\Gamma(\alpha+1)} 7.031249\ldots \\ &\quad \times 10^{-6}\hbar t^\alpha(1.\tanh(0.00625x) \\ &\quad - 88.5556)\mathrm{sech}^2(0.00625x), \\ u_2(x,t) &= \frac{1}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\hbar t^\alpha \mathrm{sech}^2(0.00625x) \\ &\quad \times (\hbar\Gamma(\alpha+1)t^\alpha \mathrm{sech}^2(0.00625x) \\ &\quad \times (-3.871838\ldots \times 10^{-6}\tanh(0.00625x) \\ &\quad + \mathrm{sech}(0.00625x)(8.739013\ldots \\ &\quad \times 10^{-8}\cosh(0.01875x) - 3.877673\ldots \\ &\quad \times 10^{-6}\sinh(0.01875x)) - 2.621704\ldots \\ &\quad \times 10^{-7}) + \Gamma(2\alpha+1)((7.031249\ldots \\ &\quad \times 10^{-6}\hbar + 7.031249\ldots \times 10^{-6}) \\ &\quad \times \tanh(0.00625x) - 6.22656 \times 10^{-4}\hbar \\ &\quad - 6.22656 \times 10^{-4})) \end{aligned}$$

and

$$\begin{aligned} v_1(x,t) &= \frac{1}{\Gamma(\alpha+1)}(\hbar t^\alpha(1.32813 \\ &\quad \times 10^{-5}\tanh(0.00625x) - 6.17969 \times 10^{-4}) \\ &\quad \times \mathrm{sech}^2(0.00625x)) \\ v_2(x,t) &= \frac{1}{\Gamma(\alpha+1)}(\hbar t^\alpha \mathrm{sech}^2(0.00625x) \\ &\quad \times ((1.32813 \times 10^{-5}\hbar + 1.32813 \times 10^{-5}) \\ &\quad \times \tanh(0.00625x))\frac{1}{\Gamma(\alpha+1)} \\ &\quad \times (-6.17969 \times 10^{-4}\hbar)\frac{1}{\Gamma(2\alpha+1)} \\ &\quad \times (\hbar^2 t^{2\alpha}\mathrm{sech}^4(0.00625x) \\ &\quad \times (-3.806372\ldots \times 10^{-6}\tanh(0.00625x) \\ &\quad + \mathrm{sech}(0.00625x) \times (1.645141\ldots \times 10^{-7} \\ &\quad \times \cosh(0.01875x) - 3.820214\ldots \times 10^{-6} \\ &\quad \times \sinh(0.01875x)) + 1.645141\ldots \times 10^{-7})). \end{aligned}$$

Test Example 2:

Let us consider the second fractional coupled Burgers' equation

$$\begin{cases} D_t^{\alpha}u - \dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial u}{\partial x} + (2.5)\dfrac{\partial uv}{\partial x} = 0 \\ t \geq 0, \quad x \in [-20, 20], \\ D_t^{\beta}v - \dfrac{\partial^2 v}{\partial x^2} - 2\dfrac{\partial v}{\partial x} + (2.5)\dfrac{\partial uv}{\partial x} = 0 \\ t \geq 0, \quad x \in [-20, 20] \end{cases} \tag{18}$$

with the initial conditions:

$$\begin{cases} u(x,0) = K\left(1 - \tanh\left(\dfrac{3}{2}Kx\right)\right) & x \in [-20, 20], \\ v(x,0) = K\left(1 - \tanh\left(\dfrac{3}{2}Kx\right)\right) & x \in [-20, 20]. \end{cases}$$

Here, the first-order terms of the homotopy series solution of the problem (18) are

$$\begin{aligned} u_1(x,t) &= \frac{1}{\Gamma(\alpha+1)}(0.003\hbar t^{\alpha}(1.\tanh(0.15x) + 7.5) \\ &\quad \times \operatorname{sech}^2(0.15x)), \\ u_2(x,t) &= \frac{1}{\Gamma(\alpha+1)} h t^{\alpha}(\operatorname{sech}^2(0.15x) \\ &\quad \times ((0.003h + 0.003)\tanh(0.15x) \\ &\quad + 0.0225h + 0.0225) \\ &\quad - \frac{1}{\Gamma(2\alpha+1)}(\hbar^2 t^{2\alpha}\operatorname{sech}^2(0.15x) \\ &\quad \times (\operatorname{sech}^2(0.15x)(0.00135 \\ &\quad - 0.00009\tanh(0.15x)) + \tanh(0.15x) \\ &\quad \times (-0.00018\tanh^2(0.15x) \\ &\quad - 0.0027\tanh(0.15x) - 0.010125)) \end{aligned}$$

and

$$\begin{aligned} v_1(x,t) &= \frac{1}{\Gamma(\alpha+1)}(\hbar t^{\alpha}(0.003\tanh(0.15x) \\ &\quad + 0.0225)\operatorname{sech}^2(0.15x)), \\ v_2(x,t) &= \frac{1}{\Gamma(\alpha+1)}(\hbar t^{\alpha}\operatorname{sech}^2(0.15x)((0.003h \\ &\quad + 0.003)\tanh(0.15x) + 0.0225h + 0.0225)) \\ &\quad \frac{1}{\Gamma(2\alpha+1)}(\hbar^2 t^{2\alpha}\operatorname{sech}^4(0.15x) \\ &\quad \times (2.48625 \times 10^{-3}\tanh(0.15x) \\ &\quad + \operatorname{sech}(0.15x)(2.57625 \times 10^{-3}\sinh(0.45x)) \\ &\quad + 6.75 \times 10^{-4}\cosh(0.45x)) \\ &\quad - 2.025 \times 10^{-3}). \end{aligned}$$

Test Example 3:

$$\begin{cases} D_t^{\alpha}u - \dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial u}{\partial x} + \dfrac{\partial uv}{\partial x} = 0 \\ \qquad t \geq 0, \quad x \in [-\pi, \pi], \\ D_t^{\beta}v - \dfrac{\partial^2 v}{\partial x^2} - 2\dfrac{\partial v}{\partial x} + \dfrac{\partial uv}{\partial x} = 0 \\ \qquad t \geq 0, \quad x \in [-\pi, \pi] \end{cases} \tag{19}$$

with the initial conditions:

$$u(x,0) = v(x,0) = \sin(x).$$

Here, the first-order terms of the homotopy series solution of the problem (19) are

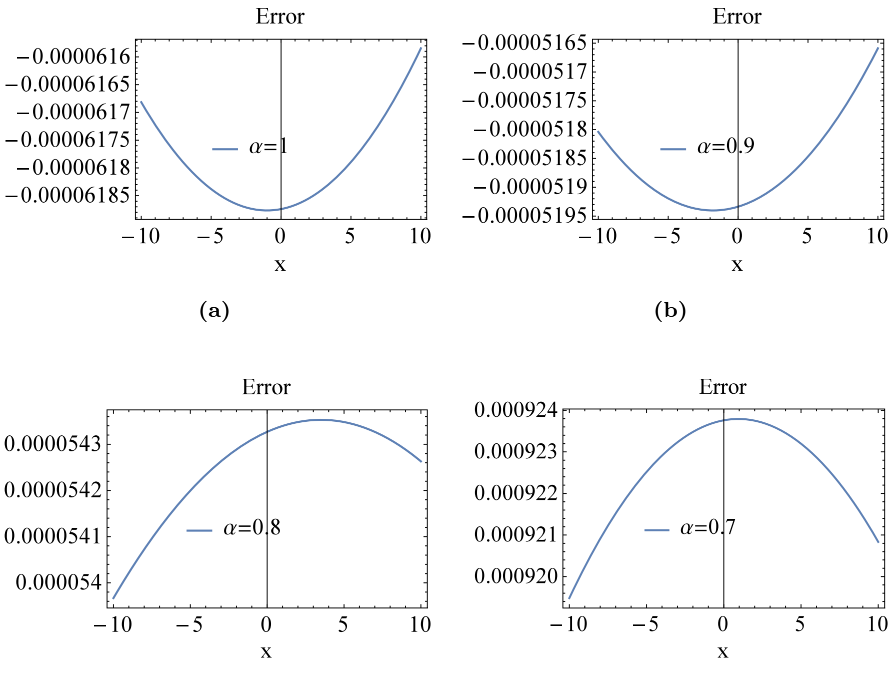
$$\begin{aligned} u_1(x,t) &= \frac{\hbar t^{\alpha}\sin(x)}{\Gamma(\alpha+1)}, \\ u_2(x,t) &= \frac{\hbar(\hbar+1)t^{\alpha}\sin(x)}{\Gamma(\alpha+1)} + \frac{\hbar^2 t^{2\alpha}\sin(x)}{\Gamma(2\alpha+1)}, \\ u_3(x,t) &= \frac{\hbar(\hbar+1)^2 t^{\alpha}\sin(x)}{\Gamma(\alpha+1)} + \frac{\hbar^2(\hbar+1)t^{2\alpha}\sin(x)}{\Gamma(2\alpha+1)} \\ &\quad + \frac{\hbar^2 t^{2\alpha}\sin(x)((\hbar+1)\Gamma(3\alpha+1) + \hbar\Gamma(2\alpha+1)t^{\alpha})}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)} \end{aligned}$$

and

$$\begin{aligned} v_1(x,t) &= \frac{\hbar t^{\alpha}\sin(x)}{\Gamma(\alpha+1)}, \\ v_2(x,t) &= \frac{\hbar(\hbar+1)t^{\alpha}\sin(x)}{\Gamma(\alpha+1)} + \frac{\hbar^2 t^{2\alpha}\sin(x)}{\Gamma(2\alpha+1)}, \\ v_3(x,t) &= \frac{\hbar(\hbar+1)^2 t^{\alpha}\sin(x)}{\Gamma(\alpha+1)} + \frac{2\hbar^2(\hbar+1)t^{2\alpha}\sin(x)}{\Gamma(2\alpha+1)} \\ &\quad + \frac{\hbar^3 t^{3\alpha}\sin(x)}{\Gamma(3\alpha+1)} \end{aligned}$$

and the solution series of the problem (19) is

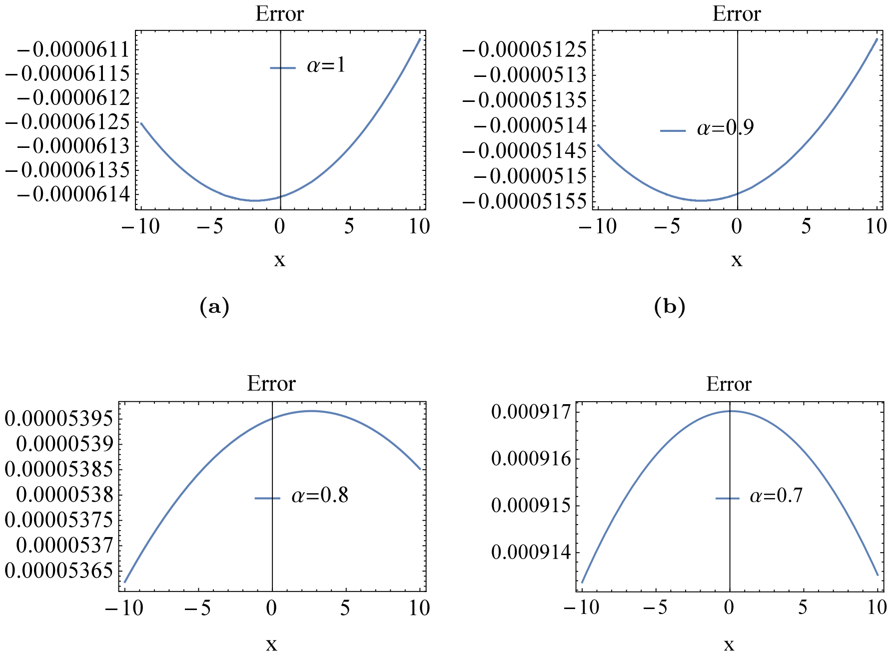
$$\begin{aligned} u(x,t) &= \sin(x) + \frac{\hbar(\hbar^3 + 4\hbar^2 + 6\hbar + 4)t^{\alpha}\sin(x)}{\Gamma(\alpha+1)} \\ &\quad + \frac{\hbar^2(\hbar^2 + 3\hbar + 3)t^{2\alpha}\sin(x)}{\Gamma(2\alpha+1)} \end{aligned}$$



(a) **(b)**

(c) **(d)**

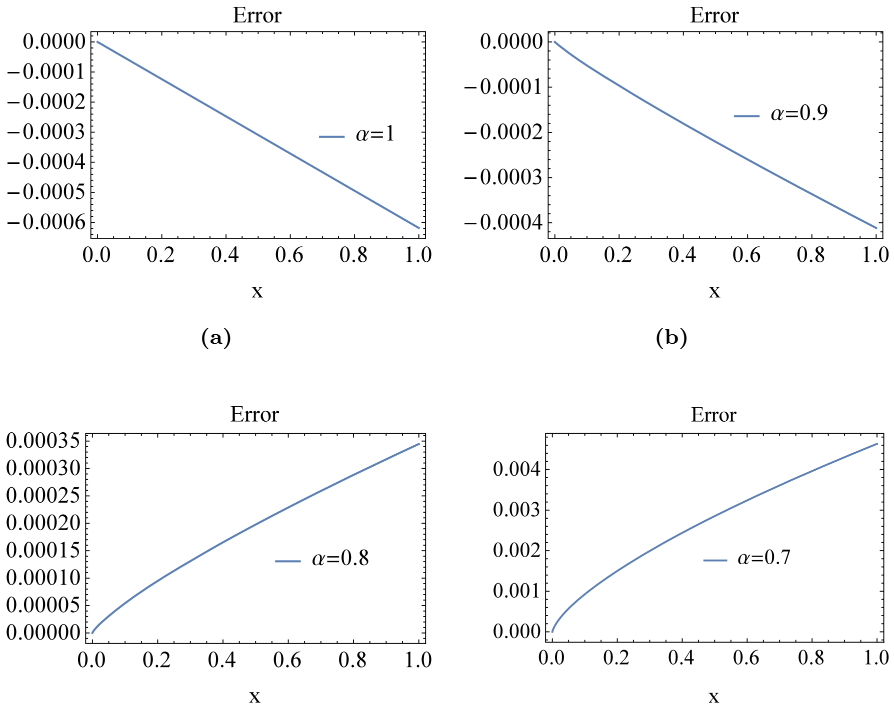
Fig. 1 The error for $t = 0.1$ and different values of α for $u(x,t)$ of Eq. (17).



(a) **(b)**

(c) **(d)**

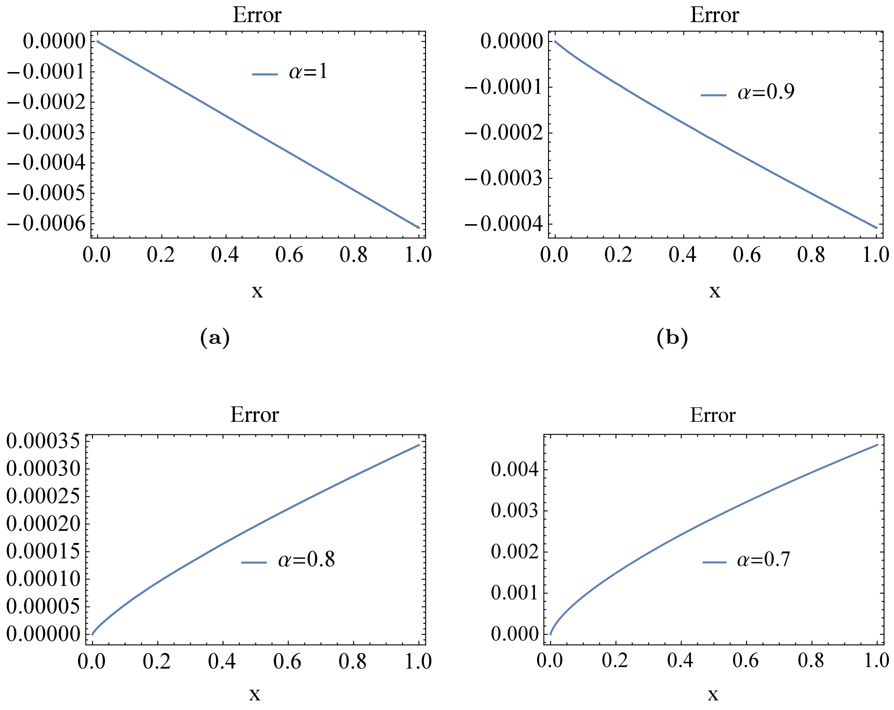
Fig. 2 The error for $t = 0.1$ and different values of α for $v(x,t)$ of Eq. (17).



(a) (b)

(c) (d)

Fig. 3 The error for $x = 1$ and different values of α of $u(x,t)$ Eq. (17).



(a) (b)

(c) (d)

Fig. 4 The error for $x = 1$ and different values of α of $v(x,t)$ Eq. (17).

$$
\begin{aligned}
&+\hbar^2 t^{2\alpha}\sin(x)\Bigg(\frac{(\hbar+1)(2\hbar+3)}{\Gamma(2\alpha+1)}\\
&+\hbar t^{\alpha}\bigg(\frac{3\hbar+4}{\Gamma(3\alpha+1)}+\frac{\hbar t^{\alpha}}{\Gamma(4\alpha+1)}\bigg)\Bigg)+\cdots,\\
v(x,t) &= \sin(x)+\frac{\hbar(\hbar^3+4\hbar^2+6\hbar+4)t^{\alpha}\sin(x)}{\Gamma(\alpha+1)}\\
&+\frac{\hbar^2(3\hbar^2+8\hbar+6)t^{2\alpha}\sin(x)}{\Gamma(2\alpha+1)}\\
&+\frac{\hbar^3(3\hbar+4)t^{3\alpha}\sin(x)}{\Gamma(3\alpha+1)}+\frac{\hbar^4 t^{4\alpha}\sin(x)}{\Gamma(4\alpha+1)}\\
&+\cdots.
\end{aligned}
$$

5. NUMERICAL RESULT AND DISCUSSION

In this section, we first plot the numerical solution of the test examples 1–3 for different values of α, Figs. 1 and 2 present the error at $t = 0.1$ for different values of α, and at $x = 1$ in Figs. 3 and 4 we plot also the error for $u(x,t)$ and $v(x,t)$. In Tables 1 and 2, we give the approximate solutions to the problem (17) for different value of α.

For $v(x,t)$, we obtain and show the approximate solution for problems (18) and (19) for different values of α in Tables 3–5 and for the problem (19)

Table 1 The Approximate Solution of the Problem (17).

x	t	Exact $u(x,t)$	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
−20	0.1	0.043782	0.043843	0.043862	0.043886
−20	0.2	0.043783	0.043905	0.043932	0.043964
−15	0.1	0.045326	0.045388	0.045407	0.045431
−15	0.2	0.045327	0.045449	0.045477	0.045509
−10	0.1	0.046879	0.046941	0.046960	0.046984
−10	0.2	0.046879	0.047003	0.047030	0.047063
−5	0.1	0.048438	0.048500	0.048519	0.048543
−5	0.2	0.048438	0.048562	0.048590	0.048622
5	0.1	0.051562	0.051624	0.051643	0.051667
5	0.2	0.051562	0.051686	0.051713	0.051746
10	0.1	0.053121	0.053182	0.053202	0.053226
10	0.2	0.053121	0.053244	0.053272	0.053304
15	0.1	0.054674	0.054735	0.054754	0.054778
15	0.2	0.054674	0.054797	0.054824	0.054856
20	0.1	0.056218	0.056278	0.056297	0.056321
20	0.2	0.056218	0.056340	0.056367	0.056398

Table 2 The Approximate Solution of the Problem (17).

x	t	Exact $v(x,t)$	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
−20	0.1	0.018782	0.018843	0.018862	0.018886
−20	0.2	0.018783	0.018904	0.018931	0.018963
−15	0.1	0.020326	0.020387	0.020406	0.020430
−15	0.2	0.020327	0.020449	0.020476	0.020508
−10	0.1	0.021879	0.021940	0.021959	0.021984
−10	0.2	0.021879	0.022002	0.022029	0.022061
−5	0.1	0.023438	0.023499	0.023518	0.023543
−5	0.2	0.023438	0.023561	0.023588	0.023621
5	0.1	0.026562	0.026623	0.026642	0.026667
5	0.2	0.026562	0.026685	0.026712	0.026744
10	0.1	0.028121	0.028182	0.028201	0.028225
10	0.2	0.028121	0.028243	0.028271	0.028303
15	0.1	0.029674	0.029734	0.029753	0.029777
15	0.2	0.029674	0.029796	0.029823	0.029854
20	0.1	0.031218	0.031278	0.031297	0.031320
20	0.2	0.031218	0.031339	0.031365	0.031397

Table 3 The Approximate Solution for Different Values of α.

x	t	Exact $u(x,t)$	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
−20	0.1	0.095267	0.095143	0.095107	0.095061
−20	0.2	0.095277	0.095027	0.094974	0.094910
−15	0.1	0.090484	0.090246	0.090178	0.090089
−15	0.2	0.090503	0.090024	0.089922	0.089800
−10	0.1	0.081791	0.081376	0.081257	0.081104
−10	0.2	0.081824	0.080989	0.080814	0.080605
−5	0.1	0.067966	0.067354	0.067178	0.066954
−5	0.2	0.068015	0.066785	0.066530	0.066227
5	0.1	0.032131	0.031500	0.031322	0.031097
5	0.2	0.032180	0.030924	0.030672	0.030379
10	0.1	0.018276	0.017840	0.017717	0.017563
10	0.2	0.018309	0.017444	0.017273	0.017075
15	0.1	0.009554	0.009300	0.009229	0.009140
15	0.2	0.009573	0.009071	0.008973	0.008859
20	0.1	0.004752	0.004619	0.004582	0.004535
20	0.2	0.004762	0.004499	0.004447	0.004389

Table 4 The Approximate Solution for Different Values of α.

x	t	Exact $v(x,t)$	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
−20	0.1	0.095267	0.095143	0.095107	0.095061
−20	0.2	0.095277	0.095027	0.094974	0.094910
−15	0.1	0.090484	0.090246	0.090178	0.090089
−15	0.2	0.090503	0.090024	0.089922	0.089800
−10	0.1	0.081791	0.081376	0.081257	0.081104
−10	0.2	0.081824	0.080989	0.080814	0.080605
−5	0.1	0.067966	0.067354	0.067178	0.066954
−5	0.2	0.068015	0.066785	0.066530	0.066227
5	0.1	0.032131	0.031500	0.031322	0.031097
5	0.2	0.032180	0.030924	0.030672	0.030379
10	0.1	0.018276	0.017840	0.017717	0.017563
10	0.2	0.018309	0.017444	0.017717	0.017075
15	0.1	0.009554	0.009300	0.009229	0.009140
15	0.2	0.009573	0.009071	0.009229	0.008859
20	0.1	0.004752	0.004619	0.004582	0.004535
20	0.2	0.004762	0.004499	0.004447	0.004389

Table 5 The Approximate Solution of the Problem (17).

x	t	Exact	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
$-\pi$	0.1	0	0	0	0
$-\pi$	0.2	0	0	0	0
−1.5	0.1	−0.902571	−0.902571	−0.875896	−0.844027
−1.5	0.2	−0.816679	−0.816679	−0.783794	−0.748745
0	0.1	0	0	0	0
0	0.2	0	0	0	0
1.5	0.1	0.902571	0.902571	0.875896	0.844027
1.5	0.2	0.816679	0.816679	0.783794	0.748745
π	0.1	0	0	0	0
π	0.2	0	0	0	0

6. CONCLUSION

In conclusion, the Homotopy Analysis Method (HAM) has demonstrated its effectiveness and promise in tackling fractional coupled Burgers' equations, as demonstrated earlier. It presents a flexible approach for resolving these intricate equations, yielding precise approximations and numerical solutions. The method's proficiency in managing fractional coupled Burgers' equations can make a substantial contribution to the comprehension and modeling of diverse physical phenomena, establishing it as a valuable asset for researchers and professionals in the field. Future investigations and the continued application of HAM in this context have the potential to generate further insights and innovations in the years to come.

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