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RECEIVED 02 August 2026
REVISED 24 August 2026
ACCEPTED 26 August 2026
PUBLISHED 14 September 2026

CITATION

Ishkhanyan A, Cesarano C and
Baboyan K (2026) Appell expansions and
general-to-confluent Heun
degeneration in a bounded recovery
diffusion.
Front. Appl. Math. Stat. 12:1956515.
doi: 10.3389/fams.2026.1956515

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Appell expansions and general-to-confluent Heun degeneration in a bounded recovery diffusion

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Corporate recovery under state-dependent uncertainty can be represented as first passage of a bounded latent financial-health diffusion between distress and recovery thresholds, with an additional killing channel for abrupt failure. When volatility is asymmetric across financial states and failure intensity is state dependent, the recovery functional satisfies a constrained general Heun equation. We specialize an origin-centered Appell F_1 expansion to this family and derive a compact four-term recurrence in four dimensionless parameters. Probabilistic admissibility ensures convergence on every compact subset of the state interval in the principal parameter region. We then track the limit $\xi \rightarrow 0$ ($|a| \rightarrow \infty$) simultaneously through the differential equation, normalized expansion basis, and coefficient recurrence. Each Appell basis element converges to an incomplete-beta function, while the recurrence converges coefficientwise to the four-term relation of a Whittaker-Ince confluent-Heun equation. Only the additional removal of state dependence from killing gives the Gauss hypergeometric equation. Thus the hierarchy corresponds economically to removing, successively, volatility asymmetry and state dependence in abrupt-failure risk. Terminating and lower-term recurrences remain exceptional. Numerical calculations verify the specialized series and illustrate an approximately first-order approach of the bounded recovery solution to its confluent limit for the selected parameter set.

Mathematics Subject Classification: 33C65; 33C75; 34B30; 60J60.

KEYWORDS

Appell F_1 function, bounded diffusion, Confluent Heun equation, corporate financial resilience, general Heun equation, Heun confluence, incomplete beta function, recurrence relation

1 Introduction

Corporate recovery is rarely a smooth transition from distress to stability. A firm's financial condition evolves under shocks whose intensity may differ between weak and strong states, while abrupt failure may occur before either formal distress or recovery thresholds are reached. These features can be represented by a bounded diffusion for latent financial health, supplemented by a state-dependent killing rate. The corresponding recovery problem provides an economically meaningful route to a constrained family of general Heun equations. Economically, the specification combines the first-passage logic

of structural credit models [1] with the unexpected-failure mechanism of intensity-based models [2, 3].

The general Heun equation is the most general second-order Fuchsian differential equation with four regular singular points [4–6]. Its additional finite singularity, relative to the Gauss hypergeometric equation, complicates local connection problems and replaces familiar two-term coefficient formulas by higher-order recurrences [4–6]. Exceptional reductions of the Heun equation to hypergeometric form are classified in [7]. Expansions in functions richer than powers provide one route through these difficulties. In particular, Appell functions of two variables encode two finite singular factors in a single basis element and can yield finite-sum, incomplete-beta, or Gauss-hypergeometric reductions on exceptional parameter sets [8].

This use of the word *Appell* should be distinguished from Appell polynomial sequences. The relevant object is the two-variable hypergeometric function F_1 , introduced within the classical theory of multiple hypergeometric series [9]. Its two arguments are especially natural for the Heun equation because the integrand generated by the derivative transformation contains both $(1-x)^{-\delta}$ and $(1-x/a)^{-\varepsilon}$. The same construction also clarifies how a two-variable basis can lose one variable under confluence without forcing the coefficient recurrence to become two- or three-term.

The present note considers a constrained Heun family generated by a bounded diffusion. The state equation has Jacobi endpoints at zero and one and a third, external zero of the variance polynomial. A state-dependent killing rate then produces the accessory and exponent products of the resulting Heun equation. This applied origin is useful mathematically because the parameter dictionary is not generic: it imposes

$$\gamma + \delta + \varepsilon = 0, \quad \alpha + \beta = -1. \quad (1)$$

These identities simplify the Appell recurrence and link convergence to probabilistic admissibility. They also create a natural confluence path. As the variance deformation disappears, the external root a recedes to infinity in modulus, but the products $\alpha\beta$ and the accessory parameter q diverge proportionally to a . The limit is therefore not, in general, the Gauss equation. It is the Whittaker-Ince form of the confluent Heun equation [4–6], whose solutions admit incomplete-beta and Appell expansions with generically higher-term recurrences [10]. The Gauss equation appears only after the state-dependent part of killing is also removed.

The article makes three contributions. First, it gives the explicit origin-centered Appell F_1 representation and a dimensionless four-term recurrence specialized to the bounded diffusion. Second, it establishes a model-specific convergence result on the unit interval. Third, it proves a coordinated general-to-confluent degeneration: the differential operator, the normalized Appell basis, and the recurrence coefficients converge together. This three-level statement is the main result. Termination conditions and a small numerical verification are included to delimit, rather than exaggerate, the exceptional solvable cases. Whereas Ishkhanyan [8] and Leroy [10] treat the general- and confluent-Heun expansions separately, the present paper identifies a diffusion-generated parameter path on which the operator limit, basis

normalization, coefficient recurrence, and convergence domain are compatible simultaneously.

The distinction between representation and reduction is important. An F_1 expansion can be analytically useful even when it is infinite and its coefficients are governed by a four-term relation. Conversely, a formal confluence of singularities does not ensure a simpler coefficient law. The relevant questions are therefore: which recurrence is selected by the diffusion constraints; on what part of the state interval does the associated series converge; and whether the expansion functions and recurrence approach the known confluent-Heun structures along the same parameter path.

Economically, X_t is a normalized latent measure of corporate financial health. Its mean-reverting drift represents adjustment toward a sustainable condition, while the signed deformation parameter ξ allows shock intensity to differ between weak and strong financial states. The lower threshold x_L marks severe distress or mandatory intervention, whereas x_H marks recovery. The killing rate $\lambda(x) = \lambda_0 - \lambda_1 x$ represents an abrupt failure channel not captured by gradual threshold passage, such as loss of market access, operational disruption, or legal prohibition. Accordingly, $V(x)$ is the discounted probability or value of reaching recovery before either threshold distress or abrupt failure. A companion application study develops interpretation, estimation, and intervention design; the present article retains the economic structure because its admissibility conditions generate the constrained Heun family analyzed below.

2 Method

2.1 Bounded recovery equation and Heun mapping

Starting from the Jacobi diffusion used for bounded stochastic transitions in econometrics [11], we introduce its lowest-degree signed variance deformation. Let $X_t \in (0, 1)$ satisfy

$$dX_t = \rho(\theta - X_t) dt + \sqrt{2D X_t(1-X_t)(1-\xi X_t)} dW_t, \quad \xi < 1, \quad (2)$$

In Equation 2, $\rho, D > 0$, $0 < \theta < 1$, and W_t is a standard Brownian motion. The factor $1 - \xi x$ is positive on the open unit interval. For $0 < \xi < 1$ the third variance root lies to the right of the interval; for $\xi < 0$ it lies to the left. The symmetric case $\xi = 0$ gives the Jacobi variance. The additional variance factor is the present model's parsimonious extension, introduced to permit signed volatility asymmetry while preserving positivity on the state interval. The specification is therefore a model construction rather than a corporate-recovery law taken unchanged from earlier work.

Economically, X_t is normalized corporate financial health, θ is its sustainable long-run level, ρ measures adjustment toward that level, and D controls the overall intensity of shocks. The signed deformation ξ determines how that intensity varies across financial states: positive ξ dampens diffusion near the strong-state boundary, negative ξ amplifies it, and $\xi = 0$ restores symmetric Jacobi variance.

Fix $0 < x_L < x_H < 1$ and define τ_L and τ_H as the first hitting times of the two thresholds. In addition, let the process be killed at rate

$$\lambda(x) = \lambda_0 - \lambda_1 x, \quad \lambda_0 \geq \lambda_1 \geq 0. \quad (3)$$

Under this Cox specification, conditional survival along a realized state path is the exponential of minus the time-integrated failure intensity [2, 3]. Integrating out abrupt failure and imposing the threshold event yields, at discount rate $r \geq 0$, the discounted recovery functional:

$$V(x) = \mathbb{E}_x \left[e^{-r\tau_H} \exp \left(- \int_0^{\tau_H} \lambda(X_t) dt \right) 1_{\{\tau_H < \tau_L\}} \right]. \quad (4)$$

In Equation 4, x_L represents severe distress or a mandatory-intervention boundary, while x_H represents recovery. The affine rate $\lambda(x)$ adds abrupt failure before either threshold is reached, and $V(x)$ measures the discounted prospect of attaining recovery first. With zero discounting, this is the probability of recovery before threshold distress or Cox failure; with positive discounting, it is the expected discounted unit recovery payoff under the stated probability measure. It is therefore interpreted as a recovery functional, not necessarily as an arbitrage-free market price. Here, recovery means first attainment of the upper stability threshold, not a post-default recovery value.

By the Feynman-Kac formula for killed diffusions [12], the recovery equation on (x_L, x_H) is

$$Dx(1-x)(1-\xi x)V'' + \rho(\theta-x)V' - (r+\lambda_0-\lambda_1 x)V = 0, \\ V(x_L) = 0, \quad V(x_H) = 1. \quad (5)$$

Introduce

$$\kappa = \frac{\rho}{D}, \quad s = \frac{r+\lambda_0}{D}, \quad h = \frac{\lambda_1}{D}, \quad a = \frac{1}{\xi}. \quad (6)$$

Using the parameters in Equation 6, for $\xi \neq 0$, division of Equation 5 by its leading coefficient gives the canonical general Heun equation

$$V'' + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\varepsilon}{x-a} \right) V' + \frac{\alpha\beta x - q}{x(x-1)(x-a)} V = 0. \quad (7)$$

The exact parameter dictionary is shown in Table 1. Direct addition of the three derivative exponents gives zero, so the Fuchs relation $\alpha + \beta + 1 = \gamma + \delta + \varepsilon$ yields $\alpha + \beta = -1$. Only the product $\alpha\beta = ah$ enters Equation 7; a convenient choice is

$$\alpha, \beta = \frac{-1 \pm \sqrt{1-4ah}}{2}. \quad (8)$$

Equation 8 fixes alpha and beta up to interchange.

TABLE 1 Dimensionless mapping and its confluence scaling.

Heun quantity	Bounded-diffusion expression	Limit as $ a \rightarrow \infty$
a	$1/\xi \in (-\infty, 0) \cup (1, \infty)$	∞
γ	$\kappa\theta$	$\kappa\theta$
δ	$\kappa a(1-\theta)/(a-1)$	$\kappa(1-\theta)$
ε	$\kappa(\theta-a)/(a-1)$	$-\kappa$
$\alpha\beta$	ah	linear in a
q	as	linear in a

2.2 Origin-centered Appell construction

Following the derivative-equation method, define

$$Y(x) = x^\gamma (x-1)^\delta (x-a)^\varepsilon V'(x). \quad (9)$$

The function Y defined in Equation 9 satisfies a second-order Fuchsian equation with the four singularities inherited from Equation 7 and an additional apparent singularity at $x_0 = q/(\alpha\beta) = s/h$ when $h \neq 0$ [8]. About $x = 0$, take a Frobenius series

$$Y(x) = x^\mu \sum_{n=0}^{\infty} c_n x^n, \quad \mu \in \{0, \gamma\}. \quad (10)$$

Termwise integration gives

$$V_\mu(x) = C_\mu + \sum_{n=0}^{\infty} c_n \Phi_{n,\mu}^{(a)}(x), \quad (11)$$

where, with $A_n = n+1-\gamma+\mu$,

$$\Phi_{n,\mu}^{(a)}(x) = \frac{(-1)^{-\delta} x^{A_n}}{(-a)^\varepsilon A_n} F_1 \left(A_n; \delta, \varepsilon; A_n+1; x, \frac{x}{a} \right). \quad (12)$$

For the normalization $c_0 = 1$ and $\text{Re}A_0 > 0$, substitution at the origin fixes the integration constant as

$$C_\mu = (-1)^{-\delta} (-a)^{-\varepsilon} \frac{a\mu}{q}. \quad (13)$$

Thus C_μ is part of the solution, not an independent integration constant. For $q \neq 0$, Equation 13 gives $C_\mu = 0$ for the $\mu = 0$ branch; when $s > 0$, $a\mu/q = \mu/s$, so the a -dependence cancels explicitly. The branch $\mu = \gamma$ always has $A_0 = 1$ and is used in the numerical check below. When γ is an integer, the second local solution generally requires the usual logarithmic modification.

The integral representation (12) for the $\mu = 0$ branch is initially valid when $\text{Re}(1-\gamma) > 0$. When $\text{Re}(1-\gamma) \leq 0$, this branch is understood by analytic continuation in γ , equivalently by continuation of the local Frobenius solution of the derivative equation, rather than by the convergent origin integral. If $\gamma \notin \mathbb{Z}$, it remains linearly independent of the $\mu = \gamma$ branch. If $\gamma \in \mathbb{Z}$, the resonant second solution generally contains a logarithm, and the determinant construction in Equation 15 is formed from the

regular solution and its logarithmic companion. In the numerical example, $\gamma = 6.5$; the $\mu = 0$ branch is therefore defined by analytic continuation, while only the $\mu = \gamma$ branch is used for the series check.

When two non-logarithmic branches V_1, V_2 are available, the recovery boundary conditions can be imposed without choosing a special normalization. Define

$$\Delta = V_1(x_L) V_2(x_H) - V_2(x_L) V_1(x_H). \quad (14)$$

Provided the determinant Delta in Equation 14 is nonzero, the boundary solution is

$$V(x) = \frac{V_1(x_L) V_2(x) - V_2(x_L) V_1(x)}{\Delta}. \quad (15)$$

The common phase and the common a -dependent factor in the local Appell basis cancel from this expression. If a logarithmic branch is required, the same determinant construction applies to any numerically or analytically continued fundamental pair.

For completeness, the Appell function of the first kind is

$$F_1(A; b_1, b_2; C; x, y) = \sum_{m, l \geq 0} \frac{(A)_{m+l} (b_1)_m (b_2)_l}{(C)_{m+l} m! l!} x^m y^l, \quad (16)$$

The series in Equation 16 is absolutely convergent for $|x| < 1$ and $|y| < 1$ [9, 13]. Equation 12 is equivalently the integral of $x^{n+\mu-\gamma} (1-x)^{-\delta} (1-x/a)^{-\varepsilon}$, up to the common branch factor. This integral form is useful for stable evaluation.

2.3 Verification procedure

Two checks are used. For the finite- a expansion, the recurrence generates c_n with $c_0 = 1$ and $c_{-1} = c_{-2} = c_{-3} = 0$. Truncations of the $\mu = \gamma$ branch are compared on $0.05 \leq x \leq 0.70$ with an adaptive Runge-Kutta integration of Equation 7, initialized from a 60-term expansion at $x = 0.05$. The parameters are $\kappa = 10$, $\theta = 0.65$, $s = 1.5$, $h = 1$, and $a = 2$. Because its initial data come from the same expansion, this is a truncation-and-propagation consistency check, not a fully independent validation of the recurrence.

The common branch factor in Equation 12 is removed before computation. For $\mu = \gamma$, one has $A_n = n + 1$. Each normalized basis element is evaluated from the real integral

$$\frac{x^{n+1}}{n+1} F_1\left(n+1; \delta, \varepsilon; n+2; x, \frac{x}{a}\right) = \int_0^x t^n (1-t)^{-\delta} (1-t/a)^{-\varepsilon} dt. \quad (17)$$

The integral representation in Equation 17 avoids artificial complex phases from powers of negative real constants and supplies the derivative of every term directly from the integrand.

For confluence, Equation 5 is solved by collocation on $[0.25, 0.75]$ with the stated boundary values. The maximum

difference from the $\xi = 0$ solution is recorded for $\xi = \pm 2^{-j}$, $j = 1, \dots, 5$. These calculations test the series representation and the limiting boundary solution without importing the empirical machinery of the companion application.

All computations were performed in Python 3.12.13 with NumPy 2.3.5 and SciPy 1.17.0 in IEEE double precision; plots used Matplotlib 3.10.8. The initial-value problem was integrated with DOP853 at relative tolerance 10^{-11} and absolute tolerance 10^{-13} . The Appell-basis integrals were evaluated by adaptive quadrature with relative and absolute tolerances 10^{-13} and a maximum of 300 subintervals. The collocation calculation used an initial uniform mesh of 401 points, tolerance 10^{-10} , and at most 20,000 nodes. Reported maximum differences were evaluated on a uniform 10,001-point grid.

3 Results

3.1 Specialized four-term recurrence

Proposition 1. Let $t_n = n + \mu$, with $\mu \in \{0, \gamma\}$. The coefficients in Equation 11 satisfy

$$S_n c_n + R_{n-1} c_{n-1} + Q_{n-2} c_{n-2} + P_{n-3} c_{n-3} = 0, \quad (18)$$

where

$$S_n = -a^2 s t_n (t_n - \gamma), \quad (19)$$

$$R_n = a^2 [s^2 + (s+h) t_n^2 - t_n \{h(1+\gamma) + s(\kappa-1)\}] + as(t_n^2 + t_n), \quad (20)$$

$$Q_n = -a^2 h (t_n^2 - \kappa t_n + 2s) - a \{(s+h) t_n^2 + 2st_n\}, \quad (21)$$

$$P_n = a^2 h^2 + ah(t_n^2 + t_n). \quad (22)$$

Derivation. The general origin-centered Appell recurrence has four successive terms [8]. Substitution of $q = as$ and $\alpha\beta = ah$ immediately gives Equation 19. In the remaining coefficients, the identities $\gamma + \delta + \varepsilon = 0$ and $\alpha + \beta = -1$ eliminate the separate dependence on $\delta, \varepsilon, \alpha$, and β . For example, the following two combinations reduce to

$$\gamma = \varepsilon - 1 + a(\gamma + \delta - 1) = a(\kappa - 1) - 1, \quad (23)$$

$$\gamma = a\gamma + a\delta + \varepsilon = a\kappa. \quad (24)$$

After the reductions in Equations 23 and 24, the product $(t_n - \alpha)(t_n - \beta) = t_n^2 + t_n + ah$ gives Equation 22. The result is a recurrence expressed entirely by the four dimensionless diffusion quantities κ, s, h, a and the origin exponent $\gamma = \kappa\theta$.

3.2 Convergence on the bounded state interval

Proposition 2. For $s, h > 0$, the convergence radius of the origin-centered series in Equation 10 is

$$R = \min \left\{ 1, |a|, \left| \frac{q}{\alpha\beta} \right| \right\} = \min \left\{ 1, |a|, \frac{s}{h} \right\}. \quad (25)$$

Under the killing-rate restrictions in Equation 3,

$$\frac{s}{h} = \frac{r + \lambda_0}{\lambda_1} \geq 1, \quad (26)$$

In Equation 26, the ratio is interpreted as infinity when $h = 0$. Hence $R = \min\{1, |a|\}$. If $|\xi| < 1$, equivalently $|a| > 1$, the expansion converges on every compact subset of $0 < x < 1$. All positive admissible deformations $0 < \xi < 1$ have this property. Negative deformations with $\xi < -1$ place the external root within one unit of the origin; continuation from another center is then required to cover the entire state interval.

Negative deformations with $\xi < -1$ give $a = 1/\xi \in (-1, 0)$, so the external singularity lies inside the unit disk centered at the origin but remains to the left of the physical interval $(0, 1)$. The boundary-value problem therefore stays regular on every interior threshold interval; only the origin-centered representation requires continuation, which may be performed through overlapping local expansions or direct integration.

The result follows from the three characteristic ratio roots 1 , $1/a$, and $\alpha\beta/q$ of the asymptotic recurrence [8]. Its significance is specific to the diffusion family: non-negativity of the killing intensity removes $q/(\alpha\beta)$ as the convergence-limiting singularity.

More explicitly, the Poincaré-Perron argument applies to the coefficient recurrence for a generic, non-minimal solution [8, 14]. The largest ratio root in modulus determines the local radius, while isolated parameter choices can have larger domains through cancellation. Proposition 2 is therefore the robust radius for the family, not a claim that analytic continuation stops at its boundary. Because the recovery thresholds are strictly internal, convergence on compact subsets is the natural statement; behavior at $x = 1$ depends separately on the exponents and is not needed for the boundary problem.

Because Equation 11 is obtained by termwise integration of the Frobenius series (10), uniform convergence on compact nonsingular subintervals within the radius in Equation 25 gives the same local radius for the integrated Appell representation, subject to the branch qualifications in Section 2.2.

3.3 Coordinated general-to-confluent degeneration

Theorem 1. Let $|a| \rightarrow \infty$ while κ, s, h , and remain fixed. Then the general Heun operator, each fixed normalized Appell basis element, and the specialized recurrence coefficients converge to the corresponding confluent-Heun objects. Under the branch and domain assumptions below, the complete summed Appell representation converges uniformly as well.

First, Equation 5 tends to

$$V'' + \left(\frac{\gamma}{x} + \frac{\delta_0}{x-1}\right)V' + \frac{s-hx}{x(x-1)}V = 0, \quad \delta_0 = \kappa(1-\theta). \quad (27)$$

In the confluent-Heun notation

$$u'' + \left(\frac{\gamma}{x} + \frac{\delta_0}{x-1} + \varepsilon_c\right)u' + \frac{\alpha_c x - q_c}{x(x-1)}u = 0, \quad (28)$$

the limiting parameters are

$$\varepsilon_c = 0, \quad \alpha_c = -h, \quad q_c = -s. \quad (29)$$

For $h \neq 0$, infinity is irregular and Equation 27 is the Whittaker-Ince limit of the confluent Heun equation [4–6, 10]. Thus $\xi = 0$ alone is not a hypergeometric restriction.

Second, remove the common a -dependent factor in Equation 12 by setting $\hat{\Phi}_{n,\mu}^{(a)} = (-a)^\varepsilon \Phi_{n,\mu}^{(a)}$. Since $\delta \rightarrow \delta_0$, $\varepsilon \rightarrow -\kappa$, and $x/a \rightarrow 0$, the double series or Euler integral gives, uniformly on compact subsets of $|x| < 1$,

$$\begin{aligned} \hat{\Phi}_{n,\mu}^{(a)}(x) &\rightarrow (-1)^{-\delta_0} \frac{x^{A_n}}{A_n} {}_2F_1(A_n, \delta_0; A_n + 1; x) \\ &= (-1)^{-\delta_0} B_x(A_n, 1 - \delta_0). \end{aligned} \quad (30)$$

The Appell basis therefore degenerates directly to the incomplete-beta basis used for confluent-Heun solutions. The normalization is common to all n and does not change a boundary solution formed from linear combinations.

To justify the limit, fix $0 < r_0 < 1$. On $0 \leq x \leq r_0$, the Euler integrands are uniformly bounded by an integrable function for all sufficiently large $|a|$, because both external factors remain separated from zero. Dominated convergence then yields Equation 30 term by term for fixed n . The identity with B_x follows from the standard incomplete-beta representation of [13]. This argument is local in n ; interchange of the $|a| \rightarrow \infty$ limit with the full infinite sum additionally requires a uniform coefficient bound, established in the next paragraph.

The preceding argument gives the limit for each fixed n . For a branch satisfying $\text{Re}A_0 > 0$, fix $0 < r_0 < \rho < 1$. After removal of the Frobenius factor, the coefficients of the resulting regularized equation for the analytic part and its recurrence-determined initial data, with $c_0 = 1$, converge uniformly on any slightly larger disk still contained in the unit disk. Continuous dependence applied to this regularized equation therefore bounds the analytic part there independently of $|a|$. Cauchy estimates on $|x| \leq \rho$ then give $|c_n(a)| \leq C\rho^{-n}$, with C independent of a . The Euler-integral representation gives $\sup_{0 \leq x \leq r_0} |\hat{\Phi}_{n,\mu}^{(a)}(x)| \leq C_1 r_0^{n+\text{Re}A_0} / (n + \text{Re}A_0)$. Hence the series terms admit a summable majorant proportional to $(r_0/\rho)^n / (n + \text{Re}A_0)$, and the Weierstrass M -test permits interchange of the $|a| \rightarrow \infty$ limit with the infinite sum, uniformly on compact subsets of $|x| < 1$.

Third, division of Equations 19–22 by a^2 produces the finite coefficientwise limits

$$\begin{aligned} \bar{S}_n &= -s t_n (t_n - \gamma), \\ \bar{R}_n &= s^2 + (s + h) t_n^2 - t_n \{h(1 + \gamma) + s(\kappa - 1)\}, \\ \bar{Q}_n &= -h(t_n^2 - \kappa t_n + 2s), \\ \bar{P}_n &= h^2 \end{aligned} \quad (31)$$

These are precisely the origin-centered incomplete-beta recurrence coefficients obtained from Equation 28 with the parameters in Equation 29. In particular, the recurrence remains four-term when $s, h > 0$; confluence does not by itself remove a term. Equations 27, 30, 31 establish the claimed coordinated degeneration.

On an interior threshold interval, the coefficients of Equation 5 are analytic in ξ near zero and the leading coefficient stays uniformly positive. Standard continuous dependence of linear boundary-value problems then gives convergence of the boundary solution whenever the limiting problem is nondegenerate. The numerical experiment below indicates the expected $O(|\xi|)$ rate for the chosen parameters; the theorem itself concerns the exact operator, basis, and recurrence limits and does not require a universal error constant.

3.4 Hypergeometric and terminating subcases

If the state-dependent killing slope is also removed, $h = 0$, Equation 27 becomes

$$x(1-x)V'' + \kappa(\theta-x)V' - sV = 0. \quad (32)$$

Equation 32 is the Gauss hypergeometric equation with $C_H = \kappa\theta$ and parameters A_H, B_H satisfying

$$A_H + B_H + 1 = \kappa, \quad A_H B_H = s. \quad (33)$$

Equation 33 determines A_H and B_H . The correct nested hierarchy is therefore

$\xi \neq 0$: generalHeun $\rightarrow \xi = 0$, $h \neq 0$: confluentHeun $\rightarrow \xi = 0$,
 $h = 0$: Gausshypergeometric.

At finite a , the origin-centered four-term recurrence reduces to three terms if $q = 0$ or $\alpha\beta = 0$, corresponding to $s = 0$ or $h = 0$. Under $r \geq 0$ and $\lambda_0 \geq \lambda_1 \geq 0$, $s = 0$ also forces $h = 0$. The exceptional two-term case $q = 0$, $a = -1$, and $\varepsilon = \delta$ is incompatible with the diffusion mapping: at $a = -1$, equality of the last two exponents would require $1 = -1$ for $\kappa > 0$ [8].

The excluded two-term case is nevertheless analytically informative because it is precisely the classical quadratic pullback for the solution normalized at the origin [8]:

$$\text{HeunG}(-1, 0; \alpha, \beta, \gamma, \delta; X) = 2F_1\left[\frac{\alpha}{2}, \frac{\beta}{2}; \frac{1+\gamma}{2}; Z^2\right], \quad \varepsilon = \delta. \quad (34)$$

In the present model, the value $a = -1$ in Equation 34 corresponds to $\xi = -1$ and is geometrically admissible, but it is not sufficient for the reduction. At this point $q = -s$, while $\delta = \kappa(1-\theta)/2$ and $\varepsilon = -\kappa(1+\theta)/2$. Thus $q = 0$ and $\varepsilon = \delta$ cannot hold nontrivially under the probabilistic admissibility conditions. The quadratic Gauss representation therefore serves as an exact external benchmark rather than an admissible subcase of the recovery model.

For a right-terminated four-term series with last nonzero coefficient c_N , the necessary condition is $\alpha = N + \mu$ or $\beta = N + \mu$,

together with two further algebraic restrictions [8]. Because $\alpha + \beta = -1$, the first condition becomes

$$ah = -(N + \mu)(N + \mu + 1). \quad (35)$$

It is impossible for $a > 1$ and $h > 0$. For $a < 0$, Equation 35 is only a necessary relation on a lower-dimensional parameter set; the two remaining restrictions must still hold. In the confluent limit, $\bar{P}_n = h^2$ never vanishes when $h > 0$, so the incomplete-beta series cannot terminate through this mechanism. Finite-sum solutions are thus exceptional rather than generic consequences of the bounded recovery model.

3.5 Numerical verification

Figure 1A reports the maximum absolute difference between the truncated $\mu = \gamma$ Appell expansion and the adaptively propagated differential-equation solution. Convergence is mildly non-monotone at low orders before settling into asymptotic decay; the error falls from 4.60×10^{-3} at $N = 8$ to 7.51×10^{-6} at $N = 20$ and 4.10×10^{-10} at $N = 32$. Figure 1B shows the maximum difference between the finite- ξ boundary solution and the confluent solution. Halving $|\xi|$ approximately halves the error for both signs: at $|\xi| = 0.5$ the differences are 4.69×10^{-2} and 4.47×10^{-2} , while at $|\xi| = 0.03125$ they are 2.96×10^{-3} and 2.95×10^{-3} . The fitted log-log slopes are close to one, empirically consistent with a regular first-order perturbation on the interior threshold interval for this parameter set.

4 Discussion

The bounded recovery diffusion selects a constrained but nontrivial general-Heun family. Its two identities in Equation 1 compress the generic Appell recurrence into Equations 18–22, while probabilistic admissibility converts a three-candidate convergence radius into the geometric condition $R = \min\{1, |a|\}$. This is more than a relabeling of known formulas: the diffusion restrictions determine which singularity controls convergence and exclude the most conspicuous two-term reduction.

The principal result is the coordinated confluence. Moving the external variance root to infinity sends the general Heun operator to a Whittaker-Ince confluent operator, the normalized F_1 basis to incomplete-beta functions, and the recurrence to the four-term confluent relation. Tracking only the differential equation would miss the behavior of the analytic representation; tracking only the basis would not establish that the coefficient sequence has the correct limiting dynamics. The three components together make the degeneration usable for approximation and computation. Together, these limits provide a controlled representation of the confluent solution on compact interior domains, not merely a limiting differential equation.

The analysis also corrects a tempting but false shortcut. Symmetric Jacobi variance does not by itself imply a Gauss-hypergeometric recovery equation. State-dependent killing leaves an irregular singularity at infinity, so the intermediate equation is

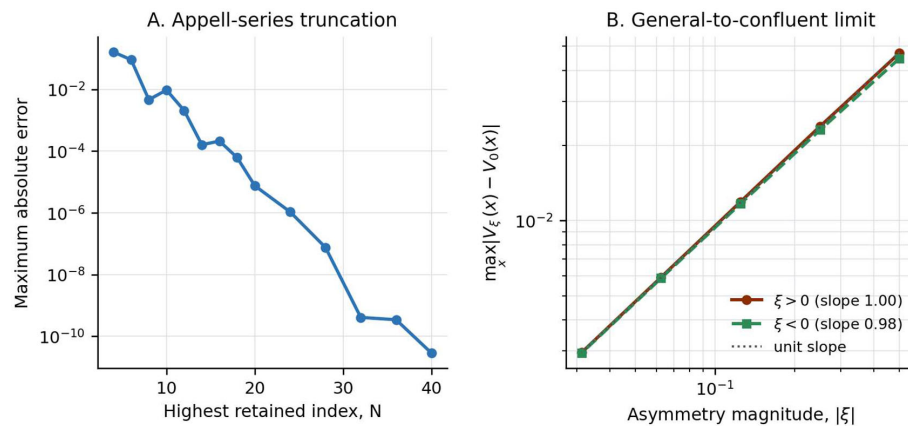


FIGURE 1

Numerical verification. (A) Maximum error of the truncated origin-centered Appell expansion relative to adaptive ODE propagation initialized from a 60-term local expansion. (B) Supremum-norm difference $\|V_\xi - V_0\|_{\infty; [0.25, 0.75]}$ between the finite-asymmetry and confluent boundary solutions; the dotted line has unit slope.

confluent Heun. Hypergeometric solvability requires the additional restriction $h = 0$. Likewise, neither confluence nor boundedness guarantees recurrence reduction or finite termination. The limiting recurrence remains four-term for $s, h > 0$, and admissible positive-asymmetry models exclude the basic termination condition.

The degeneration hierarchy also has a direct economic interpretation. When $\xi \neq 0$, uncertainty responds asymmetrically to financial health and the recovery problem belongs to the general-Heun class. Setting $\xi = 0$ restores symmetric Jacobi variance, but state-dependent abrupt-failure risk remains when $h > 0$; the equation is therefore confluent Heun rather than hypergeometric. Only when $h = 0$, so that the killing rate is state independent, does the model reduce to the Gauss equation. These nested specifications provide progressively simpler benchmarks for assessing whether volatility asymmetry and state-dependent failure intensity materially affect calculated recovery values.

The numerical experiment is deliberately limited. It tests the specialized series and the observed interior confluence rate for one parameter set; it is not a general stability comparison among Appell evaluation, continued fractions, analytic continuation, and collocation. Near $x = 1$, or for $\xi < -1$, origin-centered evaluation may require continuation or a basis centered at another singular point. Integer exponent differences also require logarithmic local solutions. These questions warrant a longer computational study but are not needed for the present analytic result.

From an approximation-theoretic viewpoint, Equation 30 identifies a basis adapted to the coalescing singular structure. A fixed power basis must reproduce the changing influence of the external root through its coefficients alone; the Appell basis carries that dependence explicitly in its second argument and exponent parameter. In the confluence limit, the basis simplifies continuously to incomplete-beta functions rather than being replaced *ad hoc*. This structural adaptation is the main potential advantage of the representation, although a rigorous complexity comparison with power-series or spectral methods remains open.

The construction extends to other killed bounded diffusions whenever the variance has a third simple root outside the

state interval and the killing term is affine. More elaborate variance polynomials lead to Fuchsian equations with additional singularities; other confluence paths may produce different Heun descendants. Fractional derivatives and orthogonal-polynomial structures are not introduced here because neither follows from the present integer-order generator. Possible fractional-calculus or orthogonal-polynomial extensions would require separate derivations and lie beyond the present integer-order diffusion model.

Fractional differential equations have also been used to represent memory-dependent effects in financial modeling [15]. Whether such formulations preserve an Appell-Heun structure or lead to a different special-function class remains an open question.

In summary, the bounded diffusion provides a concrete parameter manifold on which Appell F_1 expansions, recurrence asymptotes, and Heun confluence can be studied together. The resulting hierarchy is exact: general Heun for nonzero variance asymmetry, confluent Heun for symmetric variance with affine killing, and Gauss hypergeometric only when killing is also state independent.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

AI: Investigation, Writing – original draft, Formal analysis, Conceptualization, Writing – review & editing, Methodology, Supervision. CC: Writing – original draft, Methodology, Investigation, Writing – review & editing. KB: Methodology, Writing – original draft, Writing – review & editing, Investigation.

Funding

The author(s) declared that financial support was received for this work and/or its publication. This work was supported by the Armenian Science Committee grant 25SC-CNR-1C003.

Conflict of interest

Author KB was employed by Martin Star LLC.

The remaining author(s) declared that this work was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Generative AI statement

The author(s) declared that Generative AI was used in the creation of this manuscript. During preparation of this paper, the author(s) used OpenAI ChatGPT (GPT-5; OpenAI, <https://chatgpt.com>; accessed August 2026) to assist with drafting and language editing. The author(s) reviewed and revised all AI-assisted text and take full responsibility for the content.

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