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Research Article

Applying the monomiality principle to the new family of Apostol Hermite Bernoulli-type polynomials

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Abstract

In this article, we introduce a new class of polynomials, known as Apostol Hermite Bernoulli-type polynomials, and explore some of their algebraic properties, including summation formulas and their determinant form. The majority of our results are proven using generating function methods. Additionally, we investigate the monomiality principle related to these polynomials and identify the corresponding derivative and multiplicative operators.

Keywords: Apostol Hermite Bernoulli-type polynomials, Bernoulli-type polynomials, Hermite polynomials, monomiality principle

AMS subject classification: 33E20; 32A05; 11B83.

1. Introduction

In this document, we adhere to the standard conventions of mathematical notation. Specifically, we define: $\mathbb{N}$ as the set of natural numbers, denoted as $\{1, 2, \ldots\}$; $\mathbb{N}_0$ as the set of non-negative integers, denoted as $\{0, 1, 2, \ldots\}$; $\mathbb{Z}$ represents the set of integers; $\mathbb{R}$ represents the set of real numbers; and $\mathbb{C}$ represents the set of complex numbers.

The Appell polynomials, denoted as $\{A_n(x)\}_{n=0,1,2,\ldots}$, form a special family of functions introduced by the French mathematician Paul Appell (see [1]). These polynomials are defined by their generating function. Extensive research has been dedicated to various aspects of the Apostol–Bernoulli, and Apostol–Euler Hermite polynomials, along with their extensions and related families. These studies have explored a wide range of topics and applications, enhancing our understanding of their mathematical properties and significance. Numerous investigations, as cited in [2–7], have focused on these polynomial families, examining their properties, generating functions, and special values. In recent years, researchers have also explored modified versions of well-known polynomials such as Bernoulli, Euler, falling factorial, and Bell polynomials. These studies have employed various mathematical tools and techniques, including generating functions, umbral calculus, and p-adic integrals, to analyze and derive new properties of these generalized polynomial forms.

Notable examples of such research can be found in [8,9], where these approaches have been used to uncover intriguing connections and novel results in the study of these modified polynomial families. This includes the generalization of two-variable Hermite polynomials, originally introduced by Kampé de Fériet, as defined in [1]

$$H_n(x,y) = n! \sum_{\nu=0}^{[\frac{n}{2}]} \frac{y^\nu x^{n-2\nu}}{\nu!(n-2\nu)!}.$$

It is to be noted that [10]

$$H_n(2x,-1) = H_n(x).$$

These polynomials can be characterized by the following generating equation:

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}.$$

Let $\alpha \in \mathbb{N}$ be, the generalized Bernoulli-type polynomials $R_n^{(\alpha)}(x)$ of degree n in x are defined by means of the following generating function (see [11]):

$$\left(\frac{z^2}{2e^z-2}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} R_n^{(\alpha)}(x)\frac{z^n}{n!}, \quad |z| < 2\pi, \quad 1^{\alpha} := 1,$$

$$R_n(x) := R_n^{(1)}(x), \quad n \in \mathbb{N}_0.$$

The generalized Bernoulli-type polynomials $R_n^{(\alpha)}(x)$ satisfy the following addition formulae:

$$R_n^{(\alpha+\beta)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} R_k^{(\alpha)}(x) R_{n-k}^{(\beta)}(y), \tag{1}$$

$$R_n^{(\alpha)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} R_k^{(\alpha)}(y) x^{n-k}.$$

As an immediate consequence of (1), we have

$$\begin{aligned}
R_n(x+y) &= \sum_{k=0}^{n} \binom{n}{k} R_k(y) x^{n-k},\\
R_n(x) &= \sum_{k=0}^{n} \binom{n}{k} R_k\, x^{n-k},\\
R_n^{(\alpha)}(x) &= 0, \quad n < \alpha.
\end{aligned}$$

For $\alpha = 1$, and with the help of Software wxMaxima, it is possible to obtain first few expressions for the generalized Bernoulli-type polynomials $R_n(x)$:

$$\begin{aligned}
R_0(x) &= 0,\\
R_1(x) &= \frac{1}{2},\\
R_2(x) &= x - \frac{1}{2},\\
R_3(x) &= \frac{3}{2}x^2 - \frac{3}{2}x + \frac{1}{4},\\
R_4(x) &= 2x^3 - 3x^2 + x,\\
R_5(x) &= \frac{5}{2}x^4 - 5x^3 + \frac{5}{2}x^2 - \frac{1}{12},\\
R_6(x) &= 3x^5 - \frac{15}{2}x^4 + 5x^3 - \frac{1}{2}x.
\end{aligned}$$

For broad information on old literature and new research trends about these classes of polynomials and for the matrix approach to other classes of special polynomials, we recommend to the interested reader (see [12–16]).

2. Apostol Hermite Bernoulli-type polynomials ${}_HJ_n(x, y; \lambda)$

In this section, we explore the Apostol Hermite Bernoulli-type polynomials, a novel class that extends the classical Hermite and Bernoulli-type polynomials. We provide a comprehensive overview of their fundamental properties, including generating functions and recurrence relations. This exploration highlights the unique characteristics and potential applications of these polynomials across various areas of mathematics and its applications.

Definition 2.1. The Apostol Hermite Bernoulli-type polynomials ${}_HJ_n(x, y; \lambda)$ are defined through their generating function, which is defined in a suitable neighborhood of $t = 0$.

$$\left(\frac{t^2}{2\lambda e^t - 2}\right) e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HJ_n(x, y; \lambda)\frac{t^n}{n!}, \quad |t| < |\log(\lambda)|. \tag{2}$$

Remark 2.1. Taking $y = 0$ and $\lambda = 1$ in Equation (2) we obtain the corresponding new family of Bernoulli–type polynomials $R_n(x)$ defined as:

$$\left(\frac{t^2}{2e^z - 2}\right) e^{xt} = \sum_{n=0}^{\infty} R_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi.$$

The following are some illustrative examples showing that there exist Apostol Hermite Bernoulli-type polynomials and their respective graphs.

Example 2.1. For $\lambda = 1$, the first few Apostol Hermite Bernoulli-type polynomials are given as:

ν	${}_HJ_n(x, y; 1)$
0	$\frac{1}{2}$
1	$\frac{x}{2} - \frac{1}{4}$
2	$\frac{1}{12}\left(1 - 6x + 6x^2 + 12y\right)$
3	$\frac{1}{4}\left(x - 3x^2 + 2x^3 - 6y + 12xy\right)$
4	$\frac{1}{60}\left(-1 + 30x^2 - 60x^3 + 30x^4 + 60y - 360xy + 360x^2y + 360y^2\right)$

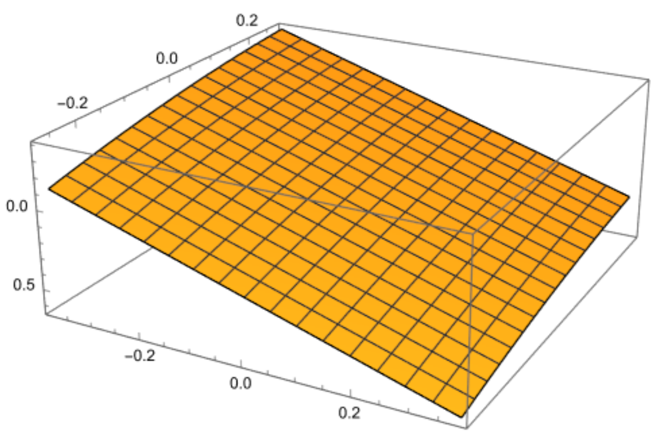

Figure 1. $\frac{1}{12}\left(1 - 6x + 6x^2 + 12y\right)$

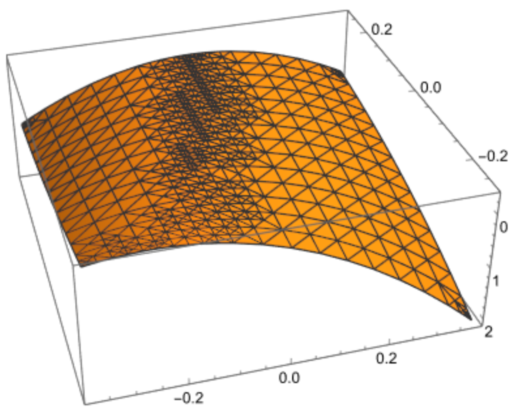

Figure 2. $\frac{1}{60}\left(-1 + 30x^2 - 60x^3 + 30x^4 + 60y - 360xy + 360x^2y + 360y^2\right)$

Theorem 2.1. *When considering real numbers x and non-negative integers n, the following relationship holds*

$$ {}_HJ_n(x+y,z+u;\lambda) = \sum_{k=0}^{n} \binom{n}{k} {}_HJ_{n-k}(y,z;\lambda) {}_HJ_k(x,u;\lambda). $$

Proof. Utilizing the expression provided in (2), we obtain

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_HJ_n(x+y,z+u;\lambda)\frac{t^n}{n!} &= \left(\frac{t^2}{2\lambda e^t - 2}\right) e^{(x+y)t+(z+u)t^2} \\ &= \left(\frac{t^2}{2\lambda e^t - 2}\right) e^{xt+yt+zt^2+ut^2} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} {}_HJ_{n-k}(y,z;\lambda) {}_HJ_k(x,u;\lambda)\frac{t^n}{n!}. \end{aligned} $$

By setting the coefficients of $\frac{t^n}{n!}$, we arrive at the result. □

Theorem 2.2. *When considering $x \in \mathbb{R}$, the following relationship is established*

$$ {}_HJ_n(x+z,y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} {}_HJ_{n-k}(z;\lambda) H_k(x,y). $$

Proof. Using the expression in (2), we get

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_HJ_n(x+z,y;\lambda)\frac{t^n}{n!} &= \left(\frac{t^2}{2\lambda e^t - 2}\right) e^{(x+z)t+yt^2} \\ &= \left(\frac{t^2}{2\lambda e^t - 2}\right) e^{zt+xt+yt^2} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} {}_HJ_{n-k}(z;\lambda) H_k(x,y)\frac{t^n}{n!}. \end{aligned} $$

By equating the coefficients of $\frac{t^n}{n!}$, we obtain the result. □

Theorem 2.3. *For $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$. Then we have*

$$ {}_HJ_n(x+z,y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} {}_HJ_k(x,y;\lambda). $$

Proof. From (2), we obtain

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_HJ_n(x+z,y;\lambda)\frac{t^n}{n!} &= \left(\frac{t^2}{2\lambda e^t - 2}\right) e^{(x+z)t+yt^2} \\ &= \left(\frac{t^2}{2\lambda e^t - 2}\right) e^{xt+yt^2+zt} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} z^{n-k} {}_HJ_k(x,y;\lambda)\frac{t^n}{n!}. \end{aligned} $$

Equating coefficients of $\frac{t^n}{n!}$ yields the result. □

For $\lambda = 1$ in (2) and following the concepts outlined in [17], we can derive the expression for the determinants of the Hermite Bernoulli-type polynomials.

Corollary 2.1. *The Hermite Bernoulli-type polynomials has the following determinantal representation:*

$$
{}_HJ_0(x,y) = \frac{1}{\delta_0}.
$$

$$
{}_HJ_n(x,y) = \frac{(-1)^n}{\delta_0^{n+1}} \begin{vmatrix} H_0(x,y) & H_1(x,y) & \cdots\cdots & H_{n-1}(x,y) & H_n(x,y) \\ \delta_0 & \delta_1 & \cdots\cdots & \delta_{n-1} & \delta_n \\ 0 & \delta_0 & \cdots\cdots & \binom{n-1}{1}\delta_{n-2} & \binom{n}{1}\delta_{n-1} \\ 0 & 0 & \ddots & \binom{n-1}{2}\delta_{n-3} & \binom{n}{2}\delta_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & & \\ 0 & \cdots & \cdots\cdots & \delta_0 & \binom{n}{n-1}\delta_1 \end{vmatrix},
$$

where the numerical sequence $\{\delta_n\}_{n\geq 0}$

$$
\left[\frac{t^2}{2e^t - 2}\right]^{-1} = \sum_{k=0}^{\infty} \delta_k \frac{t^k}{k!}.
$$

3. Monomiality Principle

The concept of monomiality traces back to 1941, with Steffenson introducing the poweroid notion [18], later refined by Dattoli [19,20]. The operators $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ serve as both multiplicative and derivative operators for a polynomial set $\{b_m(u)\}_{m\in\mathbb{N}}$, satisfying the expressions:

$$
b_{m+1}(u) = \hat{\mathcal{M}}\{b_m(u)\}
$$

and

$$
m\ b_{m-1}(u) = \hat{\mathcal{D}}\{b_m(u)\}.
$$

The set $\{b_m(u)\}_{m\in\mathbb{N}}$ manipulated by these operators is termed a quasi-monomial and must adhere to the formula:

$$
[\hat{\mathcal{D}}, \hat{\mathcal{M}}] = \hat{\mathcal{D}}\hat{\mathcal{M}} - \hat{\mathcal{M}}\hat{\mathcal{D}} = \hat{1},
$$

displaying a Weyl group structure.

The properties of $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ determine the characteristics of the quasi-monomial set $\{b_m(u)\}_{m\in\mathbb{N}}$:

(i) $b_m(u)$ satisfies the differential equation

$$
\hat{\mathcal{M}}\hat{\mathcal{D}}\{b_m(u)\} = m\ b_m(u), \tag{3}
$$

if $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ have differential realizations.

(ii) The explicit form of $b_m(u)$ is given by

$$
b_m(u) = \hat{\mathcal{M}}^m\ \{1\},
$$

with $b_0(u) = 1$.

(iii) The generating relation in exponential form for $b_m(u)$ can be expressed as

$$
e^{t\hat{\mathcal{M}}}\{1\} = \sum_{m=0}^{\infty} b_m(u)\frac{t^m}{m!}, \quad |t| < \infty \quad ,
$$

using identity (3).

The primary objective of the monomiality principle is to identify operators for multiplication and differentiation. Additionally, the monomiality principle was exploited to find these identities for hybrid special polynomials by many authors, see for example [21–23]. Thus, in the context of the monomiality principle, we establish the following outcomes to characterize the polynomials ${}_HJ_n(x,y;\lambda)$ of degree n in x,y:

Theorem 3.1. *For the polynomials ${}_HJ_n(x,y;\lambda)$ of degree n in x, the succeeding multiplicative and derivative operators hold true:*

$$\hat{\mathcal{M}}_{{}_HJ_n(x,y;\lambda)} = x + 2y\frac{\partial}{\partial x} + \frac{1}{\frac{\partial}{\partial x}} + \frac{\lambda e^{\frac{\partial}{\partial x}}}{2(\lambda e^{\frac{\partial}{\partial x}-1})} \tag{4}$$

and

$$\hat{\mathcal{D}}_{{}_HJ_n(x,y;\lambda)} = \frac{\partial}{\partial x}. \tag{5}$$

Proof. By differentiating expression (2) with respect to t, it follows that

$$\frac{\partial}{\partial t}\left[\left(\frac{t^2}{2\lambda e^t-2}\right)e^{xt+yt^2}\right] = \frac{\partial}{\partial t}\left[\sum_{n=0}^{\infty}{}_HJ_n(x,y;\lambda)\frac{t^n}{n!}\right]$$

which further gives

$$\left(x+2yt+\frac{1}{t}+\frac{\lambda e^t}{2(\lambda e^t-1)}\right)\left[\frac{t^2}{2\lambda e^t-2}e^{xt+yt^2}\right] = \sum_{n=0}^{\infty} n\ {}_HJ_n(x,y;\lambda)\frac{t^{n-1}}{n!}. \tag{6}$$

Also, differentiating (2) with respect to x, it follows that

$$\frac{\partial}{\partial x}\left[\left(\frac{t^2}{2\lambda e^t-2}\right)e^{xt+yt^2}\right] = t\left(\frac{t^2}{2\lambda e^t-2}\right)e^{xt+yt^2}. \tag{7}$$

Using expression (7) in (6), assertion (4) is proved. Again, in view of expression (7), we have

$$\frac{\partial}{\partial x}\left[\sum_{n=0}^{\infty} n\ {}_HJ_n(x,y;\lambda)\frac{t^n}{n!}\right] = \left[\sum_{n=0}^{\infty}{}_HJ_n(x,y;\lambda)\frac{t^{n+1}}{n!}\right]$$

thus, replacing $n\to n-1$, we find

$$\frac{\partial}{\partial x}\left[\sum_{n=0}^{\infty} n\ {}_HJ_n(x,y;\lambda)\frac{t^n}{n!}\right] = \left[\sum_{n=0}^{\infty} n\ {}_HJ_n(x,y;\lambda)\frac{t^{n-1}}{n!}\right]$$

which proves assertion (5) while comparing same powers of t both sides. □

Theorem 3.2. *The polynomials ${}_HJ_n(x,y;\lambda)$ of degree n in x,y satisfy the succeeding differential equation:*

$$\left[x\frac{\partial}{\partial x} + 2y\frac{\partial^2}{\partial x^2} + 1 + \frac{\lambda e^{\frac{\partial}{\partial x}}}{2(\lambda e^{\frac{\partial}{\partial x}-1})}\frac{\partial}{\partial x} - n\right]{}_HJ_n(x,y;\lambda) = 0. \tag{8}$$

Proof. Inserting expression (4) and (5) in expression (3), we obtain assertion (8). □

4. Conclusion

This article explores the properties of Apostol Hermite Bernoulli-type polynomials, offering a comprehensive framework through various characterizations, including the verification of the monomiality principle. Our findings significantly advance the theoretical development of these polynomials, paving the way for further research and applications in mathematical analysis.

Looking ahead, future research could extend the investigation to higher-dimensional analogues of Apostol Hermite Bernoulli-type polynomials, which could reveal richer structures and more complex behaviors. Furthermore, exploring the application of these polynomials in numerical methods and approximation theory could lead to practical computational tools. Closer examination of its relationships with other special functions, as well as its roles in stochastic processes and probability theory, may reveal deeper connections and broader applications. Furthermore, developing analogues of q and investigating its properties within the framework of quantum calculus could provide new perspectives and broaden the theoretical landscape of Apostol Hermite Bernoulli-type polynomials.

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