CHAPTER

12

Inverse problem for the loaded heat conductivity equation with variable coefficients

Umida Baltaeva[a,b], **Praveen Agarwal**[c,d], **Bobur Khasanov**[e], **Hamrobek Hayitbayev**[f],
Shilpi Jain[g], **Florence Hubert**[h], **and Clemente Cesarano**[i]

[a]*Department of Exact Sciences, Khorezm Mamun Academy, Uzbekistan Academy of Sciences, Khiva, Uzbekistan*
[b]*Department of Applied Mathematics, Urgench State University, Urgench, Uzbekistan*
[c]*Department of Mathematics, Anand International College of Engineering, Jaipur, India*
[d]*International Center for Basic and Applied Sciences, Jaipur, India*
[e]*Department of Exact Sciences, Khorezm Mamun Academy, Khiva, Uzbekistan*
[f]*Department of Accounting and General Professional Sciences, Mamun University, Khiva, Uzbekistan*
[g]*Department of Mathematics, Poornima College of Engineering, Jaipur, India*
[h]*Aix-Marseille Universite, CNRS, I2M, Marseille, France*
[i]*Section of Mathematics Luciano Modica, International Telematic University, Uninettuno, Rome, Italy*

12.1 Introduction and formulation of the problem

Thermal conduction refers to the transfer of internal energy, in the form of heat, between neighboring molecules within a solid, liquid, or gas, as well as between different materials in close contact, without requiring any bulk motion of the material itself [1–3]. On this basis, heat conduction has been studied for more than two centuries and remains a topic of continuing scientific interest, playing a fundamental role in both natural and engineered systems.

From both physical and mathematical viewpoints, the heat conduction equation occupies a universal position. It arises naturally in diverse contexts, including the modeling of mass diffusion processes and the description of vorticity diffusion in viscous fluids. Heat conduction problems are also of considerable interest in the theory of partial differential equations. The one-dimensional heat equation is the most thoroughly studied case and has found extensive analytical and practical applications. In contrast, heat conduction problems in three-dimensional and multi dimensional do-

Extended Hypergeometric Functions and Orthogonal Polynomials. https://doi.org/10.1016/B978-0-44-336484-6.00020-5

mains remain an active area of research, particularly when nonstandard operators are involved.

In recent years, substantial attention has been devoted to fractional differential equations, in which derivatives of non-integer order are employed. This growing interest is closely linked to advances in fractional calculus, including the systematic development of fractional integrals and derivatives and their successful application across numerous scientific disciplines [4–8].

Fractional diffusion models extend classical diffusion equations by incorporating derivatives of fractional order, thereby providing a more accurate description of anomalous diffusion processes. Such models have proven effective in representing complex transport phenomena encountered in physics, medicine, and biology [9–13].

The analysis of two-dimensional and multidimensional space-fractional diffusion equations with variable coefficients presents significant challenges from both theoretical and computational perspectives [14]. In many cases, classical methods are insufficient to establish well-posedness or to construct efficient numerical schemes. Consequently, the study of fractional diffusion equations with variable coefficients often relies on a combination of analytical techniques and advanced integration or approximation methods.

The study presented in [15] explores numerical approximation techniques for solving fractional diffusion equations with variable coefficients. The application of the method of prior estimates to boundary value problems for fractional diffusion equations, following an approach analogous to that in the classical case, is examined in [17]. In [16], a finite difference scheme is proposed for approximating solutions to spatial fractional convection-diffusion models governed by equations with variable coefficients. The homotopy analysis method and the Adomian decomposition method are utilized in [18] to address high-order time-fractional partial differential equations. Furthermore, initial and boundary value problems for fractional diffusion equations with variable coefficients have been extensively studied in [19–23].

The increased interest in the study of loaded differential equations [24] is attributed to their numerous applications and the fact that they form a distinct class of partial differential equations [25,26]. Notably, the pioneering works of Nakhushev provided one of the first generalized definitions of loaded equations, along with their classification and applications to various problems in mathematical physics and biology [27–34]. Boundary value problems for heat conduction equations with loaded terms have been extensively studied in both bounded and unbounded domains [28–31]. These investigations are particularly focused on cases where the order of the derivative in the loaded term is greater than or equal to the order of the differential operator in the equation.

In this article, we focus on both aspects, specifically examining an analogue of the fractional diffusion equation with variable coefficients under a fractional load.

Inverse problems for parabolic equations with variable coefficients have been studied extensively, and fundamental results on existence and uniqueness for corres ponding initial-boundary value problems are well established. Motivated by these developments, we consider an inverse problem for a loaded integro-differential heat conduction equation with variable coefficients. In particular, following recent progress [35–39], the present work addresses the problem of identifying an unknown coefficient in the loaded integro-differential heat conduction equation.

We introduce the following notations:

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Let $\mathbb{R}^2_T$ represent a subset of three-dimensional Euclidean space, specified by a point (x, y, z), where $(x, y) \in \mathbb{R}^2$ and $z \in (0, T]$, with $T > 0$:

$$R^2_T = \left\{ (x, y, z) \middle| (x, y) \in R^2, 0 \le z \le T \right\}.$$

Let $f(x, y)$ be a function defined on $\mathbb{R}^2$.

Definition 12.1.1. If for any $x^{(1)}, x^{(2)} \in \mathbb{R}^2$, the inequality

$$|f(x^{(1)}) - f(x^{(2)})| \le k|x^{(1)} - x^{(2)}|^l, \quad k > 0,\ l \in (0, 1), \tag{12.1.1}$$

holds, then the function $f(x)$ is said to satisfy the Holder condition with exponent l in $\mathbb{R}^2$. The class of functions satisfying condition (12.1.1) is denoted by $H^l(\mathbb{R}^2)$.

Definition 12.1.2. If for any given pair of values

$$\left(x^{(1)}, y^{(1)}, z^{(1)}\right), \left(x^{(2)}, y^{(2)}, z^{(2)}\right) \in \mathbb{R}^2_T,$$

the inequality

$$\begin{aligned} &\left| f\left(x^{(1)}, y^{(1)}, z^{(1)}\right) - f\left(x^{(2)}, y^{(2)}, z^{(2)}\right) \right| \\ &\quad \le k_1 \left|x^{(1)} - x^{(2)}\right|^l + k_2 \left|y^{(1)} - y^{(2)}\right|^l + k_3 \left|z^{(1)} - z^{(2)}\right|^{l/2}, \end{aligned} \tag{12.1.2}$$

where

$$k_i > 0\,(\text{constants}),\ i = 1, 2, 3, \quad l \in (0, 1),$$

holds, then the function $f(x, y, t)$ is said to satisfy the Holder condition with exponents l and $l/2$ in $\mathbb{R}^2_T$. The class of functions satisfying condition (12.1.2) is denoted by $H^{l,l/2}(\mathbb{R}^2_T)$.

Inverse problem. Find a pair of functions $u(x, y, z)$ and $k(x, z)$ in $(x, y, z) \in \mathbb{R}^2_T$, satisfying the following properties:

$$\begin{aligned} u_z - \gamma(z)\left(u_{xx} + u_{yy}\right) = \lambda[D^{-\alpha}_{0z} u(0, y, z) + u(0, y, z)] \\ + \int_0^z k(x, \tau)\, u(x, y, z - \tau)\, d\tau, \end{aligned} \tag{12.1.3}$$

$$u(x, y, z)|_{z=0} = \varphi(x, y), \quad (x, y) \in R^2, \tag{12.1.4}$$

$$u(x, y, z)|_{y=0} = \chi(x, z), \quad (x, z) \in R^1_T, \tag{12.1.5}$$

where $\gamma(z)$, $\varphi(x,y)$, $\chi\,(x,z)$ are given functions and

$$\gamma\,(z) \in I := \left\{\gamma\,(z)\,|\gamma\,(z) > 0,\ \gamma\,(z) \in C^1\,[0,T] \cap C(0,T)\right\}, \tag{12.1.6}$$

$$\varphi\,(x,y) \in H^{l+2}\left(R^2\right),\ \varphi\,(x,y) \le \varphi_0 = const > 0,\ \ \varphi\,(x,0) = \chi\,(x,0)\,, \tag{12.1.7}$$

$$\chi\,(x,z) \in H^{l+4,(l+4)/2}\left(\bar{R}^1_T\right),\ \ l \in (0,1)\,, \lambda \in R. \tag{12.1.8}$$

$D_{0z}^{-\alpha}$ is the Riemann–Liouville fractional integral operator [4] of order α and $\alpha > 0$. The inverse problem consists of determining the unknown functions $u(x,y,z)$ and $k\,(x,0,z)$ from the equalities (12.1.3)–(12.1.5).

12.2 Investigation of the problem

First, we will construct auxiliary problems equivalent to the inverse problem (12.1.3), (12.1.4), (12.1.5).

Let us introduce the following replacement in the problem (12.1.3)–(12.1.5):

$$\vartheta(x,y,z) = u_{yy}(x,y,z),\ \ (x,y,z) \in R^2_T. \tag{12.2.1}$$

Using the change of variable (12.2.1), the inverse problem (12.1.3), (12.1.4), (12.1.5), is equivalently reduced to the following problem.

Auxiliary problem: Find functions $\vartheta(x,y,z)$ and $k\,(x,z)$ in $(x,y,z) \in R^2_T$, possessing the following properties:

$$\vartheta_z - \gamma\,(z)\,\Delta\vartheta = \lambda[D_{0z}^{-\alpha}\vartheta\,(0,y,z) + \vartheta\,(0,y,z)] + \int_0^z k\,(x,\tau)\,\vartheta\,(x,y,z-\tau)\,d\tau, \tag{12.2.2}$$

$$\vartheta\ (x,y,z)|_{t=0} = \varphi_{yy}\,(x,y)\,,\ \ (x,y) \in R^2, \tag{12.2.3}$$

$$\begin{gathered}\vartheta\ (x,y,z)|_{y=0} = \frac{1}{\gamma\,(z)}\chi_z\,(x,z) - \chi_{xx}\,(x,z) - \\ -\frac{\lambda}{\gamma\,(z)}[D_{0z}^{-\alpha}\chi\,(0,0,z) + \chi\,(0,0,z)] - \frac{1}{\gamma\,(z)}\int_0^z k\,(x,\tau)\,\chi\,(x,0,z-\tau)\,d\tau,\end{gathered} \tag{12.2.4}$$

moreover, we assume that $\chi\,(0,0,0) = 0$, from the initial condition (12.2.3) and (12.2.4) the following condition of agreement is satisfied:

$$\varphi_{yy}\,(x,0) = \frac{1}{\gamma\,(0)}\chi_z\,(x,0) - \chi_{xx}\,(x,0)\,. \tag{12.2.5}$$

Indeed, if the compatibility conditions (12.1.5) and (12.2.5) are satisfied, and the functions φ and χ are sufficiently smooth, it can be shown that the problems (12.2.2)–(12.2.4) are equivalent to the inverse problem (12.1.3)–(12.1.5):

First, integrating twice from the (12.2.1) substitution above from 0 to y, we get:

$$u(x,y,z)=\chi\left(x,z\right)+y\varphi\left(x,0\right)+\int_0^y(y-\eta)\vartheta(x,\eta,z)d\eta \tag{12.2.6}$$

and, consequently, in (12.2.1) for the function $u(x,y,t)$, taking into account the agreement condition (12.2.5) for $z=0$, we have:

$$\begin{gathered}u\left(x,y,0\right)|_{z=0}=\chi\left(x,0\right)+yu_y\left(x,0,0\right)+\int_0^y\left(y-\xi\right)\varphi_{\xi\xi}\left(x,\xi\right)d\xi=\\=\chi\left(x,0\right)+yu_y\left(x,0,0\right)+\int_0^y\left(y-\xi\right)d\varphi_\xi=\chi\left(x,0\right)+yu_y\left(x,0,0\right)+\\+\left(y-\xi\right)\varphi_\xi\ \left(x,\xi\right)|_0^y+\int_0^y\varphi_\xi\left(x,\xi\right)d\xi=\chi\left(x,0\right)+yu_y\left(x,0,0\right)-\\-y\varphi_y\left(x,0\right)+\varphi\left(x,y\right)-\varphi\left(x,0\right)=y\left(u_y\left(x,0,0\right)-\varphi_y\left(x,0\right)\right)+\varphi\left(x,y\right)=\varphi\left(x,y\right).\end{gathered}$$

As can be seen from (12.2.6), on $y=0$ the additional condition in (12.1.5) follows.

The sequence of obtaining Eq. (12.1.3) from Eq. (12.2.2) is as follows: first, we integrate both sides of Eq. (12.2.2) twice from 0 to y:

$$\begin{gathered}\int_0^y\left(y-\xi\right)\vartheta_z\left(x,\xi,z\right)d\xi-\gamma\left(z\right)\int_0^y\left(y-\xi\right)\left(\vartheta_{xx}\left(x,\xi,z\right)+\vartheta_{\xi\xi}\left(x,\xi,z\right)\right)d\xi=\\=\int_0^y\left(y-\xi\right)[\lambda\vartheta\left(0,\xi,z\right)+\int_0^z k\left(x,\tau\right)\vartheta\left(x,\xi,z-\tau\right)d\tau]d\xi+\\+\frac{\lambda}{\Gamma\left(\alpha\right)}\int_0^y\left(y-\xi\right)d\xi\int_0^z\left(z-\tau\right)^{\alpha-1}\vartheta\left(0,\xi,\tau\right)d\tau\end{gathered}$$

and taking into account equality (12.2.6), i.e.,

$$\int_0^y(y-\xi)\vartheta(x,\xi,z)d\xi=u(x,y,z)-\chi\left(x,z\right)-y\varphi\left(x,0\right).$$

Accordingly, we establish the following relations:

$$\begin{gathered}u_z(x,y,z)-\chi_z\left(x,z\right)-\gamma\left(z\right)u_{xx}\left(x,y,z\right)+\gamma\left(z\right)\chi\left(x,z\right)+\gamma\left(z\right)y\vartheta_y\left(x,0,z\right)-\\-\gamma\left(z\right)\vartheta\left(x,y,z\right)+\gamma\left(z\right)\vartheta\left(x,0,z\right)=\\=\lambda\left(u\left(0,y,z\right)-\chi\left(0,z\right)-y\varphi_y\left(0,0\right)\right)+\\+\int_0^z k\left(x,\tau\right)\left(u(x,y,z-\tau)-\chi\left(x,z-\tau\right)-y\varphi_y\left(x,0\right)\right)d\tau+\\+\frac{\lambda}{\Gamma\left(\alpha\right)}\int_0^z\left(z-\tau\right)^{\alpha-1}\left(u\left(0,y,\tau\right)-\chi\left(0,z-\tau\right)-y\varphi_y\left(0,0\right)\right)d\tau,\end{gathered}$$

Hence, taking into account the condition (12.2.4) for $\vartheta(x, y, z)$, it is easy to see that Eq. (12.1.3) has been obtained. Thus the (12.1.3), (12.1.4), (12.1.5) inverse problem of finding functions $u(x, y, z)$ and $k(x, z)$ is equivalent to the inverse problem for finding functions $\vartheta(x, y, z)$ and $k(x, 0, z)$ from (12.2.2)–(12.2.4).

Auxiliary problem

In the second step, if we differentiate the resulting equations to t and make the replacement $\vartheta_z(x, y, z) = \rho(x, y, z)$ in (12.2.2)–(12.2.4), we obtain the following auxiliary problem for finding the functions $\vartheta(x, y, z)$, $k(x, z)$, $\rho(x, y, z)$:

$$\rho_z - \gamma(z)\left(\rho_{xx} + \rho_{yy}\right) = (\ln\gamma(z))'\,(\rho - \int_0^z k(x,\tau)\,\vartheta(x, y, z-\tau)\,d\tau) +$$
$$+\int_0^z k(x,\tau)\,\rho(x, y, z-\tau)\,d\tau - \lambda\,(\ln\gamma(z))'\,(\vartheta(0, y, z) + D_{0z}^{-\alpha}\vartheta(0, y, z) +$$
$$+\lambda(\rho(0, y, z) + (D_{0z}^{-\alpha}\rho(0, y, z)) + k(x, z)\,\varphi_{yy}(x, y) + \frac{\lambda}{\Gamma(\alpha)} z^{\alpha-1}\varphi_{yy}(0, y, 0), \tag{12.2.7}$$

$$\rho|_{z=0} = \gamma(0)\,\Delta\varphi_{yy}(x, y), \tag{12.2.8}$$

$$\rho|_{y=0} = F_z(x, z) + \frac{\gamma'(z)}{\gamma^2(z)}\int_0^z k(x,\tau)\,\chi(x, z-\tau)\,d\tau -$$
$$-\frac{1}{\gamma(z)}\int_0^z k(x,\tau)\,\chi_z(x, z-\tau)\,d\tau - \frac{1}{\gamma(z)} k(x, z)\,\varphi\left(x', 0\right), \tag{12.2.9}$$

where

$$F(x, z) = \frac{1}{\gamma(z)}\chi_z(x, z) - \chi_{xx}(x, z) - \frac{\lambda}{\gamma(z)\,\Gamma(\alpha)}\int_0^z (z-\tau)^{\alpha-1}\,\chi(0, 0, \tau)\,d\tau.$$

As a result, we obtained an auxiliary problem for finding functions $\vartheta(x, y, z)$, $k(x, 0, z)$, $\rho(x, y, z)$.

In the next step, integrating both parts of the last change of variable from 0 to t, we obtain the following equality:

$$\vartheta(x, y, z) = \varphi_{yy}(x, y) + \int_0^z \rho(x, y, \tau)\,d\tau. \tag{12.2.10}$$

If the function $\rho(x, y, z)$ is known, then the function $\vartheta(x, y, z)$ is found from (12.2.10). Thus the problem (12.2.7)–(12.2.9) leads to problems (12.2.2)–(12.2.4) and problems (12.2.2)–(12.2.4) lead to the inverse problems (12.1.3), (12.1.4), (12.1.5). Hence, finding the functions $\vartheta(x, y, z)$, $k(x, 0, z)$, $\rho(x, y, z)$ from problems (12.2.2)–(12.2.4) and (12.2.7)–(12.2.9) is equivalent to finding the functions $u(x, y, z)$, $k(x, 0, z)$, from the inverse problem (12.1.3), (12.1.4), (12.1.5).

Thus we have proved the following lemma:

Lemma 12.2.1. *Suppose that* $\gamma(z) \in I$, $\varphi(x,y) \in H^{l+6}\left(R^2\right)$, $\chi(x,z) \in H^{l+4,(l+4)/2}\left(\bar{R}^1_T\right)$, *and the matching conditions*

$$\chi(x,0) = \varphi(x,0)\,, \; \varphi_{yy}(x,0) = \frac{1}{\gamma(0)}\chi_z(x,0) - \chi_{xx}(x,0)$$

are met. Then, the problem (12.1.3), (12.1.4), (12.1.5) *is equivalent to the problem of determining the functions* $\vartheta(x,y,z)$, $k(x,0,z)$, $\rho(x,y,z)$ *from Eqs.* (12.2.2)–(12.2.4) *and* (12.2.7)–(12.2.9).

12.3 Uniqueness of solvability

Lemma 12.3.1. *The auxiliary problem* (12.2.2), (12.2.3), *and* (12.2.7)–(12.2.9) *is equivalent to finding the functions* $\vartheta(x,y,z)$, $k(x,y,z)$, $\rho(x,y,z)$ *from the following system of integral equations:*

$$\begin{aligned}
\vartheta(x,y,z) = {} & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi_{\eta\eta}(\xi,\eta)\, G d\xi d\eta + \int_0^{\sigma(z)} \frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)} \times \\
& \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\sigma^{-1}(\tau)} k(\xi,0,\alpha)\vartheta\left(\xi,\eta,\sigma^{-1}(\tau)-\alpha\right) G d\alpha d\xi d\eta + \\
& \lambda \int_0^{\sigma(z)} \frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vartheta(0,\eta,\beta)\, G d\beta d\eta \\
& + \frac{\lambda}{\Gamma(\alpha)} \int_0^{\sigma(z)} \frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)} \\
& \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\sigma^{-1}(\tau)} \left(\sigma^{-1}(\tau)-\beta\right)^{\alpha-1} \vartheta(0,\eta,\beta)\, G d\beta d\xi d\eta,
\end{aligned} \tag{12.3.1}$$

$$\begin{aligned}
\rho(x,y,z) = {} & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma(0)\, \Delta\varphi_{\eta\eta}(\xi,\eta)\, G d\xi d\eta + \int_0^{\sigma(z)} \frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)} \times \\
& \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\left(\ln\gamma\left(\sigma^{-1}(\tau)\right)\right)' \rho\left(\xi,\eta,\sigma^{-1}(\tau)\right) - \right. \\
& \left. - \left(\ln\gamma\left(\sigma^{-1}(\tau)\right)\right)' \int_0^{\sigma^{-1}(\tau)} k(\xi,0,\alpha)\vartheta\left(\xi,\eta,\sigma^{-1}(\tau)-\alpha\right) d\alpha \right) G d\xi d\eta + \\
& + \int_0^{\sigma(z)} \frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\sigma^{-1}(\tau)} k(\xi,0,\alpha)\rho\left(\xi,\eta,\sigma^{-1}(\tau)-\alpha\right) G d\alpha d\xi d\eta + \\
& + \int_0^{\sigma(t)} \frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k\left(\xi,0,\sigma^{-1}(\tau)\right) \varphi_{\eta\eta}(\xi,\eta)\, G d\xi d\eta +
\end{aligned} \tag{12.3.2}$$

$$
\begin{aligned}
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\\
&\quad\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\sigma^{-1}(\tau)}\left(\sigma^{-1}(\tau)-\beta\right)^{\alpha-1}\rho(0,\eta,\beta)\,Gd\beta d\xi d\eta-\\
&-\frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\\
&\quad\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bigg[\left(\ln\gamma\left(\sigma^{-1}(\tau)\right)\right)'\int_0^{\sigma^{-1}(\tau)}\left(\sigma^{-1}(\tau)-\beta\right)^{\alpha-1}\times\\
&\qquad\times\vartheta(0,\eta,\beta)\,d\beta-\left(\sigma^{-1}(\tau)\right)^{\alpha-1}\varphi_{\eta\eta}(\xi,\eta)\bigg]Gd\xi d\eta,
\end{aligned}
$$

$$
\begin{aligned}
k(x,0,z)=&\frac{\gamma(t)}{\varphi(x,0)}\bigg(F_z(x,z)-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\gamma(0)\,\Delta\varphi_{\eta\eta}(\xi,\eta)\,Gd\xi d\eta-\\
&-\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\ln\gamma\left(\sigma^{-1}(\tau)\right)\right)'\rho\left(\xi,\eta,\sigma^{-1}(\tau)\right)Gd\xi d\eta+\\
&+\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\times\\
&\times\int_{-\infty}^{\infty}\bigg(\left(\ln\gamma\left(\sigma^{-1}(\tau)\right)\right)'\int_0^{\sigma^{-1}(\tau)}k(\xi,0,\alpha)\vartheta\left(\xi,\eta,\sigma^{-1}(\tau)-\alpha\right)d\alpha\bigg)Gd\xi d\eta-\\
&-\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\sigma^{-1}(\tau)}k(\xi,0,\alpha)\rho\left(\xi,\eta,\sigma^{-1}(\tau)-\alpha\right)Gd\alpha d\xi d\eta-\\
&-\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}k\left(\xi,0,\sigma^{-1}(\tau)\right)\varphi_{\eta\eta}(\xi,\eta)\,Gd\xi d\eta- \qquad (12.3.3)\\
&-\frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\times\\
&\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\sigma^{-1}(\tau)}\left(\sigma^{-1}(\tau)-\beta\right)^{\alpha-1}\rho(0,\eta,\beta)\,Gd\beta d\xi d\eta+\\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\times\\
&\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bigg(\left(\ln\gamma\left(\sigma^{-1}(\tau)\right)\right)'\int_0^{\theta^{-1}(\tau)}\left(\sigma^{-1}(\tau)-\beta\right)^{\alpha-1}\vartheta(0,\eta,\beta)\,d\beta\bigg)Gd\xi d\eta-\\
&-\frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sigma^{-1}(\tau)\right)^{\alpha-1}\varphi_{\eta\eta}(\xi,\eta)\,Gd\xi d\eta\bigg)+
\end{aligned}
$$

$$+\frac{1}{\varphi(x,0)}\int_0^z \big((\ln\gamma(z))'\chi(x,z-\tau)-\chi_z(x,z-\tau)\big)k(x,0,\tau)\,d\tau,$$

where $G=G(x-\xi,y-\eta,\sigma(z))$ and, respectively, from $\sigma(z)-\tau$.

Proof. In problems (12.2.2) and (12.2.3) taking into account formula (12.2.8) as in the correct problem from [23]:

$$F(x,y,z)=\lambda D_{0z}^{-\alpha}\vartheta(0,y,z)+\int_0^z k(x,0,\tau)\vartheta(x,y,z-\tau)\,d\tau$$

in the above form, we have the integral Eq. (12.3.1) correspondingly equivalent. In the same way, taking into account (12.2.7) and (12.2.8), we obtain formula (12.3.2). Then, taking into account the resulting integral equations and using (12.2.9), we obtain the integral Eq. (12.3.3).

From problems (12.2.3), (12.2.4) and (12.2.7), (12.2.8), integral Eqs. (12.3.1) and (12.3.2) are obtained analogously to Eq. (12.1.5). Eq. (12.3.3) follows from Eqs. (12.2.9) and (12.3.2). ☐

Now, we will prove the existence and uniqueness solution of problem (12.2.1)–(12.2.3). The proof is given using the contraction mapping principle.

Theorem 12.3.2. *If conditions (12.1.6), (12.1.7), (12.1.8), and (12.2.5) are satisfied, then there exists $T_0>0$ a sufficiently small number such that for $T\in(0,T_0]$, there exists a unique solution to the integral Eqs.* (12.3.1)–(12.3.3) *belonging to the classes:*

$$\{\vartheta(x,y,z),\ \rho(x,y,z)\}\in H^{l+2,(l+2)/2}\left(\bar{R}_T^2\right),\ k(x,0,z)\in H^{l,l/2}\left(\bar{R}_T^1\right).$$

Proof. To prove the theorem using the contraction mapping principle, we write the system of Eqs. (12.3.1)–(12.3.3) as a nonlinear operator:

$$\theta=L\theta,\ \ \theta=(\theta_1,\theta_2,\theta_3)^*=(\vartheta,\rho,k(x,0,z))^*, \tag{12.3.4}$$

where $*$ is the transposition symbol, $L\theta=\left[(L\theta)_1,(L\theta)_2,(L\theta)_3\right]^*$. Thus according to the right sides of Eqs. (12.3.1)–(12.3.3), $(L\theta)_i$ $(i=1,2,3)$, we have:

$$(L\theta)_1=\theta_{01}(x,y,z)+\int_0^{\sigma(z)}\frac{d\tau}{\gamma(\widehat{\tau})}\times$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}}\theta_3(\xi,0,\alpha)\theta_1(\xi,\eta,\widehat{\tau}-\alpha)\,G\,d\alpha\,d\xi\,d\eta+$$
$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}}(\widehat{\tau}-\beta)^{\alpha-1}\theta_1(0,\eta,\beta)\,G\,d\beta\,d\xi\,d\eta,$$

where

$$(L\theta)_2=\theta_{02}(x,y,z)+\int_0^{\sigma(z)}\frac{d\tau}{\gamma(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[(\ln\gamma(\widehat{\tau}))'\theta_2(\xi,\eta,\widehat{\tau})-\right.$$

$$-\left(\ln\gamma\left(\widehat{\tau}\right)\right)'\int_0^{\widehat{\tau}}\theta_3(\xi,0,\alpha)\theta_1\left(\xi,\eta,\widehat{\tau}-\alpha\right)d\alpha\Bigg]Gd\xi d\eta+$$

$$+\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\widehat{\tau}\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}}\theta_3(\xi,0,\alpha)\theta_2\left(\xi,\eta,\widehat{\tau}-\alpha\right)Gd\alpha d\xi d\eta+$$

$$+\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\widehat{\tau}\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta_3\left(\xi,0,\widehat{\tau}\right)\varphi_{\eta\eta}\left(\xi,\eta\right)Gd\xi d\eta+$$

$$+\frac{\lambda}{\Gamma\left(\alpha\right)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\widehat{\tau}\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}}\left(\widehat{\tau}-\beta\right)^{\alpha-1}\theta_2\left(0,\eta,\beta\right)Gd\beta d\xi d\eta-$$

$$-\frac{\lambda}{\Gamma\left(\alpha\right)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\widehat{\tau}\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left(\ln\gamma\left(\widehat{\tau}\right)\right)'\int_0^{\widehat{\tau}}\left(\widehat{\tau}-\beta\right)^{\alpha-1}\theta_1\left(0,\eta,\beta\right)d\beta\right]Gd\xi d\eta.$$

$$(L\theta)_3=\theta_{03}\left(x,y,z\right)-$$

$$-\frac{\gamma\left(z\right)}{\varphi\left(x,0\right)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\widehat{\tau}\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\ln\gamma\left(\widehat{\tau}\right)\right)'\left[\theta_2\left(\xi,\eta,\widehat{\tau}\right)-\right.$$

$$\left.-\int_0^{\widehat{\tau}}\theta_3(\xi,0,\alpha)\theta_1\left(\xi,\eta,\widehat{\tau}-\alpha\right)d\alpha\right]Gd\xi d\eta-$$

$$-\frac{\gamma\left(z\right)}{\varphi\left(x,0\right)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\widehat{\tau}\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}}\theta_3(\xi,0,\alpha)\theta_2\left(\xi,\eta,\widehat{\tau}-\alpha\right)Gd\alpha d\xi d\eta-$$

$$-\frac{\gamma\left(z\right)}{\varphi\left(x,0\right)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\widehat{\tau}\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta_3\left(\xi,0,\widehat{\tau}\right)\varphi_{\eta\eta}\left(\xi,\eta\right)Gd\xi d\eta-$$

$$-\frac{\lambda\gamma\left(z\right)}{\Gamma\left(\alpha\right)\varphi\left(x,0\right)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\widehat{\tau}\right)}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}}\left(\widehat{\tau}-\beta\right)^{\alpha-1}\theta_2\left(0,\eta,\beta\right)Gd\beta d\xi d\eta-\right.$$

$$\left.-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\ln\gamma\left(\widehat{\tau}\right)\right)'\int_0^{\widehat{\tau}}\left(\widehat{\tau}-\beta\right)^{\alpha-1}\theta_1\left(0,\eta,\beta\right)d\beta Gd\xi d\eta\right)+$$

$$\frac{\lambda\gamma\left(z\right)}{\varphi\left(x,0\right)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\widehat{\tau}\right)}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\theta_2\left(0,\eta,\beta\right)+\left(\ln\gamma\left(\widehat{\tau}\right)\right)'\theta_1\left(0,\eta,\beta\right)\right)Gd\beta d\xi d\eta\right.$$

$$+\frac{1}{\varphi\left(x,0\right)}\int_0^{z}\left(\left(\ln\gamma\left(z\right)\right)'\chi\left(x,z-\tau\right)-\chi_z\left(x,z-\tau\right)\right)\theta_3\left(x,0,\tau\right)d\tau,$$

where $\sigma^{-1}(\tau)=\widehat{\tau}$, $\theta_{01}\left(x,y,z\right)$, $\theta_{02}\left(x,y,z\right)$ and $\theta_{03}\left(x,y,z\right)$ depends on the given functions, i.e.,

$$\theta_{01}\left(x,y,z\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_{\eta\eta}\left(\xi,\eta\right)Gd\xi d\eta,$$

$$\theta_{02}\left(x,y,z\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\gamma\left(0\right)\Delta\varphi_{\eta\eta}\left(\xi,\eta\right)+\lambda\varphi\left(\xi,\eta\right)\right)Gd\xi d\eta+$$

$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\widehat{\tau})^{\alpha-1}\varphi_{\eta\eta}(0,\eta)\,Gd\xi d\eta,$$

$$\theta_{03}(x,y,z)=\frac{\gamma(z)}{\varphi(x,0)}\left(F_z(x,z)-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\gamma(0)\,\Delta\varphi_{\eta\eta}(\xi,\eta)+\varphi(\xi,\eta))Gd\xi d\eta-\right.$$
$$\left.-\frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\widehat{\tau})^{\alpha-1}\varphi_{\eta\eta}(0,\eta)\,Gd\xi d\eta\right).$$

Introducing the notation $|\theta|_T^{l,l/2}=\max\left(|\theta_1|_{T_0}^{l,l/2},|\theta_2|_{T_0}^{l,l/2},|\theta_3|_{T_0}^{l,l/2}\right)$ in $H^{l,l/2}\left(R_T^2\right)$, we give the following condition:

$$S(T)=|\theta_1-\theta_0|_T^{l,l/2}\le|\theta_0|_{T_0}^{l,l/2}, \qquad (12.3.5)$$

where $\theta_0=(\theta_{01},\theta_{02},\theta_{03})$ and $|\theta_0|_{T_0}^{l,l/2}=\max\left(|\theta_{01}|_{T_0}^{l,l/2},|\theta_{02}|_{T_0}^{l,l/2},|\theta_{03}|_{T_0}^{l,l/2}\right)$. Thus for any function σ from $S(T)$, $T<T_0$, when (12.3.5) is executed, the following inequality is true:

$$|\theta_i|_T^{l,l/2}\le 2\,|\theta_0|_{T_0}^{l,l/2},\ i=1,2,3.$$

As is known, for $\varphi(x,y)\in H^{l+2}\left(R^2\right)$, the Cauchy problem for the classical heat conduction equation has a solution. In problem (12.1.3)–(12.1.5), taking into account the auxiliary problem, the initial conditions must belong to $\varphi(x,y)\in H^{l+6}$ (since the auxiliary problem involves fourth-order derivatives).

If the Cauchy condition $\varphi(x,y)\in H^{l+2}\left(\mathbb{R}^2\right)$ holds for the classical heat conduction equation, then the Cauchy problem for the classical heat diffusion equation has a solution [39]. Since we are considering the class $\varphi(x,y)\in H^{l+6}$, the fourth-order derivatives are also involved in the obtained auxiliary problem. Based on this, we introduce the following notation:

$$\gamma_0:=\max_{z\in[0,T]}\left|(\ln\gamma(t))'\right|,\ \varphi_1:=|\varphi|^{l+6},\ \chi_0:=|\chi|_T^{l+4,(l+4)/2}.$$

Contraction Mapping Principle. Any contraction mapping defined in a complete metric space has a unique fixed point; that is, the equation $x=Ax$ has a unique solution $x_0\in S$.

First, using estimates of thermal volume potentials ([38], pages 318–325), it is easy to obtain the following inequalities:

$$|(L\theta)_1-\theta_{01}|_T^{l,l/2}=$$
$$=\left|\int_0^z\frac{d\tau}{\gamma(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}}\theta_3(\xi,0,\alpha)\theta_1(\xi,\eta,\widehat{\tau}-\alpha)\,Gd\alpha d\xi d\eta\right|_T^{l,l/2}+$$
$$\left|\lambda\int_0^z\frac{d\tau}{\gamma(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta_1(0,\eta,\beta)\,Gd\xi d\eta\right|_T^{l,l/2}$$

$$+\left|\frac{\lambda}{\Gamma(\alpha)}\int_0^z \frac{d\tau}{\gamma(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}}(\widehat{\tau}-\beta)^{\alpha-1}\theta_1(0,\eta,\beta)\,Gd\beta d\xi d\eta\right|_T^{l,l/2}\le$$

$$\le\bar{\beta}_0(T)\left|(\theta_3(\xi,\eta,z_0)\,\theta_1(\xi,\eta,z-z_0))\right|_T^{l,l/2}+\bar{\beta}_1(T)\left|(\theta_1(0,\eta,z_0))\right|_T^{l,l/2}\le$$

$$\le 4\beta_0(T)\left(|\theta_0|_{T_0}^{l,l/2}\right)^2+2\beta_1(T)\left(|\theta_0|_{T_0}^{l,l/2}\right),$$

$$|(L\theta)_2-\theta_{02}|_T^{l,l/2}\le$$

$$\le 4\beta_1(T)(\gamma_0+1)\left(|\theta_0|_{T_0}^{l,l/2}\right)^2+2\beta_2(T)(2\gamma_0+\varphi_1+1)\left(|\theta_0|_{T_0}^{l,l/2}\right),$$

$$\left|(L\theta)_3-\theta_{03}\right|_T^{l,l/2}\le$$

$$\le 2\left(\beta_1(T)\,\gamma_1\varphi_0^{-1}(2\gamma_0+\varphi_1+1)+\chi_0\varphi_0^{-1}T_0(\gamma_0+1)\right)|\theta_0|_{T_0}^{l,l/2}+$$

$$+4\beta_2(T)\,\gamma_1\varphi_0^{-1}(\gamma_0+1)\left(|\theta_0|_{T_0}^{l,l/2}\right)^2.$$

As $T\to 0$, $\beta_i(T)\to 0$, $(i=0,1,2)$. Therefore if we choose T_0 so that the following inequalities should be satisfied:

$$4\beta_0(T_0)\left(|\theta_0|_{T_0}^{l,l/2}\right)^2+2\beta_1(T_0)\left(|\theta_0|_{T_0}^{l,l/2}\right)\le 1,$$

$$4\beta_1(T_0)(\gamma_0+1)\left(|\theta_0|_{T_0}^{l,l/2}\right)^2+2\beta_2(T_0)(2\gamma_0+\varphi_1+1)\left(|\theta_0|_{T_0}^{l,l/2}\right)\le 1, \tag{12.3.6}$$

$$2\left(\beta_1(T_0)\,\gamma_1\varphi_0^{-1}(2\gamma_0+\varphi_1+1)+\chi_0\varphi_0^{-1}T_0(\gamma_0+1)\right)|\theta_0|_{T_0}^{l,l/2}+$$

$$+4\beta_2(T_0)\,\gamma_1\varphi_0^{-1}(\gamma_0+1)\left(|\theta_0|_{T_0}^{l,l/2}\right)^2\le 1,$$

then the operator L for $T<T_0$ has the first property of a contraction mapping operator, that is, $L\theta\in S(T)$.

Now, consider the second property of the contraction mapping for the operator L. Let $\theta^{(1)}=\left(\theta_1^{(1)},\theta_2^{(1)},\theta_3^{(1)}\right)\in S(T)$, $\theta^{(2)}=\left(\theta_1^{(2)},\theta_2^{(2)},\theta_3^{(2)}\right)\in S(T)$, then, following evaluation

$$\left|\theta_2^{(1)}\theta_1^{(1)}-\theta_2^{(2)}\theta_1^{(2)}\right|_T^{l,l/2}=\left|\left(\theta_2^{(1)}-\theta_2^{(2)}\right)\theta_1^{(1)}+\theta_2^{(2)}\left(\theta_1^{(1)}-\theta_1^{(2)}\right)\right|_T^{l,l/2}\le$$

$$\le 2\left|\theta^{(1)}-\theta^{(2)}\right|_T^{l,l/2}\max\left(\left|\theta_1^{(1)}\right|_T^{l,l/2},\left|\theta_2^{(2)}\right|_T^{l,l/2}\right)\le 4\,|\theta_0|_T^{l,l/2}\left|\theta^{(1)}-\theta^{(2)}\right|_T^{l,l/2},$$

we have

$$\left|\left((L\theta)^{(1)}-(L\theta)^{(2)}\right)_1\right|_T^{l,l/2}=\left|\int_0^{\sigma(z)}\frac{d\tau}{\gamma(\widehat{\tau})}\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\right.$$

$$\int_0^{\widehat{\tau}} \left[\theta_3^{(1)}(\xi,0,\alpha)\theta_1^{(1)}(\xi,\eta,\widehat{\tau}-\alpha) - \theta_3^{(2)}(\xi,0,\alpha)\theta_1^{(2)}(\xi,\eta,\widehat{\tau}-\alpha)\right] G d\alpha \Big|_T^{l,l/2} +$$

$$+ \left| \int_0^{\sigma(z)} \frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\widehat{\tau}} \frac{\lambda\,(\widehat{\tau}-\beta)^{\alpha-1}}{\Gamma(\alpha)} \left[\theta_1^{(1)}(0,\eta,\beta) - \theta_1^{(2)}(0,\eta,\beta)\right] \times \right.$$

$$\times G d\beta d\xi d\eta|_T^{l,l/2} \leq \left[8\beta_0(T)\,|\theta_0|_{T_0}^{l,l/2} + 4\beta_1(T)\right] \left|\theta^{(1)} - \theta^{(2)}\right|_T^{l,l/2}.$$

Similarly, estimating the second and third components of $L\theta$ we have:

$$\left|\left((L\theta)^{(1)} - (L\theta)^{(2)}\right)_2\right|_T^{l,l/2} \leq$$

$$\leq \left[2\beta_1(T)(2\gamma_0 + \varphi_1 + 1) + 8\beta_2(T)(\gamma_0+1)\,|\theta_0|_{T_0}^{l,l/2}\right] \left|\theta^{(1)} - \theta^{(2)}\right|_{T_0}^{l,l/2},$$

$$\left|\left((L\theta)^{(1)} - (L\theta)^{(2)}\right)_3\right|_T^{l,l/2} \leq$$

$$\leq \left[2\left(\beta_1(T)\,\gamma_1\varphi_0^{-1}(2\gamma_0 + \varphi_1 + 1) + \chi_0\varphi_0^{-1}T_0(\gamma_0+1)\right)\right] \left|\theta^{(1)} - \theta^{(2)}\right|_{T_0}^{l,l/2} +$$

$$+ \left[8\beta_2(T)\,\gamma_1\varphi_0^{-1}(\gamma_0+1)\left(|\theta_0|_{T_0}^{l,l/2}\right)\right] \left|\theta^{(1)} - \theta^{(2)}\right|_{T_0}^{l,l/2}.$$

Therefore $\left|\left(L\theta^{(1)} - L\theta^{(2)}\right)\right|_T^{l,l/2} < \rho \left|\theta^{(1)} - \theta^{(2)}\right|_T^{l,l/2}$, where $\rho \leq 1$, if satisfied

$$\begin{gathered} \left[8\beta_0(T)\,|\theta_0|_{T_0}^{l,l/2} + 2\beta_1(T)\right] \leq \rho < 1, \\ \left[2\beta_1(T)(2\gamma_0 + \varphi_1 + 1) + 8\beta_2(T)(\gamma_0+1)\,|\theta_0|_{T_0}^{l,l/2}\right] \leq \rho < 1, \\ \left[2\left(\beta_1(T)\,\gamma_1\varphi_0^{-1}(2\gamma_0 + \varphi_1 + 1) + f_0\varphi_0^{-1}T_0(\gamma_0+1)\right)\right] + \\ + \left[8\beta_2(T)\,\gamma_1\varphi_0^{-1}(\gamma_0+1)\left(|\theta_0|_{T_0}^{l,l/2}\right)\right] \leq \rho < 1, \end{gathered} \tag{12.3.7}$$

then the operator L is also a contraction on $S(T)$.

From the satisfaction of inequality (12.3.7), it directly follows that (12.3.6) also holds. Furthermore, since T_0 satisfies $T < T_0$ and condition (12.3.7), the properties of a contraction mapping operator are fully satisfied. Consequently, by the Banach fixed-point theorem, Eq. (12.3.4) has a unique solution. Using the method of successive approximations for the system of Eqs. (12.3.1)–(12.3.3), we determine a unique solution within the function space $H^{l+2,(l+2)/2}\left(\bar{R}_T^2\right)$.

Thus the existence and uniqueness of a solution to the system of integral Eqs. (12.3.1)–(12.3.3) imply the existence and uniqueness of the solution to the equivalent problems (12.1.3)–(12.1.5). ☐

Declaration of competing interest

This work does not have any conflicts of interest.

References

[1] Y.A. Çengel, A.J. Ghajar, Heat and Mass Transfer: Fundamentals & Applications, 6th ed., McGraw–Hill Education, 2020, ISBN 978-1-260-22124-8.

[2] D.W. Hahn, M.N. Özisik, Heat Conduction, 3rd ed., John Wiley & Sons, Hoboken, NJ, 2012, ISBN 978-0-470-90319-2.

[3] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives. Theory and Applications, Gordon and Breach, New York, 1993.

[4] A.V. Pskhu, Partial Differential Equations of Fractional Order, Nauka, Moscow, 2005 (in Russian).

[5] R. Gorenflo, F. Mainardi, Fractional calculus: integral and differential equations of fractional order, arXiv, https://doi.org/10.48550/arXiv.0805.3823, 2008.

[6] I. Podlubny, Fractional Differential Equations, Mathematics in Science and Engineering, vol. 198, Academic Press, San Diego, 1999.

[7] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models, Imperial College Press, London, ISBN 978-1-84816-329-4, 2010.

[8] M. Mulla, Fractional calculus, fractional differential equations and applications, Open Access Library Journal 7 (2020) 1–9, https://doi.org/10.4236/oalib.1106244.

[9] F. Mainardi, A. Mura, G. Pagnini, R. Gorenflo, Time-fractional diffusion of distributed order, Journal of Vibration and Control 14 (9-10) (2008) 1267–1290, https://doi.org/10.1177/1077546307087452.

[10] J. Sabatier, O.P. Agrawal, J.A.T. Machado, Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, Springer, Dordrecht, 2007.

[11] D.A. Juraev, S. Noeiaghdam, Modern problems of mathematical physics and their applications, Axioms 11 (2) (2022) 45, https://doi.org/10.3390/axioms11020045.

[12] X.J. Yang, D. Baleanu, H.M. Srivastava, Local Fractional Integral Transforms and Their Applications, Elsevier Ltd, ISBN 978-0-12-804002-7, 2016.

[13] M. Al-Refai, T. Abdeljawad, Analysis of the fractional diffusion equations with fractional derivative of non-singular kernel, Advances in Difference Equations 315 (2017), https://doi.org/10.1186/s13662-017-1356-2.

[14] X.Y. Li, A.G. Xiao, Space-fractional diffusion equation with variable coefficients: well-posedness and Fourier pseudo spectral approximation, Journal of Scientific Computing 87 (2021) 28, https://doi.org/10.1007/s10915-021-01435-w.

[15] X. Zheng, V.J. Ervin, H. Wang, Numerical approximations for the variable coefficient fractional diffusion equations with non-smooth data, Computational Methods in Applied Mathematics 20 (3) (2020) 573–589, https://doi.org/10.1515/cmam-2019-0038.

[16] Y. Liu, M. Yamamoto, A backward Euler method for time-fractional diffusion equations with variable coefficients, Applied Mathematics and Computation 218 (6) (2011) 2742–2754, https://doi.org/10.1016/j.amc.2011.07.095.

[17] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, vol. 204, Elsevier, Amsterdam, ISBN 978-0-444-51832-3, 2006.

[18] F. Alsidrani, A. Kilicman, N. Senu, Approximate solutions for time-fractional Fornberg-Whitham equation with variable coefficients, Fractal and Fractional 7 (2023) 260, https://doi.org/10.3390/fractalfract7030260.

[19] D.K. Durdiev, J.Z. Nuriddinov, On investigation of the inverse problem for a parabolic integro-differential equation with a variable coefficient of thermal conductivity, Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki 30 (4) (2020) 572–584.

[20] A.K. Farhood, O.H. Mohammed, B.A. Taha, Solving fractional time-delay diffusion equation with variable-order derivative based on shifted Legendre-Laguerre operational matrices, Arabian Journal of Mathematics (2023), https://doi.org/10.1007/s40065-022-00416-7.

[21] F.M. Martins Nuno, Pedro Mota, An adapted plane waves method for heat conduction problems, Applied Mathematics and Computation (ISSN 0096-3003) 415 (2022) 126689, https://doi.org/10.1016/j.amc.2021.126689.

[22] F. Zeng, C. Li, F. Liu, I. Turner, The use of finite difference/element approaches for solving the time-fractional subdiffusion equation, SIAM Journal on Scientific Computing 35 (6) (2013), https://doi.org/10.1137/120889895.

[23] P. Agarwal, F. Hubert, Y. Dermenjian, U. Baltaeva, B. Hasanov, The Cauchy problem for the heat equation with a fractional load, Discrete and Continuous Dynamical Systems. Series S (2024), https://doi.org/10.3934/dcdss.2024176.

[24] A.M. Nakhushev, Loaded Equations and Their Applications, Nauka, Moscow, 2012.

[25] A.M. Krall, The development of general differential and general differential-boundary systems, The Rocky Mountain Journal of Mathematics 5 (4) (1975) 493–542, https://doi.org/10.1216/RMJ-1975-5-4-493.

[26] R.P. Agarwal, U. Baltaeva, F. Hubert, B. Khasanov, Existence and uniqueness of the solution to initial and inverse problems for integro-differential heat equations with fractional load., Electronic Journal of Differential Equations 2024 (64) (2024) 1–19.

[27] A.T. Assanova, Zh.M. Kadirbayeva, Periodic problem for an impulsive system of the loaded hyperbolic equations, Electronic Journal of Differential Equations 2018 (72) (2018) 1–8.

[28] M.T. Jenaliyev, M.I. Ramazanov, Loaded Equations as Perturbations of Differential Equations, Gylym, Almaty, 2010 (in Russian).

[29] T.K. Yuldashev, B.I. Islomov, E.K. Alikulov, Boundary-value problems for loaded third-order parabolic-hyperbolic equations in infinite three-dimensional domains, Lobachevskii Journal of Mathematics 41 (5) (2020) 926–944, https://doi.org/10.1134/S1995080220050145.

[30] M.M. Amangaliyeva, M.T. Jenaliyev, M.I. Ramazanov, S.A. Iskakov, On a boundary value problem for the heat equation and a singular integral equation associated with it, Applied Mathematics and Computation 399 (2021) 126009.

[31] M. Kosmakova, D. Akhmanova, K. Izhanova, BVP with a load in the form of a fractional integral, International Journal of Mathematics and Mathematical Sciences (2024) 7034103, https://doi.org/10.1155/2024/7034103.

[32] U. Baltaeva, Y. Babajanova, P. Agarwal, N. Ozdemir, Solvability of a mixed problem with the integral gluing condition for a loaded equation with the Riemann-Liouville fractional operator, Journal of Computational and Applied Mathematics 425 (2023) 115066, https://doi.org/10.1016/j.cam.2023.115066.

[33] K.U. Khubiev, Analogue of Tricomi problem for characteristically loaded hyperbolic-parabolic equation with variable coefficients, Ufa Mathematical Journal 9 (2) (2017) 92–101.

[34] U.I. Baltaeva, Boundary-value problem for a loaded mixed-type equation with a characteristic line of type change, Journal of Mathematical Sciences 272 (2023) 202–214, https://doi.org/10.1007/s10958-023-4.

[35] D.K. Durdiev, J.J. Jumaev, One-dimensional inverse problems of deter mining the kernel of the integro-differential heat equation in a bounded domain, Nonautonomous Dynamical Systems 10 (1) (2023) 20220163, https://doi.org/10.1515/msds-2022-0163.

[36] A.M. Denisov, A.A. Efimov, The inverse problem for an integro-differential equation and its solution method, Computational Mathematics and Modeling 30 (2019) 403–412, https://doi.org/10.1007/s10598-019-09466-7.

[37] D.K. Durdiev, Z.Zh. Zhumaev, Memory kernel reconstruction problems in the integro-differential equation of rigid heat conductor, Mathematical Methods in the Applied Sciences (2020) 1–15, https://doi.org/10.1002/mma.7133.

[38] O.A. Ladyzhenskaya, The Boundary Value Problems of Mathematical Physics, Applied Mathematical Sciences, vol. 49, Springer, Berlin, 1985.

[39] R.P. Agarwal, U. Baltaeva, F. Hubert, B. Khasanov, Existence and uniqueness of the solution to initial and inverse problems for integro-differential heat equations with fractional load, Electronic Journal of Differential Equations 2024 (64) (2024) 1–19.