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Degenerate Appell sequences via Caputo-type fractional difference operator

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Abstract

In this article, we define and study a novel class of function sequences based on their relationship with fractional-order difference operators in the Caputo sense. These sequences, known as ∇_λ^α-Appell sequences, are constructed for $0 < \alpha \leq 1$ and $\lambda > 0$. The terms of these sequences involve combinations of monomials with fractional powers and exhibit unique properties that extend classical results in the context of fractional calculus and difference operators.

Keywords Discrete fractional Calculus · Appell polynomials · Caputo operator · Mittag–Leffler function · ∇_λ-Appell sequences

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1 Introduction and preliminaries

L. Carlitz in his seminal work [10] initiated the systematic study of degenerate versions of specific cases of Appell polynomials [3]. Subsequently, many other degenerate versions of such polynomials have garnered substantial attention within the mathematical community (see [7, 13, 14, 19] and the references therein). This attention is due to both their combinatorial and arithmetic properties, as well as their applications to the theory of ODEs and PDEs, symmetry notions, and probability theory [22].

Following the classical abstract definition of Appell polynomials, F. A. Costabile and E. Longo introduced a degenerate version of them, which is characterized by the following generating function [15]:

$$f(t)e_\lambda^x(t) = \sum_{n=0}^{\infty} A_{n,\lambda}(x)\frac{t^n}{n!},$$

where $\lambda > 0$, $e_\lambda^x(t)$ denotes the degenerate exponential [19, 20, 22] given by:

$$e_\lambda^x(t) = \begin{cases} (1+\lambda t)^{\frac{x}{\lambda}}, & \text{if } \lambda \in \mathbb{R}\setminus\{0\}, \\ e^{xt}, & \text{if } \lambda = 0, \end{cases}$$

and

$$f(t) = \alpha_0 + \frac{t}{1!}\alpha_1 + \frac{t^2}{2!}\alpha_2 + \cdots + \frac{t^n}{n!}\alpha_n + \cdots, \quad \alpha_0 \neq 0.$$

Observe that $A_{0,\lambda}(x) = \alpha_0$, and

$$\Delta_\lambda A_{n,\lambda}(x) = nA_{n-1,\lambda}(x),\ n \in \mathbb{N}, \tag{1}$$

where Δ_λ is the forward difference operator defined by

$$\Delta_\lambda f(x) := \frac{f(x+\lambda) - f(x)}{\lambda}.$$

In this way, Costabile and Longo generalized the degenerate Bernoulli $\mathcal{B}_{n,\lambda}(x)$ and Euler $\mathcal{E}_{n,\lambda}(x)$ polynomials, given by the following generating functions [10, 11]:

$$\frac{te_\lambda^x(t)}{e_\lambda(t)-1} = \sum_{n=0}^{\infty} \mathcal{B}_{n,\lambda}(x)\frac{t^n}{n!},$$

$$\frac{2e_\lambda^x(t)}{e_\lambda(t)+1} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}.$$

Furthermore, these authors established that any polynomial sequence satisfying the difference equation (1) is a Δ_λ-Appell sequence.

Notice that the property of being Δ_λ-Appell sequence is satisfied by a broad class of special polynomials (see, for instance [29]). As is well known, both Appell polynomials and Δ_λ-Appell polynomials are characterized by their behavior with respect to the operators $\frac{d}{dx}$ and Δ_λ, respectively. Therefore, it makes sense to consider other operators, such as fractional operators, applied to sequences of special functions (whether polynomials or not). In this regard, it is worthy to mention the recent works [8, 9, 16], in which it can be observed how expansions of special functions in fractional powers not only provide a deeper understanding of the underlying mathematical structure but also prove highly useful for solving, in a straightforward and elementary way, various fractional differential equations. These approaches, by combining special functions with the flexibility of fractional operators, offer powerful and versatile tools that simplify complex problems, opening new perspectives both in theoretical applications and practical contexts within applied mathematics and mathematical physics.

In addition, Caputo-type discrete fractional operators have been widely investigated in discrete and time-scale settings, particularly from the viewpoint of fractional difference equations and dynamical systems (e.g., existence and uniqueness of solutions, qualitative behavior, and stability). In this direction, the research line developed by D. Mozyrska and her collaborators is especially relevant; see, for instance, [17, 24–26]. These works provide an analytical and dynamical background that complements the structural (Appell-type) approach pursued in the present paper.

Motivated by these developments, and by the analytical and dynamical studies of Caputo-type fractional operators in discrete and time-scale settings, in this paper we introduce and study sequences of functions associated with the discrete Caputo fractional operator and the discrete Mittag–Leffler (or degenerate Mittag–Leffler) function [1, 2, 5, 6, 27]. We investigate the main features and properties of these functions, which we call ∇_λ^α-Appell sequences. These special functions are combinations of monomials with fractional powers, and their behavior is analogous to that of Δ_λ- or ∇_λ-Appell polynomials [10, 12, 13, 15, 16].

The paper is organized as follows. Section 2 is devoted to preliminaries and notation, including a brief summary about novel basic properties of the ∇_λ-Appell polynomials (see Theorems 1 and 2). In Section 3, we introduce the notion of ∇_λ^α-Appell sequences of functions based on the discrete Caputo fractional operator and the degenerate Mittag–Leffler function. Furthermore, we determine an explicit expression for each term of such function sequences and obtain an associated generating function for them (see Theorems 3 and 4, respectively). At the end of this section we introduce the definition of the Bernoulli and Euler type ∇_λ^α-Appell sequences and provide inversion type formulas for them (see Theorem 5). Finally, we give some concluding remarks in Section 4.

2 ∇_λ-Appell polynomials

In this section, we characterize the families of sequences related to $e_{-\lambda}^{x}(t)$ using the backward difference operator. In what follows, let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural numbers, non-negative integers, integers, real numbers, and complex numbers, respectively. Additionally, we adopt the convention that $0^0 = 1$.

For $x \in \mathbb{C}$ and $n \in \mathbb{Z}$, we use the notations $x^{(n,\lambda)}$ and $(x)_{n,\lambda}$ for the generalized of the rising and falling factorials, respectively, i.e.,

$$x^{(n,\lambda)} = \begin{cases} 1, & \text{if } n = 0, \\ \prod_{i=1}^{n}(x + (i-1)\lambda), & \text{if } n \geq 1, \\ 0, & \text{if } n < 0, \end{cases}$$

and

$$(x)_{n,\lambda} = \begin{cases} 1, & \text{if } n = 0, \\ \prod_{i=1}^{n}(x-(i-1)\lambda), & \text{if } n \geq 1, \\ 0, & \text{if } n < 0, \end{cases}$$

where $\lambda \in \mathbb{R}$. If $\lambda = 1$, we write $x^{(n,\lambda)} := x^{(n)}$ and $(x)_{n,\lambda} := (x)_n$.

Observe that

$$(x)_{n,-\lambda} \equiv x^{(n,\lambda)}, \ x^{(n,\lambda)} \equiv \lambda^n (x/\lambda)^{(n)}.$$

For $\lambda > 0$, we recall that the degenerate exponentials have the following series of representation

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad |\lambda t| < 1,$$

see cf., [7, Eq. 10]. Now, based on the above, for $\lambda > 0$, we define the following version of the degenerate exponential function:

$$\varepsilon_\lambda^x(t) := e_{-\lambda}^x(t) = \sum_{n=0}^{\infty} x^{(n,\lambda)} \frac{t^n}{n!}, \quad |\lambda t| < 1. \tag{2}$$

Note that when we apply λ-difference operators to the aforementioned degenerate exponential functions, we obtain:

$$\Delta_\lambda e_\lambda^x(t) = t e_\lambda^x(t)$$

and

$$\nabla_\lambda \varepsilon_\lambda^x(t) = t \varepsilon_\lambda^x(t), \tag{3}$$

where

$$\nabla_\lambda f(x) := \frac{f(x) - f(x-\lambda)}{\lambda}.$$

Remark 1 In [2], the degenerate exponential function $\varepsilon_\lambda^x(t)$ is regarded as a C-semigroup. Moreover, the authors demonstrate its significant properties in solving discrete-time Cauchy problems of first-order.

In analogy to the relation (1), we introduce the definition of a family of polynomial sequences that we will call ∇_λ-Appell sequences.

Definition 1 Sequences of n-degree polynomials $\mathcal{A}_{n,\lambda}(x) \equiv \mathcal{A}_n(x)$, $n = 0, 1, \ldots$, satisfying the relation

$$\nabla_\lambda \mathcal{A}_n(x) = n\mathcal{A}_{n-1}(x), \quad n = 1, 2, \ldots \tag{4}$$

are called ∇_λ-Appell sequences.

By applying the operator ∇^{-1}, where ∇^{-1} denotes the indefinite summation operator defined in [18], to equation (4), we observe that for each $n \geq 1$, $\mathcal{A}_n(x)$ is entirely determined by $\mathcal{A}_{n-1}(x)$ and the constant α_n. In essence, there is a one-to-one correspondence between the set of sequences $\{\mathcal{A}_n(x)\}_{n\in\mathbb{N}_0}$ and the set of numerical sequences $\{\alpha_n\}_{n\in\mathbb{N}_0}$ where $\alpha_0 \neq 0$, as defined by the explicit representation

$$\mathcal{A}_{n,\lambda}(x) = \sum_{r=0}^{n} \binom{n}{r} \alpha_{n-r} x^{(r,\lambda)}, \tag{5}$$

where

$$\binom{n}{r} := \frac{n!}{(n-r)!r!}.$$

The generating function for the ∇_λ-Appell sequences is established as follows.

Theorem 1 *Given the power series*

$$f(t) = \alpha_0 + \frac{t}{1!}\alpha_1 + \frac{t^2}{2!}\alpha_2 + \cdots + \frac{t^n}{n!}\alpha_n + \cdots, \quad \alpha_0 \neq 0$$

with α_i ($i = 0, 1, \dots$) real coefficients, a ∇_λ-Appell sequence $\left\{\mathcal{A}_{n,\lambda}(x)\right\}_{n\in\mathbb{N}_0}$ is determined by the power series expansion of the product a $f(t)\varepsilon_\lambda^x(t)$, i.e.:

$$f(t)\varepsilon_\lambda^x(t) = \sum_{n=0}^{\infty} \mathcal{A}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{6}$$

Proof Using (2), we can arrange the product of the expansions of $\varepsilon_\lambda^x(t)$ and $f(t)$ in terms of the powers of t. This allows us to identify the polynomials $\mathcal{A}_{n,\lambda}(x)$, expressed in the form (5), as the coefficients of $\frac{t^n}{n!}$. □

Note that if $f(t) = 1$, then $\alpha_j := \delta_j(0)$ corresponds to the Kronecker delta. Consequently, we have $\mathcal{A}_n(x) = x^{(n,\lambda)}$ and (2) is recovered. Below, we will present two structured examples where we consider $\varepsilon_\lambda(t) := \varepsilon_\lambda^1(t)$.

Example 1 For each $n \in \mathbb{N}_0$, we consider the Bernoulli-type ∇_λ-Appell polynomials $\mathcal{W}_{n,\lambda}(x)$, which are defined by the following generating function:

$$\frac{t\,\varepsilon_\lambda^x(t)}{\varepsilon_\lambda(t) - 1} = \sum_{n=0}^{\infty} \mathcal{W}_{n,\lambda}(x)\frac{t^n}{n!}.$$

It is clear that $\mathcal{W}_{n,\lambda}(x) = \beta_{n,-\lambda}(x)$, where $\beta_{n,-\lambda}(x)$ denotes the n-th Carlitz polynomial [10, 11]. The first six Bernoulli-type ∇_λ-Appell polynomials $\mathcal{W}_{n,\lambda}(x)$ are as follows:

$$\begin{aligned}
\mathcal{W}_{0,\lambda}(x) &= 1,\\
\mathcal{W}_{1,\lambda}(x) &= x - \frac{1}{2} - \frac{\lambda}{2},\\
\mathcal{W}_{2,\lambda}(x) &= x^2 - x + \frac{1}{6} - \frac{\lambda^2}{6},\\
\mathcal{W}_{3,\lambda}(x) &= x^3 - \frac{3}{2}x^2 + \frac{3}{2}\lambda x^2 + \frac{1}{2}x - \frac{3}{2}\lambda x + \frac{1}{4}\lambda - \frac{1}{4}\lambda^3,\\
\mathcal{W}_{4,\lambda}(x) &= x^4 + x^3(-2 + 4\lambda) + x^2(1 - 6\lambda + 4\lambda^2)\\
&\quad + 2x\lambda(1 - 2\lambda) + \frac{1}{30}(-1 + 20\lambda^2 - 19\lambda^4),\\
\mathcal{W}_{5,\lambda}(x) &= x^5 + 5x^4(-1 + 3\lambda) + 5x^3(1 - 9\lambda + 11\lambda^2) + 2.5x^2\lambda(3 - 11\lambda + 6\lambda^2)
\end{aligned}$$

$$-\frac{1}{6}x(1-55\lambda^2+90\lambda^3)-\frac{1}{4}(\lambda-10\lambda^3+9\lambda^5).$$

Example 2 For each $n \in \mathbb{N}_0$, we consider the Euler-type ∇_λ-Appell polynomials $\mathcal{S}_{n,\lambda}(x)$, which are defined by the following generating function:

$$\frac{2\,\varepsilon_\lambda^x(t)}{\varepsilon_\lambda(t)+1}=\sum_{n=0}^{\infty}\mathcal{S}_{n,\lambda}(x)\frac{t^n}{n!}.$$

It is clear that $\mathcal{S}_{n,\lambda}(x)=\mathcal{E}_{n,-\lambda}(x)$, where $\mathcal{E}_{n,-\lambda}(x)$ denotes the so-called n-th degenerate Euler polynomial [21]. The first six Euler-type ∇_λ-Appell polynomials $\mathcal{S}_{n,\lambda}(x)$ are as follows:

$$\begin{aligned}
\mathcal{S}_{0,\lambda}(x)&=1,\\
\mathcal{S}_{1,\lambda}(x)&=x-\frac{1}{2},\\
\mathcal{S}_{2,\lambda}(x)&=x^2+x(\lambda-1)-\frac{\lambda}{2},\\
\mathcal{S}_{3,\lambda}(x)&=x^3+x^2\left(3\lambda-\frac{3}{2}\right)+x(2\lambda^2-3\lambda)+\frac{1}{4}-\lambda^2,\\
\mathcal{S}_{4,\lambda}(x)&=x^4+\frac{3\lambda}{2}-3\lambda^3+x^3(-2+6\lambda)+x^2\lambda(-9+11\lambda)+x\left(1-11\lambda^2+6\lambda^3\right),\\
\mathcal{S}_{5,\lambda}(x)&=x^5+\frac{5}{2}x^4(-1+4\lambda)+5x^3\lambda(-4+7\lambda)+\frac{5}{2}x^2\left(1-21\lambda^2+20\lambda^3\right)\\
&\quad+2x\lambda\left(5-25\lambda^2+12\lambda^3\right)+\frac{1}{4}\left(-2+35\lambda^2-48\lambda^4\right).
\end{aligned}$$

Theorem 2 *Let $\left\{\mathcal{A}_{n,\lambda}(x)\right\}_{n\in\mathbb{N}_0}$ be the ∇_λ-Appell sequence with generating function $f(t)\varepsilon_\lambda^x(t)$. Consider the coefficients $\beta_0,\beta_1,\beta_2,\ldots$ of the Taylor series expansion for the function $\frac{1}{f(t)}$, with $\beta_0\neq 0$. For each $n\in\mathbb{N}_0$, $\left\{\mathcal{A}_{n,\lambda}(x)\right\}_{n\in\mathbb{N}_0}$ satisfies the following determinantal form:*

$$\mathcal{A}_{0,\lambda}(x)=\frac{1}{\beta_0},$$

$$\mathcal{A}_{n,\lambda}(x)=\frac{(-1)^n}{\beta_0^{n+1}}\begin{vmatrix}
1 & x^{(1,\lambda)} & \cdots\cdots & x^{(n-1,\lambda)} & x^{(n,\lambda)}\\
\beta_0 & \beta_1 & \cdots\cdots & \beta_{n-1} & \beta_n\\
0 & \beta_0 & \cdots\cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1}\\
0 & 0 & \ddots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2}\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
\vdots & \vdots & & & \\
0 & \cdots & \cdots\cdots & \beta_0 & \binom{n}{n-1}\beta_1
\end{vmatrix}. \tag{7}$$

Proof Multiplying both sides of (6) by $\frac{1}{f(t)}$ yields:

$$\sum_{n=0}^{\infty}x^{(n,\lambda)}\frac{t^n}{n!}=\sum_{n=0}^{\infty}\mathcal{A}_{n,\lambda}(x)\frac{t^n}{n!}\sum_{n=0}^{\infty}\frac{t^n}{n!}\beta_n.$$

According to the rules of the Cauchy product, the previous equality leads to the following system of infinite equations in the unknowns $\mathcal{A}_{n,\lambda}(x)$, $n = 0, 1, \ldots$

$$\begin{cases} \mathcal{A}_{0,\lambda}(x)\beta_0 = 1, \\ \mathcal{A}_{0,\lambda}(x)\beta_1 + \mathcal{A}_{1,\lambda}(x)\beta_0 = x^{(1,\lambda)}, \\ \mathcal{A}_{0,\lambda}(x)\beta_2 + \binom{2}{1}\mathcal{A}_{1,\lambda}(x)\beta_1 + \mathcal{A}_{2,\lambda}(x)\beta_0 = x^{(2,\lambda)}, \\ \vdots \\ \mathcal{A}_{0,\lambda}(x)\beta_n + \binom{n}{1}\mathcal{A}_{1,\lambda}(x)\beta_{n-1} + \ldots + \mathcal{A}_{n,\lambda}(x)\beta_0 = x^{(n,\lambda)}, \\ \vdots \end{cases}$$

The special form of this system (lower triangular) allows us to determine the unknowns $\mathcal{A}_{n,\lambda}(x)$ using the first $n+1$ equations. By applying Cramer's rule and after some calculations, we obtain the desired result. □

3 ∇_λ^α-Appell sequences of functions

In this section, we introduce the notion of ∇_λ^α-Appell sequences of functions based on the discrete Caputo fractional operator and the degenerate Mittag–Leffler function. We determine an explicit expression for each term of such function sequences, and obtain an associated generating function for them.

In order to consider special functions and fractional operators commonly used in discrete fractional calculus, we can generalize the rising factorial as follows: for $x, v \in \mathbb{R}$, we define

$$x_\lambda^{\overline{v}} := \lambda^v \frac{\Gamma\left(\frac{x}{\lambda} + v\right)}{\Gamma\left(\frac{x}{\lambda}\right)}, \ \lambda > 0,$$

where the quotient is well-defined. Observe that, for $v \in \mathbb{N}_0$, we have $x_\lambda^{\overline{v}} = x^{(v,\lambda)}$.

Now, we recall a discrete version of the Mittag–Leffler function [1, 2, 5, 6, 27]. Let $\lambda > 0$ and $\mathbb{T} := \{0, \lambda, 2\lambda, \cdots\}$.

Definition 2 Let $\lambda, \alpha > 0$ and $t \in \mathbb{C}$. The discrete Mittag–Leffler (or degenerate Mittag–Leffler) function is defined by

$$\mathcal{E}_{\lambda,\alpha}^x(t) := \sum_{j=0}^{\infty} x_\lambda^{\overline{\alpha j}} \frac{t^j}{\Gamma(\alpha j + 1)}, \quad |t\lambda^\alpha| < 1, \ x \in \mathbb{T} \setminus \{0\}. \tag{8}$$

Using (2), we obtain

$$\mathcal{E}_{\lambda,1}^x(t) = \sum_{j=0}^{\infty} x_\lambda^{\overline{j}} \frac{t^j}{\Gamma(j+1)} = \sum_{j=0}^{\infty} x^{(j,\lambda)} \frac{t^j}{j!} = \varepsilon_\lambda^x(t), \ |t\lambda| < 1.$$

Furthermore, the degenerate Mittag–Leffler $\mathcal{E}_{\lambda,\alpha}^x(t)$ converges to the Mittag–Leffler function $E_\alpha(t, x) := \sum_{j=0}^{\infty} \frac{(tx^\alpha)^j}{\Gamma(\alpha j+1)}$ as λ approaches 0^+. In fact,

$$\lim_{\lambda \to 0^+} \mathcal{E}_{\lambda,\alpha}^x(t) = \lim_{w \to \infty} \sum_{j=0}^{\infty} \frac{(tx^\alpha)^j}{\Gamma(\alpha j + 1)} \frac{\Gamma(w + \alpha j)}{w^{\alpha j}\Gamma(w)}$$

Fig. 1 Relationship between the degenerate exponential, the exponential function, the Mittag–Leffler function and the degenerate Mittag–Leffler

$$\begin{tikzcd}
\mathcal{E}^{x}_{\lambda,\alpha}(t) \arrow[r, "\lambda \to 0^{+}"] \arrow[d, "\alpha = 1"'] & E_{\alpha}(t,x) \arrow[d, "\alpha = 1"] \\
\varepsilon^{x}_{\lambda}(t) \arrow[r, "\lambda \to 0^{+}"'] & e^{xt}
\end{tikzcd}$$

$$= \sum_{j=0}^{\infty} \frac{(tx^{\alpha})^{j}}{\Gamma(\alpha j + 1)},$$

where used

$$\lim_{z \to \infty} \frac{\Gamma(z + a)}{z^{a}\Gamma(z)} = 1.$$

For more details, see [4]. Fig. 1 illustrates the relationships among the different exponential-type functions.

Starting from generalization of rising factorials, we proceed to define the operator for arbitrary-order summation: For $0 < \alpha \leq 1$, the definition of α-th λ-fractional sum of a sequence $f : \mathbb{T} \to \mathbb{R}$ is given by

$$\nabla_{\lambda}^{-\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{x/\lambda} (x - \rho(s\lambda))_{\lambda}^{\overline{\alpha-1}} f(s\lambda)\lambda,$$

where $\rho(x) := x - \lambda$, see [6, Definition 2]. Observe that, for $\alpha = 1$, ∇_{λ} coincides with $\nabla_{\lambda}^{\alpha}$, provided that the convention $\nabla_{\lambda}^{0} f(x) = f(x)$ is adopted.

If f is a continuous function on $\mathbb{R}$, then we have

$$\lim_{\lambda \to 0^{+}} \nabla_{\lambda}^{-\alpha} f(x) = I^{-\alpha} f(x),$$

where $I^{-\alpha}$ is the Riemann-Liouville fractional integral of order $\alpha > 0$, see [6, Remark 1].

With these ideas in mind, we give the following definition.

Definition 3 The λ-fractional difference of order $0 < \alpha \leq 1$ in the Caputo sense is defined by

$$\nabla_{\lambda}^{\alpha} f(x) := \nabla_{\lambda}^{-(1-\alpha)} \nabla_{\lambda} f(x), \quad x \in \mathbb{T}.$$

Note that, for $\alpha = 1$, $\nabla_{\lambda} \equiv \nabla_{\lambda}^{\alpha}$.

For more details on fractional order difference operators see [1, 2, 4, 5, 23]. On the other hand, we recall that

$$\nabla_{\lambda}^{\alpha} \mathcal{E}_{\lambda,\alpha}^{x}(t) = t\mathcal{E}_{\lambda,\alpha}^{x}(t), \tag{9}$$

see [1]. Indeed, we recall that:

$$\nabla_{\lambda} x_{\lambda}^{\overline{r}} = r x_{\lambda}^{\overline{r-1}}, \; r \in \mathbb{R},$$

$$\nabla_\lambda^{-\alpha} x_\lambda^{\overline{\delta}} = \frac{\Gamma(\delta+1)}{\Gamma(\delta+\alpha+1)} x_\lambda^{\overline{\delta+\alpha}}, \quad x \in \mathbb{T}\setminus\{0\},\ \delta \in \mathbb{R},$$

see [6, Lemma 1] and [5, Lemma 1], respectively. Then,

$$\begin{aligned} \nabla_\lambda \mathcal{E}_{\lambda,\alpha}^x(t) &= \sum_{j=0}^{\infty} \nabla_\lambda x_\lambda^{\overline{\alpha j}} \frac{t^j}{\Gamma(\alpha j+1)} \\ &= \sum_{j=1}^{\infty} \alpha j x_\lambda^{\overline{\alpha j-1}} \frac{t^j}{\Gamma(\alpha j+1)} \\ &= t \sum_{j=0}^{\infty} x_\lambda^{\overline{\alpha j+\alpha-1}} \frac{t^j}{\Gamma(\alpha j+\alpha)}. \end{aligned}$$

Hence,

$$\begin{aligned} \nabla_\lambda^{\alpha} \mathcal{E}_{\lambda,\alpha}^x(t) &= t \sum_{j=0}^{\infty} \nabla_\lambda^{1-\alpha} x_\lambda^{\overline{\alpha j+\alpha-1}} \frac{t^j}{\Gamma(\alpha j+\alpha)} \\ &= t \sum_{j=0}^{\infty} x_\lambda^{\overline{\alpha j}} \frac{t^j}{\Gamma(\alpha j+1)} = t\, \mathcal{E}_{\lambda,\alpha}^x(t), \end{aligned}$$

where we have used the Gamma-function identity $z\Gamma(z) = \Gamma(z+1)$. Therefore, $\mathcal{E}_{\lambda,\alpha}^x(t)$ is an eigenfunction of the operator ∇_λ^α.

Definition 4 A function sequence, $\{\mathscr{C}_{n,\lambda}^{\alpha}(x)\}_{n\in\mathbb{N}_0}$, is called ∇_λ^α-Appell sequence if it satisfies the following identity

$$\nabla_\lambda^\alpha \mathscr{C}_{n,\lambda}^{\alpha}(x) = n \mathscr{C}_{n-1,\lambda}^{\alpha}(x),\ n \in \mathbb{N}, \tag{10}$$

with $\mathscr{C}_{0,\lambda}^{\alpha}(x)$ being an arbitrary non-null constant.

Remark 2 Just as in [9, Remark 1], the ∇_λ^α-Appell sequences of functions are polynomials in the strict sense only when $\alpha = 1$.

Theorem 3 *Let $0 < \alpha \leq 1$ and $\{c_{\lambda,n}\}_{n\in\mathbb{N}_0}$ be a sequence of arbitrary numbers. The sequence $\{\mathscr{C}_{n,\lambda}^{\alpha}(x)\}_{n\in\mathbb{N}_0}$ given by*

$$\mathscr{C}_{n,\lambda}^{\alpha}(x) := \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_\alpha c_{\lambda,n-r} x_\lambda^{\overline{\alpha r}},\ x \in \mathbb{T}\setminus\{0\}, \tag{11}$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix}_\alpha := \frac{n!}{\Gamma(\alpha(n-r)+1)\Gamma(\alpha r+1)}$$

satisfies the relation (10).

Proof By [6, Lemma 1] and [5, Lemma 1], we have

$$\nabla_\lambda^\alpha x_\lambda^{\overline{\alpha r}} = \frac{\alpha r \Gamma(\alpha r)}{\Gamma(\alpha(r-1)+1)} x_\lambda^{\overline{\alpha(r-1)}}.$$

Then,

$$\begin{aligned}\nabla_\lambda^\alpha \mathscr{C}_{n,\lambda}^\alpha(x) &= \sum_{r=1}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_\alpha \frac{\Gamma(\alpha r+1)}{\Gamma(\alpha(r-1)+1)} c_{\lambda,n-r} x_\lambda^{\overline{\alpha(r-1)}} \\ &= \sum_{r=1}^{n} \frac{n!}{\Gamma(\alpha(n-r)+1)} \frac{c_{\lambda,n-r} x_\lambda^{\overline{\alpha(r-1)}}}{\Gamma(\alpha(r-1)+1)} \\ &= n \sum_{r=0}^{n-1} \begin{bmatrix} n-1 \\ r \end{bmatrix}_\alpha c_{\lambda,n-1-r} x_\lambda^{\overline{\alpha r}}.\end{aligned}$$

Hence, the desired result follows from the above argument. □

Remark 3 Note that for $\alpha = 1$, the ∇_λ^α-Appell sequence of functions given in (11) becomes the ∇_λ-Appell sequence of polynomials given in (5).

The following result allows us to obtain a generating function for the ∇_λ^α-Appell sequence of functions $\{\mathscr{C}_{n,\lambda}^\alpha(x)\}_{n\in\mathbb{N}_0}$.

Theorem 4 *Let* $0 < \alpha \leq 1$ *and* $\lambda > 0$. *Given the series*

$$C_{\lambda,\alpha}(t) := \sum_{j=0}^{\infty} c_{\lambda,j} \frac{t^{\alpha j}}{\Gamma(\alpha j+1)}, \quad c_{\lambda,0} \neq 0, \tag{12}$$

with $c_{\lambda,j}$, $j = 0, 1, \ldots$, *real coefficients, the* ∇_λ^α*-Appell sequence of functions* $\{\mathscr{C}_{n,\lambda}^\alpha(x)\}_{n\in\mathbb{N}_0}$ *is determined by the following generating function:*

$$C_{\lambda,\alpha}(t)\mathcal{E}_{\lambda,\alpha}^x(t^\alpha) = \sum_{n=0}^{\infty} \mathscr{C}_{n,\lambda}^\alpha(x) \frac{t^{\alpha n}}{n!}.$$

Proof By (8) and (12), we identify the sequence $\{\mathscr{C}_{n,\lambda}^\alpha(x)\}_{n\in\mathbb{N}_0}$, given in (11), as the coefficients of $\frac{t^{\alpha n}}{n!}$. □

In what follows, we present two structured subclasses of the ∇_λ^α-Appell framework, namely the Bernoulli and Euler type sequences. They are introduced through suitable generating functions involving the discrete Mittag–Leffler function. From these generating functions, we derive recursive (triangular) systems that allow the sequences to be constructed term by term, and we compute the first few members explicitly.

3.1 Bernoulli and Euler type ∇_λ^α-Appell sequences

We introduce the Bernoulli and Euler type ∇_λ^α-Appell sequences via the following generating functions.

$$\frac{t^\alpha}{\mathcal{E}_{\lambda,\alpha}(t^\alpha)-1} \mathcal{E}_{\lambda,\alpha}^x(t^\alpha) = \sum_{n=0}^{\infty} \mathscr{B}_{n,\lambda}^\alpha(x) \frac{t^{\alpha n}}{n!}, \tag{13}$$

$$\frac{2^\alpha}{\mathcal{E}_{\lambda,\alpha}(t^\alpha)+1} \mathcal{E}_{\lambda,\alpha}^x(t^\alpha) = \sum_{n=0}^{\infty} \mathscr{E}_{n,\lambda}^\alpha(x) \frac{t^{\alpha n}}{n!}, \tag{14}$$

where $\mathcal{E}_{\lambda,\alpha}(t^\alpha) \equiv \mathcal{E}_{\lambda,\alpha}^1(t^\alpha)$.

Multiplying both sides of (13) by $\mathcal{E}_{\lambda,\alpha}(t^\alpha)-1$ and applying (8), we get

$$\begin{aligned}\sum_{n=0}^{\infty} x_\lambda^{\overline{\alpha n}} \frac{t^{\alpha(n+1)}}{\Gamma(\alpha n+1)} &= \sum_{j=1}^{\infty} \frac{1_\lambda^{\overline{\alpha j}} t^{j\alpha}}{\Gamma(\alpha j+1)} \sum_{n=0}^{\infty} \mathscr{B}_{n,\lambda}^{\alpha}(x) \frac{t^{\alpha n}}{n!} \\ &= \sum_{j=0}^{\infty} \frac{1_\lambda^{\overline{\alpha(j+1)}} t^{\alpha(j+1)}}{\Gamma(\alpha(j+1)+1)} \sum_{n=0}^{\infty} \mathscr{B}_{n,\lambda}^{\alpha}(x) \frac{t^{\alpha n}}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1_\lambda^{\overline{\alpha(j+1)}} t^{\alpha(j+1)}}{\Gamma(\alpha(j+1)+1)} \mathscr{B}_{n-j,\lambda}^{\alpha}(x) \frac{t^{\alpha(n-j)}}{(n-j)!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1_\lambda^{\overline{\alpha(j+1)}} \mathscr{B}_{n-j,\lambda}^{\alpha}(x)}{\Gamma(\alpha(j+1)+1)} \frac{t^{\alpha(n+1)}}{(n-j)!}.\end{aligned}$$

Thus, equating coefficients of $t^{\alpha(n+1)}$, it is possible to obtain a triangular system.

Similarly, multiplying both sides of (14) by $\mathcal{E}_{\lambda,\alpha}(t^\alpha)+1$ and applying (8), we get

$$2^\alpha \sum_{n=0}^{\infty} x_\lambda^{\overline{\alpha n}} \frac{t^{\alpha n}}{\Gamma(\alpha n+1)} = \sum_{j=0}^{\infty} a_{j,\lambda}^{\alpha} t^{j\alpha} \sum_{n=0}^{\infty} \mathscr{E}_{n,\lambda}^{\alpha}(x) \frac{t^{\alpha n}}{n!},$$

where the coefficients $a_{j,\lambda}^{\alpha}$ are given by

$$a_{j,\lambda}^{\alpha} = \begin{cases} 2, & \text{if } j=0, \\ \frac{1_\lambda^{\overline{\alpha j}}}{\Gamma(\alpha j+1)} & \text{if } j \geq 1. \end{cases}$$

Hence, by proceeding as before, we derive another triangular system.

More precisely, we obtain the following result.

Theorem 5 *Let $\{\mathscr{B}_{n,\lambda}^{\alpha}(x)\}_{n\in\mathbb{N}_0}$ and $\{\mathscr{E}_{n,\lambda}^{\alpha}(x)\}_{n\in\mathbb{N}_0}$ be the Bernoulli and Euler type ∇_λ^α-Appell sequences. Then, these ∇_λ^α-Appell sequences can be sequentially constructed by solving the following triangular systems:*

$$\begin{aligned} &\mathscr{B}_{0,\lambda}^{\alpha}(x) = \frac{\Gamma(\alpha+1)}{1_\lambda^{\overline{\alpha}}}, \\ &\sum_{j=0}^{n} \frac{1_\lambda^{\overline{\alpha(j+1)}} \Gamma(\alpha n+1)\, \mathscr{B}_{n-j,\lambda}^{\alpha}(x)}{\Gamma(\alpha(j+1)+1)(n-j)!} = x_\lambda^{\overline{\alpha n}}, \quad n \geq 1, \end{aligned} \tag{15}$$

and

$$\begin{aligned} &\mathscr{E}_{0,\lambda}^{\alpha}(x) = 2^{\alpha-1}, \\ &\frac{2\Gamma(\alpha n+1)}{n!} \mathscr{E}_{n,\lambda}^{\alpha}(x) + \sum_{j=1}^{n} \frac{1_\lambda^{\overline{\alpha j}} \Gamma(\alpha n+1)}{\Gamma(\alpha j+1)\,(n-j)!} \mathscr{E}_{n-j,\lambda}^{\alpha}(x) = 2^\alpha x_\lambda^{\overline{\alpha n}}, \quad n \geq 1. \end{aligned} \tag{16}$$

Remark 4 Note that (15) and (16) can also be expressed in terms of matrix-inversion formulas (cf. [28]).

Using (15), we have

$$\mathscr{B}^{\alpha}_{n,\lambda}(x)=\frac{\Gamma(\alpha+1)n!}{1^{\overline{\alpha}}_{\lambda}\Gamma(\alpha n+1)}x^{\overline{\alpha n}}_{\lambda}-\sum_{j=1}^{n-1}\frac{1^{\overline{\alpha(j+1)}}_{\lambda}\Gamma(\alpha+1)n!\mathscr{B}^{\alpha}_{n-j,\lambda}(x)}{1^{\overline{\alpha}}_{\lambda}\Gamma(\alpha(j+1)+1)(n-j)!}-\frac{1^{\overline{\alpha(n+1)}}_{\lambda}\Gamma^2(\alpha+1)n!}{1^{\overline{\alpha}}_{\lambda}\Gamma(\alpha(n+1)+1)},\quad n\geq 1.$$

As a straightforward consequence of Theorem 5, the first few terms of the Bernoulli type $\nabla^{\alpha}_{\lambda}$-Appell sequence are as follows:

$$\begin{aligned}
\mathscr{B}^{\alpha}_{0,\lambda}(x)&=\frac{\Gamma(\alpha+1)}{1^{\overline{\alpha}}_{\lambda}},\\
\mathscr{B}^{\alpha}_{1,\lambda}(x)&=\frac{\Gamma(\alpha+1)}{1^{\overline{\alpha}}_{\lambda}}\left(\frac{x^{\overline{\alpha}}_{\lambda}}{\Gamma(\alpha+1)}-\frac{1^{\overline{2\alpha}}_{\lambda}\Gamma(\alpha+1)}{\Gamma(2\alpha+1)1^{\overline{\alpha}}_{\lambda}}\right),\\
\mathscr{B}^{\alpha}_{2,\lambda}(x)&=\frac{2!\Gamma(\alpha+1)}{1^{\overline{\alpha}}_{\lambda}}\left(\frac{x^{\overline{2\alpha}}_{\lambda}}{\Gamma(2\alpha+1)}-\frac{1^{\overline{2\alpha}}_{\lambda}x^{\overline{\alpha}}_{\lambda}}{1^{\overline{\alpha}}_{\lambda}\Gamma(2\alpha+1)}+\frac{\left(1^{\overline{2\alpha}}_{\lambda}\right)^2\Gamma^2(\alpha+1)}{\left(1^{\overline{\alpha}}_{\lambda}\right)^2\Gamma^2(2\alpha+1)}-\frac{1^{\overline{3\alpha}}_{\lambda}\Gamma(\alpha+1)}{1^{\overline{\alpha}}_{\lambda}\Gamma(3\alpha+1)}\right),\\
\mathscr{B}^{\alpha}_{3,\lambda}(x)&=\frac{3!\Gamma(\alpha+1)}{1^{\overline{\alpha}}_{\lambda}}\left(\frac{x^{\overline{3\alpha}}_{\lambda}}{\Gamma(3\alpha+1)}-\frac{1^{\overline{2\alpha}}_{\lambda}\Gamma(\alpha+1)x^{\overline{2\alpha}}_{\lambda}}{1^{\overline{\alpha}}_{\lambda}\Gamma^2(2\alpha+1)}\right.\\
&\quad+\frac{\left(\Gamma(\alpha+1)\left(1^{\overline{2\alpha}}_{\lambda}\right)^2\Gamma(3\alpha+1)-1^{\overline{3\alpha}}_{\lambda}1^{\overline{\alpha}}_{\lambda}\Gamma^2(2\alpha+1)\right)x^{\overline{\alpha}}_{\lambda}}{\left(1^{\overline{\alpha}}_{\lambda}\right)^2\Gamma(3\alpha+1)\Gamma^2(2\alpha+1)}\\
&\quad\left.-\,-\frac{\left(1^{\overline{2\alpha}}_{\lambda}\right)^3\Gamma^3(\alpha+1)}{\left(1^{\overline{\alpha}}_{\lambda}\right)^3\Gamma^3(2\alpha+1)}+\frac{2\left(1^{\overline{2\alpha}}_{\lambda}1^{\overline{3\alpha}}_{\lambda}\right)\Gamma^2(\alpha+1)}{\left(1^{\overline{\alpha}}_{\lambda}\right)^2\Gamma(2\alpha+1)\Gamma(3\alpha+1)}-\frac{1^{\overline{4\alpha}}_{\lambda}\Gamma(\alpha+1)}{1^{\bar{\alpha}}_{\lambda}\Gamma(4\alpha+1)}\right).
\end{aligned}\tag{17}$$

While, the first few terms of the Euler type $\nabla^{\alpha}_{\lambda}$-Appell sequence are as follows:

$$\begin{aligned}
\mathscr{E}^{\alpha}_{0,\lambda}(x)&=2^{\alpha-1},\\
\mathscr{E}^{\alpha}_{1,\lambda}(x)&=\frac{2^{\alpha-2}}{\Gamma(\alpha+1)}\left(2x^{\overline{\alpha}}_{\lambda}-1^{\overline{\alpha}}_{\lambda}\right),\\
\mathscr{E}^{\alpha}_{2,\lambda}(x)&=2^{\alpha-2}\left(\frac{4x^{\overline{2\alpha}}_{\lambda}}{\Gamma(2\alpha+1)}-\frac{2\left(1^{\overline{\alpha}}_{\lambda}\right)x^{\overline{\alpha}}_{\lambda}}{\Gamma^2(\alpha+1)}+\frac{\left(1^{\overline{\alpha}}_{\lambda}\right)^2}{\Gamma^2(\alpha+1)}-\frac{2\left(1^{\overline{2\alpha}}_{\lambda}\right)}{\Gamma(2\alpha+1)}\right),\\
\mathscr{E}^{\alpha}_{3,\lambda}(x)&=2^{\alpha-3}\left(\frac{24x^{\overline{3\alpha}}_{\lambda}-12\left(1^{\overline{3\alpha}}_{\lambda}\right)}{\Gamma(3\alpha+1)}-\frac{12\left(1^{\overline{\alpha}}_{\lambda}\right)x^{\overline{2\alpha}}_{\lambda}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}-\frac{12\left(1^{\overline{2\alpha}}_{\lambda}\right)x^{\overline{\alpha}}_{\lambda}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right.\\
&\quad\left.+\frac{6\left(1^{\overline{\alpha}}_{\lambda}\right)^2x^{\overline{\alpha}}_{\lambda}-3\left(1^{\overline{\alpha}}_{\lambda}\right)^3}{\Gamma^3(\alpha+1)}+\frac{12\left(1^{\overline{\alpha}}_{\lambda}\right)1^{\overline{2\alpha}}_{\lambda}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right).
\end{aligned}\tag{18}$$

Remark 5 It is straightforward to verify that setting $\alpha=1$ in (17) and (18) yields the polynomials in Examples 1 and 2, respectively.

4 Conclusions

In this paper, we introduced the notion of $\nabla^{\alpha}_{\lambda}$-Appell sequences of functions based on the discrete fractional Caputo operator and the degenerate Mittag–Leffler function. We analyzed several properties of these function sequences, including explicit expressions for each term and their associated generating functions (cf. Theorems 3 and 4, respectively). In particular, we also studied some basic novel properties for the case $\alpha = 1$, in which the corresponding sequences become ∇_λ-Appell polynomial sequences.

It is worth noting that the use of fractional operators and expansions in fractional powers which mimic the structural form of Appell (or Scheffer) sequences is a recent development (cf. [8, 9] and the references therein). To the best of our knowledge, the novelty of this paper has been to study an analogous situation incorporating the notion of the degeneracy (cf. [13, 15]).

As a possible direction for future research, the Bernoulli and Euler type ∇_λ^α-Appell sequences introduced here could be explored as basis functions for representing solutions of Caputo-type fractional difference equations, or for constructing series solutions to initial value problems on discrete time scales. In particular, it would be of interest to investigate convergence properties and error estimates for such expansions, as well as potential links with qualitative features of discrete fractional dynamical systems (e.g., stability and asymptotic behavior); see, for example, [17, 24, 25].

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