PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Volume 17 · 2024 · Pages 22–31

An approximation to Appell's hypergeometric function F_2 by branched continued fraction

Tamara Antonova[a] · Clemente Cesarano[b] · Roman Dmytryshyn[c] · Serhii Sharyn[c]

Communicated by Maria Grazia Russo

Abstract

Appell's functions F_1–F_4 turned out to be particularly useful in solving a variety of problems in both pure and applied mathematics. In literature, there have been published a significant number of interesting and useful results on these functions. In this paper, we prove that the branched continued fraction, which is an expansion of ratio of hypergeometric functions F_2 with a certain set of parameters, converges uniformly to a holomorphic function of two variables on every compact subset of some domain of $\mathbb{C}^2$, and that this function is an analytic continuation of such ratio in this domain. As a special case of our main result, we give the representation of hypergeometric functions F_2 by a branched continued fraction. To illustrate this, we have given some numerical experiments at the end.

1 Introduction

Appel's functions F_1–F_4, which were introduced back in 1880 [7, 8], are probably one of the most well-known and researched families of hypergeometric functions that appear in a number of problems in applied mathematics, physics, statistics, and other fields. The study of these functions concerns, in particular, the establishment of recurrence relations [22, 36], the construction of analytic continuations [28, 34], integral representations [27, 31, 33], and branched continued fraction representations [19–21, 29, 32].

Appell's hypergeometric function F_2 is defined by double power series (see, [26])

$$F_2(a,b,b';c,c';\mathbf{z}) = \sum_{r,s=0}^{\infty} \frac{(a)_{r+s}(b)_r(b')_s}{(c)_r(c')_s} \frac{z_1^r}{r!} \frac{z_2^s}{s!}, \quad |z_1|+|z_2|<1,$$

where a, b, b', c, and c' are complex constants; c and c' are not equal to a non-positive integer; $(\cdot)_k$ is the Pochhammer symbol defined for any complex number α and non-negative integer n by $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$, $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$.

In 1993 Bodnar obtained a formal expansion ratio (see, [18])

$$\frac{F_2(a,b,b';c,c';\mathbf{z})}{F_2(a+1,b,b';c+1,c';\mathbf{z})}$$

into branched continued fraction

$$1 - \sum_{i_1=1}^{2} \cfrac{p_{i(1)} z_{i_1}}{1 - \cfrac{q_{i(1)} z_{i_1}}{1 - \sum_{i_2=1}^{2} \cfrac{p_{i(2)} z_{i_2}}{1 - \cfrac{q_{i(2)} z_{i_2}}{1 - \ddots}}}},$$

where $i(k) = (i_1, i_2, \ldots, i_k)$ is a multiindex,

$$p_1 = \frac{b(c-a)}{c(c+1)}, \quad p_2 = 1,$$

[a]Lviv Polytechnic National University
[b]Università Telematica Internazionale UNINETTUNO
[c]Vasyl Stefanyk Precarpathian National University

and for $k \geq 1$

$$p_{i(k+1)} = \begin{cases} \dfrac{(c-a+r-s)(b+r)}{(c+2r)(c+2r+1)}, \text{ if } i_k = i_{k+1} = 1, \\ \dfrac{(c'-a-r+s-1)(b'+s)}{(c'+2s)(c'+2s-1)}, \text{ if } i_k = i_{k+1} = 2, \\ \dfrac{b+r}{c+2r+1}, \text{ if } i_k = 2,\ i_{k+1} = 1, \\ \dfrac{c'+s}{c'+2s}, \text{ if } i_k = 1,\ i_{k+1} = 2, \end{cases} \qquad q_{i(k)} = \begin{cases} \dfrac{(c-b+r)(a+r+s)}{(c+2r-1)(c+2r)}, \text{ if } i_k = 1, \\ \dfrac{(c'-b'+s-1)(a+r+s)}{(c+2r-2)(c+2r-1)}, \text{ if } i_k = 2, \end{cases}$$

where r is the number of times 1 appears in the multiindex $i(k)$, $s = n - r$.

In the case $b = c$, we have the following result.

Theorem 1.1. *The ratio*

$$\frac{F_2(a,b,b';b,c';\mathbf{z})}{F_2(a+1,b,b';b,c'+1;\mathbf{z})} \tag{1}$$

has a formal branched continued fraction expansion of the form

$$1 - z_1 - \cfrac{d_1 z_2}{1 - \cfrac{d_2 z_2}{1 - z_1 - \cfrac{d_3 z_2}{1 - \cfrac{d_4 z_2}{1 - z_1 - \cfrac{d_5 z_2}{1 - \ddots}}}}}, \tag{2}$$

where

$$d_{2k-1} = \frac{(b'+k-1)(c'-a+k-1)}{(c'+2k-2)(c'+2k-1)} \quad \text{and} \quad d_{2k} = \frac{(a+k)(c'-b'+k)}{(c'+2k-1)(c'+2k)} \quad \text{for} \quad k \geq 1, \tag{3}$$

Note that (2) is a so-called 'confluent branched continued fraction with independent variables', not a continued fraction. Here, the fundamental difference between them lies in the different approach to understanding approximants. Namely, the sequence of approximants of the continued fraction for the branched continued fraction is a sequence of so-called 'figured approximants' [5,6,13–15]. More about branched continued fractions with independent variables can be found in the works [1,4,11,12,17,23,25].

In this paper, we prove that the branched continued fraction (2) uniformly converges to a holomorphic function of two variables on every compact subset of some domain of $\mathbb{C}^2$, and that this function is an analytic continuation of the ratio (1) in this domain. In Corollary 2.3, we give the representation of hypergeometric functions $F_2(1,b,b';b,c';\mathbf{z})$ by a branched continued fraction. In Section 3, we present some numerical experiments.

2 Convergence of Branched Continued Fractions

To prove our main result, we need a theorem that directly follows from [24, Theorem 1].

Theorem 2.1. *Let $g_{0,k}$, $k \geq 1$, be real numbers such that*

$$0 < g_{0,k} \leq 1 \quad \text{for all} \quad k \geq 1.$$

Then the branched continued fraction

$$1 - z_{1,0} - \cfrac{g_{0,1} z_{0,1}}{1 - \cfrac{g_{0,2}(1-g_{0,1}) z_{0,2}}{1 - (1-g_{0,2}) z_{1,2} - \cfrac{g_{0,3}(1-g_{0,2}) z_{0,3}}{1 - \cfrac{g_{0,4}(1-g_{0,3}) z_{0,4}}{1 - (1-g_{0,4}) z_{1,4} - \cfrac{g_{0,5}(1-g_{0,4}) z_{0,5}}{1 - \ddots}}}}}$$

converges if

$$|z_{1,2k}| \leq \frac{1}{2} \quad \text{and} \quad |z_{0,k+1}| \leq \frac{1}{2} \quad \text{for all} \quad k \geq 0.$$

The following is true.

Theorem 2.2. *Let a, b', and c' be real constants such that*

$$0 < d_k \leq r \quad \text{for all} \quad k \geq 1, \tag{4}$$

where d_k, $k \geq 1$, are defined by (3), r is a positive number. Then:

(A) the branched continued fraction (2) converges uniformly on every compact subset of the domain

$$\mathrm{H}_{r,s} = \bigcup_{-\pi/2<\alpha<\pi/2} \mathrm{P}_{r,s,\alpha}, \tag{5}$$

where

$$\mathrm{P}_{r,s,\alpha} = \left\{ \mathbf{z}\in\mathbb{C}^2 : \ |z_1| + \mathrm{Re}(z_1 e^{-2i\alpha}) < 2(1-s)\cos^2\alpha,\ |z_2| + \mathrm{Re}(z_2 e^{-2i\alpha}) < \frac{s\cos^2\alpha}{2r} \right\}, \quad 0<s<1, \tag{6}$$

to a function $f(\mathbf{z})$ holomorphic in $\mathrm{H}_{r,s}$;

(B) the function $f(\mathbf{z})$ is an analytic continuation of (1) in the domain (5).

Note that the assumption on the sequence $\{d_k\}$ in Theorem 2.2 involves (together with positivity) an upper bound r, and that the domain of the analytic continuation also depends on this r; and the smaller r, the larger domain.

Proof of Theorem 2.2. Let

$$F_n^{(n)}(\mathbf{z}) = 1, \quad n \ge 1, \tag{7}$$

and

$$F_{2k-1}^{(2n)}(\mathbf{z}) = 1 - \cfrac{d_{2k}z_2}{1 - z_1 - \cfrac{d_{2k+1}z_2}{1 - \ddots - z_1 - \cfrac{d_{2n-1}z_2}{1 - d_{2n}z_2}}}, \quad F_{2k-2}^{(2n)}(\mathbf{z}) = 1 - z_1 - \cfrac{d_{2k-1}z_2}{1 - \cfrac{d_{2k}z_2}{1 - \ddots - z_1 - \cfrac{d_{2n-1}z_2}{1 - d_{2n}z_2}}},$$

$$F_{2k-1}^{(2n+1)}(\mathbf{z}) = 1 - \cfrac{d_{2k}z_2}{1 - z_1 - \cfrac{d_{2k+1}z_2}{1 - \ddots - \cfrac{d_{2n}z_2}{1 - z_1 - d_{2n+1}z_2}}}, \quad F_{2k}^{(2n+1)}(\mathbf{z}) = 1 - z_1 - \cfrac{d_{2k+1}z_2}{1 - \cfrac{d_{2k+2}z_2}{1 - \ddots - \cfrac{d_{2n}z_2}{1 - z_1 - d_{2n+1}z_2}}},$$

where $n \ge 1$, $1 \le k \le n$. Then

$$F_{2k-1}^{(2n)}(\mathbf{z}) = 1 - \frac{d_{2k}z_2}{F_{2k}^{(2n)}(\mathbf{z})}, \quad F_{2k-2}^{(2n)}(\mathbf{z}) = 1 - z_1 - \frac{d_{2k-1}z_2}{F_{2k-1}^{(2n)}(\mathbf{z})}, \tag{8}$$

and

$$F_{2k-1}^{(2n+1)}(\mathbf{z}) = 1 - \frac{d_{2k}z_2}{F_{2k}^{(2n+1)}(\mathbf{z})}, \quad F_{2k}^{(2n+1)}(\mathbf{z}) = 1 - z_1 - \frac{d_{2k+1}z_2}{F_{2k+1}^{(2n+1)}(\mathbf{z})},$$

where $n \ge 1$, $1 \le k \le n$, and, thus, for each $n \ge 1$ we write the nth approximants of (2) as

$$f_n(\mathbf{z}) = 1 - z_1 - \frac{d_1 z_2}{F_1^{(n)}(\mathbf{z})}. \tag{9}$$

We show that each approximant $f_n(\mathbf{z})$ of (2) is a function holomorphic in (5). Since the numerator and denominator of the approximant are polynomials, they are entire functions of two variables. The quotient of two entire functions is a holomorphic function everywhere where the denominator does not vanish. Therefore, taking into account (9), it suffices to prove that $F_1^{(n)}(\mathbf{z}) \ne 0$ for all $n \ge 1$, and for all $\mathbf{z} \in \mathrm{H}_{r,s}$.

Let n be an arbitrary natural number, α be any real from the interval $(-\pi/2, \pi/2)$, and $\mathbf{z}$ be an arbitrary fixed point from (6). By induction on k for each index k, $1 \le k \le n$, we show that the following inequalities are valid

$$\mathrm{Re}(F_{2k-1}^{(2n)}(\mathbf{z})e^{-i\alpha}) > \frac{s\cos\alpha}{2} > 0 \tag{10}$$

and

$$\mathrm{Re}(F_{2k-1}^{(2n+1)}(\mathbf{z})e^{-i\alpha}) > \frac{s\cos\alpha}{2} > 0. \tag{11}$$

From (7) it is clear that for $k = n$ the inequality (10) holds. By the induction hypothesis that (10) holds for $k = p+1$ such that $p+1 \le n$, we prove (10) for $k = p$. Indeed, using of (8) leads to

$$F_{2p-1}^{(2n)}(\mathbf{z})e^{-i\alpha} = e^{-i\alpha} - \frac{d_{2p}z_2 e^{-2i\alpha}}{F_{2p}^{(2n)}(\mathbf{z})e^{-i\alpha}}$$

and

$$F_{2p}^{(2n)}(\mathbf{z})e^{-i\alpha} = e^{-i\alpha} - \frac{z_1 e^{-2i\alpha}}{e^{-i\alpha}} - \frac{d_{2p+1}z_2 e^{-2i\alpha}}{F_{2p+1}^{(2n)}(\mathbf{z})e^{-i\alpha}}.$$

In the proof of Lemma 4.41 in [30] it is shown that if $x \geq c > 0$ and $v^2 \leq 4u+4$, then

$$\min_{-\infty<y<+\infty} \operatorname{Re}\left(\frac{u+iv}{x+iy}\right) = -\frac{\sqrt{u^2+v^2}-u}{2x}. \tag{12}$$

Using (4), (6), (12) and the induction hypothesis, we have

$$\begin{aligned}\operatorname{Re}(F_{2p}^{(2n)}(\mathbf{z})e^{-i\alpha}) &\geq \cos\alpha - \frac{|z_1 e^{-2i\alpha}| + \operatorname{Re}(z_1 e^{-2i\alpha})}{2\operatorname{Re}(e^{-i\alpha})} - \frac{d_{2p+1}(|z_2 e^{-2i\alpha}| + \operatorname{Re}(z_2 e^{-2i\alpha}))}{2\operatorname{Re}(F_{2p+1}^{(2n)}(\mathbf{z})e^{-i\alpha})} \\ &> \cos\alpha - (1-s)\cos\alpha - \frac{s\cos\alpha}{2} \\ &= \frac{s\cos\alpha}{2} > 0\end{aligned}$$

and

$$\begin{aligned}\operatorname{Re}(F_{2p-1}^{(2n)}(\mathbf{z})e^{-i\alpha}) &\geq \cos\alpha - \frac{d_{2p}(|z_2 e^{-2i\alpha}| + \operatorname{Re}(z_2 e^{-2i\alpha}))}{2\operatorname{Re}(F_{2p}^{(2n)}(\mathbf{z})e^{-i\alpha})} \\ &> \cos\alpha - \frac{\cos\alpha}{2} \\ &= \frac{\cos\alpha}{2} > 0.\end{aligned}$$

Similarly, we obtain inequalities (11).

Therefore,

$$F_1^{(n)}(\mathbf{z}) \neq 0 \quad \text{for all} \quad n \geq 1 \quad \text{and} \quad \mathbf{z} \in \mathrm{P}_{r,s,\alpha}.$$

It follows that the approximants $f_n(\mathbf{z})$, $n \geq 1$, of (2) form a sequence of functions holomorphic in (6), and, consequently, in domain $\mathrm{H}_{r,s}$ by virtue of arbitrariness α.

Let K be an arbitrary compact subset of $\mathrm{H}_{r,s}$. Then there exists an open bi-disk

$$\mathrm{D}_L = \{\mathbf{z} \in \mathbb{C}^2 : \ |z_1| < L, \ |z_2| < L\},$$

containing K. Let us cover K by domains of the form

$$\mathrm{G}_{r,s,\alpha,L} = \mathrm{P}_{r,s,\alpha} \bigcap \mathrm{D}_L.$$

From this cover we choose a finite subcover

$$\mathrm{G}_{r,s,\alpha_1,L}, \ \mathrm{G}_{r,s,\alpha_2,L}, \ \ldots, \ \mathrm{G}_{r,s,\alpha_k,L}.$$

Using (9), (10), and (11), for the arbitrary $p \in \{1, \ 2, \ \ldots, \ k\}$ we obtain for any $\mathbf{z} \in \mathrm{G}_{r,s,\alpha_p,L}$ and $n \geq 1$

$$\begin{aligned}|f_n(\mathbf{z})| &\leq 1 + |z_1| + \frac{d_1|z_2|}{\operatorname{Re}(F_1^{(n)}(\mathbf{z})e^{-i\alpha_p})} \\ &< 1 + L + \frac{2rL}{\cos\alpha_p} \\ &= M(\mathrm{G}_{r,s,\alpha_p,L}).\end{aligned}$$

We set

$$M(\mathrm{K}) = \max_{1 \leq p \leq k} M(\mathrm{G}_{r,s,\alpha_p,L}).$$

Then for arbitrary $\mathbf{z} \in \mathrm{K}$ we have

$$|f_n(\mathbf{z})| \leq M(\mathrm{K}), \quad \text{for} \quad n \geq 1,$$

i.e., the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded on every compact subset of the domain $\mathrm{H}_{r,s}$.

We assume that the domain

$$\mathrm{Q}_R = \left\{\mathbf{z} \in \mathbb{R}^2 : \ -\frac{1}{4} < -R < z_1 < 0, \ -\frac{1}{8r} < -R < z_2 < 0\right\}$$

is contained in $\mathrm{H}_{r,s}$ for each $0 < R < 1/(8r)$, in particular, say $\mathrm{Q}_{1/(16r)} \subset \mathrm{H}_{r,s}$. Using (4) it is easy to show that for arbitrary $\mathbf{z} \in \mathrm{Q}_R$, $\mathrm{Q}_R \subset \mathrm{H}_{r,s}$, the following inequalities are valid

$$|z_1| < \frac{1}{4} \quad \text{and} \quad |d_k z_2| < \frac{1}{8} \quad \text{for all} \quad k \geq 1,$$

i.e. the elements of (11) satisfy the conditions of Theorem 2.1, with

$$g_{0,k} = \frac{1}{2} \quad \text{for all} \quad k \geq 1.$$

It follows from Theorem 2.1 that (2) converges in Q_R, $\mathrm{Q}_R \subset \mathrm{H}_{r,s}$. Hence by [2, Theorem 3] (see also [16, Theorem 2.17] and [35, Theorem 24.2]), the branched continued fraction (2) converges uniformly on compact subsets of $\mathrm{H}_{r,s}$ to a holomorphic function. These completes the proof of *(A)*.

The proof of *(B)* is similar to the proof of Theorem 3 in [3]; hence it is omitted. □

Setting $a = 0$ and replacing c' by $c' - 1$ in Theorem 2.2, we obtain the following result.

Corollary 2.3. *Let b' and c' be real constants such that*

$$0 < \frac{b'}{c'} \leq r, \quad 0 < \frac{(b'+k)(c'+k-1)}{(c'+2k-1)(c'+2k)} \leq r \quad \text{and} \quad 0 < \frac{k(c'-b'+k-1)}{(c'+2k-2)(c'+2k-1)} \leq r \quad \text{for all} \quad k \geq 1,$$

where r is a positive number. Then:

(A) the branched continued fraction

$$\cfrac{1}{1 - z_1 - \cfrac{\frac{b'}{c'} z_2}{1 - \cfrac{\frac{(c'-b')}{c'(c'+1)} z_2}{1 - z_1 - \cfrac{\frac{(b'+1)c'}{(c'+1)(c'+2)} z_2}{1 - \cfrac{\frac{2(c'-b'+1)}{(c'+2)(c'+3)} z_2}{1 - \ddots}}}}}$$

converges uniformly on every compact subset of the domain (5) to a function $f(\mathbf{z})$ holomorphic in $H_{r,s}$;

(B) the function $f(\mathbf{z})$ is an analytic continuation of $F_2(1, b, b'; b, c'; \mathbf{z})$ in the domain (5).

3 Numerical Experiments

By Corollary 2.3 we have

$$\begin{aligned} \ln\left(1 + \frac{z_2}{1+z_1}\right) &= z_2 F_2(1, b, 1; b, 2; -z_1, -z_2) \\ &= \cfrac{z_2}{1 + z_1 + \cfrac{\frac{1}{2} z_2}{1 + \cfrac{\frac{1}{6} z_2}{1 + z_1 + \cfrac{\frac{1}{3} z_2}{1 + \cfrac{\frac{1}{5} z_2}{1 + \ddots}}}}}. \end{aligned} \tag{13}$$

The branched continued fraction in (13) converges and represents a single-valued branch of the analytic function

$$\ln\left(1 + \frac{z_2}{1+z_1}\right) \tag{14}$$

in the domain

$$\mathrm{H}_s = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ |\arg(z_1 + 1 - s)| < \pi, \ \left|\arg\left(z_2 + \frac{s}{2}\right)\right| < \pi \right\}, \quad 0 < s < 1.$$

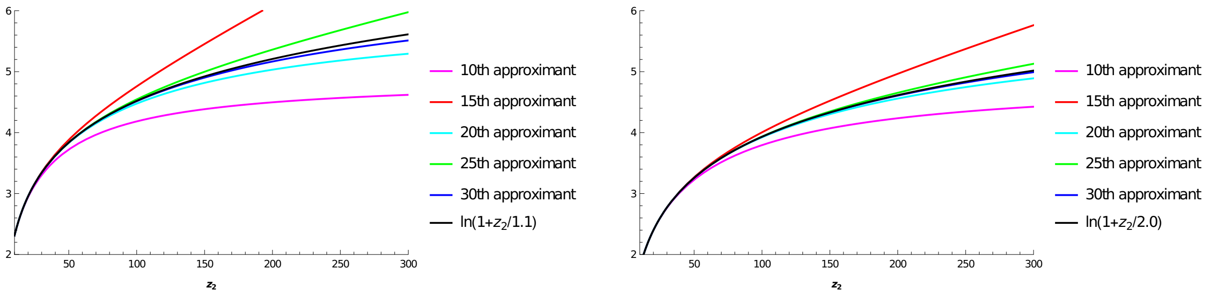
The results of evaluations (13) and

$$\begin{aligned} \ln\left(1 + \frac{z_2}{1+z_1}\right) &= z_2 F_2(1, b, 1; b, 2; -z_1, -z_2) \\ &= -\sum_{r,s=0}^{\infty} \frac{(1)_{r+s}(1)_s}{(2)_s} \frac{(-z_1)^r}{r!} \frac{(-z_2)^{s+1}}{s!} \end{aligned} \tag{15}$$

Table 1: Relative error of 10th partial sum and 10th approximants for $\ln(1+z_2/(1+z_1))$.

z	**(14)**	**(13)**	**(15)**
$(-0.01, 0.1)$	0.096228	2.8844×10^{-16}	2.2613×10^{-13}
$(0.1+0.01i, 0.1+0.01i)$	$0.0871089+0.00757447i$	9.5235×10^{-16}	3.4973×10^{-09}
$(1.0, 1.0)$	0.40546511	1.7002×10^{-12}	$2.9649\times10^{+03}$
$(1+i, 1-i)$	$0.2938933-0.4636476i$	6.9884×10^{-11}	$2.2319\times10^{+02}$
$(3.0, 5.0)$	0.81093022	6.0879×10^{-09}	$8.4668\times10^{+08}$
$(6.0, 30.0)$	1.66500777	1.9580×10^{-05}	$6.9735\times10^{+15}$
$(10+10i, -10-10i)$	$-2.6990814-0.7378151i$	3.6187×10^{-03}	$1.3918\times10^{+11}$
$(10.0, 100.0)$	2.311634929	5.3044×10^{-04}	$1.1109\times10^{+21}$
$(1+100i, 1+100i)$	$0.6930597+0.0049985i$	9.7041×10^{-10}	$2.6863\times10^{+24}$
$(1000, 1000)$	0.6926476	9.6335×10^{-10}	$2.6852\times10^{+35}$
$(1-1000i, 1-1000i)$	$0.6931463-0.0004999i$	9.7152×10^{-10}	$2.6847\times10^{+35}$
$(-1000+1000i, 1000-1000i)$	$-7.2538289-2.3556942i$	4.2875×10^{-01}	$5.3972\times10^{+32}$
$(10000, 10000)$	0.6930972	9.7071×10^{-10}	$2.6848\times10^{+46}$

are displayed in the Table 1. Plots of the values of the nth approximants of (13) are shown in Figure 1 (a)–(b). Here we can see the so-called 'fork property' for a branched continued fraction with positive elements (see [16, p. 29]). That is, the plots of the values of even (odd) approximations of (13) approaches from below (above) to the plot of the function $\ln(1+z_2/(1+z_1))$ at fixed values of z_1. The plots at fixed values of z_2 are similar. Figure 2 (a)–(d) shows the plots where the 10th approximant of (13) guarantees certain truncation error bounds for function $\ln(1+z_2/(1+z_1))$.



(a) At $z_1 = 0.1$.
(b) At $z_1 = 1.0$.

Figure 1: The plots of values of the nth approximants of (13) for $\ln(1+z_2/(1+z_1))$.

One more example, by Corollary 2.3 we obtain

$$\arctan\sqrt{\frac{z_2}{1+z_1}} = \sqrt{z_2(1+z_1)}F_2\left(1,b,\frac{1}{2};b,\frac{3}{2};-z_1,-z_2\right) = \cfrac{\sqrt{z_2(1+z_1)}}{1+z_1+\cfrac{\frac{1}{3}z_2}{1+\cfrac{\frac{4}{15}z_2}{1+z_1+\cfrac{\frac{9}{35}z_2}{1+\cfrac{\frac{16}{63}z_2}{1+\ddots}}}}}, \tag{16}$$

where the branched continued fraction converges and represents a single-valued branch of the analytic function of two variables

$$\arctan\sqrt{\frac{z_2}{1+z_1}} \tag{17}$$

in the domain

$$\mathrm{R}_s = \left\{\mathbf{z}\in\mathbb{C}^2 : \ |\arg(z_1+1-s)|<\pi,\ \left|\arg\left(z_2+\frac{3s}{4}\right)\right|<\pi\right\}, \quad 0<s<1.$$

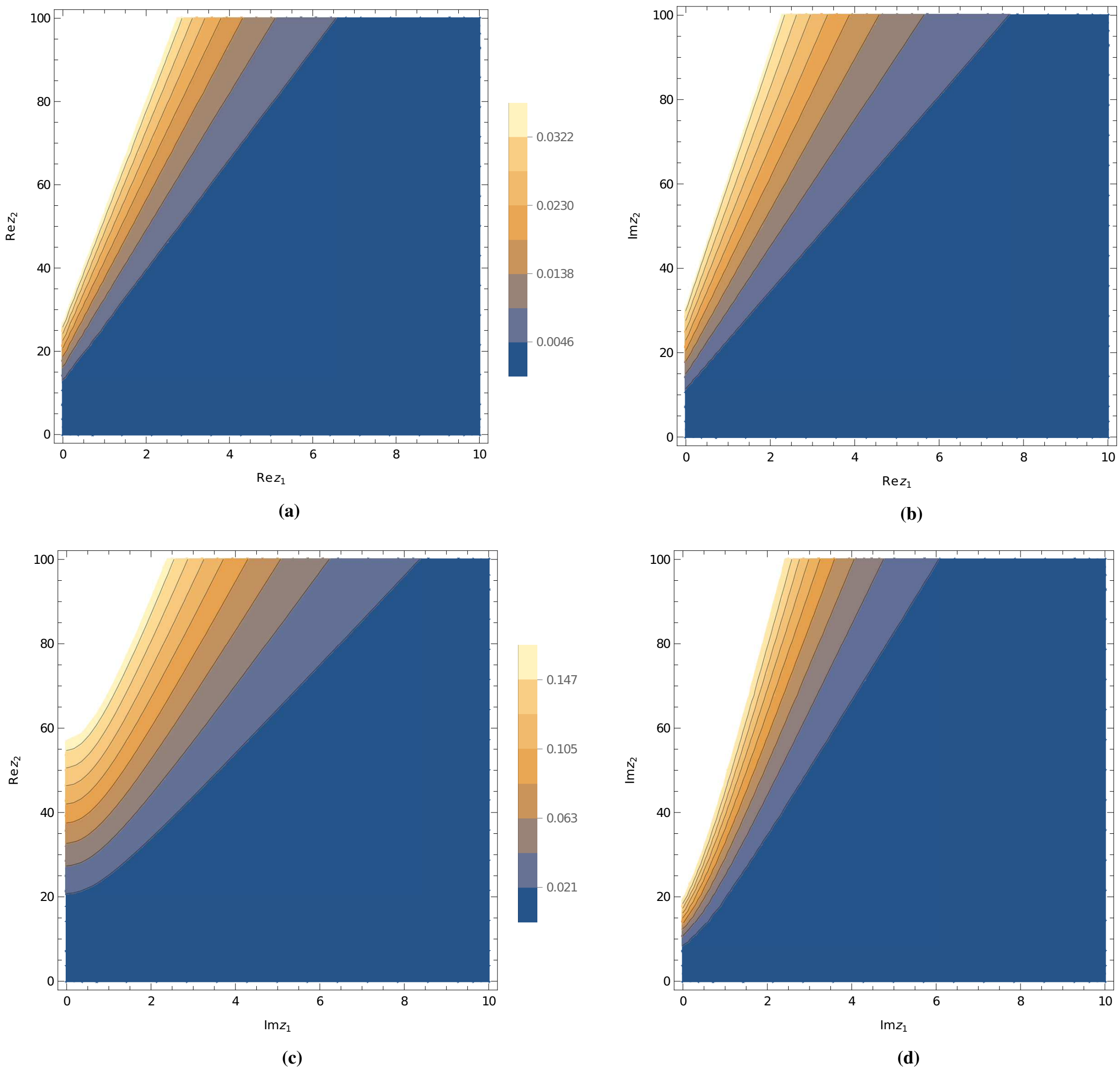


Figure 2: The plots where the 10th approximant of (13) guarantees certain truncation error bounds for $\ln(1+z_2/(1+z_1))$.

The numerical illustration of (16) and

$$\begin{aligned}\arctan\sqrt{\frac{z_2}{1+z_1}} &= \sqrt{z_2(1+z_1)}F_2\left(1,b,\frac{1}{2};b,\frac{3}{2};-z_1,-z_2\right)\\ &= \sqrt{z_2(1+z_1)}\sum_{r,s=0}^{\infty}\frac{(1)_{r+s}(1/2)_s}{(3/2)_s}\frac{(-z_1)^r}{r!}\frac{(-z_2)^s}{s!} \end{aligned}\tag{18}$$

is given in the Table 2. The graphical illustrations of (16) and (17) are given in Figures 3 (a)–(b) and 4 (a)–(d). Here we have results like to the results in the previous example.

Thus, numerical experiments confirmed the expediency and effectiveness of using branched continued fractions as an approximation tool, in particular, of analytic functions,

Calculations and plots were made using Wolfram Mathematica software.

Acknowledgments

This work is part of the Erasmus+ project KA131-2021-1-IT02-KA131-HED-000009659.

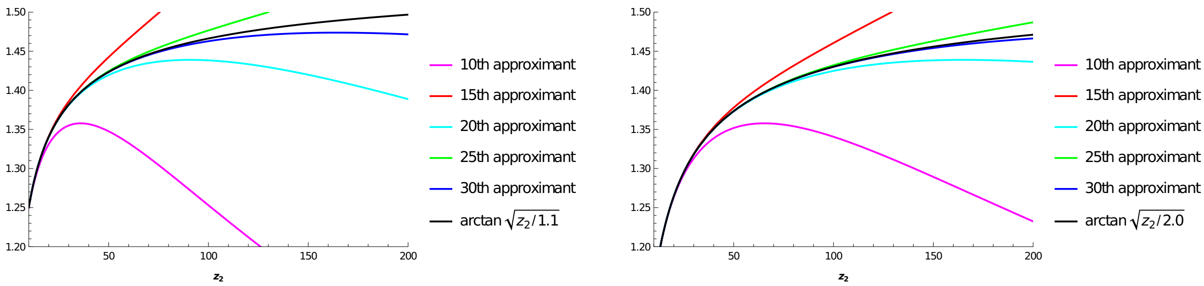
The authors are grateful to anonymous referees for important suggestions and guidelines in order to make this paper more coherent.

References

[1] T. Antonova, R. Dmytryshyn, V. Goran, On the analytic continuation of Lauricella-Saran hypergeometric function $F_K(a_1,a_2,b_1,b_2;a_1,b_2,c_3;\mathbf{z})$. *Mathematics*, 11(21): 4487, 2023.

Table 2: Relative error of 10th partial sum and 10th approximants for arctan $\sqrt{z_2/(1+z_1)}$.

z	(17)	(16)	(18)
$(-0.01, 0.01)$	0.100167	0	1.3855×10^{-16}
$(-0.01, 0.1)$	0.307725	0	1.1545×10^{-13}
$(0.1+0.01i, 0.1+0.01i)$	$0.293289+0.0125413i$	3.7861×10^{-16}	1.9078×10^{-09}
$(0.2, 0.3)$	0.463648	1.4367×10^{-15}	3.2554×10^{-05}
$(0.9, 0.1)$	0.225513	1.2308×10^{-16}	4.9359×10^{-01}
$(-0.001, 0.9)$	0.75932	4.3178×10^{-10}	9.2141×10^{-03}
$(1.0, 1.0)$	0.61547979	1.8029×10^{-12}	$1.5376\times10^{+02}$
$(1+i, 1-i)$	$0.6466485-0.3235375i$	7.3031×10^{-11}	$1.2140\times10^{+02}$
$(4.0, 10.0)$	0.95531662	2.3049×10^{-07}	$1.3529\times10^{+11}$
$(10+10i, -10-10i)$	$0.3806548-2.0302143i$	3.1341×10^{-03}	$5.7608\times10^{+11}$
$(10.0, 100.0)$	1.25055029	7.2305×10^{-04}	$3.5865\times10^{+20}$
$(1+100i, 1+100i)$	$0.7853607+0.0024994i$	1.0618×10^{-09}	$1.3261\times10^{+24}$
$(1000, 1000)$	0.7851483	1.0539×10^{-09}	$1.3257\times10^{+35}$
$(1-1000i, 1-1000i)$	$0.7853978-0.0002499i$	1.0629×10^{-09}	$1.3254\times10^{+35}$
$(-1000+1000i, 1000-1000i)$	$1.1779722-4.3201866i$	3.7200×10^{-01}	$2.7321\times10^{+33}$
$(10000, 10000)$	0.7853732	1.0621×10^{-09}	$1.3254\times10^{+46}$



(a) At $z_1 = 0.1$. **(b)** At $z_1 = 1.0$.

Figure 3: The plots of values of the nth approximants of (16) for arctan $\sqrt{z_2/(1+z_1)}$.

[2] T. Antonova, R. Dmytryshyn, V. Kravtsiv. Branched continued fraction expansions of Horn's hypergeometric function H_3 ratios. *Mathematics*, 9(2): 148, 2021.

[3] T. Antonova, R. Dmytryshyn, I.-A. Lutsiv, S. Sharyn. On some branched continued fraction expansions for Horn's hypergeometric function $H_4(a,b;c,d;z_1,z_2)$ ratios. *Axioms*, 12(3): 299, 2023.

[4] T. Antonova, R. Dmytryshyn, S. Sharyn. Branched continued fraction representations of ratios of Horn's confluent function H_6. *Constr. Math. Anal.*, 6(1): 22–37, 2023.

[5] T. M. Antonova, R. I. Dmytryshyn. Truncation error bounds for branched continued fraction $\sum_{i_1=1}^{N}\frac{a_{i(1)}}{1}+\sum_{i_2=1}^{i_1}\frac{a_{i(2)}}{1}+\sum_{i_3=1}^{i_2}\frac{a_{i(3)}}{1}+\cdots$. *Ukr. Math. J.*, 72(7), 1018–1029, 2020.

[6] T. M. Antonova, R. I. Dmytryshyn. Truncation error bounds for branched continued fraction whose partial denominators are equal to unity. *Mat. Stud.*, 54(1): 3–14, 2020.

[7] P. Appell. Sur les séries hyper géométriques de deux variables et sur des équations différentielles lineaires aux dérivées partielles. *C. R. Acad. Sci. Paris*, 90: 296–298, 731–734, 1880.

[8] P. Appell. Sur les fonctions hyperg´eom´etriques de deux variables. *J. Math. Pures Appl.*, 8(3): 173–216, 1882.

[9] W. N. Bailey. Generalised Hypergeometric Series. Cambridge University Press: Cambridge, 1935.

[10] I. B. Bilanyk. A truncation error bound for some branched continued fractions of the special form. *Mat. Stud.*, 52(2): 115–123, 2019.

[11] D. I. Bodnar, I. B. Bilanyk. On the convergence of branched continued fractions of a special form in angular domains. *J. Math. Sci.*, 246(2): 188–200, 2020.

[12] D. I. Bodnar, I. B. Bilanyk. Parabolic convergence regions of branched continued fractions of the special form. *Carpathian Math. Publ.*, 13(3): 619–630, 2021.

[13] D. I. Bodnar, I. B. Bilanyk. Two-dimensional generalization of the Thron-Jones theorem on the parabolic domains of convergence of continued fractions. *Ukr. Math. J.*, 74(9): 1317–1333, 2023.

[14] D. I. Bodnar, I. B. Bilanyk. Estimation of the rates of pointwise and uniform convergence of branched continued fractions with inequivalent variables. *J. Math. Sci.*, 265(3): 423–437, 2022.

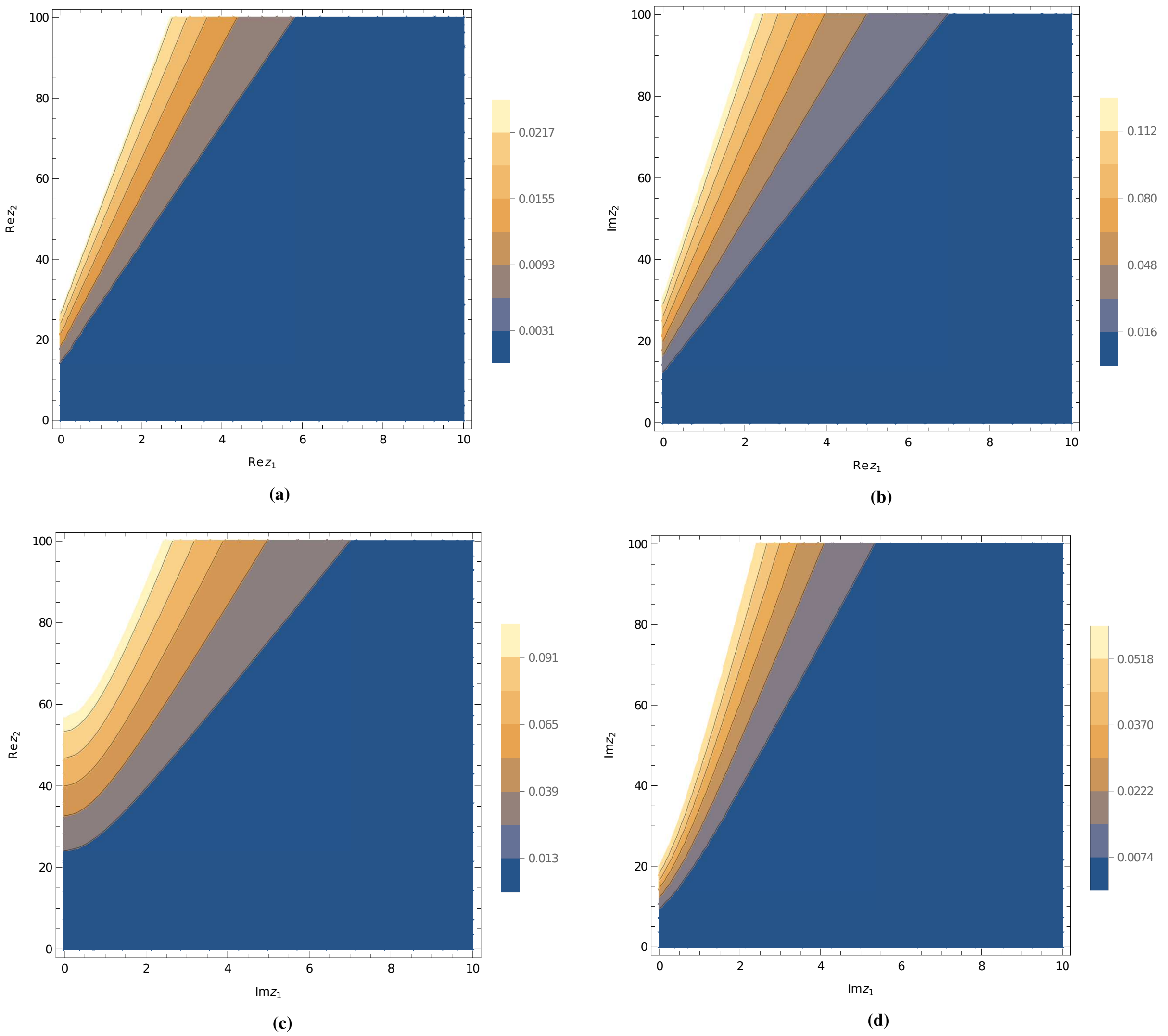


Figure 4: The plots where the 10th approximant of (16) guarantees certain truncation error bounds for arctan $\sqrt{z_2/(1+z_1)}$.

[15] D. I. Bodnar, O. S. Bodnar, I. B. Bilanyk. A truncation error bound for branched continued fractions of the special form on subsets of angular domains. *Carpathian Math. Publ.*, 15(2): 437–448, 2023.

[16] D. I. Bodnar. Branched Continued Fractions. Naukova Dumka: Kyiv, 1986. (In Russian)

[17] O. S. Bodnar, R. I. Dmytryshyn, S. V. Sharyn. On the convergence of multidimensional S-fractions with independent variables. *Carpathian Math. Publ.*, 12(2): 353–359, 2020.

[18] D. I. Bodnar. Expansion of a ratio of hypergeometric functions of two variables in branching continued fractions. *J. Math. Sci.*, 64(5): 1155–1158, 1993.

[19] D. I. Bodnar, O. S. Manzii. Expansion of the ratio of Appel hypergeometric functions F_3 into a branching continued fraction and its limit behavior. *J. Math. Sci.*, 107(1): 3550–3554, 2001.

[20] D. I. Bodnar. Multidimensional C-fractions. *J. Math. Sci.*, 90(5): 2352–2359, 1998.

[21] P. I. Bodnarchuk, V. Y. Skorobogatko. Branched Continued Fractions and Their Applications. Naukova Dumka: Kyiv, 1974. (In Ukrainian)

[22] Yu. A. Brychkov, N. Saad. On some formulas for the Appell function $F_2(a,b,b';c,c';w;z)$. *Integral Transforms Spec. Funct.*, 25(2): 111–123, 2013.

[23] R. Dmytryshyn, C. Cesarano, I.-A. Lutsiv, M. Dmytryshyn. Numerical stability of the branched continued fraction expansion of Horn's hypergeometric function H_4. *Mat. Stud.*, 61(1): 51–60, 2024.

[24] R. I. Dmytryshyn. Convergence of multidimensional A- and J-fractions with independent variables. *Comput. Methods Funct. Theory*, 22(2): 229–242, 2022.

[25] R. Dmytryshyn, I.-A. Lutsiv, M. Dmytryshyn. On the analytic extension of the Horn's hypergeometric function H_4. *Carpathian Math. Publ.*, 16(1): 32–39, 2024.

[26] A. Erdélyi, M. Magnus, F. Oberhettinger, F. G. Tricomi. Higher Transcendental Functions. Vol. 1. McGraw-Hill Book Co.: New York, NY, 1953.

[27] Ch. Ferreira, J. L. López. Asymptotic expansions of the Appell's function F_1. *Q, Appl. Math.*, 62(2): 235–257, 2004.

[28] G. E. Hahne. Analytic continuation of Appell's hypergeometric series F_2 to the vicinity of the singular point $x = 1$, $y = 1$. *J. Math. Phys.*, 10(3): 524–531, 1969.

[29] V. R. Hladun, N. P. Hoyenko, O. S. Manzij, L. Ventyk. On convergence of function $F_4(1,2;2,2;z_1,z_2)$ expansion into a branched continued fraction. *Math. Model. Comput.*, 9(3): 767–778, 2022.

[30] W. B. Jones, W. J. Thron. Continued Fractions: Analytic Theory and Applications. Addison-Wesley Pub. Co.: Reading, Mass., 1980.

[31] J. L. López, P. Pagola, E. P. Sinusia. Asymptotics of the first Appell function F_1 with large parameters II. *Integral Transforms Spec. Funct.*, 24(12): 982–999, 2013.

[32] O. S. Manzii. On the approximation of an Appell hypergeometric function by a branched continued fraction. *J. Math. Sci.*, 90(5): 2376–2380, 1998.

[33] K. Mimachi. Integral representations of Appell's F_2, F_3, Horn's H_2 and Olsson's F_p functions. *Kyushu J. Math.*, 74(1): 1–13, 2020.

[34] K. Sud, L. E. Wright. A new analytic continuation of Appell's hypergeometric series F_2. *J. Math. Phys.*, 17(9): 1719–1721, 1976.

[35] H. S. Wall. Analytic Theory of Continued Fractions. D. Van Nostrand Co.: New York, NY, 1948.

[36] X. Wang. Recursion formulas for Appell functions. *Integral Transforms Spec. Funct.*, 236: 421–433, 2012.