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About Fourier expansions and integral representation of new parametric trigonometric-type *U*-Fubini-Bernoulli polynomials

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Abstract

In this paper, we introduce the parametric trigonometric-type *U*-Fubini-Bernoulli polynomials and the *U*-Fubini-Bernoulli-type numbers. Then, we explain their properties and their relationships to other families of polynomials, such as Pollaczek polynomials, Laguerre polynomials, and the Hermite polynomials. In the same way, we also examine the Fourier expansions and derive the integral representation of this family of polynomials.

Keywords New *U*-Fubini-Bernoulli polynomials · Pollaczek polynomial · Laguerre polynomials L · Fourier expansion · Integral representations · Hermite polynomials

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1 Introduction

Throughout this article, $\mathbb{N}$ will mean the set of natural numbers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, likewise $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$ will denote the set of real numbers, positive real numbers, and the set of complex numbers. The space of all polynomials in one variable with real or complex coefficients is denoted by $\mathbb{P}$, and $\log(\xi)$ denotes the principal value of the multi-valued logarithm function. (see [9, 22]).

It is well known that the classical Bernoulli polynomials, with $x \in \mathbb{R}$, are defined by the following generating function (see [8, 11, 12, 15, 19]) :

$$\frac{\xi e^{\xi x}}{e^{\xi}-1} = \sum_{n=0}^{\infty} B_n(x) \frac{\xi^n}{n!}, \quad (|\xi| < 2\pi). \tag{1}$$

For $x = 0$, the numbers $B_n(0) = B_n$ appear. The generalized Fubini type polynomial $a_n^{(\alpha)}(x)$ of order α are defined by (see [17, 20]):

$$\left(\frac{2}{(2-e^{\xi})^2}\right)^{\alpha} e^{x\xi} = \sum_{n=0}^{\infty} a_n^{(\alpha)}(x) \frac{\xi^n}{n!}, \quad (|\xi| < \ln 2), \tag{2}$$

with $\alpha \in \mathbb{N}$. For $a_n^{(\alpha)}(0) = a_n^{(\alpha)}$, denotes the Fubini type numbers of order α.

Many authors have studied the generation of functions for special polynomials with their congruence properties, recurrence relations, computational formulas, and other properties (cf. [5, 6, 14, 16, 23]). Fourier series and generating function theory are active branches of modern analysis that has gained importance due to its applications in methods of analysis for mathematical solutions to boundary value problems, engineering, and signal processing in communications. Several authors have used the Lipschitz summation formula (see [13]) to introduce the Fourier sequence of several families of polynomials. In [13], by applying the Lipschitz summation formula the author presents the Fourier series of the classical Bernoulli polynomials given in (1) as follows: for $n = 1$, $0 < x < 1$, and $n > 1$, $0 \leq x \leq 1$, we have

$$B_n(x) = -\frac{2\,n!}{(2\pi)} \sum_{k=1}^{\infty} \frac{\cos(2\pi\, k\, x - \frac{n\,\pi}{2})}{k^n}. \tag{3}$$

Another method used is the Cauchy residue theorem. Several authors have alternatively used this method to continue working on Fourier series representations of different families of polynomials, for example, see [3, 4, 7, 10–13, 21]. Recently in [20] the authors presented the Fourier series for the Apostol Fubini-Euler type polynomials, using precisely the Cauchy residue theorem. In the present work, we introduce the new trigonometric-type U-Fubini-Bernoulli polynomials and investigate some of their properties, such as recurrence relations and connection formulas with other polynomial families. On the other hand, we study its Fourier series and integral representation. In the next section, we present some known results that will be used in the rest of the paper. In Sect. 3, we introduce the new family of polynomials, parametric trigonometric-type U-Fubini-Bernoulli polynomials, and the U-Fubini-Bernoulli-type numbers, we study some of their properties. Throughout Sect. 4, we study some connections between these polynomials with other families of polynomials such as Pollaczek polynomials, Laguerre polynomials, and the Hermite polynomials. Finally, in Sect. 5, we develop the Fourier expansion and the integral representation of these new polynomials.

2 Preliminary and basic results

We begin by recalling that the falling factorial n of order k; $\langle n\rangle_k$, is (see [10]):

$$\langle n\rangle_k = n(n-1)\cdots(n-k+1), \quad k \geq 1;\ \langle n\rangle_0 = 1. \tag{4}$$

It's known that the function $v(x) = (\sin(x))^k$ as a real function has a domain

$$\{x \in \mathbb{R} : (\sin(x) \neq 0 \text{ and } k \in \mathbb{Z}) \text{ or } (k \geq 1 \text{ and } k \in \mathbb{Z})\ \mathit{or}\ (\sin(x) \geq 0 \text{ and } k > 0) \text{ or } \sin(x) > 0\}.$$

Also, it has an integral representation given by [1]

$$(\sin(x))^k = \left(x \int_0^1 \cos(t\,x)\,dt\right)^k. \tag{5}$$

The Laplace transform of the function t^n is given by (see, [13, p. 2198 Eq.(3.2)])

$$\int_0^\infty t^n e^{-at} dt = \frac{n!}{a^{n+1}}, \qquad n \in \mathbb{N}_0; \quad \Re(a) > 0. \tag{6}$$

For $n \in \mathbb{N}$, the Pollaczek polynomials $P_n(x;a,b)$ are defined as the coefficients of the following generating function (see [18])

$$f(x,\xi) = f(\cos(\theta),\xi) = \left(1 - \xi\, e^{i\,\theta}\right)^{-\frac{1}{2}+i\,h(\theta)} \left(1 - \xi\, e^{-i\,\theta}\right)^{-\frac{1}{2}-i\,h(\theta)} = \sum_{n=0}^{\infty} P_n(x;a,b)\frac{\xi^n}{n!}, \tag{7}$$

where a and b are real parameters, $a > |b|$, and $h(\theta) = \dfrac{a\,\cos(\theta) + b}{2\,\sin(\theta)}$.

The Laguerre polynomials $L_n^{(\alpha)}(x)$ in the variable x and parameter $\alpha > -1$ are defined by means of the generating function (cf. [18])

$$\mathcal{D}(x,\xi) = \left(\frac{1}{(1-\xi)^{\alpha+1}}\right) \exp\left(\frac{-x\xi}{1-\xi}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\,\frac{\xi^n}{n!}. \tag{8}$$

For $n \in \mathbb{N}$ the Hermite polynomials $H_n(x)$ are defined as the coefficients of the following generating function (cf. [18]):

$$\mathcal{H}(x,\xi) = e^{(2\,x\,\xi-\xi^2)} = \sum_{n=0}^{\infty} H_n(x)\,\frac{\xi^n}{n!}. \tag{9}$$

The particular choices $\alpha = \pm\frac{1}{2}$ in (8) allow the reduction of Hermite polynomials to Laguerre polynomials (cf. [18])

$$H_{2n}(x) = (-1)^n\,2^{2n}\,n!\,L_n^{(-\frac{1}{2})}(x^2), \tag{10}$$

$$H_{2n+1}(x) = (-1)^n\,2^{n+1}\,n!\,x\,L_n^{(\frac{1}{2})}(x^2), \tag{11}$$

$$H_n(x) = \frac{n!}{2\,i\pi}\int \xi^{-n-1}\,\exp(2x\xi-\xi^2)\,d\xi, \quad |\xi|<1. \tag{12}$$

We observed that the functions $\mathfrak{L}(x,\xi) = \exp\left(\frac{x\xi}{1-\xi}\right)$, and $\mathscr{U}(x,\xi) = \exp\left(-2\,x\,\xi+\xi^2\right)$ have the following alternative representation:

$$\mathfrak{L}(x,\xi) = \exp\left(\frac{x\xi}{1-\xi}\right) = 1+\frac{2}{-1+\coth\left(\frac{x\,\xi}{2(1-\xi)}\right)}, \tag{13}$$

$$\mathscr{U}(x,\xi) = \exp\left(-2\,x\,\xi+\xi^2\right) = 1+\frac{2}{-1+\coth\left(\frac{-2x\,\xi+\xi^2}{2}\right)}. \tag{14}$$

The Euler formula is given by [22]

$$e^{ix} = \cos(x)+i\sin(x). \tag{15}$$

3 New parametric trigonometric-type *U*-Fubini-Bernoulli polynomials $\mathfrak{A}_n^{[s]}(x;\psi)$ and some of their properties

We focus our attention first on defining in $\mathbb{P}$ the new parametric trigonometric-type U-Fubini-Bernoulli polynomials, which we denote by $\mathfrak{A}_n^{[s]}(x;\psi)$. Furthermore, we study some of their properties.

Definition 3.1 For $\psi\in\mathbb{R}-\{0\}$, the new parametric trigonometric-type U-Fubini-Bernoulli polynomials $\mathfrak{A}_n^{[s]}(x;\psi)$ in the variable $x\in\mathbb{R}$ are defined by the means of the power series expansion at 0 of the following generating function:

$$\eta(x,\xi;\psi) = \left(\frac{\xi}{2\xi^2-\xi+2}\right)e^{\psi\,\xi\,sin(x)} = \sum_{n=0}^{\infty}\mathfrak{A}_n^{[s]}(x;\psi)\frac{\xi^n}{n!}, \quad (|\xi|<1)\,. \tag{16}$$

A few values of the $\mathfrak{A}_n^{[s]}(x;\psi)$ polynomials can be computed for $n=0,1,2,3,4$, and 5 of (16), as follows:

$$\begin{aligned}
\mathfrak{A}_0^{[s]}(x;\psi) =&0,\\
\mathfrak{A}_1^{[s]}(x;\psi) =&\frac{1}{2},\\
\mathfrak{A}_2^{[s]}(x;\psi) =&\psi\, sin(x) + \frac{1}{2},\\
\mathfrak{A}_3^{[s]}(x;\psi) =&\frac{3\,\psi^2\,(sin(x))^2}{2} + \frac{3\,\psi\, sin(x)}{2} - \frac{9}{4},\\
\mathfrak{A}_4^{[s]}(x;\psi) =&3\,\psi^2\,(sin(x))^2 + 2\,\psi^3\,(sin(x))^3 - 9\,\psi\, sin(x) - \frac{21}{2},\\
\mathfrak{A}_5^{[s]}(x;\psi) =&5\,\psi^3\,(sin(x))^3 - \frac{45\,\psi^2\,(sin(x))^2}{2} + \frac{5\,\psi^4\,(sin(x))^4}{2} - \frac{105\,\psi\, sin(x)}{2} + \frac{75}{4}.
\end{aligned}$$

For $x = 0$, in (16) corresponds to the generating function of the U-Fubini-Bernoulli-type numbers, $\mathfrak{A}_n^{[s]}(\psi) := \mathfrak{A}_n^{[s]} = \mathfrak{A}^{[s]}(0;\mathfrak{A}^{[s]})$ given by

$$\eta(\xi) = \left(\frac{\xi}{2\xi^2 - \xi + 2}\right) = \sum_{n=0}^{\infty} \mathfrak{A}_n^{[s]}(\psi)\frac{\xi^n}{n!}, \quad |\xi| < 1. \tag{17}$$

From (17), we get the following numbers:

$$\begin{aligned}
&\mathfrak{A}_0^{[s]}(\psi) = 0; \quad \mathfrak{A}_1^{[s]}(\psi) = \frac{1}{2}; \quad \mathfrak{A}_2^{[s]}(\psi) = \frac{1}{2};\\
&\mathfrak{A}_3^{[s]}(\psi) = -\frac{9}{4}; \quad \mathfrak{A}_4^{[s]}(\psi) = -\frac{21}{2}; \quad \mathfrak{A}_5^{[s]}(\psi) = \frac{75}{4}.
\end{aligned}$$

The next result is an immediate consequence of Definition 3.1, and (17).

Proposition 3.1 *Let $\psi \in \mathbb{R} - \{0\}$, $n \in \mathbb{N}_0$, and $\{\mathfrak{A}_n^{[s]}(x;\psi)\}_{n\geq 0}$ be a sequence of the new parametric trigonometric-type U-Fubini-Bernoulli polynomials. Then, the following relations hold:*

$$\mathfrak{A}_n^{[s]}(x;\psi) = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{A}_{n-k}^{[s]}(\psi)\psi^k \sin^k(x), \tag{18}$$

$$\mathfrak{A}_n^{[s]}(x;\psi) - \mathfrak{A}_n^{[s]}(\psi) = \sum_{k=2}^{n} \frac{\psi^k \,\langle n\rangle_k\, \mathfrak{A}_{k-1}^{[s]}(\psi)}{\Gamma(k+1)} \left(x\int_0^1 \cos(t\,x)\,dt\right)^k, \quad n \geq 2, \tag{19}$$

with $\langle n\rangle_k$ given in (4).

Proof We have the following:

$$\begin{aligned}\sum_{k=0}^{\infty}\mathfrak{A}_n^{[s]}(x;\psi)\frac{\xi^n}{n!}&=\left(\sum_{n=0}^{\infty}\mathfrak{A}_n^{[s]}(\psi)\frac{\xi^n}{n!}\right)\left(\sum_{n=0}^{\infty}\psi^n\,\sin^n(x)\frac{\xi^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)\psi^k\mathfrak{A}_n^{[s]}(\psi)_{n-k}(\psi)\,\sin^k(x)\,\frac{\xi^n}{n!}.\end{aligned}\tag{20}$$

Because of (20), we obtain (18).

To prove (19), we apply (5), (16), (17), and the Cauchy product rule, which yields

$$\begin{aligned}\sum_{n=0}^{\infty}\left[\mathfrak{A}_n^{[s]}(x;\psi)-\mathfrak{A}_n^{[s]}(\psi)\right]\frac{\xi^n}{n!}&=\left(\sum_{n=1}^{\infty}\mathfrak{A}_n^{[s]}(\psi)\frac{\xi^n}{n!}\right)\left(\sum_{n=1}^{\infty}(\psi\,\sin(x))^n\,\frac{\xi^n}{n!}\right)\\&=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\psi^k\binom{n}{k}\mathfrak{A}_{k-1}^{[s]}(\psi)\,\sin^k(x)\right)\frac{\xi^n}{n!}\\&=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\psi^k\langle n\rangle_k\frac{\mathfrak{A}_{k-1}^{[s]}(\psi)}{k!}\,(\sin(x))^k\right)\frac{\xi^n}{n!},\end{aligned}\tag{21}$$

by comparing coefficients in (21), we obtain (19). □

Proposition 3.2 *Let* $\psi\in\mathbb{R}-\{0\}$, *and* $\{\mathfrak{A}_n^{[s]}(x;\psi)\}_{n\geq 0}$ *be a sequence of the parametric trigonometric-type U-Fubini-Bernoulli polynomials. Then, the following relation holds:*

$$\begin{aligned}\mathfrak{A}_n^{[s]}(x;\psi)=&\sum_{k=1}^{n}\left(\begin{array}{c}n\\k\end{array}\right)\Gamma(n-k+1)\,\psi^k\,(\sin(x))^k i\,\mathbb{U}_n^k\\&+\Gamma(n+1)\,i\,4^{-n}\left(\frac{(1-i\sqrt{15})^n-(1+i\sqrt{15})^n}{\sqrt{15}}\right),\end{aligned}\tag{22}$$

with $\Gamma(n)$, *the gamma function and*

$$\mathbb{U}_n^k=4^{-n+k}\left(\frac{(1-i\sqrt{15})^{n-k}-(1+i\sqrt{15})^{n-k}}{\sqrt{15}}\right).\tag{23}$$

Proof It is noteworthy that the following series representation follows:

$$\left(\frac{\xi}{2\xi^2-\xi+2}\right)=\sum_{n=0}^{\infty}i\,4^{-n}\left(\frac{(1-i\sqrt{15})^n-(1+i\sqrt{15})^n}{\sqrt{15}}\right)\xi^n.\tag{24}$$

Now, starting at (16) and (24), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\mathfrak{A}_n^{[s]}(x;\psi)\frac{\xi^n}{n!} &= \left(\sum_{n=1}^{\infty}\Gamma(n+1)\,i\,4^{-n}\,\left(\frac{(1-i\sqrt{15})^n-(1+i\sqrt{15})^n}{\sqrt{15}}\right)\frac{\xi^n}{n!}\right)\\ &\quad\times\left(1+\sum_{n=1}^{\infty}(\psi\,\sin(x))^n\,\frac{\xi^n}{n!}\right)\\ &=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\left(\begin{array}{c} n\\ k\end{array}\right)\Gamma(n-k+1)\,\psi^k\,(\sin(x))^k\,i\,\mathbb{U}_n^k\right.\\ &\quad\left.+\,\Gamma(n+1)\,i\,4^{-n}\,\left(\frac{(1-i\sqrt{15})^n-(1+i\sqrt{15})^n}{\sqrt{15}}\right)\right)\frac{\xi^n}{n!},\end{aligned} \tag{25}$$

from (25) we arrived at (22) being $\mathbb{U}_n^k$ as in (23). Proposition 3.2 is proven. □

Theorem 3.1 *For every $n\in\mathbb{N}$ and $\psi\in\mathbb{R}-\{0\}$, the new parametric trigonometric-type U-Fubini-Bernoulli polynomials satisfy*

$$\begin{aligned}&(n-1)\,\mathfrak{A}_n^{[s]}(x;\psi)+n\,\psi\,\sin(x)\,\mathfrak{A}_{n-1}^{[s]}(\psi)\\ &\quad=4\sum_{k=1}^{n}\binom{n}{k}\left(k\,\mathfrak{A}_{k-1}^{[s]}(\psi)\,\mathfrak{A}_{n-k}^{[s]}(x;\psi)-\mathfrak{A}_{n-k}^{[s]}(\psi)\,\mathfrak{A}_{k}^{[s]}(x;\psi)\right)\ \ n>1,\end{aligned} \tag{26}$$

$$n\,\mathfrak{A}_n^{[s]}(x;\psi)+\Theta(x,\xi)\,\frac{\partial}{\partial x}\mathfrak{A}_n^{[s]}(x;\psi)-\mathfrak{A}_n^{[s]}(x;\psi)=0;\quad n>2, \tag{27}$$

with

$$\Theta(x,\xi;\psi)=\frac{\sec(x)\,\left(\psi\,(-2\xi^2+\xi-2)\,\sin(x)+(4\xi-1)\right)}{\psi\,(2\xi^2-\xi+2)},\ \ \xi\in\mathbb{C}-\left\{\frac{1}{4}\pm i\,\frac{\sqrt{15}}{4}\right\},$$

and $x\neq\dfrac{(2k+1)\,\pi}{2},\ \ k\in\mathbb{Z}.$

Proof By differentiating both sides of (16) with respect to ξ, we get

$$\begin{aligned}\frac{\partial}{\partial\xi}\eta(x,\xi;\psi) &=\frac{e^{\psi\,\xi\,\sin(x)}}{2\xi^2-\xi+2}-\frac{\xi\,e^{\psi\,\xi\,\sin(x)}\,(4\xi-1)}{(2\xi^2-\xi+2)^2}+\frac{\psi\,\xi\,e^{\psi\,\xi\,\sin(x)}\,\sin(x)}{2\xi^2-\xi+2}\\ &=\sum_{n=0}^{\infty}\mathfrak{A}_n^{[s]}(x;\psi)\,n\,\frac{\xi^{n-1}}{n!}.\end{aligned} \tag{28}$$

So,

$$\frac{\xi\, e^{\psi\,\xi\,\sin(x)}}{2\xi^2-\xi+2}-\frac{\xi\,(4\xi-1)}{2\xi^2-\xi+2}\sum_{n=0}^{\infty}\mathfrak{A}_n^{[s]}(x;\psi)\,\frac{\xi^n}{n!}+\psi\,\sin(x)\sum_{n=0}^{\infty}\mathfrak{A}_n^{[s]}(x;\psi)\frac{\xi^{n+1}}{n!}$$
$$=\sum_{n=0}^{\infty}\mathfrak{A}_n^{[s]}(x;\psi)\,n\,\frac{\xi^n}{n!},$$

in view of the above equation, we have

$$\sum_{n=0}^{\infty}\mathfrak{A}_n^{[s]}(x;\psi)\,\frac{\xi^n}{n!}-\left(4\sum_{n=0}^{\infty}\mathfrak{A}_n^{[s]}(\psi)\frac{\xi^{n+1}}{n!}-\sum_{n=0}^{\infty}\mathfrak{A}_n^{[s]}(\psi)\frac{\xi^n}{n!}\right)\sum_{n=0}^{\infty}\mathfrak{A}_n^{[s]}(x;\psi)\,\frac{\xi^n}{n!}$$
$$=\left(\sum_{n=0}^{\infty}\mathfrak{A}_n^{[s]}(x;\psi)\,n\,\frac{\xi^n}{n!}\right)-\left(\psi\,\sin(x)\sum_{n=0}^{\infty}\mathfrak{A}_n^{[s]}(x;\psi)\frac{\xi^{n+1}}{n!}\right).$$

Hence,

$$\sum_{n=1}^{\infty}\left[(n-1)\mathfrak{A}_n^{[s]}(x;\psi)+\psi\,\sin(x)\,n\,\mathfrak{A}_{n-1}^{[s]}(\psi)\right]\frac{\xi^n}{n!}=\sum_{n=1}^{\infty}\left(4\sum_{n=1}^{n}\binom{n}{k}\mathfrak{A}_{k-1}^{[s]}(\psi)\,k\,\mathfrak{A}_{n-k}^{[s]}(x;\psi)\right.$$
$$\left.-\sum_{k=1}^{n}\binom{n}{k}\mathfrak{A}_{n-k}^{[s]}(\psi)\,\mathfrak{A}_k^{[s]}(x;\psi)\right)\frac{\xi^n}{n!},$$

which proves the result (26).

To prove (27), we differentiate (16) concerning *x* as follows:

$$\frac{\partial}{\partial x}\eta(x,\xi;\psi)=\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\mathfrak{A}_n^{[s]}(x;\psi)\,\frac{\xi^n}{n!},\tag{29}$$

$$\frac{\partial}{\partial x}\eta(x,\xi;\psi)=\frac{\psi\,\xi^2\,\cos(x)e^{\psi\,\xi\,\sin(x)}}{2\xi^2-\xi+2}.\tag{30}$$

Combining (28) with (29) and (30) allows us to write

$$\begin{aligned}&\frac{\partial}{\partial\xi}\eta(x,\xi;\alpha)-\frac{\xi\,e^{\psi\,\xi\,sin(x)}}{\xi(2\xi^2-\xi+2)}+\left(\frac{(4\xi-1)}{\xi\,\psi\,\cos(x)\,(2\xi^2-\xi+2)}\right)\frac{\partial}{\partial x}\eta(x,\xi;\psi)\\&-\frac{\tan(x)}{\xi}\frac{\partial}{\partial x}\eta(x,\xi;\psi)=0.\end{aligned}\tag{31}$$

Therefore, from (31) we have

$$\xi\frac{\partial}{\partial\xi}\eta(x,\xi;\psi)-\left(\frac{(4\xi-1)}{\psi\,(2\xi^2-\xi+2)\,\cos(x)}-\tan(x)\right)\frac{\partial}{\partial x}\eta(x,\xi;\alpha)-\eta(x,\xi;\psi)=0.\tag{32}$$

Hence, from (29) and (32), and after simplifying, we can get

$$\sum_{n=1}^{\infty} n\,\mathfrak{A}_n^{[s]}(x;\psi)\,\frac{\xi^n}{n!} + \sum_{n=1}^{\infty}\left(\frac{(4\xi-1)}{\psi\,\cos(x)\,(2\xi^2-\xi+2)} - \tan(x)\right)\frac{\partial}{\partial x}\mathfrak{A}_n^{[s]}(x;\psi)\frac{\xi^n}{n!}$$
$$-\sum_{n=1}^{\infty}\mathfrak{A}_n^{[s]}(x;\psi)\,\frac{\xi^n}{n!} = 0,$$

and consequently

$$n\,\mathfrak{A}_n^{[s]}(x;\psi) + \left(\frac{\sec(x)\,\left(\psi\,(-2\xi^2+\xi-2)\,\sin(x)+(4\xi-1)\right)}{\psi\,(2\xi^2-\xi+2)}\right)\frac{\partial}{\partial x}\mathfrak{A}_n^{[s]}(x;\psi) \tag{33}$$
$$-\,\mathfrak{A}_n^{[s]}(x;\psi) = 0.$$

In (33) doing $\Theta(x,\xi;\psi) = \left(\frac{\sec(x)\,\left(\psi\,(-2\xi^2+\xi-2)\,\sin(x)+(4\xi-1)\right)}{\psi\,(2\xi^2-\xi+2)}\right)$ follows (27).

Theorem 3.1 is proved. ☐

4 A few connection formulas for the polynomials $\mathfrak{A}_n^{[s]}(x;\psi)$ and the numbers $\mathfrak{A}_n^{[s]}(\psi)$

From definitions (16) and (17), it is possible to draw some relationships between the polynomials $\mathfrak{A}_n^{[s]}(x;\psi)$, the numbers, $\mathfrak{A}_n^{[s]}(\psi)$, and other families of polynomials, such as

Pollaczek polynomials, Laguerre polynomials, and the Hermite polynomials. In this section, we will study these relations. As well as an addition formula for the numbers $\mathfrak{A}_n^{[s]}(\psi)$ is established.

Theorem 4.1 *For every $n\in\mathbb{N}$, and $\psi\in\mathbb{R}-\{0\}$, the U-Fubini-Bernoulli-type numbers, $\mathfrak{A}_n^{[s]}(\psi)$, and the parametric trigonometric-type U-Fubini-Bernoulli polynomials, $\mathfrak{A}_n^{[s]}(x;\psi)$, are related with the Pollaczek polynomials $P_n(x;a,b)$, respectively, by means of the following identities:*

$$\mathfrak{A}_n^{[s]}(\psi) = \frac{\mathcal{V}(\theta)}{\sqrt{15}}\sum_{k=1}^{n}\langle n\rangle_k\, i\,4^{-k}\left((1-i\sqrt{15})^k-(1+i\sqrt{15})^k\right)\,P_{n-k}(x;a,b), \tag{34}$$

where

$$\mathcal{V}(\theta) = \left(\sum_{n=0}^{\infty}\frac{(i\,h(\theta)(\log(\xi)-\log(\gamma)))^n}{n!}\right) \tag{35}$$
$$\times\,(1-|\xi|)^{\frac{1}{2}}(1-|\xi|\,e^{2\,i\,\theta})^{\frac{1}{2}}.$$

And

$$\mathfrak{A}_n^{[s]}(x,\psi) = \mathcal{V}(\theta)\sum_{q=1}^{n}\sum_{k=1}^{q}\left(\begin{array}{c} n \\ q \end{array}\right)\left(\begin{array}{c} q \\ n \end{array}\right)\Gamma(q-k+1)\,(\sin(x))^k\,P_{n-q}(x;a,b)i\,\mathbb{U}_n^k, \tag{36}$$

with $\mathcal{V}(\theta)$ given in (35), and $\mathbb{U}_n^k$ defined in (23).

Proof Using appropriately (7) and (17), we see that

$$\sum_{n=0}^{\infty}\mathfrak{A}_n^{[s]}(\psi)\,\frac{\xi^n}{n!} = \left(\frac{\xi\,\left(1-\xi\,e^{-i\,\theta}\right)^{\frac{1}{2}+i\,h(\theta)}\left(1-\xi\,e^{i\,\theta}\right)^{\frac{1}{2}-i\,h(\theta)}}{(2\xi^2-\xi+2)}\right)\sum_{n=0}^{\infty}P_n(x;a,b)\frac{\xi^n}{n!}. \tag{37}$$

Now if $\xi=|\xi|\,e^{i\,\theta}$, then

$$\begin{aligned}\left(1-\xi\,e^{-i\,\theta}\right)^{\frac{1}{2}+i\,h(\theta)}\left(1-\xi\,e^{i\,\theta}\right)^{\frac{1}{2}-i\,h(\theta)} =&(1-|\xi|)^{\frac{1}{2}}(1-|\xi|)^{i\,h(\theta)}(1-|\xi|e^{2i\,\theta})^{\frac{1}{2}} \\ &\times(1-|\xi|e^{2i\,\theta})^{-i\,h(\theta)},\end{aligned} \tag{38}$$

with $|\xi|\neq 1,0$, and $\theta\neq\frac{1}{2}(2\pi\,n+i\log|\xi|),\ n\in\mathbb{Z}$.

Note that using (15), we can deduce

$$\begin{aligned}(1-|\xi|)^{i\,h(\theta)} =&e^{i\,h(\theta)\log(1-|\xi|)}, \\ (1-|\xi|\,e^{2\,i\,\theta}) =&\left[(1-|\xi|\cos(2\theta))-i|\xi|\sin(2\theta)\right], \\ |1-|\xi|\,e^{2\,i\,\theta}| =&\sqrt{(1-2|\xi|\cos(2\theta))+|\xi|^2}, \\ (1-|\xi|e^{2i\,\theta})^{-i\,h(\theta)} =&\frac{1}{e^{i\,h(\theta)\log(\gamma)}},\end{aligned}$$

with

$$\gamma=(1-|\xi|\,e^{2\,i\,\theta}),\ \text{and}\ \log(\gamma)=\ln|\gamma|+i\,Arg(\gamma).$$

Also,

$$\xi=1-|\xi|,\ \text{and}\ \log(\xi)=\ln|\xi|+i\,Arg(\xi).$$

From the above expressions in (38), we receive

$$\begin{aligned}\left(1-\xi\,e^{-i\,\theta}\right)^{\frac{1}{2}+i\,h(\theta)}\left(1-\xi\,e^{i\,\theta}\right)^{\frac{1}{2}-i\,h(\theta)} =&\frac{(1-|\xi|)^{\frac{1}{2}}e^{i\,h(\theta)\log(1-|\xi|)}(1-|\xi|\,e^{2\,i\,\theta})^{\frac{1}{2}}}{e^{i\,h(\theta)\log(\gamma)}} \\ =&\left(\sum_{n=0}^{\infty}\frac{(i\,h(\theta)(\log(\xi)-\log(\gamma)))^n}{n!}\right) \\ &\times(1-|\xi|)^{\frac{1}{2}}(1-|\xi|\,e^{2\,i\,\theta})^{\frac{1}{2}}.\end{aligned}$$

Denoted by $\mathcal{V}(\theta)$, the above expression, according to (35), and using (24), and (37), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\mathfrak{A}_n^{[s]}(\psi)\,\frac{\xi^n}{n!} &= \left(\frac{(1-|\xi|)^{\frac{1}{2}}\,e^{i\,h(\theta)\log(1-|\xi|)}(1-|\xi|\,e^{2\,i\,\theta})^{\frac{1}{2}}}{e^{i\,h(\theta)\log(\gamma)}}\right)\left\{\left(\sum_{n=1}^{\infty}P_n(x;a,b)\frac{\xi^n}{n!}\right)\right.\\ &\quad\times\left.\left(\sum_{k=1}^{\infty}\left(\frac{n!\,i\,4^{-n}\left((1-i\sqrt{15})^n-(1+i\sqrt{15})^n\right)}{\sqrt{15}}\right)\frac{\xi^n}{n!}\right)\right\}\\ &=\mathcal{V}(\theta)\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{n}\binom{n}{k}\,k!\left(\frac{i\,4^{-k}\left((1-i\sqrt{15})^k-(1+i\sqrt{15})^k\right)}{\sqrt{15}}\right)\right.\\ &\quad\times\left.(P_{n-k}(x;a,b))\,\frac{\xi^n}{n!}\right\}.\end{aligned}\tag{39}$$

Now in (39), using the falling factorial defined in (4), we obtain (34). The result (36) can be similarly proved, using appropriately (7), (16), (24), and the Cauchy product rule. Also taking into account (23), and (35). This completes the proof of the Theorem 4.1. ☐

Theorem 4.2 *For $n \in \mathbb{N}$, the following summation formula for the U-Fubini-Bernoulli-type numbers, $\mathfrak{A}_n^{[s]}(\psi)$, hold true:*

$$\sum_{q=0}^{n}(-1)^q\langle n\rangle_q\binom{-1-\alpha}{q}\mathfrak{A}_{n-q}^{[s]}(\psi)=\sum_{l=0}^{n}\binom{n}{l}\mathfrak{A}_{n-l}^{[s]}(\psi)L_l^{(\alpha)}(x)(1+\mathfrak{L}_1(x,\xi));\ \ n>2,\tag{40}$$

with $L_n^{(\alpha)}(x)$, the Laguerre polynomials given in (8), and

$$\mathfrak{L}_1(x,\xi)=\frac{2}{-1+\coth\left(\frac{x\,\xi}{2(1-\xi)}\right)},\quad \xi\neq 1.\tag{41}$$

Proof Taking (8), (13), and (17), we obtain the following results:

$$\begin{aligned}&\left(\sum_{n=0}^{\infty}\mathfrak{A}_n^{[s]}(\psi)\,\frac{\xi^n}{n!}\right)\left(\sum_{n=0}^{\infty}n!\,(-1)^n\binom{-1-\alpha}{n}\frac{\xi^n}{n!}\right)\\ &\quad=\mathfrak{L}(x,\xi)\left(\sum_{n=0}^{\infty}\mathfrak{A}_n^{[s]}(\psi)\,\frac{\xi^n}{n!}\right)\\ &\qquad\times\left(\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)\,\frac{\xi^n}{n!}\right)\\ &\quad\Leftrightarrow\sum_{n=0}^{\infty}\sum_{l=0}^{n}\left(\binom{n}{l}\mathfrak{A}_{n-l}^{[s]}(\psi)L_l^{(\alpha)}(x)+\binom{n}{l}\mathfrak{A}_{n-l}^{[s]}L_l^{(\alpha)}(x)\left(\frac{2}{-1+\coth\left(\frac{x\,\xi}{2(1-\xi)}\right)}\right)\right)\frac{\xi^n}{n!}\\ &\quad=\sum_{n=0}^{\infty}\left(\sum_{q=0}^{n}(-1)^q\binom{n}{q}\,q(q-1)!\,\mathfrak{A}_{n-q}^{[s]}(\psi)\binom{-1-\alpha}{q}\right)\frac{\xi^n}{n!}\\ &\quad\Leftrightarrow\sum_{q=0}^{n}(-1)^q\langle n\rangle_q\binom{-1-\alpha}{q}\mathfrak{A}_{n-q}^{[s]}(\psi)=\sum_{l=0}^{n}\binom{n}{l}\mathfrak{A}_{n-l}^{[s]}(\psi)L_l^{(\alpha)}(x)(1+\mathfrak{L}_1(x,\xi)),\end{aligned}\tag{42}$$

in (42), we have taken into account $\mathfrak{L}_1(x,\xi)$ given in (41), and (4), so we arrive at (40), which completes the proof of Theorem 4.2. □

Theorem 4.3 *For $\psi \in \mathbb{R}-\{0\}$, the parametric trigonometric-type U-Fubini-Bernoulli polynomials, $\mathfrak{A}_n^{[s]}(x;\psi)$, are related with the Hermite polynomials $H_n(x)$, by means of the following identities:*

$$\mathfrak{A}_n^{[s]}(x;\psi) = (1+\mathscr{U}_1(x,\xi)) \sum_{j=1}^{n}\sum_{k=1}^{j} \binom{n}{j}\binom{j}{k}\, \psi^k\, \mathfrak{A}_{j-k}^{[s]}(\psi)\, \sin^k(x)\, H_{n-j}(x), \tag{43}$$

where

$$\mathscr{U}_1(x,\xi) = \frac{2}{-1+\coth\left(\frac{-2x\,\xi+\xi^2}{2}\right)}. \tag{44}$$

Proof Applying (9), (14), (16), and (17), we get

$$\begin{aligned}
\left(\sum_{n=1}^{\infty}\mathfrak{A}_n^{[s]}(x;\psi)\,\frac{\xi^n}{n!}\right) =& \mathscr{U}(x,\xi)\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\binom{n}{k}\mathfrak{A}_{n-k}^{[s]}(\psi)\,\psi^k\,\sin^k(x)\frac{\xi^n}{n!}\right)\\
&\times\left(\sum_{n=1}^{\infty}H_n(x)\,\frac{\xi^n}{n!}\right)\\
&\Leftrightarrow \sum_{n=1}^{\infty}\mathfrak{A}_n^{[s]}(x;\psi)\,\frac{\xi^n}{n!} = (1+\mathscr{U}_1(x,\xi))\sum_{n=1}^{\infty}\sum_{j=1}^{n}\left(\sum_{k=1}^{j}\binom{n}{j}\binom{j}{k}\mathfrak{A}_{j-k}^{[s]}(\psi)\,\psi^k\,\sin^k(x)\,H_{n-j}(x)\right)\frac{\xi^n}{n!}\\
&\Leftrightarrow \sum_{n=1}^{\infty}\mathfrak{A}_n^{[s]}(x;\psi)\,\frac{\xi^n}{n!} = \sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\sum_{k=1}^{j}\binom{n}{j}\mathfrak{A}_{j-k}^{[s]}(\psi)\,\psi^k\,\sin^k(x)\,H_{n-j}(x)\right.\\
&\left.\times\left((1+\mathscr{U}_1(x,\xi))\binom{j}{k}\right)\right)\frac{\xi^n}{n!},
\end{aligned} \tag{45}$$

in the above expression, we have considered $\mathscr{U}_1(x,\xi)$ as in (44), so (45) leads to (43). Theorem 4.3 is proved. □

Taking into account relations (10), (11), and (12) and making the corresponding modifications in (43), we obtain

Corollary 4.1 *For $\psi \in \mathbb{R}-\{0\}$ and $n \in \mathbb{N}$, the parametric trigonometric-type U-Fubini-Bernoulli polynomials, $\mathfrak{A}_n^{[s]}(x;\psi)$, are related to the Laguerre polynomials with parameters $\alpha = \pm\frac{1}{2}$ by means of the following identities.*

$$\begin{aligned}
\mathfrak{A}_{2n+j}^{[s]}(x;\psi) =&(1+\mathscr{U}_1(x,\xi))\sum_{j=1}^{2n+j}\sum_{k=1}^{j}\left\{(-1)^n\,2^{2n}\,n!\binom{2n+j}{j}\psi^k\,\sin^k(x)\right.\\
&\left.\times\binom{j}{k}\mathfrak{A}_{j-k}^{[s]}(\psi)\,L_n^{(-\frac{1}{2})}(x^2)\right\},
\end{aligned} \tag{46}$$

$$\begin{aligned}
\mathfrak{A}_{2n+j+1}^{[s]}(x;\psi) =&(1+\mathscr{U}_1(x,\xi))\sum_{j=1}^{2n+j+1}\sum_{k=1}^{j}\left\{(-1)^n\,2^{2n+1}\,n!\,x\binom{2n+1+j}{j}\psi^k\,\sin^k(x)\right.\\
&\left.\times\binom{j}{k}\mathfrak{A}_{j-k}^{[s]}(\psi)\,L_n^{(\frac{1}{2})}(x^2)\right\},
\end{aligned} \tag{47}$$

$$\mathfrak{A}^{[s]}_{n+j}(x;\psi) = -\left(1+\mathscr{U}_1(x,\xi)\right)\sum_{j=1}^{n+j}\sum_{k=1}^{j}\left\{\frac{i\,\Gamma(n+1)}{2\pi}\left(\begin{array}{c} n+j \\ j \end{array}\right)\psi^k\,\sin^k(x)\right. \tag{48}$$

$$\left.\times\left(\begin{array}{c} j \\ k \end{array}\right)\mathfrak{A}^{[s]}_{j-k}(\psi)\int \xi^{-n-1}\,e^{2x\xi-\xi^2}\,d\xi\right\}. \tag{49}$$

5 Fourier expansions and integral representations for the trigonometric-type *U*-Fubini-Bernoulli polynomials of parameter ψ

To arrive at the Fourier series of the polynomials $\mathfrak{A}^{[s]}_n(x;\psi)$, we consider the function

$$\eta_n(\xi) = \left(\frac{\xi}{2\xi^2-\xi+2}\right)\frac{e^{\psi\,\xi\,sin(x)}}{\xi^{n+1}}, \tag{50}$$

this function has a pole of order $n+1$ at $\xi = 0$, and single poles in

$$\xi_1 = \frac{1}{4}+i\,\frac{\sqrt{15}}{4}, \quad \text{and} \quad \frac{1}{4}-i\,\frac{\sqrt{15}}{4}.$$

We then integrate $\eta_n(\xi)$ around the circle Υ_N with radius $\mathcal{R} = \frac{1}{2}(N+\varrho)\pi$, $0<\varrho<1$. It is ensured that the circle Υ_N does not pass any of the poles ξ_1 and ξ_2.

We prove the following lemma, needed to establish the Fourier expansion of the new parametric trigonometric-type *U*-Fubini-Bernoulli polynomials.

Lemma 5.1 *Let Υ_N be a circle centered about the origin, positively-oriented with radius $\frac{1}{2}(N+\varrho)\pi$, $N\in\mathbb{N}$, $\varrho\in\mathbb{R}$, fixed such that $0<\varrho<1$. Then for $0<\psi\leq\frac{1}{|\Re(\xi)|}$, and $n\in\mathbb{N},\ n\geq 1$*

$$\lim_{N\to\infty}\int_{\Upsilon_N}\eta_n(\xi)\,d\xi = \lim_{N\to\infty}\int_{\Upsilon_N}\left(\frac{\xi}{2\xi^2-\xi+2}\right)\frac{e^{\psi\,\xi\,sin(x)}}{\xi^{n+1}}\,d\xi = 0. \tag{51}$$

Proof Applying the module, we can estimate the integral on Υ_N as follows:

$$\left|\int_{\Upsilon_N}\left(\frac{\xi}{2\xi^2-\xi+2}\right)\frac{e^{\psi\,\xi\,sin(x)}}{\xi^{n+1}}\,d\xi\right| \leq \sup_{\xi\in\Upsilon_N}\left|\left(\frac{\xi}{2\xi^2-\xi+2}\right)\frac{e^{\psi\,\xi\,sin(x)}}{\xi^{n+1}}\right|\left[\frac{1}{2}(N+\varrho)\pi\right]2\pi, \tag{52}$$

with $|\xi| = \mathcal{R} = \frac{1}{2}(N+\varrho)\pi$.

Note that, for all $\xi\in\Upsilon_N$ with $\xi = u+i\,v$, $\Re(\xi) = u$, and $\Im(\xi) = v$, $|\Re(\xi)| = |\Im(\xi)| = \mathcal{R}$, we get

$$\left|e^{\psi\,\xi\,\sin(x)}\right| = \left|e^{\psi\,(u+iv)\,\sin(x)}\right| = e^{\psi\,u\,\sin(x)} = e^{\psi\,|\xi|\cos(\theta)\,\sin(x)}, \text{ with } \theta \text{ argument of } \xi. \tag{53}$$

Also,

$$\begin{aligned}\left|2\xi^2-\xi+2\right| &= \left|2(\xi^2+1)-\xi\right| \geq \left||2(\xi^2+1)|-|\xi|\right| \\ &\geq \left|2(|\xi|^2-1)-|\xi|\right| = |2\mathcal{R}^2-\mathcal{R}-2| \\ &\geq \frac{\pi(N+\varrho)(\pi(N+\varrho)-1)-4}{2}.\end{aligned} \tag{54}$$

Then, using (53) and (54), we have

$$\frac{\left|\xi\, e^{\psi\,\xi\, sen(x)}\right|}{|2\xi^2-\xi+2|} \leq \frac{2|\xi|e^{\psi\,|\xi|\,\cos(\theta)\,\sin(x)}}{\pi(N+\varrho)(\pi(N+\varrho)-1)-4} \leq \left(\frac{4\,e^{\psi\,|\xi|\,\cos(\theta)\sin(x)}}{(\pi(N+\varrho)-1)-\dfrac{4}{\pi(N+\varrho)}}\right)\cdot \tag{55}$$

By replacing (55) in (52), we get

$$\begin{aligned}\left|\int_{\Upsilon_N}\left(\frac{\xi}{2\xi^2-\xi+2}\right)\frac{e^{\psi\,\xi\,sin(x)}}{\xi^{n+1}}\,d\xi\right| &\leq \left(\frac{4\,e^{\psi\,|\xi|\,\cos(\theta)\sin(x)}}{(\pi(N+\varrho)-1)-\dfrac{4}{\pi(N+\varrho)}}\right)\frac{2\pi}{|\xi|^n} \\ &\leq \left(\frac{e^{\psi\,|\Re(\xi)|\,\sin(x)}\,2^{n+2}\pi}{((N+\varrho)\pi-1)-\dfrac{4}{(N+\varrho)\pi}}\right)\frac{1}{[(N-\varrho)\pi]^n} \\ &\leq \left(\frac{e^{\sin(x)}\,2^{n+2}\pi}{((N+\varrho)\pi-1)-\dfrac{4}{(N+\varrho)\pi}}\right)\frac{1}{[(N-\varrho)\pi]^n} \\ &\leq \left(\frac{e\,2^{n+2}\pi}{((N+\varrho)\pi-1)-\dfrac{4}{(N+\varrho)\pi}}\right)\frac{1}{[(N-\varrho)\pi]^n}.\end{aligned} \tag{56}$$

Now, (56) goes to 0 as $N\to\infty$ for $n\geq 1$. The proof of the lemma 5.1 is complete. □

We are now going to talk about the paper's main results, which are the Fourier series expansion and the integral representation of the new parametric trigonometric-type U-Fubini-Bernoulli polynomials. The following theorem contains such an expansion of the Fourier series.

Theorem 5.2 For $n\in\mathbb{N}$, $n\geq 1$, and $0<\psi\leq\frac{1}{|\Re(\xi)|}$. Then

$$\begin{aligned}\mathfrak{A}_n^{[s]}(x;\psi) =&\frac{2\,n!\,e^{i\pi}}{\sqrt{15}}\left[\sin\left(\frac{\psi}{4}\sqrt{15}\sin(x)-n\,\arctan(\sqrt{15})\right)\right]\\&\times\left(\sum_{k=0}^{\infty}\frac{4^{-1-k}(\psi\,\sin(x))^{2k}(4+8\,k+\psi\,\sin(x))}{(1+2k)!}\right)\\=&\frac{2\,n!\,e^{i\pi}}{\sqrt{15}}\left(\sin\left(\frac{\psi}{4}\sqrt{15}\sin(x)-n\,\arctan(\sqrt{15})\right)\right)\\&\times\left(\sum_{k=0}^{\infty}\frac{4^{-k}(\psi\,\sin(x))^{k}}{k!}\right).\end{aligned}\tag{57}$$

Proof Let $\eta_n(\xi)$ be the function defined as in (50), we start with $\int_{\Upsilon_N}\eta_n(\xi)\,d\xi$, over the circle $\Upsilon_N=\left\{\xi:|\xi|=\frac{1}{2}(N+\varrho)\pi; N\in\mathbb{N}, \varrho\in\mathbb{R},\ 0<\varrho<1\right\}$. We now consider the poles of the function $\eta_n(\xi)$ as follows:

$$\xi_k=\frac{1}{4}\left(1+(-1)^k i\sqrt{15}\right),\ \ k=1,2,$$

and $\xi=0$ is a pole of order $n+1$. The poles ξ_k are simple poles. We now turn to the application of the Cauchy residue theorem. It follows that

$$\int_{\Upsilon_N}\eta_n(\xi)\,d\xi=2\pi\,i\left(\mathfrak{R}es(\eta_n(\xi),\xi=0)+\sum_{k=1,2}\mathfrak{R}es(\eta_n(\xi),\xi=\xi_k)\right),\tag{58}$$

where the sum is over all those poles of $\eta_n(\xi)$ that lie inside Υ_N.

We evaluate $\mathfrak{R}es(\eta_n(\xi),\xi=0)$ and $\mathfrak{R}es(\eta_n(\xi),\xi=\xi_k)$ as follows:

$$\begin{aligned}\mathfrak{R}es(\eta_n(\xi),\xi=0)=&\frac{1}{n!}\lim_{\xi\to 0}\frac{d^n}{d\xi^n}\left(\sum_{m=0}^{\infty}\mathfrak{A}_m^{[s]}(x;\psi)\frac{\xi^m}{m!}\right)\\=&\frac{1}{n!}\lim_{\xi\to 0}\left(\sum_{m=0}^{\infty}\mathfrak{A}_m^{[s]}(x;\psi)\frac{\xi^{m-n}}{(m-n)!}\right)\\=&\frac{1}{n!}\lim_{\xi\to 0}\left(\sum_{m=n}^{\infty}\mathfrak{A}_m^{[s]}(x;\psi)\frac{\xi^{m-n}}{(m-n)!}\right)=\frac{1}{n!}\mathfrak{A}_n^{[s]}(x;\psi).\end{aligned}\tag{59}$$

On the other hand, for $\xi=\xi_1$, we get

$$
\begin{aligned}
\mathfrak{Res}(\eta_n(\xi),\xi=\xi_1) &= \lim_{\xi\to\xi_1}\frac{e^{\psi\,\xi\sin(x)}}{2(\xi-\xi_2)\xi^n} = \lim_{\xi\to\left(\frac{1}{4}+i\frac{\sqrt{15}}{4}\right)}\frac{e^{\psi\,\xi\sin(x)}}{2\left(\xi-\left(\frac{1}{4}-i\frac{\sqrt{15}}{4}\right)\right)\xi^n}\\
&=\frac{-\left(i2^{2n}\sqrt{15}\,e^{\psi\ \sin(x)\left(\frac{1}{4}+i\frac{15}{4}\right)}\right)}{15\,(1+i\sqrt{15})^n}\\
&=\frac{-i\sqrt{15}\,e^{\psi\ \sin(x)\left(\frac{1}{4}+i\frac{15}{4}\right)}}{15\,e^{i\,n\ \arctan(\sqrt{15})}}\\
&=-\,i\frac{\sqrt{15}\,e^{\frac{\psi}{4}\ \sin(x)}}{15}\ \left(\frac{e^{i\left(\frac{\psi}{4}\sqrt{15}\ \sin(x)\right)}}{e^{i\,n\ \arctan(\sqrt{15})}}\right).
\end{aligned}
\tag{60}
$$

Also, for $\xi=\xi_2$, we have

$$
\begin{aligned}
\mathfrak{Res}(\eta_n(\xi),\xi=\xi_2) &= \lim_{\xi\to\xi_2}\frac{e^{\psi\,\xi\sin(x)}}{2(\xi-\xi_1)\xi^n} = \lim_{\xi\to\left(\frac{1}{4}-i\frac{\sqrt{15}}{4}\right)}\frac{e^{\psi\,\xi\sin(x)}}{2\left(\xi-\left(\frac{1}{4}+i\frac{\sqrt{15}}{4}\right)\right)\xi^n}\\
&=\frac{\left(i2^{2n}\sqrt{15}\,e^{-\psi\ \sin(x)\left(-\frac{1}{4}+i\frac{\sqrt{15}}{4}\right)}\right)}{15\,(1-i\sqrt{15})^n}\\
&=\frac{i\sqrt{15}\,e^{-\psi\ \sin(x)\left(-\frac{1}{4}+i\frac{\sqrt{15}}{4}\right)}}{15\,e^{i\,n\ \arctan(-\sqrt{15})}}\\
&=i\frac{\sqrt{15}\,e^{\frac{\psi}{4}\ \sin(x)}}{15}\ \left(\frac{e^{-i\left(\frac{\psi}{4}\sqrt{15}\ \sin(x)\right)}}{e^{i\,n\ \arctan(-\sqrt{15})}}\right).
\end{aligned}
\tag{61}
$$

Hence, by (58), (59), (60), and (61), we find that

$$
\begin{aligned}
\int_{\Upsilon_N}\eta_n(\xi)\,d\xi =&2\pi\,i\left(\frac{1}{n!}\mathfrak{A}_n^{[s]}(x;\psi)-i\frac{e^{\frac{\psi}{4}\ \sin(x)}}{\sqrt{15}}\left(\frac{e^{i\left(\frac{\psi}{4}\ \sqrt{15}\ \sin(x)\right)}}{e^{i\,n\ \arctan(\sqrt{15})}}\right)\right.\\
&\left.+\,i\frac{e^{\frac{\psi}{4}\ \sin(x)}}{\sqrt{15}}\left(\frac{e^{-i\left(\frac{\psi}{4}\ \sqrt{15}\ \sin(x)\right)}}{e^{i\,n\ \arctan(-\sqrt{15})}}\right)\right)\\
=&2\pi\,i\frac{1}{n!}\mathfrak{A}_n^{[s]}(x;\psi)+\frac{2\,\pi\,e^{\frac{\psi}{4}\ \sin(x)}}{\sqrt{15}}\left(\left(\frac{e^{i\left(\frac{\psi}{4}\ \sqrt{15}\ \sin(x)\right)}}{e^{i\,n\ \arctan(\sqrt{15})}}\right)\right.\\
&\left.-\left(\frac{e^{-i\left(\frac{\psi}{4}\ \sqrt{15}\ \sin(x)\right)}}{e^{i\,n\ \arctan(-\sqrt{15})}}\right)\right).
\end{aligned}
$$

So, taking $N\to\infty$, we can use the Lemma 5.1, this yields

$$\begin{aligned}
\mathfrak{A}_n^{[s]}(x;\psi) =& \frac{1}{i} n!\, \frac{e^{\frac{\psi}{4}\,\sin(x)}}{\sqrt{15}} \left(\left(\frac{e^{-i\left(\frac{\psi}{4}\,\sqrt{15}\,\sin(x)\right)}}{e^{i\,n\,\arctan(-\sqrt{15})}} \right) - \left(\frac{e^{i\left(\frac{\psi}{4}\,\sqrt{15}\,\sin(x)\right)}}{e^{i\,n\,\arctan(\sqrt{15})}} \right) \right) \\
=& n!\, i\, \frac{e^{\frac{\psi}{4}\,\sin(x)}}{\sqrt{15}} \left(\left(\frac{e^{i\left(\frac{\psi}{4}\,\sqrt{15}\,\sin(x)\right)}}{e^{i\,n\,\arctan(\sqrt{15})}} \right) - \left(\frac{e^{-i\left(\frac{\psi}{4}\,\sqrt{15}\,\sin(x)\right)}}{e^{i\,n\,\arctan(-\sqrt{15})}} \right) \right) \\
=& n!\, i\, \frac{e^{\frac{\psi}{4}\,\sin(x)}}{\sqrt{15}} \left(e^{i\left(\left(\frac{\psi}{4}\,\sqrt{15}\,\sin(x)\right)-\left(n\,\arctan(\sqrt{15})\right)\right)} - e^{-i\left(\left(\frac{\psi}{4}\,\sqrt{15}\,\sin(x)\right)+\left(n\,\arctan(-\sqrt{15})\right)\right)} \right) \\
=& n!\, i\, \frac{e^{\frac{\psi}{4}\,\sin(x)}}{\sqrt{15}} \left(e^{i\left(\left(\frac{\psi}{4}\,\sqrt{15}\,\sin(x)\right)-\left(n\,\arctan(\sqrt{15})\right)\right)} - e^{i\left(\left(-\frac{\psi}{4}\,\sqrt{15}\,\sin(x)\right)+\left(n\,\arctan(\sqrt{15})\right)\right)} \right),
\end{aligned}$$

therefore

$$\mathfrak{A}_n^{[s]}(x;\psi) = \frac{n!\, e^{i\pi/2}}{\sqrt{15}} \left(e^{i\left(\left(\frac{\psi}{4}\,\sqrt{15}\,\sin(x)\right)-\left(n\,\arctan(\sqrt{15})\right)\right)} - e^{i\left(\left(-\frac{\psi}{4}\,\sqrt{15}\,\sin(x)\right)+\left(n\,\arctan(\sqrt{15})\right)\right)} \right) e^{\frac{\psi}{4}\,\sin(x)}.$$

In the above expression by applying (15) and simplifying we arrive at

$$\mathfrak{A}_n^{[s]}(x;\psi) = \frac{2\, n!\, e^{i\pi}}{\sqrt{15}} \left(\sin\left(\frac{\psi}{4}\sqrt{15}\sin(x) - n\,\arctan(\sqrt{15}) \right) \right) e^{\frac{\psi}{4}\,\sin(x)}. \tag{62}$$

Since,

$$\begin{aligned}
e^{\frac{\psi}{4}\,\sin(x)} =& \sum_{k=0}^{\infty} \frac{4^{-1-k}(\psi\,\sin(x))^{2k}(4+8\,k+\psi\,\sin(x))}{(1+2k)!} \\
=& \sum_{k=0}^{\infty} \frac{4^{-k}(\psi\,\sin(x))^{k}}{k!}.
\end{aligned} \tag{63}$$

Then, from (62), and (63), we can find (57), which completes the proof of Theorem 5.2. □

Theorem 5.3 *For $n \in \mathbb{N}$ and $0 < x < \pi$, we have*

$$\mathfrak{A}_m^{[s]}(x;\psi) = \frac{2\,n!}{\sqrt{15}} \mathrm{U}(n,x,t) \left(1 + \int_0^{\infty} \mathrm{J}(n,x,t)\, t^n\, e^{t\left(\frac{i}{2}\,e^{i\,x}-1\right)}\, dt \right), \tag{64}$$

where

$$\mathrm{U}(n,x,t) = \sin\left(n\,\arctan(\sqrt{15}) - \left(1 + x\int_0^1 \cos(t\,x) \right)^{\frac{-(2n+1)}{k}} \right), \tag{65}$$

$$\mathrm{J}(n,x,t) = \frac{\left(\sqrt[\sqrt{15}]{e} - 1 \right) e^{\left(\frac{-i\,t\,e^{-ix}}{2}\right)}}{\Gamma(2n+1)}. \tag{66}$$

Proof Taking into account (6), and (57) with $\psi = \dfrac{4\,(1+\sin(x))^{\frac{-[2n+1]}{k}}}{\sin(x)\,\sqrt{15}}$, $k = 1, 2, \cdots$, we have

$$\begin{aligned}
\mathfrak{A}_m^{[s]}(x;\psi) =& \frac{2\,n!}{\sqrt{15}} \sin\left(n\, \arctan(\sqrt{15}) - \frac{\psi}{4}\sqrt{15}\sin(x)\right) \\
&\times \left(1 + \sum_{k=1}^{\infty} \frac{4^{-k}(\psi\, \sin(x))^k}{k!}\right) \\
=& \frac{2\,n!}{\sqrt{15}} \sin\left(n\, \arctan(\sqrt{15}) - (1+\sin(x))^{\left(\frac{-(2n+1)}{k}\right)}\right) \\
&\times \left(1 + \sum_{k=1}^{\infty} \frac{(1+\sin(x))^{-(2n+1)}}{(\sqrt{15})^k\, k!}\right) \\
=& \frac{2\,n!}{\sqrt{15}} \sin\left(n\, \arctan(\sqrt{15}) - (1+\sin(x))^{\left(\frac{-(2n+1)}{k}\right)}\right) \\
&\times \left(1 + \sum_{k=1}^{\infty} \frac{15^{-k/2}}{k!\,(2n)!}\, \frac{(2n)!}{(1+\sin(x))^{(2n+1)}}\right) \\
=& \frac{2\,n!}{\sqrt{15}} \sin\left(n\, \arctan(\sqrt{15}) - (1+\sin(x))^{\left(\frac{-(2n+1)}{k}\right)}\right) \\
&\times \left(1 + \sum_{k=1}^{\infty} \frac{15^{-k/2}}{k!\,(2n)!} \int_0^{\infty} t^{2n}\, e^{-(1+\sin(t))\,t}\, d\,t\right),
\end{aligned} \tag{67}$$

using (5), and

$$\sum_{k=1}^{\infty} \frac{15^{-k/2}}{k!\,(2n)!} = \frac{\left(\sqrt[\sqrt{15}]{e} - 1\right)}{\Gamma(2n+1)},$$

in (67) follows

$$\begin{aligned}
\mathfrak{A}_m^{[s]}(x;\psi) =& \frac{2\,n!}{\sqrt{15}} \sin\left(n\, \arctan(\sqrt{15}) - \left(1 + x\int_0^1 \cos(t\,x)\,d\,t\right)^{\left(\frac{-(2n+1)}{k}\right)}\right) \\
&\times \left(1 + \int_0^{\infty} \left(\frac{\left(\sqrt[\sqrt{15}]{e} - 1\right)}{\Gamma(2n+1)}\right) t^{2n}\, e^{-(1+\sin(t))\,t}\, d\,t\right) \\
=& \frac{2\,n!}{\sqrt{15}} \sin\left(n\, \arctan(\sqrt{15}) - \left(1 + x\int_0^1 \cos(t\,x)\,d\,t\right)^{\left(\frac{-(2n+1)}{k}\right)}\right) \\
&\times \left(1 + \int_0^{\infty} \left(\frac{\left(\sqrt[\sqrt{15}]{e} - 1\right)\, e^{-t\,\frac{i}{2}e^{-i\,x}}}{\Gamma(2n+1)}\right)\right)\, e^{t\left(\frac{i}{2}e^{i\,x}-1\right)}\, t^{2n}\, dt
\end{aligned} \tag{68}$$

It turns out that when considering in (68) the substitutions (65), and (66), we obtain (64). Theorem 5.3 is proved. □

6 Conclusions

In this work, we introduced a new family of trigonometric U–Fubini–Bernoulli–type polynomials and established several fundamental algebraic identities, connection formulas with classical polynomial families, and analytic representations. We also derived their Fourier expansions and obtained an explicit integral representation, providing additional tools for their analysis and potential applications. The results reveal a rich internal structure that naturally links these polynomials with well-known special functions and suggests promising directions for further research in harmonic analysis and the theory of special functions.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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