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Impact of White Noise on the Exact Solutions of the Stochastic Riemann Wave Equation in Quantum Mechanics

Wael W. Mohammed [1,2,*], Clemente Cesarano [3], Doaa Rizk [4], Elkhateeb S. Aly [5] and Mahmoud El-Morshedy [6]

1 Department of Mathematics, College of Science, University of Ha'il, Ha'il 2440, Saudi Arabia
2 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
3 Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy; c.cesarano@uninettunouniversity.net
4 Department of Mathematics, College of Science and Arts in Al-Asyah, Qassim University, Buraidah 6640, Saudi Arabia
5 Department of Mathematics, College of Science, Jazan University, Jazan 45142, Saudi Arabia
6 Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia
* Correspondence: wael.mohammed@mans.edu.eg

Abstract: In this article, the stochastic Riemann wave equation (SRWE) forced by white noise in the Itô sense is considered. The extended tanh function and mapping methods are applied to obtain new elliptic, rational, hyperbolic, and trigonometric stochastic solutions. Furthermore, we generalize some previous studies. The obtained solutions are important in explaining some exciting physical phenomena, since the SRWE is required for describing wave propagation. We plot numerous 3D and 2D graphical representations to explain how the multiplicative white noise influences the exact solutions of the SRWE. We can infer that the introduction of multiplicative white noise disrupts the symmetry of the solutions and serves to stabilize the solutions of the SRWE.

Keywords: partial differential equations; nonlinear systems; modified extended tanh function method; mapping method; stability by noise

MSC: 60H15; 60H10; 83C15; 35A20; 35Q51

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1. Introduction

Stochastic nonlinear evolution equations (SNLEEs) play a crucial role in various fields of science and engineering. These equations take into account both randomness and nonlinearity, making them essential for modeling complex systems and phenomena [1–3]. SNLEEs are vital for understanding the behavior of dynamic systems. Unlike deterministic equations, SNLEEs take into account inherent uncertainties and fluctuations, which are often observed in real-world systems. For instance, in climate modeling, SNLEEs help incorporate random variations and nonlinearity to model the complex interactions between various climatic factors accurately. By considering the inherent stochastic nature of these systems, scientists can obtain more realistic predictions and improve their understanding of the underlying mechanisms.

Moreover, SNLEEs find applications in physics and chemistry. For instance, in statistical physics, these equations are employed to understand the behavior of complex systems, such as turbulent flow or phase transitions. By accounting for the stochastic nature of these phenomena, scientists can study the emergence of ordered structures from random initial conditions. In chemical kinetics, SNLEEs are used to model reaction processes involving random fluctuations, assisting in the design of efficient chemical reactors and optimizing reaction conditions.

In many practical situations, finding exact solutions to these equations is extremely challenging due to the presence of both nonlinearity and stochasticity. However, researchers have used several techniques and methods to obtain exact solutions for different kinds of SNLEEs, such as the modified mapping method [4], the ϕ^6-expansion scheme [5], the (G'/G)-expansion method [6], He's semi-inverse techniques [7], the new auxiliary equation method [8], the tanh–coth method [9], etc.

In this paper, we consider the stochastic Riemann wave equation (SRWE) [10]:

$$\begin{aligned} &\mathcal{U}_t + \gamma_1 \mathcal{U}_{xxy} + \gamma_2 \mathcal{U}\mathcal{V}_x + \gamma_3 \mathcal{V}\mathcal{U}_x = \sigma \mathcal{U} W_t, \\ &\mathcal{U}_y = \mathcal{V}_x, \end{aligned} \tag{1}$$

where γ_1, γ_2, and γ_3 are real constants; σ is the intensity of noise; $W(t)$ is the white noise (Gaussian process); and $\mathcal{U}W_t$ is a multiplicative white noise in the Itô sense.

The Riemann wave equation (RWE) finds numerous applications in various scientific disciplines. In acoustics, it helps in the study of sound waves and their propagation in different media. By solving the Riemann wave equation, one can determine the behavior of sound waves in air, water, or other materials, aiding in the design of acoustic systems, such as speakers and musical instruments.

In optics, the RWE is essential for understanding the behavior of light waves. It allows scientists and engineers to analyze the properties of light, such as refraction, diffraction, and interference, and enables the design of optical systems, like lenses and cameras.

In quantum mechanics, the RWE plays a central role in describing the behavior of particles at the quantum level. The equation represents the wave function of a particle, which provides information about its position, momentum, and energy. The Riemann wave equation, along with the principles of quantum mechanics, allows us to understand phenomena such as wave-particle duality and quantum tunneling.

Due to the importance of the RWE, many researchers have obtained the solution of the Riemann wave equation using different methods, such as the generalized (G'/G)-expansion method [11], generalized Kudryashov method [12], extended tanh function technique [13], Wronskian method [14], generalized exponential rational function approach [15], modified $\exp(-\varphi(\xi))$-function method [16], and new extended direct algebraic method [17].

The novelty of this article is to acquire the exact stochastic solutions of the SRWE (1). To obtain these solutions, we use two different approaches, including the extended tanh–coth method and the mapping method. Also, we expand upon some earlier research, such as the results presented in [13]. The solutions that were produced are important in explaining some exciting physical phenomena since the SRWE is required for describing wave propagation. Additionally, we explore the impact of noise on the exact solutions of the SRWE (1) by presenting various figures using the MATLAB software 2018.

This article is organized as follows: In Section 2, the wave equation for the SRWE (1) is derived. In Section 3, the exact stochastic solution of the SRWE (1) is achieved via the extended tanh function and mapping methods. In Section 4, the effect of white noise on the obtained solutions of the SRWE may be detected. Finally, conclusions of the paper are offered.

2. Wave Equation for the SRWE

To derive the wave equation of the SRWE (1), we use

$$\mathcal{U}(x,y,t) = \Psi(\xi) e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]},\ \mathcal{V}(x,y,t) = \Phi(\xi) e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]} \text{ and } \xi = \xi_1 x + \xi_2 y + \lambda t, \tag{2}$$

where Ψ and Φ are real deterministic functions and ξ_1, ξ_2, and λ are nonzero constants. We observe that

$$\mathcal{U}_x = \xi_1 \Psi' e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \mathcal{U}_y = \xi_2 \Psi' e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{3}$$

$$\mathcal{U}_{xxy} = \xi_1^2 \xi_2 \Psi^{(3)} e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]},\ \mathcal{V}_x = \xi_1 \Phi' e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{4}$$

and

$$\begin{aligned} \mathcal{U}_t &= [\lambda\Psi' + \sigma\Psi W_t + \frac{1}{2}\sigma^2\Psi - \frac{1}{2}\sigma^2\Psi]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]} \\ &= [\lambda\Psi' + \sigma\Psi W_t]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \end{aligned} \tag{5}$$

where $+\frac{1}{2}\sigma^2\Psi$ is the Itô correction term. Inserting Equation (2) into Equation (1) and utilizing (3)–(5), we obtain

$$\begin{aligned} &\lambda\Psi' + \gamma_1\xi_1^2\xi_2\Psi^{(3)} + (\gamma_2\xi_1\Psi\Phi' + \gamma_3\xi_1\Phi\Psi')e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]} = 0, \\ &\xi_2\Psi' = \xi_1\Phi'. \end{aligned} \tag{6}$$

Integrating the second equation of (6), we obtain

$$\Phi = \frac{\xi_2}{\xi_1}\Psi + \frac{\rho}{\xi_1}, \tag{7}$$

where ρ is a constant of the integral. Plugging Equation (7) into the first equation of (6), we have

$$\lambda\Psi' + \gamma_1\xi_1^2\xi_2\Psi^{(3)} + (\gamma_2\xi_2\Psi\Psi' + \gamma_3\xi_2\Psi\Psi' + \rho\gamma_3\Psi')e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]} = 0.$$

Taking the expectations $\mathbb{E}(\cdot)$ on both sides, we obtain

$$\lambda\Psi' + \gamma_1\xi_1^2\xi_2\Psi^{(3)} + (\gamma_2\xi_2\Psi\Psi' + \gamma_3\xi_2\Psi\Psi' + \rho\gamma_3\Psi')e^{-\frac{1}{2}\sigma^2 t}\mathbb{E}(e^{\sigma W(t)}) = 0. \tag{8}$$

Since $W(t)$ is a Gaussian process, then $\mathbb{E}(e^{\sigma W(t)}) = e^{\frac{1}{2}\sigma^2 t}$. Hence, Equation (8) becomes

$$\gamma_1\xi_1^2\xi_2\Psi^{(3)} + (\lambda + \rho\gamma_3)\Psi' + (\gamma_2\xi_2 + \gamma_3\xi_2)\Psi\Psi' = 0. \tag{9}$$

Integrating Equation (9) once with respect to ξ, we have

$$\Psi'' + \ell_1\Psi + \ell_2\Psi^2 = 0, \tag{10}$$

where

$$\ell_1 = \frac{\lambda + \rho\gamma_3}{\gamma_1\xi_1^2\xi_2} \text{ and } \ell_2 = \frac{\gamma_2 + \gamma_3}{2\gamma_1\xi_1^2}.$$

3. Exact Stochastic Solutions of SRWE

To obtain exact stochastic solutions for the SRWE (1), we use two alternative methods: the extended tanh–coth approach and the mapping method.

3.1. Modified Extended tanh Function Method

We apply here the modified extended tanh function method (for more information, see [18]). Let us assume the solution Ψ of Equation (10) has the form

$$\Psi(\xi) = \sum_{j=0}^{M} a_j\phi^j + \sum_{j=1}^{M} b_j\phi^{-j}, \tag{11}$$

where ϕ solves

$$\phi' = \phi^2 + k. \tag{12}$$

The solutions of Equation (12) are as follows: If $k > 0$, we obtain

$$\phi(\xi) = \sqrt{k}\tan(\sqrt{k}\xi) \text{ or } \phi(\xi) = -\sqrt{k}\cot(\sqrt{k}\xi). \tag{13}$$

If $k < 0$, we obtain

$$\phi(\xi) = -\sqrt{-k}\tanh(\sqrt{-k}\xi) \ \text{ or } \phi(\xi) = -\sqrt{-k}\coth(\sqrt{-k}\xi). \tag{14}$$

If $k = 0$, we obtain

$$\phi(\xi) = \frac{-1}{\xi}. \tag{15}$$

To find the value of M, we balance Ψ^2 with Ψ'' in Equation (10) to have

$$2M = M + 2,$$

then

$$M = 2. \tag{16}$$

Rewriting Equation (11) and using Equation (16), we obtain

$$\Psi(\xi) = a_0 + a_1\phi + a_2\phi^2 + b_1\phi^{-1} + b_2\phi^{-2}. \tag{17}$$

Substituting Equation (17) into Equation (10), we obtain

$$\begin{aligned}
&(6ka_2 + \ell_2 a_2^2)\phi^4 + (2a_1 + 2\ell_2 a_1 a_2)\phi^3 + (8ka_2 + 2a_0 a_2 \ell_2 + a_1^2\ell_2 + \ell_1 a_2)\phi^2 \\
&(2ka_1 + \ell_1 a_1 + 2\ell_2 a_0 a_1 + 2a_2 b_1)\phi + (2k^2 a_2 + 2b_2 + \ell_1 a_0 + \ell_2 a_0^2 + 2\ell_2 a_1 b_1 \\
&+2\ell_2 a_2 b_2) + (2kb_1 + 2\ell_2 a_0 b_1 + 2\ell_2 a_1 b_2 + \ell_1 b_1)\phi^{-1} + (8kb_2 + 2a_0 b_2 \ell_2 \\
&+b_1^2\ell_2 + \ell_1 b_2)\phi^{-2} + (2b_1 k^2 + 2\ell_2 b_1 b_2)\phi^{-3} + (6k^2 b_2 + \ell_2 b_2^2)\phi^4 = 0.
\end{aligned}$$

The coefficients of each power of ϕ are set to zero as follows:

$$6a_2 + \ell_2 a_2^2 = 0,$$

$$\begin{gathered}
2a_1 + 2\ell_2 a_1 a_2 = 0, \\
8ka_2 + 2a_0 a_2 \ell_2 + a_1^2\ell_2 + \ell_1 a_2 = 0, \\
2ka_1 + \ell_1 a_1 + 2\ell_2 a_0 a_1 + 2a_2 b_1 = 0, \\
2k^2 a_2 + 2b_2 + \ell_1 a_0 + \ell_2 a_0^2 + 2\ell_2 a_1 b_1 + 2\ell_2 a_2 b_2 = 0, \\
2kb_1 + 2\ell_2 a_0 b_1 + 2\ell_2 a_1 b_2 + \ell_1 b_1 = 0, \\
8kb_2 + 2a_0 b_2 \ell_2 + b_1^2\ell_2 + \ell_1 b_2 = 0, \\
2b_1 k^2 + 2\ell_2 b_1 b_2 = 0.
\end{gathered}$$

and

$$6k^2 b_2 + \ell_2 b_2^2 = 0.$$

We obtain the four separate sets by solving these equations as follows:
First set:

$$a_0 = \frac{-6k}{\ell_2},\ a_1 = 0,\ a_2 = \frac{-6}{\ell_2},\ b_1 = 0,\ b_2 = 0,\ \lambda = 4k\gamma_1\zeta_1^2\zeta_2 - \rho\gamma_3. \tag{18}$$

Second set:

$$a_0 = \frac{-2k}{\ell_2},\ a_1 = 0,\ a_2 = \frac{-6}{\ell_2},\ b_1 = 0,\ b_2 = 0,\ \lambda = -4k\gamma_1\zeta_1^2\zeta_2 - \rho\gamma_3. \tag{19}$$

Third set:

$$a_0 = \frac{-12k}{\ell_2},\ a_1 = 0,\ a_2 = \frac{-6}{\ell_2},\ b_1 = 0,\ b_2 = \frac{-6k^2}{\ell_2},\ \lambda = 16k\gamma_1\zeta_1^2\zeta_2 - \rho\gamma_3. \tag{20}$$

Fourth set:

$$a_0 = \frac{8k}{\ell_2},\ a_1 = 0,\ a_2 = \frac{-6}{\ell_2},\ b_1 = 0,\ b_2 = \frac{-6k^2}{\ell_2},\ \lambda = -14k\gamma_1\xi_1^2\xi_2 - \rho\gamma_3. \tag{21}$$

First set: Using (18), the solution of Equation (10) takes the form

$$\Psi(\xi) = \frac{-6k}{\ell_2} - \frac{6}{\ell_2}\phi^2(\xi).$$

There are three cases for $\phi(\xi)$:
Case 1: When $k > 0$, we have the following, using (13):

$$\Psi(\xi) = \frac{-6k}{\ell_2} - \frac{6k}{\ell_2}\tan^2(\sqrt{k}\xi) = -\frac{6k}{\ell_2}\sec^2(\sqrt{k}\xi),$$

and

$$\Psi(\xi) = \frac{-6k}{\ell_2} - \frac{6k}{\ell_2}\cot^2(\sqrt{k}\xi) = \frac{-6k}{\ell_2}\csc^2(\sqrt{k}\xi).$$

Hence, the exact solutions of the SRWE (1), using (7), are

$$\mathcal{U}(x,y,t) = -\frac{6k}{\ell_2}\sec^2(\sqrt{k}\xi)e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{22}$$

$$\mathcal{V}(x,y,t) = [-\frac{6k\xi_2}{\ell_2\xi_1}\sec^2(\sqrt{k}\xi) + \frac{\rho}{\xi_1}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{23}$$

and

$$\mathcal{U}(x,y,t) = \frac{-6k}{\ell_2}\csc^2(\sqrt{k}\xi)e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{24}$$

$$\mathcal{V}(x,y,t) = [-\frac{6k\xi_2}{\ell_2\xi_1}\csc^2(\sqrt{k}\xi) + \frac{\rho}{\xi_1}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{25}$$

where $\xi = \xi_1 x + \xi_2 y + [4k\gamma_1\xi_1^2\xi_2 - \rho\gamma_3]t$.
Case 2: When $k < 0$, we have the following, using (14):

$$\Psi(\xi) = \frac{-6k}{\ell_2} + \frac{6k}{\ell_2}\tanh^2(\sqrt{-k}\xi) = \frac{-6k}{\ell_2}\operatorname{sech}^2(\sqrt{-k}\xi),$$

and

$$\Psi(\xi) = \frac{-6k}{\ell_2} + \frac{6k}{\ell_2}\coth^2(\sqrt{-k}\xi) = \frac{6k}{\ell_2}\operatorname{csch}^2(\sqrt{-k}\xi).$$

Hence, the exact solutions of the SRWE (1), using (7), are

$$\mathcal{U}(x,y,t) = \frac{-6k}{\ell_2}\operatorname{sech}^2(\sqrt{-k}\xi)e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{26}$$

$$\mathcal{V}(x,y,t) = [-\frac{6k\xi_2}{\ell_2\xi_1}\operatorname{sech}^2(\sqrt{-k}\xi) + \frac{\rho}{\xi_1}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{27}$$

and

$$\mathcal{U}(x,y,t) = \frac{6k}{\ell_2}\operatorname{csch}^2(\sqrt{-k}\xi)e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{28}$$

$$\mathcal{V}(x,y,t) = [\frac{6k\xi_2}{\ell_2\xi_1}\operatorname{csch}^2(\sqrt{-k}\xi) + \frac{\rho}{\xi_1}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}. \tag{29}$$

Case 3: When $k = 0$, we have the following, using (15):

$$\Psi(\xi) = -\frac{6}{\ell_2}\frac{1}{\xi^2}.$$

Therefore, the exact solutions of the SRWE (1), using (7), are

$$\mathcal{U}(x,y,t) = [\frac{-6}{\ell_2}\frac{1}{\xi^2}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{30}$$

$$\mathcal{V}(x,y,t) = [\frac{-6\xi_2}{\ell_2\xi_1}\frac{1}{\xi^2} + \frac{\rho}{\xi_1}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{31}$$

where $\xi = \xi_1 x + \xi_2 y - \rho\gamma_3 t$.

Second set: Using (19), the solution of Equation (10) takes the form

$$\Psi(\xi) = \frac{-2k}{\ell_2} - \frac{6}{\ell_2}\phi^2(\xi).$$

There are three cases for $\phi(\xi)$:

Case 1: When $k > 0$, we have the following, using (13):

$$\Psi(\xi) = \frac{-2k}{\ell_2} - \frac{6k}{\ell_2}\tan^2(\sqrt{k}\xi),$$

and

$$\Psi(\xi) = \frac{-2k}{\ell_2} - \frac{6k}{\ell_2}\cot^2(\sqrt{k}\xi).$$

Therefore, the exact solutions of the SRWE (1), using (7), are

$$\mathcal{U}(x,y,t) = [\frac{-2k}{\ell_2} - \frac{6k}{\ell_2}\tan^2(\sqrt{k}\xi)]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{32}$$

$$\mathcal{V}(x,y,t) = [\frac{-2k\xi_2}{\ell_2\xi_1} - \frac{6k\xi_2}{\ell_2\xi_1}\tan^2(\sqrt{k}\xi) + \frac{\rho}{\xi_1}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{33}$$

and

$$\mathcal{U}(x,y,t) = [\frac{-2k}{\ell_2} - \frac{6k}{\ell_2}\cot^2(\sqrt{k}\xi)]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{34}$$

$$\mathcal{V}(x,y,t) = [\frac{-2k\xi_2}{\ell_2\xi_1} - \frac{6k\xi_2}{\ell_2\xi_1}\cot^2(\sqrt{k}\xi) + \frac{\rho}{\xi_1}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{35}$$

where $\xi = \xi_1 x + \xi_2 y - [4k\gamma_1\xi_1^2\xi_2 + \rho\gamma_3]t$.

Case 2: When $k < 0$, we have the following, using (14):

$$\Psi(\xi) = \frac{-2k}{\ell_2} + \frac{6k}{\ell_2}\tanh^2(\sqrt{-k}\xi),$$

and

$$\Psi(\xi) = \frac{-2k}{\ell_2} + \frac{6k}{\ell_2}\coth^2(\sqrt{-k}\xi).$$

Therefore, the exact solutions of the SRWE (1), using (7), are

$$\mathcal{U}(x,y,t) = \frac{-2k}{\ell_2} + \frac{6k}{\ell_2}\tanh^2(\sqrt{-k}\xi)e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{36}$$

$$\mathcal{V}(x,y,t) = [\frac{-2k\xi_2}{\ell_2\xi_1} + \frac{6k\xi_2}{\ell_2\xi_1}\tanh^2(\sqrt{k}\xi) + \frac{\rho}{\xi_1}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{37}$$

and

$$\mathcal{U}(x,y,t)=\frac{-2k}{\ell_2}+\frac{6k}{\ell_2}\coth^2(\sqrt{-k}\xi)e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{38}$$

$$\mathcal{V}(x,y,t)=[\frac{-2k\xi_2}{\ell_2\xi_1}+\frac{6k\xi_2}{\ell_2\xi_1}\coth^2(\sqrt{k}\xi)+\frac{\rho}{\xi_1}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{39}$$

where $\xi=\xi_1 x+\xi_2 y-[4k\gamma_1\xi_1^2\xi_2+\rho\gamma_3]t$.

Case 3: When $k=0$, we have the following, using (15):

$$\Psi(\xi)=\frac{-6}{\ell_2}\frac{1}{\xi^2}.$$

Thus, the exact solutions of the SRWE (1), using (7), are

$$\mathcal{U}(x,y,t)=\frac{-6}{\ell_2}\frac{1}{\xi^2}e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{40}$$

$$\mathcal{V}(x,y,t)=[\frac{-6\xi_2}{\ell_2\xi_1}\frac{1}{\xi^2}+\frac{\rho}{\xi_1}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{41}$$

where $\xi=\xi_1 x+\xi_2 y-\rho\gamma_3 t$.

Third set: Using (20), the solution of Equation (10) takes the form

$$\Psi(\xi)=\frac{-12k}{\ell_2}-\frac{6}{\ell_2}\phi^2(\xi)-\frac{6k^2}{\ell_2}\phi^{-2}(\xi).$$

For $\phi(\xi)$, there are three cases:

Case 1: When $k>0$, we have the following, using (13):

$$\begin{aligned}\Psi(\xi) &= \frac{-12k}{\ell_2}-\frac{6k}{\ell_2}\tan^2(\sqrt{k}\xi)-\frac{6k}{\ell_2}\cot^2(\sqrt{k}\xi)\\ &= -\frac{6k}{\ell_2}[\sec^2(\sqrt{k}\xi)+\csc^2(\sqrt{k}\xi)].\end{aligned}$$

Hence, the exact solutions of the SRWE (1), using (7), are

$$\mathcal{U}(x,y,t)=-\frac{6k}{\ell_2}[\sec^2(\sqrt{k}\xi)+\csc^2(\sqrt{k}\xi)]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{42}$$

$$\mathcal{V}(x,y,t)=[\frac{-6k\xi_2}{\ell_2\xi_1}\sec^2(\sqrt{k}\xi)-\frac{6k\xi_2}{\ell_2\xi_1}\csc^2(\sqrt{k}\xi)+\frac{\rho}{\xi_1}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{43}$$

where $\xi=\xi_1 x+\xi_2 y+[16k\gamma_1\xi_1^2\xi_2-\rho\gamma_3]t$.

Case 2: When $k<0$, we have the following, using (14):

$$\begin{aligned}\Psi(\xi) &= \frac{-12k}{\ell_2}+\frac{6k}{\ell_2}\tanh^2(\sqrt{-k}\xi)+\frac{6k}{\ell_2}\coth^2(\sqrt{-k}\xi)\\ &= \frac{-6k}{\ell_2}[\operatorname{sech}^2(\sqrt{-k}\xi)-\operatorname{csch}^2(\sqrt{-k}\xi)].\end{aligned}$$

Thus, the exact solutions of the SRWE (1), using (7), are

$$\mathcal{U}(x,y,t)=\frac{-6k}{\ell_2}[\operatorname{sech}^2(\sqrt{-k}\xi)-\operatorname{csch}^2(\sqrt{-k}\xi)]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{44}$$

$$\mathcal{V}(x,y,t)=[\frac{-6k\xi_2}{\ell_2\xi_1}\operatorname{sech}^2(\sqrt{-k}\xi)+\frac{6k\xi_2}{\ell_2\xi_1}\operatorname{csch}^2(\sqrt{-k}\xi)+\frac{\rho}{\xi_1}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{45}$$

where $\xi=\xi_1 x+\xi_2 y+[16k\gamma_1\xi_1^2\xi_2-\rho\gamma_3]t$.

Case 3: When $k = 0$, we have the following, using (15):

$$\Psi(\xi) = \frac{6}{\ell_2}\frac{1}{\xi^2} + \frac{6}{\ell_2}\xi^2.$$

Thus, the exact solutions of the SRWE (1), using (7), are

$$\mathcal{U}(x,y,t) = (\frac{6}{\ell_2}\frac{1}{\xi^2} + \frac{6}{\ell_2}\xi^2)e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{46}$$

$$\mathcal{V}(x,y,t) = [\frac{6\xi_2}{\ell_2\xi_1}\frac{1}{\xi^2} + \frac{6\xi_2}{\ell_2\xi_1}\xi^2 + \frac{\rho}{\xi_1}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}. \tag{47}$$

Fourth set: Using (21), the solution of Equation (10) takes the form

$$\Psi(\xi) = \frac{8k}{\ell_2} - \frac{6}{\ell_2}\phi^2(\xi) - \frac{6k^2}{\ell_2}\phi^{-2}(\xi).$$

There are three cases for $\phi(\xi)$:
Case 1: When $k > 0$, we have the following, using (13):

$$\Psi(\xi) = \frac{8k}{\ell_2} - \frac{6k}{\ell_2}\tan^2(\sqrt{k}\xi) - \frac{6k}{\ell_2}\cot^2(\sqrt{k}\xi).$$

Thus, the exact solutions of the SRWE (1), using (7), are

$$\mathcal{U}(x,y,t) = [\frac{8k}{\ell_2} - \frac{6k}{\ell_2}\tan^2(\sqrt{k}\xi) - \frac{6k}{\ell_2}\cot^2(\sqrt{k}\xi)]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{48}$$

$$\mathcal{V}(x,y,t) = [\frac{8k\xi_2}{\ell_2\xi_1} - \frac{6k\xi_2}{\ell_2\xi_1}\tan^2(\sqrt{k}\xi) - \frac{6k\xi_2}{\ell_2\xi_1}\cot^2(\sqrt{k}\xi) + \frac{\rho}{\xi_1}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{49}$$

where $\xi = \xi_1 x + \xi_2 y - [14k\gamma_1\xi_1^2\xi_2 + \rho\gamma_3]t$.
Case 2: When $k < 0$, we have the following, using (14):

$$\Psi(\xi) = \frac{8k}{\ell_2} + \frac{6k}{\ell_2}\tanh^2(\sqrt{-k}\xi) + \frac{6k}{\ell_2}\coth^2(\sqrt{-k}\xi).$$

Therefore, the exact solutions of the SRWE (1), using (7), are

$$\mathcal{U}(x,y,t) = [\frac{8k}{\ell_2} + \frac{6k}{\ell_2}\tanh^2(\sqrt{-k}\xi) + \frac{6k}{\ell_2}\coth^2(\sqrt{-k}\xi)]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{50}$$

$$\mathcal{V}(x,y,t) = [\frac{8k\xi_2}{\ell_2\xi_1} + \frac{6k\xi_2}{\ell_2\xi_1}\tanh^2(\sqrt{-k}\xi) + \frac{6k\xi_2}{\ell_2\xi_1}\coth^2(\sqrt{-k}\xi) + \frac{\rho}{\xi_1}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{51}$$

where $\xi = \xi_1 x + \xi_2 y - (14k\gamma_1\xi_1^2\xi_2 + \rho\gamma_3)t$.
Case 3: When $k = 0$, we have the following, using (15):

$$\Psi(\xi) = \frac{6}{\ell_2}\frac{1}{\xi^2} + \frac{6}{\ell_2}\xi^2.$$

Thus, the exact solutions of the SRWE (1), using (7), are

$$\mathcal{U}(x,y,t) = [\frac{6}{\ell_2}\frac{1}{\xi^2} + \frac{6}{\ell_2}\xi^2]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{52}$$

$$\mathcal{V}(x,y,t) = [\frac{6\xi_2}{\ell_2\xi_1}\frac{1}{\xi^2} + \frac{6\xi_2}{\ell_2\xi_1}\xi^2 + \frac{\rho}{\xi_1}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{53}$$

where $\xi = \xi_1 x + \xi_2 y - \rho\gamma_3 t$.

Remark 1. *If we set* $\sigma = 0$ *in Equations (26), (27), (36)–(39), (50), and (51), then we obtain the same results, (30)–(35), (28), and (29), respectively, that were reported in [13].*

3.2. Mapping Method

The mapping method [19] is used here. With $M = 2$, the solutions of Equation (10) have the following form:

$$\Psi(\xi) = \hbar_0 + \hbar_1 Z(\xi) + \hbar_2 Z^2(\xi), \tag{54}$$

where $\hbar_0$, $\hbar_1$, and $\hbar_2$ are unknown constants and $Z(\xi)$ is the solution of

$$Z' = \sqrt{r + qZ^2 + pZ^4}, \tag{55}$$

where r, q, and p are real parameters. We obtain the following by differentiating Equation (54) twice and using (55):

$$\Psi'' = \hbar_1(qZ + 2pZ^3) + 2\hbar_2(r + 2qZ^2 + 3pZ^4). \tag{56}$$

Putting Equations (54) and (56) into Equation (10), we obtain

$$(6\hbar_2 p + \ell_2\hbar_2^2)Z^4 + (2p\hbar_1 + 2\ell_2\hbar_1\hbar_2)Z^3$$
$$+(4\hbar_2 q + 2\ell_2\hbar_0\hbar_2 + \ell_2\hbar_1^2 + \ell_1\hbar_2)Z^2$$
$$+(\hbar_1 q + 2\ell_2\hbar_0\hbar_1 + \ell_1\hbar_1)Z + (2r\hbar_2 + \ell_2\hbar_0^2 + \ell_1\hbar_0) = 0.$$

Each Z power's coefficients are set to zero, as follows:

$$6\hbar_2 p + \ell_2\hbar_2^2 = 0,$$

$$2p\hbar_1 + 2\ell_2\hbar_1\hbar_2 = 0,$$
$$4\hbar_2 q + 2\ell_2\hbar_0\hbar_2 + \ell_2\hbar_1^2 + \ell_1\hbar_2 = 0,$$
$$\hbar_1 q + 2\ell_2\hbar_0\hbar_1 + \ell_1\hbar_1 = 0,$$

and

$$2r\hbar_2 + \ell_2\hbar_0^2 + \ell_1\hbar_0 = 0.$$

When we solve these equations, we obtain

$$\hbar_0 = \frac{-(\ell_1 + 4q)}{2\ell_2},\ \ \hbar_1 = 0,\ \hbar_2 = \frac{-6p}{\ell_2},\ \lambda = -4q\gamma_1\xi_1^2\xi_2 - \rho\gamma_3.$$

Therefore, Equation (10) has the following solution:

$$\Psi(\xi) = \frac{-(\ell_1 + 4q)}{2\ell_2} - \frac{6p}{\ell_2}Z^2(\xi). \tag{57}$$

To find the the solutions $Z(\xi)$ of Equation (55), there are many cases, depending on r, q, and p, as follows:

Case 1: When $r = 1$, $q = -(1+m^2)$, and $p = m^2$, then $Z(\xi) = sn(\xi)$. In this case, the exact solutions of the SRWE (1), found by utilizing Equations (2), (57), and (7), are

$$\mathcal{U}(x,y,t) = [\frac{-(\ell_1 - 4(1+m^2))}{2\ell_2} - \frac{6m^2}{\ell_2}sn^2(\xi)]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{58}$$

$$\mathcal{V}(x,y,t) = [\frac{-\xi_2(\ell_1 - 4(1+m^2))}{2\ell_2\xi_1} - \frac{6\xi_2 m^2}{\ell_2\xi_1} sn^2(\xi) + \frac{\rho}{\xi_1}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{59}$$

where $\xi = \xi_1 x + \xi_2 y + [4(1+m^2)\gamma_1\xi_1^2\xi_2 - \rho\gamma_3]t$. If $m \to 1$, then Equation (58) changes to

$$\mathcal{U}(x,y,t) = [\frac{-(\ell_1 - 8)}{2\ell_2} - \frac{6}{\ell_2}\tanh^2(\xi)]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{60}$$

$$\mathcal{V}(x,y,t) = [\frac{-\xi_2(\ell_1 - 8)}{2\ell_2\xi_1} - \frac{6\xi_2}{\ell_2\xi_1}\tanh^2(\xi) + \frac{\rho}{\xi_1}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}. \tag{61}$$

Case 2: If $r = -m^2(1-m^2)$, $q = 2m^2 - 1$, and $p = 1$, then $Z(\xi) = ds(\xi)$. In this case, the exact solutions of the SRWE (1) found by utilizing Equations (2), (57), and (7) are

$$\mathcal{U}(x,y,t) = [\frac{-(\ell_1 + 4(2m^2 - 1))}{2\ell_2} - \frac{6}{\ell_2} ds^2(\xi)]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{62}$$

$$\mathcal{V}(x,y,t) = [\frac{-\xi_2(\ell_1 + 4(2m^2 - 1))}{2\ell_2\xi_1} - \frac{6\xi_2}{\ell_2\xi_1} ds^2(\xi) + \frac{\rho}{\xi_1}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}. \tag{63}$$

where $\xi = \xi_1 x + \xi_2 y - [4(2m^2 - 1)\gamma_1\xi_1^2\xi_2 + \rho\gamma_3]t$. If $m \to 1$, then Equation (62) changes to

$$\mathcal{U}(x,y,t) = [\frac{-(\ell_1 + 4)}{2\ell_2} - \frac{6}{\ell_2}\text{csch}^2(\xi)]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{64}$$

$$\mathcal{V}(x,y,t) = [\frac{-\xi_2(\ell_1 + 4)}{2\ell_2\xi_1} - \frac{6\xi_2}{\ell_2\xi_1}\text{csch}^2(\xi) + \frac{\rho}{\xi_1}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}. \tag{65}$$

When $m \to 0$, then Equation (62) changes to

$$\mathcal{U}(x,y,t) = [\frac{-(\ell_1 - 4)}{2\ell_2} - \frac{6}{\ell_2}\csc^2(\xi)]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{66}$$

$$\mathcal{V}(x,y,t) = [\frac{-\xi_2(\ell_1 - 4)}{2\ell_2\xi_1} - \frac{6\xi_2}{\ell_2\xi_1}\csc^2(\xi) + \frac{\rho}{\xi_1}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}. \tag{67}$$

Case 3: If $r = (1-m^2)$, $q = 2 - m^2$, and $p = 1$, then $Z(\xi) = cs(\xi)$. In this case, the exact solutions of the SRWE (1) found by utilizing Equations (2), (57), and (7) are

$$\mathcal{U}(x,y,t) = [\frac{-(\ell_1 + 4(2 - m^2))}{2\ell_2} - \frac{6}{\ell_2} cs^2(\xi)]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{68}$$

$$\mathcal{V}(x,y,t) = [\frac{-\xi_2(\ell_1 + 4(2 - m^2))}{2\ell_2\xi_1} - \frac{6\xi_2}{\ell_2\xi_1} cs^2(\xi) + \frac{\rho}{\xi_1}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{69}$$

where $\xi = \xi_1 x + \xi_2 y - [4(2 - m^2)\gamma_1\xi_1^2\xi_2 + \rho\gamma_3]t$. If $m \to 1$, then Equation (68) transfers to Equation (64).

When $m \to 0$, Equation (68) changes to

$$\mathcal{U}(x,y,t) = [\frac{-(\ell_1 + 8)}{2\ell_2} - \frac{6}{\ell_2}\cot^2(\xi)]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{70}$$

$$\mathcal{V}(x,y,t) = [\frac{-\xi_2(\ell_1 + 8)}{2\ell_2\xi_1} - \frac{6\xi_2}{\ell_2\xi_1}\cot^2(\xi) + \frac{\rho}{\xi_1}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}. \tag{71}$$

Case 4: If $r=(1-m^2)$, $q=2m^2-1$, and $p=-m^2$, then $Z(\xi)=cn(\xi)$. In this case, the exact solutions of the SRWE (1) found by utilizing Equations (2), (57), and (7), are

$$\mathcal{U}(x,y,t)=[\frac{-(\ell_1+4(2m^2-1))}{2\ell_2}-\frac{6m^2}{\ell_2}cn^2(\xi)]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{72}$$

$$\mathcal{V}(x,y,t)=[\frac{-\xi_2(\ell_1+4(2m^2-1))}{2\ell_2\xi_1}-\frac{6m^2\xi_2}{\ell_2\xi_1}cn^2(\xi)+\frac{\rho}{\xi_1}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{73}$$

where $\xi=\xi_1 x+\xi_2 y-[4(2m^2-1)\gamma_1\xi_1^2\xi_2+\rho\gamma_3]t$. If $m\to 1$, then Equation (72) tends to turn into

$$\mathcal{U}(x,y,t)=[\frac{-(\ell_1+4)}{2\ell_2}+\frac{6}{\ell_2}\text{sech}^2(\xi)]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{74}$$

$$\mathcal{V}(x,y,t)=[\frac{-\xi_2(\ell_1+4)}{2\ell_2\xi_1}+\frac{6\xi_2}{\ell_2\xi_1}\text{sech}^2(\xi)+\frac{\rho}{\xi_1}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}. \tag{75}$$

Case 5: If $r=(m^2-1)$, $q=2-m^2$, and $p=-1$, then $Z(\xi)=dn(\xi)$. In this case, the exact solutions of the SRWE (1) found by utilizing Equations (2), (57), and (7) are

$$\mathcal{U}(x,y,t)=[\frac{-(\ell_1+4(2-m^2))}{2\ell_2}+\frac{6}{\ell_2}dn^2(\xi)]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{76}$$

$$\mathcal{V}(x,y,t)=[\frac{-\xi_2(\ell_1+4(2-m^2))}{2\ell_2\xi_1}+\frac{6\xi_2}{\ell_2\xi_1}dn^2(\xi)+\frac{\rho}{\xi_1}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{77}$$

where $\xi=\xi_1 x+\xi_2 y-[4(2-m^2)\gamma_1\xi_1^2\xi_2+\rho\gamma_3]t$. If $m\to 1$, then Equation (76) changes to Equation (74).

Case 6: If $r=\frac{1}{4}$, $q=\frac{(m^2-2)}{2}$, and $p=\frac{m^2}{4}$, then $Z(\xi)=\frac{sn(\xi)}{1+dn(\xi)}$. In this case, the exact solutions of the SRWE (1) found by utilizing Equations (2), (57), and (7) are

$$\mathcal{U}(x,y,t)=[\frac{-(\ell_1+2(m^2-2))}{2\ell_2}+\frac{3m^2}{2\ell_2}\frac{sn^2(\xi)}{[1+dn(\xi)]^2}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{78}$$

$$\mathcal{V}(x,y,t)=[\frac{-\xi_2(\ell_1+2(m^2-2))}{2\ell_2\xi_1}+\frac{3m^2\xi_2}{2\ell_2\xi_1}\frac{sn^2(\xi)}{[1+dn(\xi)]^2}+\frac{\rho}{\xi_1}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{79}$$

where $\xi=\xi_1 x+\xi_2 y-[2(m^2-2)\gamma_1\xi_1^2\xi_2+\rho\gamma_3]t$. If $m\to 1$, then Equation (78) tends to turn into

$$\mathcal{U}(x,y,t)=[\frac{-(\ell_1-2)}{2\ell_2}+\frac{3}{2\ell_2}\frac{\tanh^2(\xi)}{[1+\text{sech}(\xi)]^2}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{80}$$

$$\mathcal{V}(x,y,t)=[\frac{-(\ell_1-2)\xi_2}{2\ell_2\xi_1}+\frac{3\xi_2}{2\ell_2\xi_1}\frac{\tanh^2(\xi)}{[1+\text{sech}(\xi)]^2}+\frac{\rho}{\xi_1}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}. \tag{81}$$

Case 7: If $p=\frac{-1}{4}$, $q=\frac{(m^2+1)}{2}$, and $r=\frac{-(1-m^2)^2}{4}$, then $Z(\xi)=mcn(\xi)\pm dn(\xi)$. In this case, the exact solutions of the SRWE (1) found by utilizing Equations (2), (57), and (7) are

$$\mathcal{U}(x,y,t)=[\frac{-(\ell_1+2(1+m^2))}{2\ell_2}+\frac{3}{2\ell_2}(mcn(\xi)\pm dn(\xi))^2]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{82}$$

$$\mathcal{V}(x,y,t)=[\frac{-\xi_2(\ell_1+2(1+m^2))}{2\ell_2\xi_1}+\frac{3\xi_2}{2\ell_2\xi_1}(mcn(\xi)\pm dn(\xi))^2+\frac{\rho}{\xi_1}]e^{[\sigma W(t)-\frac{1}{2}\sigma^2 t]}, \tag{83}$$

where $\xi=\xi_1 x+\xi_2 y-[2(m^2+1)\gamma_1\xi_1^2\xi_2+\rho\gamma_3]t$. If $m\to 1$, then Equation (82) transfers to Equation (74).

Case 8: If $p = \frac{m^2-1}{4}$, $q = \frac{(m^2+1)}{2}$, and $r = \frac{m^2-1}{4}$, then $Z(\mu) = \frac{dn(\xi)}{1+sn(\xi)}$. In this case, the exact solutions of the SRWE (1) found by utilizing Equations (2), (57), and (7) are

$$\mathcal{U}(x,y,t) = [\frac{-(\ell_1 + 2(1+m^2))}{2\ell_2} - \frac{3(m^2-1)}{2\ell_2}\frac{dn^2(\xi)}{[1+sn(\xi)]^2}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{84}$$

$$\mathcal{V}(x,y,t) = [\frac{-\xi_2(\ell_1 + 2(1+m^2))}{2\ell_2\xi_1} - \frac{3\xi_2(m^2-1)}{2\ell_2\xi_1}\frac{dn^2(\xi)}{[1+sn(\xi)]^2} + \frac{\rho}{\xi_1}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{85}$$

where $\xi = \xi_1 x + \xi_2 y - [2(m^2+1)\gamma_1\xi_1^2\xi_2 + \rho\gamma_3]t$. When $m \to 0$, then Equation (84) transfers to

$$\mathcal{U}(x,y,t) = [\frac{-(\ell_1+2)}{2\ell_2} + \frac{3}{2\ell_2}\frac{1}{[1+\sin(\xi)]^2}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{86}$$

$$\mathcal{V}(x,y,t) = [\frac{-\xi_2(\ell_1+2)}{2\ell_2\xi_1} + \frac{3\xi_2}{2\ell_2\xi_1}\frac{1}{[1+\sin(\xi)]^2} + \frac{\rho}{\xi_1}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}. \tag{87}$$

Case 9: If $p = \frac{1-m^2}{4}$, $q = \frac{(1-m^2)}{2}$, and $r = \frac{1-m^2}{4}$, then $Z(\mu) = \frac{cn(\mu)}{1+sn(\mu)}$. In this case, the exact solutions of the SRWE (1) found by utilizing Equations (2), (57), and (7) are

$$\mathcal{U}(x,y,t) = [\frac{-(\ell_1 + 2(1-m^2))}{2\ell_2} - \frac{3(1-m^2)}{2\ell_2}\frac{cn^2(\xi)}{[1+sn(\xi)]^2}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{88}$$

$$\mathcal{V}(x,y,t) = [\frac{-\xi_2(\ell_1 + 2(1-m^2))}{2\ell_2\xi_1} - \frac{3\xi_2(1-m^2)}{2\ell_2\xi_1}\frac{cn^2(\xi)}{[1+sn(\xi)]^2} + \frac{\rho}{\xi_1}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{89}$$

where $\xi = \xi_1 x + \xi_2 y - [2(1-m^2)\gamma_1\xi_1^2\xi_2 + \rho\gamma_3]t$. When $m \to 0$, then Equation (84) transfers to

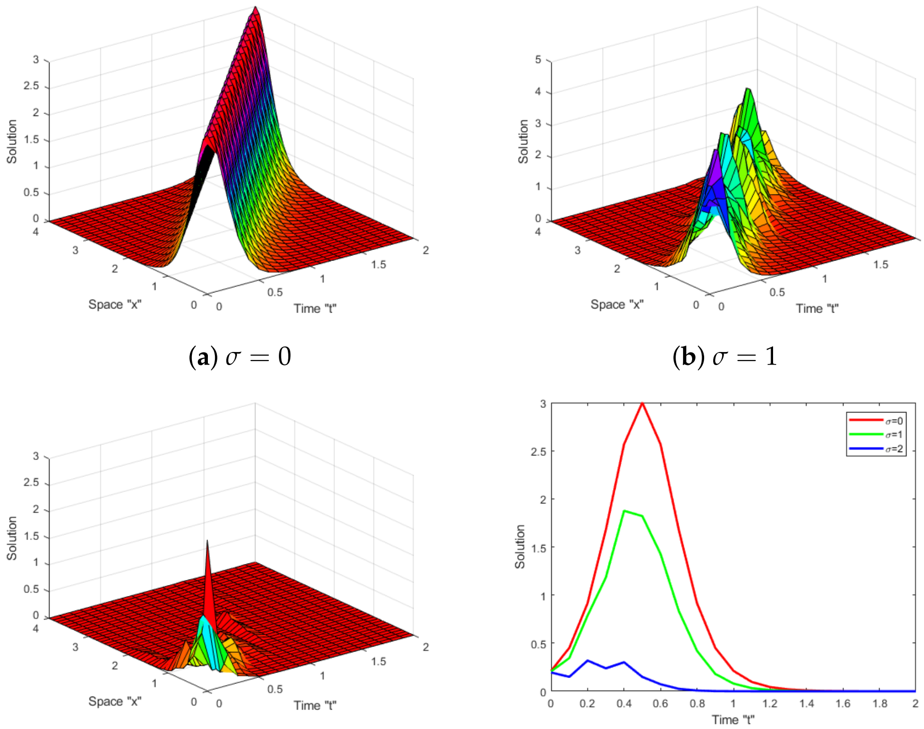
$$\mathcal{U}(x,y,t) = [\frac{-(\ell_1+2)}{2\ell_2} - \frac{3}{2\ell_2}\frac{\cos^2(\xi)}{[1+\sin(\xi)]^2}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}, \tag{90}$$

$$\mathcal{V}(x,y,t) = [\frac{-\xi_2(\ell_1+2)}{2\ell_2\xi_1} - \frac{3\xi_2}{2\ell_2\xi_1}\frac{\cos^2(\xi)}{[1+\sin(\xi)]^2} + \frac{\rho}{\xi_1}]e^{[\sigma W(t) - \frac{1}{2}\sigma^2 t]}. \tag{91}$$

4. The Impact of White Noise

We discuss here the impact of white noise on the acquired solutions of the SRWE (1). Different graphical representations are provided to illustrate the behavior of these solutions. For $\gamma_1 = 0.5$, $\gamma_2 = \gamma_3 = \xi_1 = 1$, and for varying values of ξ_2 and σ (noise intensity), we simulate the graphical representations for some attained solutions, including Equations (26), (27), (58), and (59), as follows:

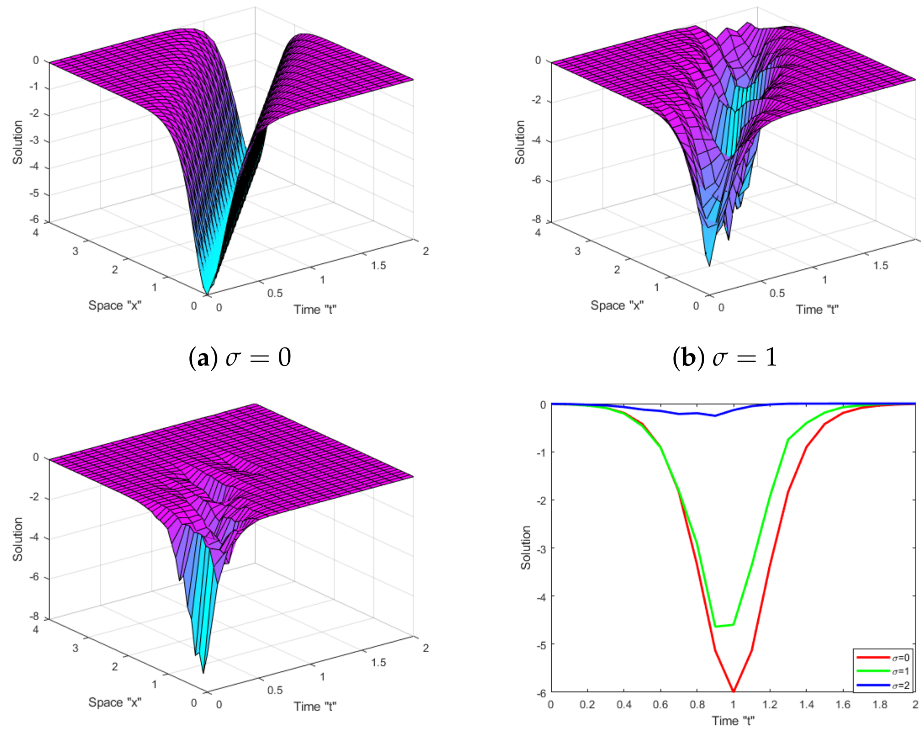
Finally, we can see in the preceding Figures 1–4 that when the noise is absent (i.e., $\sigma = 0$) there are several types of the solutions of the SRWE (1), including the bright bill, dark bill, periodic, and so on. However, when the noise occurs, all of these solutions begin to deteriorate, and the surface becomes flat as the noise amplitude rises. This indicates that the noise influences the solutions' behavior and it makes them stable.



(**a**) $\sigma = 0$ (**b**) $\sigma = 1$

(**c**) $\sigma = 2$ (**d**) $\sigma = 0,\ 1,\ 2$

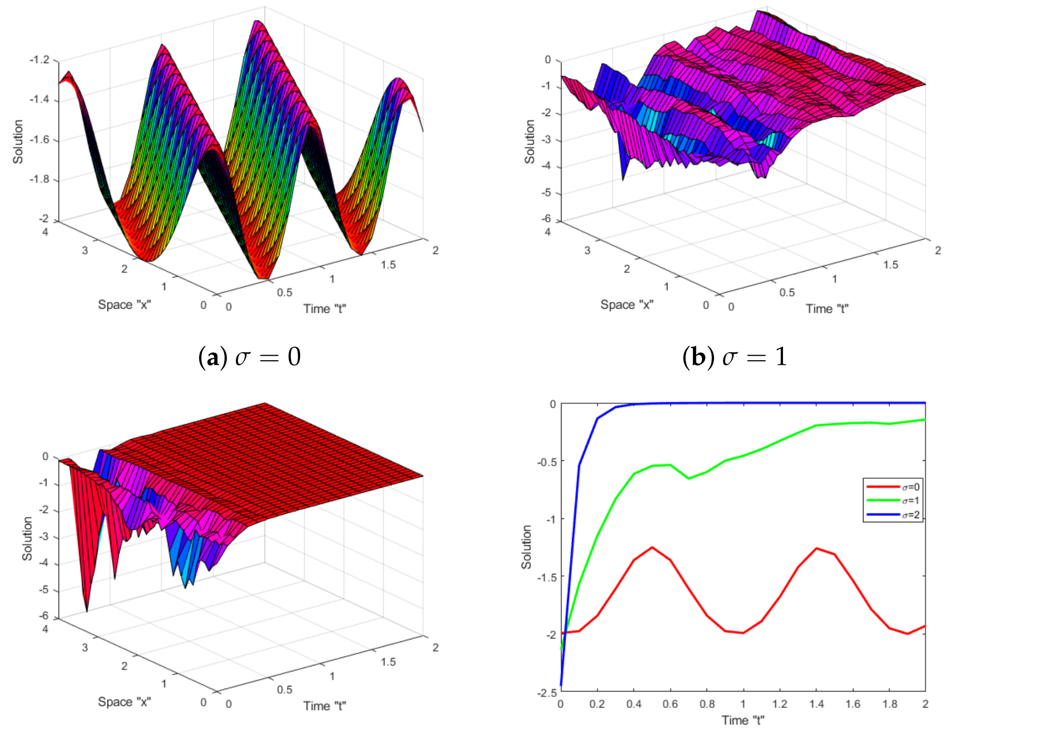
Figure 1. (**a**–**c**) present 3D profiles of $\mathcal{U}(x, y, t)$ given in Equation (26) for $\xi_2 = -2$ and for different $\sigma = 0,\ 1,\ 2$; (**d**) shows a 2D profile for these values of σ.



(**a**) $\sigma = 0$ (**b**) $\sigma = 1$

(**c**) $\sigma = 2$ (**d**) $\sigma = 0,\ 1,\ 2$

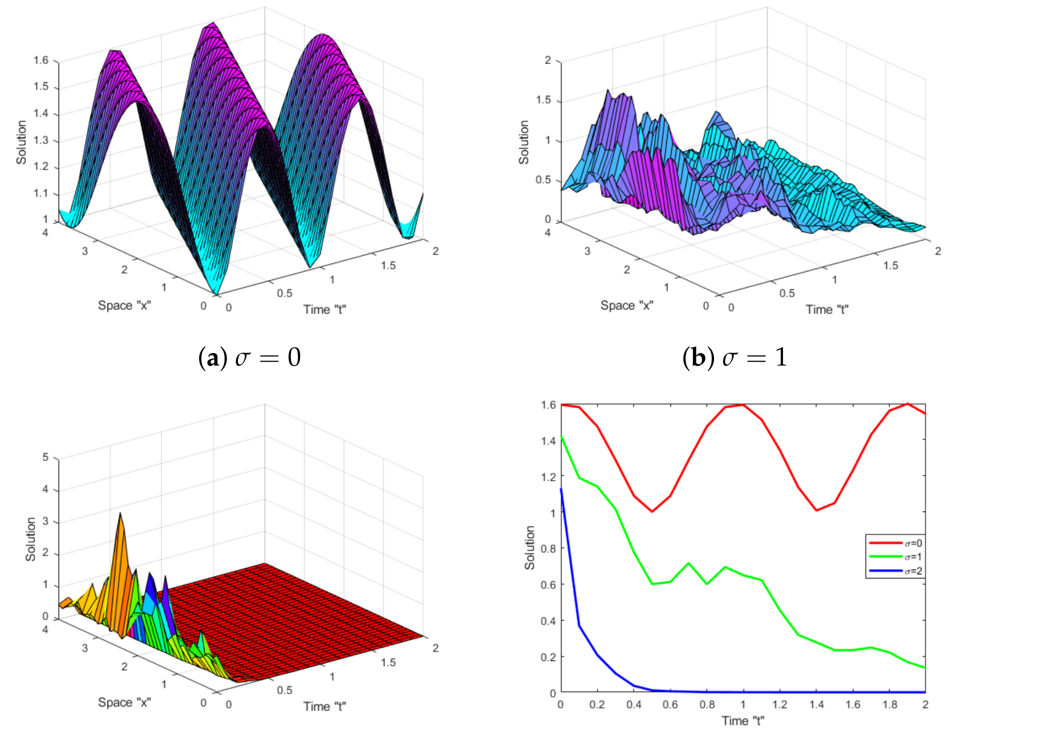
Figure 2. (**a**−**c**) present 3D profiles of $\mathcal{V}(x, y, t)$ given in Equation (27) for $\xi_2 = -2$ and for different $\sigma = 0,\ 1,\ 2$; (**d**) shows 2D shapes for these values of σ.



(**a**) $\sigma = 0$ (**b**) $\sigma = 1$

(**c**) $\sigma = 2$ (**d**) $\sigma = 0,\ 1,\ 2$

Figure 3. (**a**–**c**) present 3D shapes of the solution $\mathcal{U}(x, y, t)$ given in Equation (58) for $m = 0.5$, $\xi_2 = -0.8$ and different $\sigma = 0,\ 1,\ 2$; (**d**) shows a 2D profile for these values of σ.



(**a**) $\sigma = 0$ (**b**) $\sigma = 1$

(**c**) $\sigma = 2$ (**d**) $\sigma = 0,\ 1,\ 2$

Figure 4. (**a**–**c**) present 3D shapes of the solution $\mathcal{V}(x, y, t)$ given in Equation (59) for $m = 0.5$, $\xi_2 = -0.8$ and for various $\sigma = 0,\ 1,\ 2$; (**d**) shows a 2D profile for these values of σ.

5. Conclusions

In this study, we took into account the stochastic Riemann wave equation (SRWE) forced by white noise in the Itô sense (1). Using the extended tanh function approach and the mapping method, we were able to acquire exact solutions for the SRWE. Since the SRWE is required for describing wave propagation, the produced solutions are crucial for explaining a number of fascinating physical phenomena. Moreover, we expanded several previous results, such as those published in [13]. Finally, the MATLAB software was used to demonstrate the effect of multiplicative white noise on the exact solutions of the SRWE (1). We discovered that the stochastic term stabilizes the solutions of the SRWE around zero. In the future, we may investigate the SRWE with fractional derivatives or additive noise.

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