A Research Announcement on New Parametric U–Charlier–Poisson Polynomials and Their Szász–Type Operators

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Abstract

In this work, we present a new family of parametric U–Charlier–Poisson–type polynomials, denoted by $G_n^{[2+J]}(x;\alpha,\beta,\lambda)$. We establish several foundational properties of this family, including explicit representations, an associated orthogonality structure, and a link with derivatives of harmonic functions. Furthermore, we introduce Szász–type operators constructed from these polynomials and investigate their approximation behavior by means of Korovkin's theorem, thereby obtaining convergence results for the proposed operators.

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1. Introduction

Throughout this article, $\mathbb{N}$ will mean the set of natural numbers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$,likewise $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$ will denote the set of real numbers, positive real numbers, and the set of complex numbers. As usual, will denote by $C[0,\infty)$ the set of all functions f continuous in the interval $[0,\infty)$. The notation $UC[0,\infty)$ will denote the space of functions uniformly continuous on $[0,\infty)$. The space of all polynomials in one variable with real coefficients is denoted by $\mathbb{P}$, and $\log(z)$ denotes the principal value of the multi-valued logarithm function. In [5], a famous theorem about linear operators is published, known as the Korovkin Theorem, which states that a sequence of linear operators under certain conditions converges uniformly in each subset of the locally compact domain. Korovkin Theorem, in its many applications, was also used to demonstrate the convergence of Szász operators, which are defined by (see [9, p. 239, Eq. (2)]):

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \tag{1}$$

where $f \in C[0,\infty)$, $n \in \mathbb{N}$, and $x \geq 0$. The generalizations of Szász operators by using polynomials, especially defined via generating functions, have been frequently studied lately. These kinds

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of generalizations provide a range of new sequences of operators to approximation theory highly advantageous when interpolating positive continuous functions [9]. A known generalization of (1) can be obtained using the Appell polynomials given by (cf. [12]):

$$P_n(f;x) := \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{2}$$

considering that $p_k(x) \geq 0$, for $x \in [0, \infty)$ and $g(1) \neq 0$.

Some time later, Serhan Varma, et al., in ([12, p.122 Eq. (1.7)]), generalized (1) in the following way: first, they use the Brenke-type polynomials, which are defined by the following generating function:

$$\zeta(z)\xi(xz) = \sum_{k=0}^{\infty} p_k(x) \frac{z^k}{k!}, \tag{3}$$

where ζ and ξ are analytical functions. Second, they introduce the linear positive operators including the Brenke-type, polynomials which are given by (see [12, p. 121, Eq. (1.12)]):

$$L_n(f;x) := \frac{1}{\zeta(1)\xi(nx)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{4}$$

where $x \geq 0$ and $n \in \mathbb{N}$. It is observed that if $\xi(z) = e^z$ in (3), then the operators (4) concerning (3) lead to (2) with respect to the Appell polynomials, and if $\xi(z) = e^z$ and $\zeta(z) = 1$ in (4), we have (1).

On the other hand, when using the Brenke-type polynomials given in (3), with $\xi(z) = e^z$ and $\zeta(t) = \left(1 - \frac{z}{a}\right)^u$, we have the classic Charlier-Poisson polynomials, which are defined by (see [10, p. 458, Eq. (1.2)]):

$$e^z \left(1 - \frac{z}{a}\right)^u = \sum_{k=0}^{\infty} C_k(a,u) \frac{z^k}{k!}, \qquad a \neq 0. \tag{5}$$

Then, Serhan Varma, et al., introduce the positive linear operators involving Charlier-Poisson polynomials (see [11, p. 119, Eq. (1.6)]) by replacing $\xi(z) = e^z$ and $\zeta(z) = \left(1 - \frac{z}{a}\right)^u$ in (4), as follows:

$$L_n(f;x,a) = e^{-1}\left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k(a, -(a-1)nx)}{k!} f\left(\frac{k}{n}\right), \tag{6}$$

where $a > 1$ and $x \geq 0$. We see that if in (6) we take on both sides $a \to \infty$ and $x \to x - \frac{1}{n}$, then we get the Szász operators given in (1). The convergence and bounding properties of these operators were also investigated [11]. Furthermore, in [1], a study of Charlier-Poisson polynomials is presented, in particular, their explicit representation given by

$$C_n(x,\alpha) = \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} k! (-\alpha)^{n-k}. \tag{7}$$

The Charlier-Poisson polynomials $C_n(x,\alpha)$, $x \in \mathbb{N}_0$, and, $\alpha > 0$, are orthogonal with respect to the Poisson distribution with mean α, that is,

$$\sum_{x=0}^{\infty} C_m(x,\alpha)C_n(x,\alpha)\frac{e^{-\alpha}\beta^x}{x!} = \alpha^n n!\delta_{mn}, \quad m,n \in \mathbb{N}_0, \tag{8}$$

where δ_{mn} is the Kronecker delta.

Our contribution aims to introduce a new family of discrete polynomials, called new parametric U-Charlier-Poisson type polynomials, and we investigate some of their properties. Thus, the operators obtained from Brenke-type polynomials are applied to the said polynomials. The outline of this work is as follows: In Section 2, we study some basic results of operators obtained from Brenke-type polynomials applied to Charlier-Poisson polynomials and other results necessary for developing this work. In Section 3, we introduce the new parametric U-Charlier-Poisson type polynomials and explore some of their properties. In Section 4, we investigate the orthogonality relation. Finally, in Section 5, we apply the Szász-type operators (4), obtained from Brenke-type polynomials to the new family of polynomials to study the convergence and bounding properties.

2. Background and previous results

Let f be some function of a real variable x. The backward and forward difference operators Δ and ∇ respectively, are defined as (see [6, p. 19–20]):

$$\nabla f(x) := f(x) - f(x-1), \tag{9}$$
$$\Delta f(x) := f(x+1) - f(x). \tag{10}$$

Given two real sequences $\{a_k\}$ and $\{b_k\}$, if $b_{-1} = 0$, then (see [6, p. 20])

$$\sum_{k=0}^{\infty}(\Delta a_k)b_k = -\sum_{k=0}^{\infty} a_k \nabla b_k. \tag{11}$$

Furthermore, for $f_1(x)$ and $f_2(x)$ with real values, the following property is satisfied (cf. [3]):

$$\nabla(f_1(x)f_2(x)) = f_1(x)\nabla f_2(x) + f_2(x-1)\nabla f_1(x). \tag{12}$$

The falling factorial x of order n is (see [4])

$$\langle x\rangle_n := x(x-1)\cdots(x-n+1), \text{ with } \langle x\rangle_0 = 1, \tag{13}$$

and the rising factorial x of order n is (see [4])

$$(x)_n := x(x+1)\cdots(x+n-1), \quad (x)_0 := 1. \tag{14}$$

The rising factorial and the falling factorial fulfill the following relationship (see [3]):

$$(x)_n = \frac{\Gamma(n+x)}{\Gamma(x)}, \tag{15}$$
$$\langle x\rangle_n = \frac{x!}{(x-n)!}, \tag{16}$$

where $\Gamma(x)$ is the Gamma function.

On the other hand, the digamma function $\psi(x)$ is defined as the logarithmic derivative of the gamma function $\Gamma(x)$ (see [7])

$$\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{17}$$

The generalized harmonic numbers function is given by (see [7])

$$H_n^m(x) = \sum_{k=0}^{n-1} \frac{1}{(k+x)^m}, \quad n, m \in \mathbb{N}. \tag{18}$$

If $m = 1$ in (18), then

$$H_n^{(1)}(x) = \sum_{k=0}^{n-1} \frac{1}{k+x}. \tag{19}$$

If $x = 0$ in (18), we have

$$H_n^m(0) = H_n^m = \sum_{k=1}^{n} \frac{1}{k^m}, \tag{20}$$

where H_n^m denotes the so-called n-th harmonic numbers of order m.

Notice that from (15) and (17) follows

$$\begin{aligned} \frac{d}{dx}(x)_n &= \frac{\Gamma(n+x)}{\Gamma(x)} \left(\frac{d}{dx} \ln(\Gamma(n+x) - \frac{d}{dx} \ln(\Gamma(x) \right) \\ &= \frac{\Gamma(n+x)}{\Gamma(x)} \left(\psi(n+x) - \psi(x) \right). \end{aligned} \tag{21}$$

By (15), (19), and (21), we obtain

$$(x)_n = \frac{1}{H_n^{(1)}(x)} \frac{d}{dx}(x)_n. \tag{22}$$

The Stirling numbers of the first kind, $s(n,k)$, appear as the coefficients in the following generating function (see [8]):

$$\frac{(\log(1+z))^k}{k!} = \sum_{n=k}^{\infty} s(n,k) \frac{z^n}{n!}. \tag{23}$$

In addition, they also satisfy

$$\langle x \rangle_n = \sum_{k=0}^{n} s(n,k) x^k. \tag{24}$$

Note that from (24), we can write

$$(1+z)^x = \sum_{n=0}^{\infty} \binom{x}{n} z^n = \sum_{n=0}^{\infty} \langle x \rangle_n \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} s(n,k) \frac{z^n}{n!} \right) x^k. \tag{25}$$

Now, (cf. [1, p. 170 Eq. (1.1)]) it is possible to represent the Charlier-Poisson polynomials given in (5) as follows:

$$e^{-\alpha w}(1+w)^x = \sum_{n=0}^{\infty} C_n(x,\alpha)\frac{w^n}{n!}, \tag{26}$$

with $\alpha \neq 0$. Note that by taking $\alpha = a$, $w = -\frac{z}{a}$, $x = u$ in (26) we have (5).

It is worth noting that the classical orthogonal polynomials possess a weight function that conforms to the Pearson equation of the form

$$\nabla\left[\sigma(x)\omega(x)\right] = \tau(x)\omega(x), \tag{27}$$

whit σ a polynomial of degree ≤ 2 and τ a polynomial of degree ≤ 1. We note that in (27) the backward difference operator ∇, given in (9), is used for orthogonal polynomials on the lattice and it is replaced by differentiation in the case of orthogonal polynomials on an interval of the real line. The Pearson equation is an important part of the theory of classical orthogonal polynomials because it lets us find many useful properties of these polynomials.

Let $f \in UC[0,\infty)$, If $\delta > 0$, the modulus of continuity of the function f, denoted by $\omega(f;\delta)$ is defined by (cf. [11])

$$\omega(f;\delta) := \sup_{x,y\in[0,\infty)} |f(x)-f(y)|, \;\; where \;\; |x-y| < \delta. \tag{28}$$

Additionally, it is well known that,

$$|f(x)-f(y)| \leq \omega(f,\delta)\left(\frac{|x-y|}{\delta}+1\right). \tag{29}$$

Also, we have if f is uniformly continuous, then

$$|f(x)-f(y)| \leq \omega(f,\delta). \tag{30}$$

The following Proposition summarizes some properties of the operators defined in (6).

Proposition 2.1. *For $n \in \mathbb{N}$, let $L_n(f;x,a)$ the positive linear operators involving Charlier-Poisson polynomials in the variable $x \geq 0$. Then, the following statements hold.*

1. *[11, Lemma 1] The operators defined in (6) satisfy the following identities:*
   (i) $L_n(1;x,a) = 1$.
   (ii) $L_n(s;x,a) = x + \frac{1}{n}$.
   (iii) $L_n(s^2;x,a) = x^2 + \frac{x}{n}\left(3+\frac{1}{a-1}\right) + \frac{2}{n^2}$.
2. *[11, Theorem 1] Let E be the set given by*

$$S := \left\{f : [0,\infty) \to \mathbb{R} : |f(x)| \leq Me^{Ax}, \;\; A \in \mathbb{R} \; and \; M \in \mathbb{R}^+\right\}.$$

If $f \in C[0,\infty) \cap S$, then

$$\lim_{n\to\infty} L_n(f;x,a) = f(x). \tag{31}$$

That is, the operators defined in (6) converge uniformly on every compact subset of $[0,\infty)$.

3. *[11, Theorem 2] Let $f \in UC[0,\infty) \cap S$. Then the operators L_n given in (6) satisfy*

$$|L_n(f;x,a) - f(x)| \leq \left\{1 + \sqrt{x\left(1+\frac{1}{a-1}\right)+\frac{2}{n}}\right\} \omega\left(f;\frac{1}{\sqrt{n}}\right), \tag{32}$$

with ω given by (28).

3. New parametric U-Charlier-Poisson type polynomials and some of their properties

In this section, we shall introduce a new class of discrete polynomials, which we denote by $G_n^{[2+J]}(x;\alpha,\beta,\lambda)$ and will we call new parametric U-Charlier-Poisson type polynomials. Furthermore, we obtain some of their properties.

Definition 3.1. *For a fixed $J \in \mathbb{N}$, $\beta, \lambda \in \mathbb{R}$ and $\alpha \neq 0$, the new family of parametric U-Charlier-Poisson type polynomials $G_n^{[2+J]}(x;\alpha,\beta,\lambda)$ in the variable $x \in \mathbb{N}_0$ are defined by the means of the power series expansion at 0 of the following generating function:*

$$u(x;z;\alpha,\beta,\lambda) = \left[\beta e^{-\alpha z} + \lambda(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}\right](1+z)^x = \sum_{n=0}^{\infty} G_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!}. \tag{33}$$

From (33) and taking $A_j(\lambda,\alpha) = \lambda(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}$, the first parametric U-Charlier-Poisson type polynomials are obtained, which are:

$$\begin{aligned}
G_0^{[2+J]}(x;\alpha,\beta,\lambda) &= \beta + A_j(\lambda,\alpha),\\
G_1^{[2+J]}(x;\alpha,\beta,\lambda) &= -\alpha\beta + A_j(\lambda,\alpha) + x(\beta + A_j(\lambda,\alpha)),\\
G_2^{[2+J]}(x;\alpha,\beta,\lambda) &= \alpha^2\beta - 2\alpha\beta x + x(x-1)(\beta + A_j(\lambda,\alpha)),\\
G_3^{[2+J]}(x;\alpha,\beta,\lambda) &= -\alpha^3\beta + \alpha^2\beta x - 2\alpha\beta x(x-1) + x(x-1)(x-2)(-\alpha\beta + \beta + A_j(\lambda,\alpha)).
\end{aligned}$$

Note that if $\beta = 1$ and $\lambda = 0$, $z = w$ in (33), we have the classic Charlier-Poisson polynomials given in (26). Therefore, the generating function of $G_n^{[2+J]}(x;\alpha,\beta,\lambda)$ in (33) includes, as its special cases, the generating function of the Charlier-Poisson polynomials, i.e., $C_n(x,\alpha) = G_n^{[2+J]}(x;\alpha,1,0)$.

Substituting $x = 0$ in (33), we have

$$u(0;z;\alpha,\beta,\lambda) = \left[\beta e^{-\alpha z} + \lambda(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}\right] = \sum_{n=0}^{\infty} G_n^{[2+J]}(0,\alpha,\beta,\lambda)\frac{z^n}{n!}, \tag{34}$$

where $G_n^{[2+J]}(0,\alpha,\beta,\lambda) = G_n^{[2+J]}(\alpha,\beta,\lambda)$ denotes the parametric U-Charlier-Poisson type numbers. In view of (34), we can compute a few values of the numbers $G_n^{[2+J]}(\alpha,\beta,\lambda)$ as follows:

$$G_0^{[2+J]}(\alpha,\beta,\lambda) = \beta + A_j(\lambda,\alpha), \qquad G_3^{[2+J]}(\alpha,\beta,\lambda) = -\beta\alpha^3,$$
$$G_1^{[2+J]}(\alpha,\beta,\lambda) = -\beta\alpha, \qquad G_4^{[2+J]}(\alpha,\beta,\lambda) = \beta\alpha^4,$$
$$G_2^{[2+J]}(\alpha,\beta,\lambda) = \alpha^2\beta, \qquad G_5^{[2+J]}(\alpha,\beta,\lambda) = -\beta\alpha^5.$$

One can use $A_j(\lambda,\alpha)$ in the following manner:

$$A_j(\lambda,\alpha) = \lambda(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2} = \sum_{n=0}^{\infty} U_n^{[2+J]}(\alpha)\frac{\lambda^n}{n!}. \tag{35}$$

Whereby some $U_n^{[2+J]}(\alpha)$ are

$$U_o^{[2+J]}(\alpha) = (-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}, \qquad U_2^{[2+J]}(\alpha) = 2(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2},$$
$$U_1^{[2+J]}(\alpha) = (-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}, \qquad U_3^{[2+J]}(\alpha) = 6(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}.$$

Proposition 3.1. *Let $\beta \in \mathbb{R} - \{0\}$, $J \in \mathbb{N}$ fixed, and $\left\{G_k^{[2+J]}(\alpha,\beta,\lambda)\right\}_{k=0}^{\infty}$ be a parametric U-Charlier-Poisson type sequence of numbers defined as in (34). Then, the following relationship is fulfilled:*

$$G_n^{[2+J]}(\alpha,\beta,\lambda) = (-1)^n\beta\alpha^n, \tag{36}$$
$$\textit{with} \quad G_0^{[2+J]}(\alpha,\beta,\lambda) = \beta + A_j(\lambda,\alpha). \tag{37}$$

Proof. By using (34) follows

$$\sum_{n=0}^{\infty} G_n^{[2+J]}(\alpha,\beta,\lambda)\frac{z^n}{n!} = A_j(\lambda,\alpha) + \beta\sum_{n=0}^{\infty}(-1)^n\alpha^n\frac{z^n}{n!}$$
$$\Leftrightarrow \quad G_0^{[2+J]}(\alpha,\beta,\lambda) + \sum_{n=1}^{\infty} G_n^{[2+J]}(\alpha,\beta,\lambda)\frac{z^n}{n!} = \beta\sum_{n=0}^{\infty}(-1)^n\alpha^n\frac{z^n}{n!} + A_j(\lambda,\alpha).$$

With what we have,

$$G_0^{[2+J]}(\alpha,\beta,\lambda) = \sum_{n=1}^{\infty}\left[(-1)^n\alpha^n\beta - G_n^{[2+J]}(\alpha,\beta,\lambda)\right]\frac{z^n}{n!} + (\beta + A_j(\lambda,\alpha)). \tag{38}$$

From (38) follows (36) and (37). Proposition 3.1 is proved.

With its proof, the following proposition provides a concise formula for the parametric U-Charlier-Poisson type polynomials $G_n^{[2+J]}(x;\alpha,\beta,\lambda)$.

Proposition 3.2. *Given a fixed* $J \in \mathbb{N}$, *let* $\left\{G_n^{[2+J]}(x;\alpha,\beta,\lambda)\right\}_{n=0}^{\infty}$ *be a parametric* U*-Charlier-Poisson type sequence of polynomials, defined as in (33). Then, the following explicit representation hold:*

$$G_n^{[2+J]}(x;\alpha,\beta,\lambda) = \beta C_n(x,\alpha) + \lambda(-\alpha)^{-J}\left[\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}\right]\langle x\rangle_n, \qquad (39)$$

where $\langle x\rangle_n$, *is the falling factorial defined in (13).*

Proof. Using the generating function of the parametric U-Charlier-Poisson type polynomials given in (33), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!} &= \left[\beta e^{-\alpha z} + \lambda(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}\right](1+z)^x \\
&= \beta\sum_{n=0}^{\infty} C_n(x,\alpha)\frac{z^n}{n!} + \lambda(-\alpha)^{-J}\left[\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}\right]\sum_{n=0}^{\infty}\binom{x}{n}z^n \\
&= \beta\sum_{n=0}^{\infty} C_n(x,\alpha)\frac{z^n}{n!} + \lambda(-\alpha)^{-J}\left[\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}\right]\sum_{n=0}^{\infty}\langle x\rangle_n\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\left[\beta C_n(x,\alpha) + \lambda(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}\langle x\rangle_n\right]\frac{z^n}{n!}.
\end{aligned}$$

Considering the above expression, we thus have (39). Proposition 3.2 is demonstrated.

Proposition 3.3. *For a fixed* $J \in \mathbb{N}$, *let* $\left\{G_k^{[2+J]}(x;\alpha,\beta,\lambda)\right\}_{k=0}^{\infty}$ *be a parametric* U*-Charlier-Poisson type sequence of polynomials defined by (33). If* $\beta \to 0$, *and* $\lambda \to 1$, *then the following identity holds:*

$$\sum_{k=0}^{n}\binom{n}{k}G_k^{[2+J]}(x;\alpha,0,1)C_{n-k}(-\alpha,-x) = \alpha^n\sum_{n=0}^{\infty}n!\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}. \qquad (40)$$

Proof. Let us write (33) as

$$\begin{aligned}
\left[\beta e^{-\alpha z}(1+z)^x + A_j(\lambda,\alpha)(1+z)^x\right]e^{\alpha z}(1+z)^{-x} &= \left(\sum_{n=0}^{\infty}G_n^{[2+J]}(x;\beta,\alpha,\lambda)\frac{z^n}{n!}\right)e^{\alpha z}(1+z)^{-x} \\
&= \left(\sum_{n=0}^{\infty}G_n^{[2+J]}(x;\beta,\alpha,\lambda)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}C_n(-\alpha,-x)\frac{z^n}{n!}\right).
\end{aligned}$$

From the above expression and (35), we have

$$\beta + e^{\alpha z}A_j(\lambda,\alpha) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}G_k^{[2+J]}(x;\beta,\alpha,\lambda)C_{n-k}(-x,-\alpha)\frac{z^n}{n!}.$$

$$\Leftrightarrow \quad \beta + \sum_{n=0}^{\infty} \alpha^n \left(\sum_{n=0}^{\infty} U_n^{[2+J]}(\lambda;\alpha) \frac{\lambda^n}{n!} \right) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} G_k^{[2+J]}(x;\beta,\alpha,\lambda) C_{n-k}(-x,-\alpha) \frac{z^n}{n!}.$$

Then, taking $\beta \to 0$, and $\lambda \to 1$, follows

$$\sum_{n=0}^{\infty} \alpha^n \left(\sum_{n=0}^{\infty} n! \prod_{m=2}^{2+J} \frac{2^m - 1}{2^m - 2} \right) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} G_k^{[2+J]}(x;\beta,\alpha,\lambda) C_{n-k}(-\alpha,-x) \frac{z^n}{n!},$$

from which (40) follows. Proposition 3.3 is demonstrated.

Proposition 3.4. *For a fixed $J \in \mathbb{N}$, $\beta \in \mathbb{R} - \{0\}$, let $\left\{ G_k^{[2+J]}(x;\alpha,\beta,\lambda) \right\}_{k=0}^{\infty}$ be a parametric U-Charlier-Poisson type sequence of polynomials defined by (33). Then, we have the following relationship:*

$$A_j(\lambda,\alpha) C_n(x,-\alpha) + \sum_{k=0}^{n} \beta s(n,k) x^k = \sum_{l=0}^{n} \binom{n}{l} G_l^{[2+j]}(x;\alpha,\beta,\lambda) \alpha^{n-l}, \tag{41}$$

where $s(n,k)$ is defined by (24).

Proof. From (33), implies that

$$\begin{aligned} \beta(1+z)^x &= \left[\sum_{n=0}^{\infty} G_n^{[2+J]}(x;\alpha,\beta,\lambda) \frac{z^n}{n!} - A_j(\lambda,\alpha)(1+z)^x \right] \frac{1}{e^{-\alpha z}} \\ &= \left(\sum_{n=0}^{\infty} G_n^{[2+J]}(x;\alpha,\beta,\lambda) \frac{z^n}{n!} \right) \left(\sum_{n=0}^{\infty} \alpha^n \frac{z^n}{n!} \right) - A_j(\lambda,\alpha) \sum_{n=0}^{\infty} C_n(x,-\alpha) \frac{z^n}{n!}. \end{aligned}$$

Now, using (25) follows:

$$\begin{aligned} \beta \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} s(n,k) x^k \right) \frac{z^n}{n!} &= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} G_l^{[2+J]}(x;\alpha,\beta,\lambda) \alpha^{n-l} \frac{z^n}{n!} - A_j(\lambda,\alpha) \sum_{n=0}^{\infty} C_n(x,-\alpha) \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} G_l^{[2+J]}(x;\alpha,\beta,\lambda) \alpha^{n-l} - A_j(\lambda,\alpha) C_n(x,-\alpha) \right) \frac{z^n}{n!}, \end{aligned}$$

which yields (41). Our Proposition 3.4 is proven.

Proposition 3.5. *For a fixed $J \in \mathbb{N}$, $\beta \in \mathbb{R} - \{0\}$ the following relations hold for the parametric U-Charlier-Poisson type polynomials defined by (33):*

$$n \frac{\partial}{\partial x} G_{n-1}^{[2+J]}(x;\alpha,\beta,\lambda) = \sum_{k=1}^{n} (-1)^k (n-k) \langle n \rangle_k G_{n-k-1}^{[2+J]}(x;\alpha,\beta,\lambda), \quad (G_{-n}^{[2+J]} \equiv 0), \tag{42}$$

$$\frac{1}{\alpha} G_{n+1}^{[2+J]}(x;\alpha,\beta,\lambda) - \aleph(x;z;\alpha) \frac{\partial}{\partial x} G_n^{[2+J]}(x;\alpha,\beta,\lambda) + G_n^{[2+J]}(x;\alpha,\beta,\lambda) - A_j(\lambda,\alpha) \langle x \rangle_n = 0, \tag{43}$$

where $\alpha \in \mathbb{R} - \{0\}$, $z \in \mathbb{C} - \{0,-1\}$, $n \in \mathbb{N}$ with

$$\aleph(x;z;\alpha) = \frac{x}{\alpha} \left[\frac{1}{(1+z)\log(1+z)} \right], \tag{44}$$

and $A_j(\lambda,\alpha)$ given in (35).

PROOF. To prove (42), we note that by differentiating (33) with respect to x, we can write

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\partial}{\partial x} G_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!} &= \left(\sum_{n=1}^{\infty}(-1)^n \frac{z^n}{n}\right)\left(\sum_{n=0}^{\infty} G_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!}\right)\\
\Leftrightarrow \sum_{n=1}^{\infty} \frac{\partial}{\partial x} G_{n-1}^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^{n-1}}{(n-1)!} &= \left(\sum_{n=1}^{\infty}(-1)^n (n-1)!\frac{z^n}{n}\right)\left(\sum_{n=0}^{\infty} G_{n-1}^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^{n-1}}{(n-1)!}\right)\\
\Leftrightarrow \sum_{n=1}^{\infty} n\, \frac{\partial}{\partial x} G_{n-1}^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!} &= \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^k (k-1)!\binom{n}{k}(n-k)G_{n-1-k}^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!}\\
\Leftrightarrow n\, \frac{\partial}{\partial x} G_{n-1}^{[2+J]}(x;\alpha,\beta,\lambda) &= \sum_{k=1}^{n}(-1)^k (k-1)!\binom{n}{k}(n-k)G_{n-1-k}^{[2+J]}(x;\alpha,\beta,\lambda).
\end{aligned}$$

Of the above expression and applying (16) follows (42).

Now to prove (43), we derive (33) with respect to z as follows:

$$\frac{\partial}{\partial z}u(x;z;\alpha,\beta,\lambda) = \sum_{n=0}^{\infty} G_{n+1}^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!}, \tag{45}$$

and

$$\frac{\partial}{\partial z}u(x;z;\alpha,\beta,\lambda) = \frac{x}{(1+z)}\left[(1+z)^x(\beta\, e^{-\alpha z} + A_j(\lambda,\alpha))\right] - \alpha\,\beta\, e^{-\alpha z}(1+z)^x. \tag{46}$$

Likewise, if we derive (33) with respect to x, we have the following:

$$\frac{\partial}{\partial x}u(x;z;\alpha,\beta,\lambda) = \sum_{n=0}^{\infty} \frac{\partial}{\partial x} G_n^{[2+J]}(x;\alpha,\beta,\lambda)\,\frac{z^n}{n!}, \tag{47}$$

$$\frac{\partial}{\partial x}u(x;z;\alpha,\beta,\lambda) = (1+z)\log(1+z)(\beta\, e^{-\alpha z} + A_j(\lambda,\alpha)). \tag{48}$$

By using (45), (46), (47), and (48), we obtain

$$\begin{aligned}
&\frac{1}{\alpha}\frac{\partial}{\partial z}u(x;z;\alpha,\beta,\lambda) - \frac{1}{\alpha}\left[\frac{x}{(1+z)\log(1+z)}\right]\frac{\partial}{\partial x}u(x;z;\alpha,\beta,\lambda)\\
&\qquad + u(x;z;\alpha,\beta,\lambda) - (1+z)^x A_j(\lambda,\alpha) = 0\\
\Leftrightarrow\ & \frac{1}{\alpha}\sum_{n=0}^{\infty} G_{n+1}^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!} - \frac{1}{\alpha}\sum_{n=0}^{\infty}\left[\frac{x}{(1+z)\log(1+z)}\right]\frac{\partial}{\partial x}G_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!}\\
&\qquad + \sum_{n=0}^{\infty} G_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!} - A_j(\lambda,\alpha)\sum_{n=0}^{\infty}\binom{x}{n}z^n = 0\\
\Leftrightarrow\ & \sum_{n=0}^{\infty}\frac{1}{\alpha} G_{n+1}^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!} - \sum_{n=0}^{\infty}\frac{1}{\alpha}\left[\frac{x}{(1+z)\log(1+z)}\right]\frac{\partial}{\partial x}G_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!}\\
&\qquad + \sum_{n=0}^{\infty} G_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!} - \sum_{n=0}^{\infty} A_j(\lambda,\alpha)\langle x\rangle_n\frac{z^n}{n!} = 0.
\end{aligned}$$

Of the previous equation taking $\aleph(x;z;\alpha)$ as in (44), (43) follows. Proposition 3.5 is proved.

Proposition 3.6. *For a fixed $J \in \mathbb{N}$, $\beta \in \mathbb{R} - \{0\}$, let $G_n^{[2+J]}(x;\alpha,\beta,\lambda)$ be the parametric U-Charlier-Poisson type polynomials. Then the following statement holds:*

$$\frac{d}{dx}(x)_n = \frac{H_n(x)}{\beta}\left[-\lambda(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2} + \sum_{k=0}^{n}(-1)^n\alpha^k G_{n-k}^{[2+J]}(-x;\alpha,\beta,\lambda)\right], \tag{49}$$

using $(x)_n$ given by (14), and $H_n(x) = H_n^{(1)}(x)$ given in (19).

Proof. Taking $z \to -z$, and $x \to -x$ in (33), we have

$$\begin{aligned}
\beta(1-z)^{-x} &= e^{-\alpha z}\sum_{n=0}^{\infty}(-1)^n G_n^{[2+J]}(-x;\alpha,\beta,\lambda)\frac{z^n}{n!} - A_j e^{-\alpha z}(1-z)^{-x}\\
&= \sum_{n=0}^{\infty}(-1)^n\alpha^n\frac{z^n}{n!}\sum_{n=0}^{\infty}(-1)^n G_n^{[2+J]}(-x;\alpha,\beta,\lambda)\frac{z^n}{n!} - A_j(\lambda,\alpha)\sum_{n=0}^{\infty}(-1)^n C_n(-x,\alpha)\frac{z^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(-1)^n\alpha^k G_{n-k}^{[2+J]}(-x;\alpha,\beta,\lambda)\frac{z^n}{n!} - A_j(\lambda,\alpha)\sum_{n=0}^{\infty}(-1)^n C_n(-x,\alpha)\frac{z^n}{n!}.
\end{aligned}$$

Then, for $|z| < 1$, using the Binomial Theorem, we have

$$\begin{aligned}
\frac{1}{(1-z)^x} &= \sum_{n=0}^{\infty}\beta^{-1}\left[\sum_{k=0}^{n}\binom{n}{k}(-1)^n\alpha^k G_{n-k}^{[2+J]}(-x;\alpha,\beta,\lambda) - A_j(\lambda,\alpha)(-1)^n C_n(-x,\alpha)\right]\frac{z^n}{n!}\\
\sum_{n=0}^{\infty}(x)_n\frac{z^n}{n!} &= \sum_{n=0}^{\infty}\beta^{-1}\left[\sum_{k=0}^{n}\binom{n}{k}(-1)^n\alpha^k G_{n-k}^{[2+J]}(-x;\alpha,\beta,\lambda) - A_j(\lambda,\alpha)(-1)^n C_n(-x,\alpha)\right]\frac{z^n}{n!}\\
(x)_n &= \sum_{k=0}^{n}(-1)^n\beta^{-1}\alpha^k G_{n-k}^{[2+J]}(-x;\alpha,\beta,\lambda) - \beta^{-1}A_j(-1)^n C_n(-x,\alpha).
\end{aligned}$$

This way, using (22) follows (49). This completes the demonstration of Proposition 3.6.

4. Orthogonality relationship of the polynomials $G_n^{[2+J]}(x;\alpha,\beta,\lambda)$

The main aim of this section is to obtain the relation of orthogonality satisfied by the polynomials $G_n^{[2+J]}(x;\alpha,\beta,\lambda)$, and apply it to study a relationship between these polynomials and the operator ∇ given in (9).

For a fixed $J \in \mathbb{N}$, we define the parametric U-Charlier-Poisson discrete weight function ω^α as

$$\omega^\alpha(x;\beta,\lambda) = \frac{e^{-\alpha}\alpha^x}{x!}\left|\mathscr{M}(\beta,\lambda,\alpha) + i\Theta(\beta,\lambda,\alpha)\right|^{-2}, \tag{50}$$

where $\alpha \in \mathbb{R}$, $\alpha < 0$; β, $\lambda \in \mathbb{R} - \{0\}$, on the lattice $\mathbb{N}$; $z, w \in \mathbb{C}$; $z = a_1 + ia_2$, $w = c_1 + ic_2$ in the circle $C(0,|\eta|)$ and $|\eta| = \min\{|z|,|w|\}$. While $\mathscr{M}(\beta,\lambda,\alpha)$ and $\Theta(\beta,\lambda,\alpha)$ are given by

$$\begin{aligned}
\mathscr{M}(\beta,\lambda,\alpha) &= \beta\left(\beta + A_j(\lambda,\alpha)(\varepsilon_2\cos(c_2\alpha) + \varepsilon_1\cos(a_2\alpha))\right)\\
&\quad + \left[A_j(\lambda,\alpha)\right]^2\varepsilon_1\varepsilon_2\cos(\alpha(a_2+c_2)), \qquad (51)\\
\Theta(\beta,\lambda,\alpha) &= \beta A_j(\lambda,\alpha)\left(\varepsilon_2\sin(c_2\alpha) + \varepsilon_1\sin(a_2\alpha)\right)\\
&\quad + \left[A_j(\lambda,\alpha)\right]^2\varepsilon_1\varepsilon_2\sin(\alpha(a_2+c_2)), \qquad (52)
\end{aligned}$$

where $A_j(\lambda,\alpha)$ given in (35), $\varepsilon_1 = e^{a_1\alpha}$, and $\varepsilon_2 = e^{c_1\alpha}$.

With the weight $\omega^{\alpha}(x;\beta,\lambda)$ given in (50), we can introduce on $\mathbb{P}$ the inner product as follows:

$$\langle f,g\rangle_{\omega^{\alpha}} = \sum_{x=0}^{\infty} f(x)g(x)\omega^{\alpha}(x;\beta,\lambda), \tag{53}$$

where $f,g\in\mathbb{P}$. Which has positive weights for every $\alpha<0$

The Pearson equation concerning (27) for weight (50) is now of the form

$$\nabla\omega^{\beta}(x;\alpha,\beta,\lambda) = \left(\frac{\alpha-x}{\alpha}\right)\omega^{\beta}(x;\alpha,\beta,\lambda). \tag{54}$$

Theorem 4.1. *For a fixed $J\in\mathbb{N}$, if $\alpha\in\mathbb{R}$, $\alpha<0$, β, $\lambda\in\mathbb{R}-\{0\}$, and $m,n\in\mathbb{N}_{\not{\nu}}$. Then, the parametric U-Charlier-Poisson type polynomials for the weight (50) satisfy the following orthogonality relation:*

$$\sum_{x=0}^{\infty} G_m^{[2+J]}(x;\alpha,\beta,\lambda)G_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{e^{-\alpha}\alpha^x}{x!}\,|\Omega(\beta,\lambda,\alpha)|^{-2} = \frac{\Gamma(n+1)\alpha^n}{\Omega(\beta,\lambda,\alpha)}\delta_{m,n}. \tag{55}$$

Whit $\Omega(\beta,\lambda,\alpha)=\mathscr{M}(\beta,\lambda,\alpha)+i\Theta(\beta,\lambda,\alpha)$.

Proof. One can see that from (33) follows:

$$\begin{aligned} L_G(x;z,\alpha,\beta,\lambda) &= \beta\left(\sum_{n=0}^{\infty}(-\alpha)^n\frac{z^n}{n!}\right)\sum_{n=0}^{\infty}\binom{x}{n}z^n + A_j(\lambda,\alpha)\sum_{n=0}^{\infty}\binom{x}{n}z^n \\ &= \beta\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(-\alpha)^{n-k}\langle x\rangle_k\frac{z^n}{n!} + A_j(\lambda,\alpha)\sum_{n=0}^{\infty}\langle x\rangle_n\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\left[\beta\sum_{k=0}^{n}\binom{n}{k}(-\alpha)^{n-k}\langle x\rangle_k + A_j(\lambda,\alpha)\langle x\rangle_n\right]\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\beta\binom{n}{k}(-\alpha)^{n-k}\frac{\langle x\rangle_k}{n!} + A_j(\lambda,\alpha)\frac{\langle x\rangle_n}{n!}\right]z^n. \end{aligned}$$

This implies that

$$L_G(x;z,\alpha,\beta,\lambda) = \sum_{n=0}^{\infty} D_n^{[2+J]}(x;\alpha,\beta,\lambda)z^n, \tag{56}$$

where

$$D_n^{[2+J]}(x;\alpha,\beta,\lambda) = \sum_{k=0}^{n}\beta\binom{n}{k}(-\alpha)^{n-k}\frac{\langle x\rangle_k}{n!} + A_j(\lambda,\alpha)\frac{\langle x\rangle_n}{n!}. \tag{57}$$

Similarly, we obtain

$$L_G(x;w,\alpha,\beta,\lambda) = \sum_{n=0}^{\infty} D_m^{[2+J]}(x;\alpha,\beta,\lambda)w^n, \tag{58}$$

with

$$D_n^{[2+J]}(x;\alpha,\beta,\lambda) = \sum_{k=0}^{m} \beta \binom{m}{k} (-\alpha)^{m-k} \frac{\langle x \rangle_k}{m!} + A_j(\lambda,\alpha) \frac{\langle x \rangle_m}{m!}. \quad (59)$$

On the other hand, we have

$$\begin{aligned} L_G(x;z,\alpha,\beta,\lambda)L_G(x;w,\alpha,\beta,\lambda) &= \left[\beta e^{-\alpha z} + A_j(\lambda,\alpha)\right]\left[\beta e^{-\alpha w} + A_j(\lambda,\alpha)\right](1+z)^x(1+w)^x \\ &= e^{-\alpha z}e^{-\alpha w}(\beta + A_j(\lambda,\alpha)e^{\alpha z})(\beta + A_j(\lambda,\alpha)e^{\alpha w})(1+z)^x(1+w)^x, \end{aligned}$$

and so, we have that

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\alpha^k L_G(x;z,\alpha,\beta,\lambda)L_G(x;w,\alpha,\beta,\lambda)}{k!} &= (\beta + A_j(\lambda,\alpha)e^{\alpha z})(\beta + A_j(\lambda,\alpha)e^{\alpha w})e^{-\alpha z}e^{-\alpha w}e^{\alpha(1+z)(1+w)} \\ &= (\beta + A_j(\lambda,\alpha)e^{\alpha z})(\beta + A_j(\lambda,\alpha)e^{\alpha w})e^{\alpha}e^{\alpha z w}. \end{aligned}$$

So,

$$\sum_{k=0}^{\infty} L_G(x;z,\alpha,\beta,\lambda)L_G(x;w,\alpha,\beta,\lambda)\frac{e^{-\alpha}\alpha^k}{k!} = (\beta + A_j(\lambda,\alpha)e^{\alpha z})(\beta + A_j(\lambda,\alpha)e^{\alpha w}) \sum_{n=0}^{\infty} \alpha^n \frac{z^n w^n}{n!}. \quad (60)$$

By using (56) and (58) on the left side of (60), we found

$$\begin{aligned} \sum_{k=0}^{\infty} L_G(x;z,\alpha,\beta,\lambda)L_G(x;w,\alpha,\beta,\lambda)\frac{e^{-\alpha}\alpha^k}{k!} &= \sum_{k=0}^{\infty} \frac{e^{-\alpha}\alpha^k}{k!} \sum_{n=0}^{\infty} D_n^{[2+J]}(x;\alpha,\beta,\lambda)z^n \sum_{m=0}^{\infty} D_m^{[2+J]}(x;\alpha,\beta,\lambda)w^n \\ &= \sum_{m,n=0}^{\infty} \sum_{k=0}^{\infty} D_m^{[2+J]}(x;\alpha,\beta,\lambda)D_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{e^{-\alpha}\alpha^k}{k!} z^n w^n. \quad (61) \end{aligned}$$

By combining Equation (60) with Equation (61), we have that.

$$\sum_{n=0}^{\infty} (\beta + e^{\alpha z} b)(\beta + e^{\alpha w} b)\frac{\alpha^n z^n w^n}{n!} = \sum_{m,n=0}^{\infty} \sum_{k=0}^{\infty} D_m^{[2+J]}(x;\alpha,\beta,\lambda)D_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{e^{-\alpha}\alpha^k}{k!} z^n w^n.$$

Which results in

$$\sum_{k=0}^{\infty} D_m^{[2+J]}(k;\alpha,\beta,\lambda)D_n^{[2+J]}(k;\alpha,\beta,\lambda)\frac{e^{-\alpha}\alpha^k}{k!} = \begin{cases} \left[\dfrac{\alpha^n(\beta + A_j(\lambda,\alpha)e^{\alpha z})(\beta + A_j(\lambda,\alpha)e^{\alpha w})}{n!}\right], & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases}$$

$$\Leftrightarrow \sum_{k=0}^{\infty} D_m^{[2+J]}(k;\alpha,\beta,\lambda)D_n^{[2+J]}(k;\alpha,\beta,\lambda)\frac{e^{-\alpha}\alpha^k}{k!} = \left[\frac{\alpha^n(\beta + A_j(\lambda,\alpha)e^{\alpha z})(\beta + A_j(\lambda,\alpha)e^{\alpha w})}{n!}\right]\delta_{m,n}.$$

And so we can see that

$$\sum_{k=0}^{\infty} G_m^{[2+J]}(x,\alpha,\beta,\lambda)G_n^{[2+J]}(x,\alpha,\beta,\lambda)\frac{e^{-\alpha}\alpha^k}{k!} = n!\alpha^n(\beta+A_j(\lambda,\alpha)e^{\alpha z})(\beta+A_j(\lambda,\alpha)e^{\alpha w})\delta_{m,n}. \quad (62)$$

Now, from Equation (62), we consider the following product:

$$(\beta+A_j(\lambda,\alpha)e^{\alpha z})(\beta+A_j(\lambda,\alpha)e^{\alpha w}) = \beta^2+\beta\varepsilon_2 A_j(\lambda,\alpha)e^{ic_2\alpha}+\beta\varepsilon_1 A_j(\lambda,\alpha)e^{ia_2\alpha} + [A_j(\lambda,\alpha)]^2\,\varepsilon_1\varepsilon_2 e^{ia_2\alpha}e^{ic_2\alpha}. \quad (63)$$

Finally, we take into consideration the following: we develop the calculations in (63) and substitute Equations (51) and (52) into the result, then organizing (62) with these calculations we get (55), which completes the proof of Theorem 4.1.

Again employing (63) with certain conditions provides

Corollary 4.1. *For a fixed $J\in\mathbb{N}$, if $\alpha\in\mathbb{R}$, $\alpha<0$, $\beta,\ \lambda\in\mathbb{R}-\{0\}$, and $m,n\in\mathbb{N}_{\nvdash}$. Assume that $z_1=a_1+ia_2$, $z_2=c_1+ic_2$, with $a_1,c_1\to 0$ and $a_2\to c_1=\zeta$ in the circle $C(0,|\eta|)$. Then, the parametric U-Charlier-Poisson type polynomials satisfy the following orthogonality relation*

$$\sum_{x=0}^{\infty} G_m^{[2+J]}(x,\alpha,\beta,\lambda)G_n^{[2+J]}(x,\alpha,\beta,\lambda)\frac{e^{-\alpha}\alpha^x}{x!}\,|\Omega_1(\beta,\lambda,\alpha)|^{-2} = \frac{\Gamma(n+1)\alpha^n}{\Omega_1(\beta,\lambda,\alpha)}\delta_{m,n}.$$

Whit $\Omega_1(\beta,\lambda,\alpha)=\mathscr{M}_1(\beta,\lambda,\alpha)+i\Theta_1(\beta,\lambda,\alpha)$. Also $\mathscr{M}_1(\beta,\lambda,\alpha)$ and $\Theta_1(\beta,\lambda,\alpha)$ are given by

$$\mathscr{M}_1(\beta,\lambda,\alpha) = \beta\left(\beta+2A_j(\lambda,\alpha)\cos(\zeta\alpha)+[A_j(\lambda,\alpha)]^2\cos(2\alpha\zeta)\right), \quad (64)$$

$$\Theta_1(\beta,\lambda,\alpha) = 2\beta A_j(\lambda,\alpha)\sin(\zeta\alpha)+[A_j(\lambda,\alpha)]^2\sin(2\zeta\alpha). \quad (65)$$

Using the orthogonality property of the polynomials $G_n^{[2+J]}(x,\alpha,\beta,\lambda)$, the summation by parts given in (11), and the Pearson equation given in (54), we can see the following structure relation:

Proposition 4.1. *The parametric U-Charlier-Poisson type polynomials given in (33), satisfy*

$$\Delta G_n^{[2+J]}(x;\alpha,\beta,\lambda) = a_{n-1,n}^{\alpha} G_{n-1}^{[2+J]}(x;\alpha,\beta,\lambda), \quad (66)$$

where $a_{n-1,n}^{\alpha}$ are the Fourier coefficients.

Proof. Writing the polynomials $\Delta G_n^{[2+J]}(x;\alpha,\beta,\lambda)=G_n^{[2+J]}(x+1;\alpha,\beta,\lambda)-G_n^{[2+J]}(x;\alpha,\beta,\lambda)$ in terms of $\left\{G_n^{[2+J]}(x,\alpha,\beta,\lambda)\right\}_{n\geq 0}$, we have

$$G_n^{[2+J]}(x+1;\alpha,\beta,\lambda)-G_n^{[2+J]}(x;\alpha,\beta,\lambda) = \sum_{k=0}^{n-1} a_{k,n}^{\alpha} G_k^{[2+J]}(x;\alpha,\beta,\lambda), \quad (67)$$

where

$$a_{k,n}^{\alpha} = \frac{\left\langle \Delta G_n^{[2+J]}(x;\alpha,\beta,\lambda), G_k^{[2+J]}(x;\alpha,\beta,\lambda)\right\rangle_{\omega^\alpha}}{\left\langle G_k^{[2+J]}(x;\alpha,\beta,\lambda), G_k^{[2+J]}(x;\alpha,\beta,\lambda)\right\rangle_{\omega^\alpha}}, \qquad k=0,1,\ldots,n-1.$$

This way, applying (11) and (12) follows

$$\begin{aligned}
\left\langle G_k^{[2+J]}, G_k^{[2+J]} \right\rangle_{\omega^\alpha} a_{k,n}^\alpha &= \sum_{L=0}^{\infty} \left(\Delta G_n^{[2+J]}(L;\alpha,\beta,\lambda) G_k^{[2+J]}(L;\alpha,\beta,\lambda) \right) \omega^\alpha(L,\beta,\lambda) \\
&= -\sum_{L=0}^{\infty} G_n^{[2+J]}(L;\alpha,\beta,\lambda) \nabla \left(\omega^\alpha(L,\beta,\lambda) G_k^{[2+J]}(L;\alpha,\beta,\lambda) \right) \\
&= -\sum_{L=0}^{\infty} G_n^{[2+J]}(L;\alpha,\beta,\lambda) \left[\omega^\alpha(L,\beta,\lambda) \nabla G_k^{[2+J]}(L;\alpha,\beta,\lambda) + G_k^{[2+J]}(L-1;\alpha,\beta,\lambda) \nabla \omega^\alpha(L) \right] \\
&= -\sum_{L=0}^{\infty} G_n^{[2+J]}(L;\alpha,\beta,\lambda) \omega^\alpha(L,\beta,\lambda) \nabla G_k^{[2+J]}(L;\alpha,\beta,\lambda) \\
&\quad -\sum_{L=0}^{\infty} G_n^{[2+J]}(L-1;\alpha,\beta,\lambda) G_k^{[2+J]}(L-1;\alpha,\beta,\lambda) \nabla \omega^\alpha(L;\beta,\lambda).
\end{aligned}$$

Now, due to orthogonality of $G_n^{[2+J]}(L-1;\alpha,\beta,\lambda)$, since ∇G_k^α has degree $k-1$, we have the first sum is zero. For the second sum, substituting (54) yields

$$\begin{aligned}
\left\langle G_k^{[2+J]}, G_k^{[2+J]} \right\rangle_{\omega^\alpha} a_{k,n}^\alpha &= -\sum_{L=0}^{\infty} G_n^{[2+J]}(L;\alpha,\beta,\lambda) G_k^{[2+J]}(L-1;\alpha,\beta,\lambda) \omega^\alpha(L,\beta,\lambda) \frac{\alpha - L}{\alpha} \\
&= -\frac{1}{\alpha} \sum_{L=0}^{\infty} G_n^{[2+J]}(L;\alpha,\beta,\lambda) G_k^{[2+J]}(L-1;\alpha,\beta,\lambda) \omega^\alpha(L,\beta,\lambda). \qquad (68)
\end{aligned}$$

This sum is zero for $k+1 < n$, so only $a_{n-1,n}^\alpha$ can be non-zero. Therefore, from (68) follows (66). Proposition 4.1 is proved.

5. Szász-type operators including the parametric U-Charlier-Poisson type polynomials

In this section, we present a linear positive Szász-type operator given by (4) involving the U-Charlier-Poisson type polynomials. With the help of the Korovkin Theorem, we study the convergence and some properties.

We define the Szász-type operators, including the generating function of the parametric U-Charlier-Poisson type polynomials given in (33), with $\alpha = a$, $z = -\dfrac{1}{a}$, and $x = -(a-1)nx$ as follows:

$$J_n(f,x) = (\beta e + A_j(\lambda,\alpha))^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{G_k^{[2+J]}(-(a-1)nx, a, \beta, \lambda)}{k!} f\left(\frac{k}{n}\right), \qquad (69)$$

where $f \in C[0,\infty)$, $n \in \mathbb{N}$, $\beta e \neq A_j(\lambda,\alpha)$, and $x \geq 0$.

Lemma 5.1. *For $n \in \mathbb{N}$ and $x \geq 0$, the operators J_n defined by (69), satisfy the following identities:*

1. $J_n(1,x) = 1$,
2. $J_n(s,x) = x + \dfrac{\beta e}{n\left(\beta e + A_j(\lambda,\alpha)\right)}$,

3. $J_n(s^2,x) = x^2 + x\left(\dfrac{1}{n(a-1)+\dfrac{2\beta e}{n(\beta e + A_j(\lambda,\alpha))}+\dfrac{1}{n}}\right) + \dfrac{2\beta e}{n^2(\beta e + A_j(\lambda,\alpha))}$,

with $\beta e \neq A_j(\lambda,\alpha)$.

Proof. Using the generating function of the parametric U-Charlier-Poisson type polynomials, given by (33), we can see that

$$\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!} = \left(1-\frac{1}{a}\right)^{-(a-1)nx}(\beta e + A_j(\lambda,\alpha)), \tag{70}$$

$$\sum_{k=0}^{\infty}\frac{kG_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!} = \left(1-\frac{1}{a}\right)^{-(a-1)nx}[\beta e + nx(\beta e + A_j(\lambda,\alpha))]. \tag{71}$$

$$\sum_{k=0}^{\infty}\frac{k^2G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!} = \left(1-\frac{1}{a}\right)^{-(a-1)nx} \times \left[n^2x^2(\beta e + A_j(\lambda,\alpha)) + n^2x(\beta e + A_j(\lambda,\alpha))\Phi + 2\beta e\right] \tag{72}$$

where

$$\Phi = \left(\frac{1}{n(a-1)+\dfrac{2\beta e}{n(\beta e + A_j(\lambda,\alpha))}+\dfrac{1}{n}}\right).$$

Then, multiplying in each of the equations (70)-(72) by the right multiplicative inverses and using the Definition of the operators (69) the assertions of the lemma are obtained.

Theorem 5.2. *Let* $S := \{f : [0,\infty) \to \mathbb{R} : |f(x)| \leq Me^{Ax}\}$, *where* $A \in \mathbb{R}$. *If* $f \in C[0,\infty) \cap S$, *then*

$$\lim_{n\to\infty} J_n(f,x) = f. \tag{73}$$

That is, the operators defined in (69) converge uniformly on every compact subset of $[0,\infty)$.

Proof. By using the Lemma 5.1, we have

$$\lim_{n\to\infty} J_n(s^i;x,a) = x^i, \quad i = 0,1,2.$$

In this way, using Korovkin's Theorem [2], convergence is guaranteed in each compact subset of $[0,\infty)$.

The next result gives the rate of convergence of the sequence J_n to f by means of the modulus of continuity.

Theorem 5.3. *Let* $f \in UC[0,\infty) \cap S$. *Then the operators* J_n *satisfy the inequality that follows:*

$$|J_n(f,x) - f(x)| \leq \left\{1 + \sqrt{\Upsilon_n(x;\beta,\lambda)}\right\}\omega\left(f;\frac{1}{\sqrt{n}}\right), \tag{74}$$

with

$$\Upsilon_n(x;\beta,\lambda)=\left[\frac{x}{n}\left[\frac{n^2(\beta e+H)^2-2\beta e}{n^3(a-1)(\beta e+H)^2+3\beta e+H}\right]+\frac{2\beta e}{n^2(\beta e+H)}\right], \quad a\neq 1$$

where $H=\left(\beta e+\lambda(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}\right)^{-1}$.

Proof. By using (28), (30), the Definition of the new operators given in (69), and identity 1 of the Lemma 5.1, we can write

$$|J_n(f,x)-f(x)|=\left|H\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}f\left(\frac{k}{n}\right)-1\cdot f(x)\right|.$$

Thereupon

$$\begin{aligned}|J_n(f,x)-f(x)| &= \left|H\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left(f\left(\frac{k}{n}\right)-f(x)\right)\right|\\ &\leq H\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left|f\left(\frac{k}{n}\right)-f(x)\right|.\end{aligned}$$

This way of (29) follows:

$$\begin{aligned}|J_n(f,x)-f(x)| &\leq \left\{H\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left(\frac{1}{\delta}\left|\frac{k}{n}-x\right|+1\right)\omega(f,\delta)\right\}\\ &\leq \left\{1+\frac{1}{\delta}H\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left|\frac{k}{n}-x\right|\right\}\omega(f,\delta)\end{aligned} \quad (75)$$

On the other hand, it holds by Cauchy-Schwarz inequality for series, and Lemma 5.1 the following:

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left|\frac{k}{n}-x\right| &\leq \left(H^{-1}\left(1-\frac{1}{a}\right)^{-(a-1)nx}\right)^{1/2}\\ &\times\left(\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left(\frac{k}{n}-x\right)^2\right)^{1/2}.\end{aligned}$$

Then, taking

$$\phi=\left(H^{-1}\left(1-\frac{1}{a}\right)^{-(a-1)nx}\right)^{1/2}\left(\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left(\frac{k}{n}-x\right)^2\right)^{1/2},$$

is fulfilled

$$\begin{aligned}
\phi &= \left(H^{-1}\left(1-\frac{1}{a}\right)^{-(a-1)nx}\right)^{1/2} \\
&\times \left(H^{-1}\left(1-\frac{1}{a}\right)^{-(a-1)nx} H\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left(\frac{k}{n}-x\right)^2\right)^{1/2} \\
&= H^{-1}\left(1-\frac{1}{a}\right)^{-(a-1)nx}\left[H\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left(\frac{k^2}{n^2}-2\frac{kx}{n}+x^2\right)\right]^{1/2} \\
&= H^{-1}\left(1-\frac{1}{a}\right)^{-(a-1)nx}\left[J_n(s^2,x)-2xJ_n(s,x)+x^2J_n(1,x)\right]^{1/2}.
\end{aligned}$$

So, of (75) and the above expression, we get

$$|J_n(f,x)-f(x)| \le \left\{1+\frac{1}{\delta}\sqrt{\Upsilon_n(x;\beta,\lambda)}\right\}\omega(f,\delta).$$

By choosing $\delta := \delta_n = \dfrac{1}{\sqrt{n}}$, we arrive at the desired result. Theorem 5.3 is proved.

Lemma 5.4. *For $x \in [0,\infty)$, the sequence of operators J_n given in (69), satisfy the following property*

$$J_n((s-x)^2,x) = x\left(\frac{1}{n(a-1)+\dfrac{2\beta e}{n(\beta e+A_j(\lambda,\alpha))}+\dfrac{1}{n}}-\frac{2\beta e}{n(\beta e+A_j(\lambda,\alpha))}\right)+\frac{2\beta e}{n^2(\beta e+A_j(\lambda,\alpha))},$$

with $\beta e \neq A_j(\lambda,\alpha)$.

Proof. By taking advantage of the linearity property of J_n operators, we have

$$J_n((s-x)^2,x) = J_n(s^2,x)-2xJ_n(s,x)-x^2J_n(1,x).$$

Next, we apply Lemma 5.1, we obtain the desired outcome.

Theorem 5.5. *Let $f \in C[0,\infty)\cap S$ and $x \in [0,\infty)$. The operators J_n satisfy the inequality that follows:*

$$|J_n(f,x)-f(x)| \le 2\omega(f;\sqrt{\tau_n}), \tag{76}$$

where

$$\tau_n = x\left(\frac{1}{n(a-1)+\dfrac{2\beta e}{n(\beta e+A_j(\lambda,\alpha))}+\dfrac{1}{n}}-\frac{2\beta e}{n(\beta e+A_j(\lambda,\alpha))}\right)+\frac{2\beta e}{n^2(\beta e+A_j(\lambda,\alpha))}.$$

PROOF. By the identity 1 of the Lemma 5.1, and using the modulus of continuity property, it is fulfilled

$$\begin{aligned} |J_n(f,x) - f(x)| &\leq H\left(1-\frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{G_k^{[2+J]}(-(a-1)nx, a, \beta, \lambda)}{k!} \left| f\left(\frac{k}{n}\right) - f(x) \right| \\ &\leq \left\{ 1 + H\left(1-\frac{1}{a}\right)^{(a-1)nx} \frac{1}{\delta} \sum_{k=0}^{\infty} \frac{G_k^{[2+J]}(-(a-1)nx, a, \beta, \lambda)}{k!} \left| \frac{k}{n} - x \right| \right\} \omega(f, \delta). \end{aligned}$$

On the other hand, by the Lemma 5.4, and the Cauchy-Schwarz inequality, it holds

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{G_k^{[2+J]}(-(a-1)nx, a, \beta, \lambda)}{k!} \left| \frac{k}{n} - x \right| &\leq \sqrt{(H)^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}} \times \\ &\quad \left\{ \sum_{k=0}^{\infty} \frac{G_k^{[2+J]}(-(a-1)nx, a, \beta, \lambda)}{k!} \left| \frac{k}{n} - x \right| \right\}^{1/2} \\ &\leq H\left(1-\frac{1}{a}\right)^{-^{-1}(a-1)nx} \{\tau_n\}^{1/2}. \end{aligned}$$

This way, $|J_n(f,x) - f(x)| \leq \left\{1 + \frac{1}{\delta}\sqrt{\tau_n}\right\}^{1/2}$. Thus, by taking $\delta = \sqrt{\tau_n}$, we have the desired result.

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