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Analytic Approach Solution to Time-Fractional Phi-4 Equation with Two-Parameter Fractional Derivative

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Abstract: This paper is devoted to building a general framework for constructing a solution to fractional Phi-4 differential equations using a Caputo definition with two parameters. We briefly introduce some definitions and properties of fractional calculus in two parameters and the Phi-4 equation. By investigating the homotopy analysis method, we built the solution algorithm. The two parameters of the fractional derivative gain vary the behavior of the solution, which allows the researchers to fit their data with the proper parameter. To evaluate the effectiveness and accuracy of the proposed algorithm, we compare the results with those obtained using various numerical methods in a comprehensive comparative study.

Keywords: Phi-4 equation; generalized Caputo fractional derivative; homotopy analysis method

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1. Introduction

Fractional differential equations provide a versatile framework for describing complex phenomena in various scientific and engineering disciplines. These equations involve fractional derivatives, enabling a more nuanced representation of processes with memory and non-local effects. Several physical applications are studied based on fractional derivatives; for instance, Gómez-Aguilar et al. [1] analyze the RC circuit in terms of delay, rise, and settling times, and the Lyapunov-like functions with some applications are also presented based on the fractional derivative [2]. One notable example within this realm is the fractional Phi-4 equation [3], which extends the traditional Phi-4 equation by incorporating fractional calculus concepts. This inclusion allows for a more accurate modeling of intricate physical and chemical systems, making fractional differential equations a valuable tool in understanding the intricacies of dynamic processes:

$$u_{tt}(x,t) = u_{xx}(x,t) - m^2 u(x,t) - \lambda u^3(x,t), \tag{1}$$

where m is a constant representing the wave's propagation speed. The equation is widely studied in nonlinear dynamics and field theory. It incorporates nonlinear terms (u^3), which leads to interesting phenomena, such as soliton solutions.

The solution of the Phi-4 equation is explored with various numerical methods, aiming to achieve precise approximations and elucidate their properties. These methods provide insights into key aspects, such as the behavior of solutions over time, stability characteristics, convergence properties, and the impact of different parameters on the system dynamics. The utilization of diverse numerical techniques enhances our understanding of the Phi-4 equation's complex behavior and contributes to the development of robust computational tools for investigating its properties under varying conditions, such as the spectral collection method [4], Natural transform decomposition method [5] and Jacobi elliptic sine–cosine expansion method used by M. Alquran [6], the B-spline collocation method by W. K. Zahra [7], and Bhrawy, A. H. [8]. A. K. Alomari used the homotopy Sumudu approach [3]; further examples are the modified residual power series method used by M. Alquran [9], the homotopy perturbation method in [10] by Ehsani, the tanh method [11] by Hira Tariq, the q-homotopy analysis transform method (q-HATM) by Gau et al. [12], and, very recently, the Yang transform decomposition method (YTDM) in [13].

The homotopy analysis method (HAM) has been developed as a crucial and adaptable strategy for approximating solutions to both linear and nonlinear differential equations, and its scope extends to fractional differential equations. Its inception can be traced back to Shijun Liao's Ph.D. dissertation in 1992 [14–16]. Since then, the HAM methodology has gained widespread adoption, effectively addressing various classes of well-known differential equations [17].

The HAM is one of the most powerful methods for solving differential equations because of several features, such as the freedom of choosing the initial function, the fact that it does not depend on the large and small parameters of the equations, and the fact that it has a convergent control parameter that can enlarge the convergent region and give accurate results with few terms of the series solution.

Usually, fractional differential equations typically involve a single fractional parameter. However, recent advancements have brought forth formulations incorporating multiple fractional parameters. Notably, the Caputo–Katugampola derivative is introduced by Almeida [18], characterized by two parameters. The Caputo–Katugampola derivative is presented in diverse cases by Odibat and Baleanu [19], while Abdeljawad [20] presents generalized Mittag–Leffler kernel fractional operators (GMLKs), encompassing three fractional parameters. The incorporation of these additional parameters introduces a heightened level of complexity to the equations, influencing the behaviors of their solutions. The impact of these new parameters on solution behaviors is evident in various studies. For instance, Alomari et al. [21] investigated the effect of GMLKs on fractional parabolic equations.

In this paper, our focus is on applying the HAM to a fractional differential equation with two parameters. This exploration aims to unravel the solution characteristics and gain insights into the behavior of the system concerning the introduced fractional parameters [3].

$$ {}^{C}D^{\alpha,\rho}_{a^+,t}u(x,t) = u_{xx}(x,t) - m^2u(x,t) - \lambda u^3(x,t), \tag{2} $$

where ${}^{C}D^{\alpha,\rho}_{a^+,t}$ is the Caputo fractional derivative (CFD) operator with parameters $1 < \alpha \leq 2$ and $\rho > 0$. We extend the fractional derivative for more generalized parameters to understand how the field's dynamics evolve under different conditions and provide a more comprehensive and flexible modeling approach. The consideration of generalized derivatives allows us to delve into the nuances of behavioral changes, offering a more nuanced and versatile understanding of system dynamics. By incorporating these generalized parameters, we aim to enhance the accuracy and applicability of our model, paving the way for a more robust exploration of the diverse phenomena that may emerge in the studied system.

This paper is meticulously structured to offer a comprehensive overview of the homotopy analysis method (HAM) as applied to the resolution of fractional Phi-4 equations featuring two parameters. The document meticulously delineates the essential components of the study, underscoring the efficacy of the HAM as a potent analytical tool. The outline encapsulates the fundamental concepts and definitions pertinent to the Phi-4 equation, incorporating the generalized fractional derivative in the Caputo sense, which is presented in Section 2. This strategic inclusion establishes the foundation for systematically exploring the proposed methodology in Section 3. Section 4 gives numerical simulations of the solution. Then, some conclusions are drawn in Section 5. Throughout this paper, readers are afforded valuable insights into the deliberate application of the HAM and its numerical significance in tackling intricate fractional Phi-4 equations distinguished by dual parameters.

2. Some Definitions and Theorems of Fractional Calculus

The R-L fractional operator for a function f of order $\alpha \geq 0$ is defined as

$$\begin{aligned} I^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} f(\zeta) d\zeta, \\ I^0 f(t) &= f(t). \end{aligned}$$

The CFD of a given function f for $n-1 < \alpha < n, n \in \mathbb{N}$, is

$$D_C^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\zeta)^{n-\alpha-1} f^{(n)}(\zeta) d\zeta.$$

The generalized fractional integral (GFI) of f, $I_{a^+}^{\alpha,\rho} f(t)$, of order $\alpha > 0$ and $\rho > 0$ is [19]

$$\left(I_{a^+}^{\alpha,\rho} f\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \zeta^{\rho-1} (t^{\rho} - \zeta^{\rho})^{\alpha-1} f(\zeta) d\zeta, \;\; \alpha > 0, t > a \geq 0,$$

and the Caputo–Katugampola derivative with two parameters is defined by Almeida et al. [18], with $0 < \alpha \leq 1$ and $\rho > 0$, as

$$\left({}^{C}D_{a^+}^{\alpha,\rho} f\right)(t) = \frac{\rho^{\alpha}}{\Gamma(1-\alpha)} \int_a^t (t^{\rho} - \zeta^{\rho})^{-\alpha} f'(\zeta) d\zeta, \;\; 0 < \alpha \leq 1, t > a \geq 0.$$

Recently, the definition of CFD with two parameters for $n-1 < \alpha \leq n$ was modified by Odibat and Baleanu [19,22].

Definition 1. *The generalized Caputo derivative (GCFD) of* $f : [0,\infty) \longrightarrow \mathbb{R}$, ${}^{C}D_{a^+}^{\alpha,\rho} f(t)$, *of order* $\alpha > 0$ *is given by*

$$\left({}^{C}D_{a^+}^{\alpha,\rho} f\right)(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_a^t \zeta^{\rho-1} (t^{\rho} - \zeta^{\rho})^{n-\alpha-1} \left(\zeta^{1-\rho} \frac{d}{d\zeta}\right)^n f(\zeta) d\zeta, \quad n-1 < \alpha \leq n.$$

whenever it exists, where $n = \lceil \alpha \rceil$ *and* $t > a \geq 0$.

It is worthy to mention that the GCFD gives the Caputo derivative when $\rho = 1$ and the Hadamard derivative whenever $\lim_{\rho \to 0^+}$ [23]. This is one of the features of using this kind of derivative.

Theorem 1. ([19]). *Let* $n-1 < \alpha \leq n, s \geq 0, \rho > 0$, *and* $f \in C^n[a,b]$. *Then, for* $a < t \leq b$,

$$I_{s^+}^{\alpha,\rho} D_{s^+}^{\alpha,\rho} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{1}{\rho^k k!} (t^{\rho} - s^{\rho})^k \left[\left(x^{1-\rho} \frac{d}{dx} \right)^k f(x) \right]_{x=s}. \tag{3}$$

Theorem 2. ([19]). *For* $n-1 < \alpha \leq n, a \geq 0, \rho > 0$, *and* $f \in C^n[a,b]$,

$$D_{a^+}^{\alpha,\rho} I_{a^+}^{\alpha,\rho} f(t) = f(t), \tag{4}$$

where $a < t \leq b$.

3. Analytic Approach

To commence, we establish the overarching framework for solving FDEs with two parameters using the HAM. The equation under consideration takes the following form:

$$ {}^{C}D^{\alpha,\rho}_{a^+}u(x,t) + \mathfrak{R}u(x,t) + \mathfrak{N}u(x,t) = f(x,t), \tag{5} $$

where ${}^{C}D^{\alpha,\rho}_{a^+}u(x,t)$ is the GCFD of $u(x,t)$, with $0 < \alpha \leq 1, \rho > 0$; the initial guess, denoted by $u_0(x,t)$, satisfies the initial or boundary conditions; $\mathfrak{R}$ is the linear operator; $\mathfrak{N}$ is a nonlinear operator, such as u^3; and $f(x,t)$ is the source term, which might be zero. To implement the HAM, as detailed in references [14–16], we introduce the nonlinear operator as follows:

$$ N[\psi(x,t,q)] = {}^{C}D^{\alpha,\rho}_{a^+}\psi(x,t,q) + \mathfrak{R}\psi(x,t,q) + \mathfrak{N}\psi(x,t,q) - f(x,t), \tag{6} $$

where the function $\psi(x,t,q)$ is a real-valued function dependent on x,t, and $q \in [0,1]$. Liao's zeroth-order deformation [14,15] is

$$ (1-q)\mathbb{L}[\psi(x,t,q) - u_0(x,t)] = \hbar q N[\psi(x,t,q)]. \tag{7} $$

where the parameter $\hbar \neq 0$ serves as a nonzero convergent control parameter, N denotes the nonlinear operator, and $\mathbb{L}$ is an injective linear operator. For our purposes, we define $\mathbb{L} = D^{\alpha,\rho}_{a^+}$. Notably, $\psi(x,t,0) = u_0(x,t)$, and $\psi(x,t,1) = u(x,t)$. By expanding $\psi(x,t,q)$ in a Taylor series with respect to q, we have

$$ \psi(x,t,q) = \sum_{i=0}^{\infty} u_i(x,t)q^i, $$

where

$$ u_i(x,t) = \frac{1}{i!}\frac{\partial^i \psi(x,t,q)}{\partial q^i}|_{q=0}, \tag{8} $$

the m-th order Liao's deformation equation is

$$ \mathbb{L}[u_m(x,t) - \lambda_m u_{m-1}(x,t)] = \hbar R_m\left(\vec{u}_{m-1}(x,t)\right), \tag{9} $$

where $\vec{u}_{m-1} = \{u_0, u_1, u_2, \cdots, u_{m-1}\}$, and

$$ R_m(u_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\psi(x,t,q)]}{\partial q^{m-1}}\Bigg|_{q=0}. $$

By applying $\mathbb{L}^{-1} = I^{\alpha,\rho}_{a^+}$ in (9), we obtain

$$ \begin{aligned} u_m(x,t) &= \lambda_m h_{m-1}(x,t) + \hbar I^{\alpha,\rho}_{a^+}[R_m(u_{m-1}(x,t))] \\ &+ \sum_{k=0}^{\lceil\alpha\rceil-1} \tfrac{1}{\rho^k k!}(t^\rho - a^\rho)^k \left[\left(s^{1-\rho}\tfrac{d}{ds}\right)^k (u_m(x,s) - \lambda_m u_{m-1}(x,s))\right]_{s=a}, \end{aligned} \tag{10} $$

where

$$ \lambda_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1 \end{cases}. $$

Now, considering our Equation (2) along with I.C $u(x,0) = f_1(x), v(x,0) = f_2(x)$, we have N as follows:

$$ N\Big[\psi(x,t,q) = D^{\alpha,\rho}_{a^+}\psi(x,t,q) - m^2\psi(x,t,q) - \lambda\psi^3(x,t,q), \tag{11} $$

where

$$\psi(x,t,q) = \sum_{m=0}^{n} u_m(x,t)q^m, u_m(x,t) = \frac{1}{m!}\frac{\partial^m \psi(x,t,q)}{\partial q^m}|_{q=0}. \tag{12}$$

So, the m-th order deformation equation is

$$D_{a^+}^{\alpha,\rho}[u_m(x,t) - \lambda_m u_{m-1}(x,t)] = \hbar\Big(D_{a^+}^{\alpha,\rho} u_{m-1} + R_m\Big[\overrightarrow{u}_{m-1}(x,t)\Big]\Big), \tag{13}$$

with

$$R_m = \frac{\partial^2 u_{n-1}}{\partial x^2} - m^2 u_{n-1} - \lambda \sum_{i=0}^{n-1} u_{n-1-i} \sum_{j=0}^{i} u_j u_{i-j},$$

subject to the initial conditions $u_m(x,0) = 0$ for $m = 1,2,3,\cdots$ At this point, we apply $\mathbb{L}^{-1} = I_{0^+}^{\alpha,\rho}$, the inverse operator, to obtain

$$\begin{aligned} u_m(x,t) &= (\lambda_m + \hbar)u_{m-1} + \hbar I_{0^+}^{\alpha,\rho} R_m\Big[\overrightarrow{u}_{m-1}(x,t)\Big] \\ &\quad +(u_m(x,0) - (1+\lambda_m)u_{m-1}(x,0)), \end{aligned}$$

for $m = 1,2,3,\cdots$ So, the $M-$th order of series solutions is

$$u(x,t) = u_0(x,t) + \textstyle\sum_{i=1}^{M} u_i(x,t). \tag{14}$$

As $M \to \infty$, the series solutions converge to the exact solution.

4. Numerical Experiment

In this section, we apply the HAM to examine fractional Phi-4 Equation (2) under the specified initial conditions. The equation is considered with $m = 1$ and $\lambda = -1$, and the initial conditions are imposed as follows:

$$u(x,0) = \tanh\big(\tfrac{x}{4}\big), \qquad u_t(x,0) = -\tfrac{3}{4}\sec\mathrm{h}^2\big(\tfrac{x}{4}\big). \tag{15}$$

By applying the HAM algorithm as outlined in Section 3, we obtain the first terms of the approximations as follows:

$$\begin{aligned} u_0(x,t) &= \tanh\big(\tfrac{x}{4}\big) - \tfrac{3t}{4}\sec\mathrm{h}^2\big(\tfrac{x}{4}\big), \\ u_1(x,t) &= \frac{27\hbar\rho^{-\alpha}\Gamma\left(\frac{\rho+3}{\rho}\right)\sec\mathrm{h}^6\left(\frac{x}{4}\right)t^{\alpha\rho+3}}{64\Gamma\left(\alpha+\frac{3}{\rho}+1\right)} - \frac{3\hbar\rho^{-\alpha-1}\Gamma\left(\frac{1}{\rho}\right)\sec\mathrm{h}^4\left(\frac{x}{4}\right)t^{\alpha\rho+1}}{32\Gamma\left(\alpha+\frac{1}{\rho}+1\right)} - \frac{3\hbar\rho^{-\alpha-1}\Gamma\left(\frac{1}{\rho}\right)\sec\mathrm{h}^2\left(\frac{x}{4}\right)t^{\alpha\rho+1}}{4\Gamma\left(\alpha+\frac{1}{\rho}+1\right)} + \\ &\quad \frac{h\rho^{-\alpha}\tanh\left(\frac{x}{4}\right)t^{\alpha\rho} - \hbar\rho^{-\alpha}\tanh^3\left(\frac{x}{4}\right)t^{\alpha\rho}}{\alpha\Gamma(\alpha)} + \frac{39h\rho^{-\alpha-1}\Gamma\left(\frac{1}{\rho}\right)\tanh^2\left(\frac{x}{4}\right)\sec\mathrm{h}^2\left(\frac{x}{4}\right)t^{\alpha\rho+1}}{16\Gamma\left(\alpha+\frac{1}{\rho}+1\right)} + \\ &\quad \frac{h\rho^{-\alpha}\tanh\left(\frac{x}{4}\right)\sec\mathrm{h}^2\left(\frac{x}{4}\right)t^{\alpha\rho}}{8\alpha\Gamma(\alpha)} - \frac{27\hbar\rho^{-\alpha}\Gamma\left(\frac{\rho+2}{\rho}\right)\tanh\left(\frac{x}{4}\right)\sec\mathrm{h}^4\left(\frac{x}{4}\right)t^{\alpha\rho+2}}{16\Gamma\left(\alpha+\frac{2}{\rho}+1\right)}, \\ u_2(x,t) &= -\frac{3\hbar\rho^{-\alpha-1}\Gamma\left(\frac{1}{\rho}\right)\sec\mathrm{h}^2\left(\frac{x}{4}\right)t^{\alpha\rho+1}}{4\Gamma\left(\alpha+\frac{1}{\rho}+1\right)} - \frac{3\hbar^2\rho^{-\alpha-1}\Gamma\left(\frac{1}{\rho}\right)\sec\mathrm{h}^2\left(\frac{x}{4}\right)t^{\alpha\rho+1}}{4\Gamma\left(\alpha+\frac{1}{\rho}+1\right)} - \frac{\hbar^2\rho^{-\alpha}\tanh^3\left(\frac{x}{4}\right)t^{\alpha\rho}}{\alpha\Gamma(\alpha)} \\ &\quad \frac{\hbar^2\rho^{-2\alpha}\tanh\left(\frac{x}{4}\right)\sec\mathrm{h}^2\left(\frac{x}{4}\right)\Gamma\left(\frac{\alpha\rho+\rho}{\rho}\right)t^{2\alpha\rho}}{4\alpha\Gamma(\alpha)\Gamma(2\alpha+1)} + \frac{\hbar^2\rho^{-\alpha}\tanh\left(\frac{x}{4}\right)\sec\mathrm{h}^2\left(\frac{x}{4}\right)t^{\alpha\rho}}{8\alpha\Gamma(\alpha)} + \cdots \end{aligned}$$

In this way, we can find the approximate solution with M terms in Equation (14). To determine the value of the convergent parameter, $\hbar$, we plot the $\hbar$-curve in Figures 1 and 2 with $\alpha = 2, \rho = 1, m = 1$, and $\lambda = -1$. It is clear that in the region $-1.2 \leq \hbar \leq -0.8$, the derivatives do not depend on $\hbar$. For simplicity, we choose $\hbar = -1$. So, the HAM solution gives

$$u(x,t) = -\frac{t^9\sec\mathrm{h}^2\left(\frac{x}{4}\right)}{483,840} + \frac{t^7\sec\mathrm{h}^2\left(\frac{x}{4}\right)}{6720} - \frac{1}{160}t^5\sec\mathrm{h}^2\left(\frac{x}{4}\right) + \frac{1}{8}t^3\sec\mathrm{h}^2\left(\frac{x}{4}\right) - \frac{3}{4}t\sec\mathrm{h}^2\left(\frac{x}{4}\right) + \cdots \tag{16}$$

The exact solution is

$$u(x,t) = \tanh\left(\frac{x-3t}{4}\right).$$

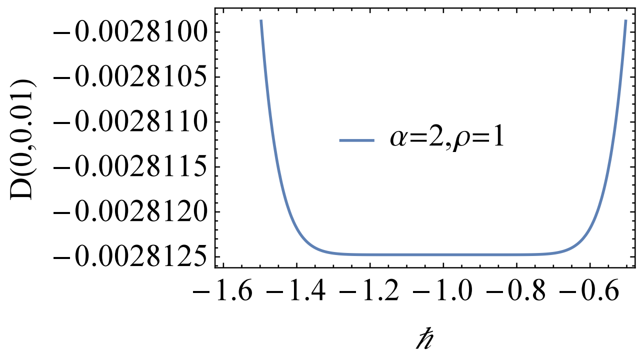

Figure 1. $\hbar$—Curve using $D(0, 0.01)$ for (2) with $\alpha = 2$ and $\rho = 1$.

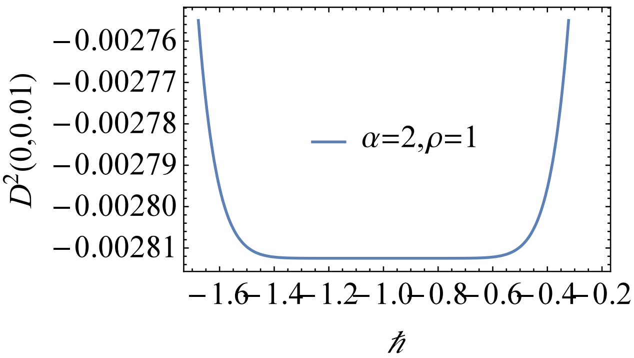

Figure 2. $\hbar$—Curve using $D^2(0, 0.01)$ for (2) with $m = 1, \lambda = -1, \alpha = 2$, and $\rho = 1$.

At this point, we present a comprehensive analysis of the 10th-order homotopy analysis method (HAM) solutions for the time-fractional Phi-4 equation. Our findings, illustrated in Figure 3, highlight the convergence of the approximate solution towards the exact solution, as depicted in Figure 4. It is clear that the solution satisfies the boundary conditions $\lim_{x\to\infty} u(x,t) = 1$ and $\lim_{x\to-\infty} u(x,t) = -1$. Furthermore, Figure 5 provides the absolute error between the exact solution and the 10th-order HAM solution. The convergence and the stability of the solution are experimentally presented in Table 1, which gives the absolute error for $t = 0.25$, different values of x, and a number of series terms M. It is clear that as M increases, the absolute error becomes closer to zero. Similarly, as x increases, the error can fit as a linearly increase, which indicates a little about stability.

Table 1. The approach using 2, 4, and 6 terms of the series solution at $t = 0.25$ and varying x with $\alpha = 2, \rho = 1$.

x	$M = 2$	$M = 4$	$M = 6$
-5	1.2694×10^{-5}	1.8080×10^{-8}	2.2880×10^{-11}
-3	1.6010×10^{-5}	1.6540×10^{-8}	1.3700×10^{-11}
-1	8.0970×10^{-6}	5.8930×10^{-9}	3.0910×10^{-12}
1	4.2140×10^{-6}	2.0120×10^{-9}	3.4200×10^{-12}
3	1.4235×10^{-5}	1.2540×10^{-8}	8.2550×10^{-12}
5	1.3608×10^{-5}	1.7770×10^{-8}	42.0550×10^{-11}

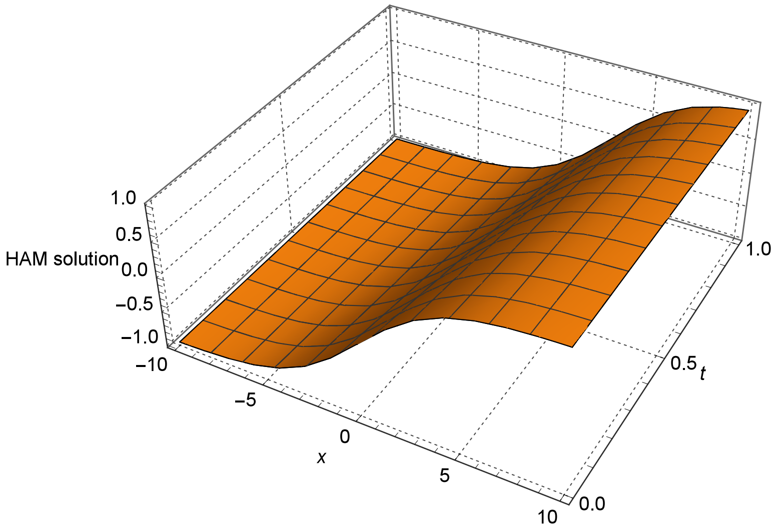

Figure 3. The HAM solution ($u(x,t)$) for (2) with $\alpha = 2, \rho = 1, m = 1$, and $\lambda = -1$.

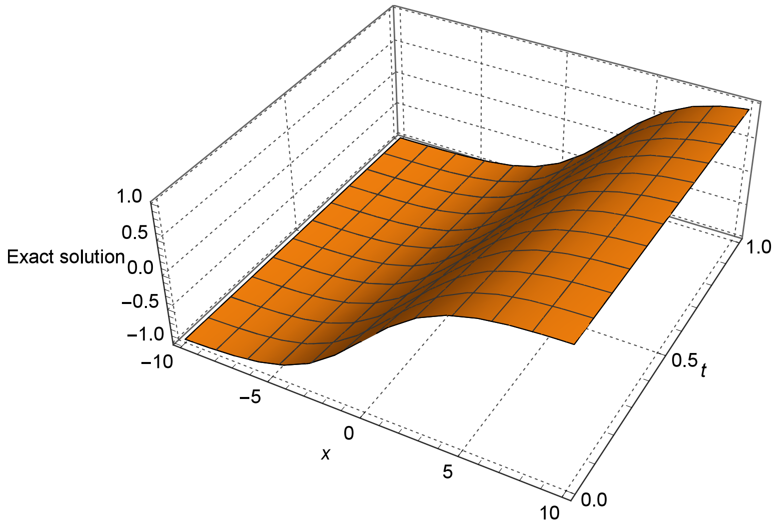

Figure 4. The exact solution ($u(x,t)$) for (2) with $\alpha = 2, \rho = 1, m = 1$, and $\lambda = -1$.

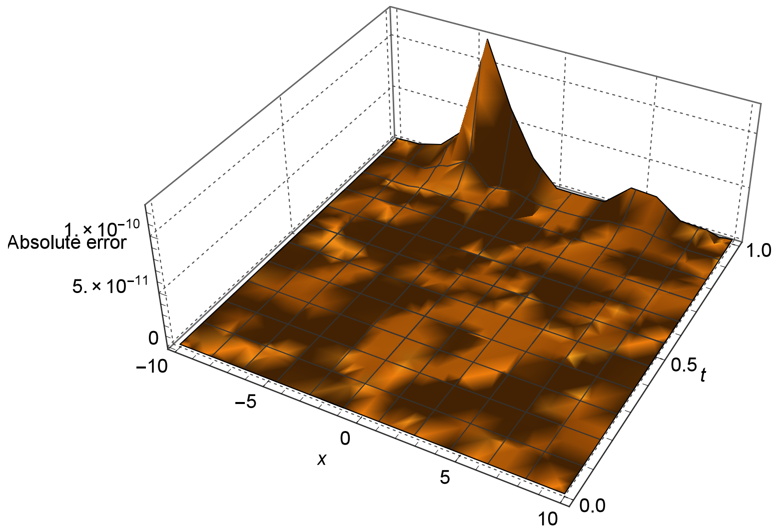

Figure 5. The absolute error of (2) with $\alpha = 2, \rho = 1, m = 1$, and $\lambda = -1$.

To investigate the effect of the two-parameter fractional derivative on the time-fractional Phi-4 equation, initially, we optimize the choice of $\hbar$ as

$$Res = {}^{C}D^{\alpha,\rho}_{a^+,t}u(x,t) - \left(u_{xx}(x,t) - m^2u(x,t) - \lambda u^3(x,t)\right),$$

where $u(x,t)$ represents the HAM solutions. Now, $\hbar$ is determined by minimizing $\Delta(\hbar)$ (the least square error):

$$\Delta(\hbar) = \frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}\left(Res\left(\frac{i}{m},\frac{j}{n}\right)\right)^2. \tag{17}$$

Exploring the 10th-order HAM solutions for the time-fractional Phi-4 equation needs optimal values of $\hbar$; to find them, we plot the square residual error for different values of ρ and α. Next, Figure 6 delves into the characterization of $\Delta(\hbar)$ for $\rho = 1$ and varying α, shedding light on the optimal values that can be obtained by minimizing the $\Delta(\hbar)$ function, which gives the optimal values of $\hbar$ of $-0.963186, -0.960893$, and -0.854131 for $\alpha = 1.95, 1.9$, and 1.5, respectively. Similarly, by fixing $\alpha = 1.95$ and varying ρ, we obtain $-0.772763, -0.963187$, and -0.240798 as optimal values for $\rho = 0.75, 1$, and 1.2, respectively.

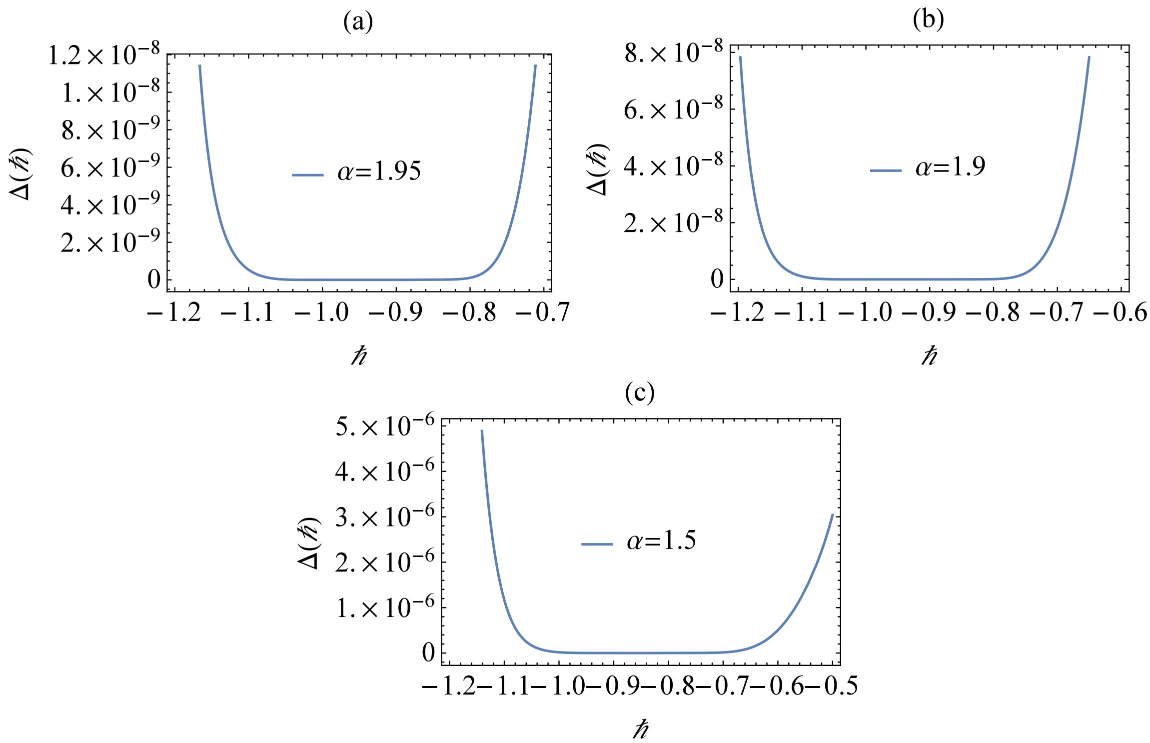

Figure 6. Square residual error for (2) with $\rho = 1; m = 1; \lambda = -1$; and (**a**) $\alpha = 1.95$, (**b**) $\alpha = 1.9$, and (**c**) $\alpha = 1.5$.

Based on the optimal values of $\hbar$, the detailed analysis contributes valuable insights into the behavior and sensitivity of the solutions under different parameter settings, which can be seen in Figure 7. It presents the HAM solution with fixed $\alpha = 1$ and $x = 1$ and varying ρ. Moreover, Figure 8 shows the effect of α with fixed = 1 and $x = 1$. Both parameters affect the solution behavior of the equation.

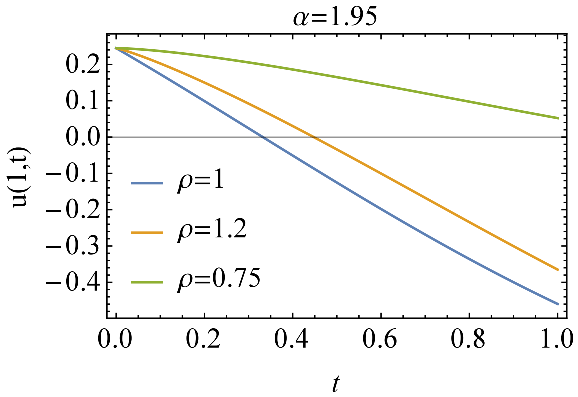

Figure 7. The HAM solution for (2) with $\alpha = 1.95$ and several values of ρ at $x = 1$.

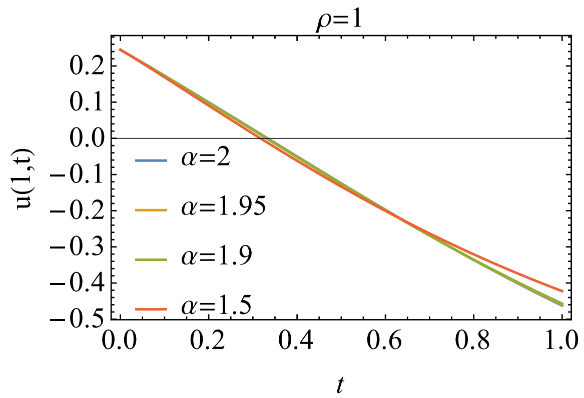

Figure 8. The HAM solution for (2) with $\rho = 1$ and several values of α at $x = 1$.

To demonstrate the effect of the fractional parameters numerically, Table 2 illustrates the HAM approximate solution for different values of α and ρ, and varying x and t, which gives the effect of the fractional parameters of the solution approaches. Finally, Table 3 compares the absolute error of the HAM with those of q-HAM and YTDM using three terms of the series solution. Clearly, the HAM gives a lower absolute error than the other used methods.

Table 2. HAM solutions to (3) with different values of α and ρ.

x	t	Exact Solution	$\alpha = 2$ $\rho = 1$	$\alpha = 1.95$ $\rho = 1$	$\alpha = 1.95$ $\rho = 0.75$	$\alpha = 1.95$ $\rho = 1.2$	$\alpha = 1.9$ $\rho = 0.75$
0.5	0.1	0.049958	0.049958	0.049864	0.120031	0.080269	0.119504
	0.2	−0.024994	−0.024994	−0.025195	0.112612	0.025977	0.111543
	0.3	−0.099667	−0.099667	−0.099869	0.103485	−0.033128	0.101977
1	0.1	0.173235	0.173235	0.173032	0.236786	0.202643	0.235793
	0.2	0.099667	0.099667	0.099119	0.222787	0.150031	0.220764
	0.3	0.024994	0.024994	0.024136	0.205509	0.092041	0.202636
1.5	0.1	0.291312	0.291312	0.291022	0.347309	0.318994	0.345957
	0.2	0.221278	0.221278	0.220445	0.328219	0.269515	0.325444
	0.3	0.148885	0.148885	0.147466	0.304531	0.214313	0.300555
2	0.1	0.401134	0.401134	0.400787	0.449239	0.426461	0.447658
	0.2	0.336375	0.336375	0.335344	0.426881	0.381221	0.447658
	0.3	0.268271	0.268271	0.266441	0.398952	0.330165	0.394209
2.5	0.1	0.500520	0.500520	0.500146	0.540966	0.523097	0.539286
	0.2	0.442230	0.442230	0.441094	0.517169	0.482787	0.513659
	0.3	0.379948	0.379948	0.377876	0.487222	0.436813	0.482069

Table 3. Comparison of the absolute error for $\alpha = 2$ and $\rho = 1$ with the present, q-HAM, and YTDM methods using three terms of the series solution.

	HAM Error	q-HATM Error	YTDM Error	HAM Error	q-HATM Error	YTDM Error
$x \mid t$	**0.01**	**0.01**	**0.01**	**0.05**	**0.05**	**0.05**
−5	5.4852×10^{-10}	1.96030×10^{-5}	2.47795×10^{-6}	1.3806×10^{-8}	3.99056×10^{-2}	2.47883×10^{-3}
−3	8.7531×10^{-10}	3.17168×10^{-4}	2.40477×10^{-5}	2.1824×10^{-8}	3.84183×10^{-2}	2.77402×10^{-3}
−1	5.3623×10^{-10}	4.33502×10^{-3}	2.28902×10^{-4}	1.3557×10^{-8}	1.83324×10^{-2}	2.97842×10^{-3}
1	5.2732×10^{-10}	1.65573×10^{-3}	2.14103×10^{-3}	1.2443×10^{-8}	4.27869×10^{-2}	2.56482×10^{-3}
3	8.7674×10^{-10}	1.97624×10^{-2}	1.96890×10^{-3}	2.1962×10^{-8}	5.88089×10^{-2}	1.96730×10^{-3}
5	5.5227×10^{-10}	1.99868×10^{-2}	1.96890×10^{-3}	1.4265×10^{-8}	5.99307×10^{-2}	1.75689×10^{-3}

5. Conclusions

In this study, we have successfully developed the homotopy analysis method (HAM) for solving the fractional Phi-4 equation with two parameters. Our results underscore the effectiveness and versatility of this method when applied to the realm of generalized fractional differential equations. Through meticulous comparisons with the exact solution, we have validated the precision inherent in our proposed approach. Furthermore, the determination of residual error, achieved by optimizing the convergent control parameter, $\hbar$, adds an extra layer of scrutiny, affirming the accuracy of our algorithm. Upon comparing the results obtained using this method with those previously published, it becomes evident that the approach offers enhanced accuracy and faster convergence. The integration of two fractional parameters with the Phi-4 equation can give researchers a wide range of utilized materials that align seamlessly with this broad spectrum of solutions and enable the expansion of experiments. The robustness and efficacy demonstrated by this methodology underscore its aptitude for addressing equations of similar complexity and nature.

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