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A note on q-truncated exponential polynomials

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In this paper, we introduce the q-truncated exponential polynomials by means of the integral form. Certain properties of the q-truncated exponential polynomials like series definition, recurrence relations, q-differential equations and integral representations are obtained. Also, we introduce the associated q-truncated exponential polynomials, higher order q-truncated exponential polynomials and higher order associated q-truncated exponential polynomials. Furthermore, we obtain their integral forms, generating functions, series definitions, summation and operational formulas.

Key words and phrases: quantum calculus, truncated exponential polynomials, recurrence relations, summation and integral formulas, q-Hermite polynomials, q-dilatation operator.

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Introduction

Quantum calculus, briefly called q-calculus, is an emerging field of research. For $q \to 1^-$, the quantum calculus led to the usual calculus. Recently, the field of quantum calculus has been proven instrumental in several areas like mathematical sciences, quantum physics, quantum mechanics, quantum algebra, approximation theory and operator theory etc. The q-analogue of several special functions like q-Hermite polynomials, q-Laguerre polynomials, q-Appell polynomials and q-Sheffer polynomials are established and studied. Very recently, the quantum algebra representations of certain q-special functions like q-Tricomi functions, 2-variable q-Bessel functions, 2-variable q-Hermite polynomials, 2-variable q-Laguerre polynomials, family of q-modified-Laguerre-Appell polynomials, characterizing q-Bessel functions of the first kind and a review on q-difference equations for Al-Salam-Carlitz polynomials are obtained [4, 5, 11, 12, 24–26].

Currently, we review some fundamental notions, symbols and conclusions from our results in quantum mathematics that are required for the rest of this paper discussion.

The q-analogue of a complex number α is defined by $[\alpha]_q = (1-q^\alpha)/(1-q)$, $0<q<1$ (see [2]). The q-factorial is defined by

$$[n]_q! = \begin{cases} [1]_q[2]_q\dots[n]_q, & n \geq 1, \quad 0<q<1, \\ 1, & n=0. \end{cases}$$

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The Gauss q-binomial coefficient (see [2]) is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \quad k = 0, 1, \ldots, n.$$

The raising and lowering q-powers (see [2]) are defined by

$$(x \pm a)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} x^{n-k} (\pm a)^k. \tag{1}$$

For $n = 1$, it is obvious that $(x \pm a)_q^1 = (x \pm a)$. The two q-exponential functions (see [2]), denoted by $e_q(x)$ and $E_q(x)$, are defined by

$$e_q(x) = \frac{1}{(x(1-q);q)_\infty} = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \quad |x| < \frac{1}{1-q}, \quad 0 < q < 1, \tag{2}$$

and

$$E_q(x) = (-x(1-q);q)_\infty = \sum_{n=0}^{\infty} q^{\binom{2}{n}} \frac{x^n}{[n]_q!}, \quad x \in \mathbb{C}, \quad 0 < q < 1, \tag{3}$$

respectively. The relations between both q-exponential functions, namely

$$e_q(x)E_q(-x) = 1, \quad |x| < \frac{1}{1-q} \tag{4}$$

and

$$e_q(x)E_q(y) = \sum_{k=0}^{\infty} \frac{(x+y)_q^k}{[k]_q!},$$

are established in [2] and [10], respectively. The q-derivative of a function f with respect to x (see [17]), denoted by $D_{q,x}f(x)$, is defined by

$$D_{q,x}f(x) = \frac{f(qx) - f(x)}{qx - x}, \quad 0 < q < 1, \quad x \neq 0.$$

Also, for any two functions $f(x)$ and $g(x)$, we have

$$D_{q,x}\big(f(x)g(x)\big) = f(x)D_{q,x}g(x) + g(qx)D_{q,x}f(x). \tag{5}$$

In particular, we have

$$D_{q,x}E_q(\alpha x) = \alpha E_q(\alpha q x). \tag{6}$$

By mathematical induction, it is easy to verify that the kth order q-derivative of the q-exponential functions are

$$D_{q,x}^k e_q(\alpha x) = \alpha^k e_q(\alpha x), \quad k \geq 1, \tag{7}$$

and

$$D_{q,x}^k E_q(\alpha x) = \alpha^k q^{\binom{k}{2}} E_q(\alpha q^k x), \quad k \geq 1, \tag{8}$$

where $D_{q,x}^k$ denotes the kth order q-derivative with respect to x.

The Heine's binomial formula

$$\frac{1}{(1-t)_q^m} = \sum_{k=0}^{\infty} \begin{bmatrix} m+k-1 \\ k \end{bmatrix}_q t^k, \quad m \in \mathbb{C}, \tag{9}$$

is given in [19], where

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_q = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1)\Gamma_q(\alpha-\beta+1)}, \quad 0<q<1, \quad \alpha,\beta\in\mathbb{C}, \tag{10}$$

(see [10]). The q-Gamma function $\Gamma_q(\alpha)$ (see [2,19]) is given by

$$\int_0^{\frac{1}{1-q}} x^{\alpha-1}E_q(-qx)d_qx = \Gamma_q(\alpha), \quad \alpha>0, \quad 0<q<1, \quad x\in\mathbb{C}. \tag{11}$$

From equation (9), it can easily be verified (see [2]) that

$$D_{q,t}\frac{1}{(1-t)_q^m} = \frac{[m]_q}{(1-t)_q^{m+1}}. \tag{12}$$

The q-Hermite polynomials, which are defined in several ways, has vast literature due to its wide applications in various fields of mathematics and physics; therefore, a lot has been written about them (see, e.g., [3,16,22,23,29]). Recently, N. Raza et al. [24] defined the 2-variable q-Hermite polynomials by means of the following generating function

$$e_q(xt)e_q(yt^2) = \sum_{k=0}^{\infty} H_{n,q}(x,y)\frac{t^n}{[n]_q!} \tag{13}$$

and series definition

$$H_{n,q}(x,y) = [n]_q!\sum_{k=0}^{[n/2]}\frac{y^kx^{n-2k}}{[k]_q![n-2k]_q!}, \quad 0\le k\le n. \tag{14}$$

Truncated exponential polynomials (TEP for short) have been proven to play a significant role in the evaluating integrals involving products of special functions in the physical sciences. TEP are important in the applied mathematics as they may be characterized using several methods such as orthogonality criteria, generating functions, differential equations, integral transformations, recurrence relations, and operational formulae. Mathematical and physical science researchers value the helpful qualities of generalizations and extensions in their applications. For instance, the generalizations of TEP via monomiality principle, convolution of the 2-variable truncated-exponential polynomials with Sheffer polynomials by using operational methods, truncated exponential-based Appell polynomials, 3-variable Legendre-truncated-exponential-based Sheffer sequences, truncated-exponential based Apostol-type polynomials, and hybrid family of truncated exponential-Gould-Hopper polynomials are introduced and studied in [8,20,21,27,28,30]. The properties of the family of truncated special polynomials are relatively little known.

The TEP $e_n(x)$ are defined (see [1]) by the consequence series

$$e_n(x) = \sum_{k=0}^{n}\frac{x^k}{k!}, \tag{15}$$

which is the sum of first $(n+1)$ terms of the Maclaurin's series of e^x. These polynomials appear in many problems of optical and quantum mechanics. The first systematic study of properties of these polynomials is given by G. Dattoli et al. [9]. Most of the properties of TEP

$e_n(x)$ can be derived from its series definition, given by equation (15). We note that the $e_n(x)$ have (see [9]) the following integral representation

$$e_n(x) = \frac{1}{n!}\int_0^{+\infty} exp(-\zeta)(x+\zeta)^n d\zeta. \tag{16}$$

G. Dattoli et al. [9] defined the 2nd order TEP by the integral representation

$$_{[2]}e_n(x) = \frac{1}{n!}\int_0^{\infty} exp(-\zeta)H_n(x,\zeta)d\zeta$$

and the mth order TEP $_{[m]}e_n(x)$ by the integral representation as

$$_{[m]}e_n(x) = \frac{1}{n!}\int_0^{\infty} exp(-\zeta)H_n^{(m)}(x,\zeta)d\zeta. \tag{17}$$

The integral representation (see [9]) of mth order associated TEP $_{[m]}e_n^{(\alpha)}(x)$ is given by

$$_{[m]}e_n^{(\alpha)}(x) = \frac{1}{n!}\int_0^{\infty} exp(-\zeta)\zeta^{\alpha}H_n^{(m)}(x,\zeta)d\zeta. \tag{18}$$

We motivated by the fact that the TEP $e_n(x)$ has applications in different fields of mathematics and sciences and by the work of G. Dattoli and his co-authors on characteristics of the TEP $e_n(x)$. The rest of the paper is organized as follows. In Section 1, we introduce the q-TEP by means of the integral form. Certain properties for the q-TEP like series definition, recurrence relations, q-differential equations and integral representations are obtained. In Section 2, we introduce the associated q-TEP, higher order q-TEP and higher order associated q-TEP. Also, we obtain their integral forms, generating functions and series definitions. In Section 3, summation and operational formulas are established.

1 The q-truncated exponential polynomials

In this section, we introduce the q-TEP $E_{n,q}(x)$ by means of integral representation and obtain their generating function, series definition, recurrence relations, differential equations.

In view of equation (16), we define the $E_{n,q}(x)$ as

$$E_{n,q}(x) = \frac{1}{[n]_q!}\int_0^{\frac{1}{1-q}} E_q(-q\zeta)(\zeta+x)_q^n d_q\zeta, \tag{19}$$

which on using equation (1), gives

$$E_{n,q}(x) = \frac{1}{[n]_q!}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} x^k \int_0^{\frac{1}{1-q}} E_q(-q\zeta)\zeta^{n-k} d_q\zeta. \tag{20}$$

Therefore, using (11) in equation (20) and then simplifying, we get the following series definition of $E_{n,q}(x)$:

$$E_{n,q}(x) = \sum_{k=0}^{n} q^{\binom{k}{2}}\frac{x^k}{[k]_q!}, \quad x\in\mathbb{C}, \quad 0<q<1, \quad 0\le k\le n. \tag{21}$$

The subsequent theorem is used to prove the generating function of $E_{n,q}(x)$.

Theorem 1. *The q-TEP $E_{n,q}(x)$ satisfy the following generating function*

$$\frac{1}{(1-t)}E_q(xt) = \sum_{n=0}^{\infty} E_{n,q}(x)t^n, \quad x \in \mathbb{C}, \quad |t| < \frac{1}{1-q}, \quad 0 < q < 1, \tag{22}$$

where $E_q(xt)$ is the q-exponential function, given by equation (3).

Proof. In view of (21), we have

$$\sum_{n=0}^{\infty} E_{n,q}(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} q^{\binom{k}{2}} \frac{x^k}{[k]_q!} t^n,$$

which on using the following series rearrangement technique (see [1])

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k) \tag{23}$$

gives

$$\sum_{n=0}^{\infty} E_{n,q}(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{x^k}{[k]_q!} t^{n+k} = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{x^k}{[k]_q!} t^k,$$

which on using equation (9) for $m = 1$ and (3), gives assertion (22). The proof of Theorem 1 is completed. □

Replacing x by ax in (22), then taking q-partial derivative of both sides of resultant equation with respect to x by using (6) and then again using (22) in the resultant equation, we get

$$D_{q,x}E_{n,q}(ax) = aE_{n-1,q}(aqx), \quad a, x \in \mathbb{C}, \quad 0 < q < 1, \quad n \geq 1, \tag{24}$$

which, for $q \to 1^-$ and $a = 1$, gives $\frac{d}{dx}e_n(x) = e_{n-1}(x)$ (see [9]). Again, taking q-partial derivative of both sides of equation (24) with respect to x and then using (24) in the right hand side of resultant equation, we get

$$D^2_{q,x}E_{n,q}(ax) = a^2qE_{n-2,q}(aq^2x), \quad a, x \in \mathbb{C}, \quad 0 < q < 1, \quad n \geq 2.$$

Following the same steps $k-1$ times, equation (24) gives the following k^{th} order q-partial derivative of $E_{n,q}(x)$ with respect to x:

$$D^k_{q,x}E_{n,q}(ax) = a^k q^{\binom{k}{2}} E_{n-k,q}(aq^kx), \quad a, x \in \mathbb{C}, \quad 0 < q < 1, \quad n \geq k.$$

Remark 1. *To establish the q-differential recurrence relations and q-operational differential equation for q-TEP $E_{n,q}(x)$, we recall the definition of q-dilatation operator T^k_z, which acts on any function of the complex variable z in the following manner (see [13]):*

$$T^k_z f(z) = f(q^kz), \quad z \in \mathbb{C}, \quad k \in \mathbb{R}, \quad 0 < q < 1, \tag{25}$$

which satisfy the property $T^{-1}_z T^1_z f(z) = f(z)$.

Also, it is worth to recall [2], that

$$(x-a)_q^{m+n} = (x-a)_q^m (x-q^m a)_q^n \tag{26}$$

and $(x-a)_q^{-n} = 1/(x-q^{-n}a)_q^n$. In particular, for $x=1$ and $a=t$, equation (26) gives

$$(1-t)_q^{m+n} = (1-t)_q^m (1-q^m t)^n. \tag{27}$$

The subsequent theorem is used to prove the existence of the pure and q-differential recurrence relations for $E_{n,q}(x)$.

Theorem 2. *The q-TEP $E_{n,q}(x)$ satisfy the following pure and q-differential recurrence relations:*

$$([n]_q + xq^n)E_{n,q}(x) + q^n E_{n,q}\Big(\frac{x}{q}\Big) - xq^n E_{n-1,q}(x) - [n+1]_q E_{n+1,q}(x) = 0, \quad n \geq 1, \tag{28}$$

$$(1+x)E_{n,q}(qx) + q[n]_q E_{n,q}(x) - qxE_{n-1,q}(qx) - [n+1]_q E_{n+1,q}(x) = 0, \quad n \geq 1, \tag{29}$$

$$([n]_q + xq^n)E_{n,q}(x) + q^n E_{n,q}\Big(\frac{x}{q}\Big) - xq^n D_{q,x} T_x^{-1} E_{n,q}(x) - [n+1]_q E_{n+1,q}(x) = 0, \tag{30}$$

$$(1+x)E_{n,q}(qx) + q[n]_q E_{n,q}(x) - qxD_{q,x} E_{n,q}(x) - [n+1]_q E_{n+1,q}(x) = 0. \tag{31}$$

Proof. Taking q-partial derivative of (22) with respect to t by taking $f_q(t) = E_q(xt)$ and $g_q(t) = 1/(1-t)$ and then using (5), we get

$$\sum_{n=0}^{\infty} D_{q,t} E_{n,q}(x) t^n = D_{q,t}\Big(\frac{1}{1-t}\Big) E_q(xt) + \frac{1}{(1-qt)} D_{q,t} E_q(xt),$$

which on using equations (6) and (12), gives

$$\sum_{n=1}^{\infty} E_{n,q}(x)[n]_q t^{n-1} = \frac{1}{(1-t)_q^2} E_q(xt) + \frac{x}{(1-qt)} E_q(qxt).$$

Using (27) for $m=n=1$ in the right hand side of preceding formula and then simplifying, we get

$$(1-t)\sum_{n=1}^{\infty} E_{n,q}(x)[n]_q t^{n-1} = \frac{1}{(1-qt)} E_q(xt) + \frac{x(1-t)}{(1-qt)} E_q(qxt).$$

Using (22) in the right hand side of preceding equation, we get

$$\sum_{n=1}^{\infty} E_{n,q}(x)[n]_q t^{n-1} - \sum_{n=0}^{\infty} E_{n,q}(x)[n]_q t^n = \sum_{n=0}^{\infty} q^n E_{n,q}\Big(\frac{x}{q}\Big) t^n + x\sum_{n=0}^{\infty} q^n E_{n,q}(x) t^n - x\sum_{n=0}^{\infty} q^n E_{n,q}(x) t^{n+1}.$$

Comparing the coefficients of t from both sides of preceding equation then simplifying, we get assertion (28).

Also, taking q-partial derivative of equation (22) with respect to t by taking $g_q(t) = E_q(xt)$ and $f_q(t) = 1/(1-t)$ then using (5), we get

$$\sum_{n=0}^{\infty} D_{q,t} E_{n,q}(x) t^n = \frac{1}{(1-t)} D_{q,t} E_q(xt) + D_{q,t}\Big(\frac{1}{1-t}\Big) E_q(qxt),$$

which on using equations (6) and (12), gives

$$\sum_{n=1}^{\infty} E_{n,q}(x)[n]_q t^{n-1} = \frac{x}{(1-t)} E_q(qxt) + \frac{1}{(1-t)_q^2} E_q(qxt).$$

Using (27) for $m = n = 1$ in the right hand side of preceding equation, we get

$$\sum_{n=1}^{\infty} E_{n,q}(x)[n]_q t^{n-1} = \frac{x}{(1-t)} E_q(qxt) + \frac{1}{(1-t)(1-qt)} E_q(qxt),$$

which on using (22) in the right hand side of preceding equation and then simplifying, gives

$$(1-qt) \sum_{n=1}^{\infty} E_{n,q}(x)[n]_q t^{n-1} = (x+1) \sum_{n=0}^{\infty} E_{n,q}(qx) t^n - qx \sum_{n=0}^{\infty} E_{n,q}(qx) t^{n+1}.$$

Comparing the coefficients of t from both sides of preceding equation then simplifying, we get the assertion (29).

Using (25) in the left hand side of equation (28), gives

$$([n]_q + xq^n) E_{n,q}(x) + q^n E_{n,q}\left(\frac{x}{q}\right) - xq^n T_x^{-1} E_{n-1,q}(qx) - [n+1]_q E_{n+1,q}(x) = 0.$$

Using (24) in the left hand side of preceding equation, gives assertion (30). Finally, using (24) in the left hand side of equation (29), gives assertion (31). The proof of Theorem 2 is completed. □

Example 1. *Applying formulas* (28)–(31)*, we have the following:*

$$\left([2]_{2/3} + x\left(\frac{2}{3}\right)^2\right) E_{2,2/3}(x) + \left(\frac{2}{3}\right)^2 E_{2,2/3}\left(\frac{3x}{2}\right) - x\left(\frac{2}{3}\right)^2 E_{1,2/3}(x) - [3]_{2/3} E_{3,2/3}(x) = 0,$$

$$(1+x) E_{3,3/4} \frac{3x}{4} + \frac{3}{4}[3]_{3/4} E_{3,3/4}(x) - \frac{3x}{4} E_{2,3/4}\left(\frac{3x}{4}\right) - [4]_{3/4} E_{4,3/4}(x) = 0,$$

$$\left([4]_{4/5} + x\left(\frac{4}{5}\right)^4\right) E_{4,4/5}(x) + \left(\frac{4}{5}\right)^4 E_{4,4/5}\left(\frac{5x}{4}\right) - x\left(\frac{4}{5}\right)^4 D_{4/5,x} T_x^{-1} E_{4,4/5}(x) = [5]_{4/5} E_{5,4/5}(x),$$

$$(1+x) E_{5,5/6}\left(\frac{5x}{6} x\right) + 5/6[5]_{5/6} E_{5,5/6}(x) - (5/6) x D_{5/6,x} E_{5,5/6}(x) = [6]_{5/6} E_{6,5/6}(x).$$

The following theorem about the q-differential equations for q-TEP $E_{n,q}(x)$.

Theorem 3. *The q-TEP $E_{n,q}(x)$ satisfy the following q-differential equations*

$$\left[xq^{n-1} D_{q,x}^2 T_x^{-2} - (xq^{n-1} + q^{n-1} T_x^{-1} + [n-1]_q) D_{q,x} T_x^{-1} + [n]_q\right] E_{n,q}(x) = 0, \tag{32}$$

$$\left[qx D_{q,x}^2 T_x^{-1} - (q[n-1]_q + xT_x + T_x) D_{q,x} T_x^{-1} + [n]_q\right] E_{n,q}(x) = 0, \tag{33}$$

where $n \geq 1$, $0 < q < 1$.

Proof. Substituting n by $n-1$ into the expression (30), we get

$$([n-1]_q + xq^{n-1}) E_{n-1,q}(x) + q^{n-1} E_{n-1,q}\left(\frac{x}{q}\right) - xq^{n-1} D_{q,x} T_x^{-1} E_{n-1,q}(x) - [n]_q E_{n,q}(x) = 0.$$

Using (25), preceding equation gives

$$xq^{n-1} D_{q,x} T_x^{-2} E_{n-1,q}(qx) - xq^{n-1} T_x^{-1} E_{n-1,q}(qx) - q^{n-1} T_x^{-2} E_{n-1,q}(qx)$$
$$- [n-1]_q T_x^{-1} E_{n-1,q}(qx) + [n]_q E_{n,q}(x) = 0,$$

where $n \geq 1$, $0 < q < 1$. Using (24) for $a = 1$ in the left hand side of the preceding equation and then simplifying the resultant equation, we get assertion (32).

Similarly, substituting n by $n-1$ into the expression (31), we get

$$(1+x)E_{n-1,q}(qx) + q[n-1]_q E_{n-1,q}(x) - qxD_{q,x}E_{n-1,q}(x) - [n]_q E_{n,q}(x) = 0,$$

where $n \geq 1$, $0 < q < 1$. Using (25), aforementioned equation gives

$$qxD_{q,x}T_x^{-1}E_{n-1,q}(qx) - xE_{n-1,q}(qx) - E_{n-1,q}(qx) \\ - q[n-1]_q T_x^{-1}E_{n-1,q}(qx) + [n]_q E_{n,q}(x) = 0,$$

where $n \geq 1$, $0 < q < 1$. Using (24) for $a = 1$ in the left hand side of the preceding equation and then simplifying the resultant equation, we get assertion (33). The proof of Theorem 3 is completed. □

Example 2. *Applying formulas* (32) *and* (33)*, we have the following equations:*

$$[x(2/3)^2 D^2_{2/3,x}T_x^{-2} - (x(2/3)^2 + (2/3)^2 T_x^{-1} + [2]_{2/3})D_{2/3,x}T_x^{-1} + [3]_{2/3}]E_{3,2/3}(x) = 0,$$

$$[3/4xD^2_{3/4,x}T_x^{-1} - (3/4[3]_{3/4} + xT_x + T_x)D_{3/4,x}T_x^{-1} + [4]_{3/4}]E_{4,3/4}(x) = 0.$$

To demonstrate the second result, it is worth recalling the definition of the factorization method, which is used to study the decreasing and increasing operators and certain properties of special polynomials (see [15]). The factorization method can be treated equivalent to the monomiality principle (see [6]).

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials such that $deg(p_n(x)) = n$ with $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$. Then the differential operators ϕ_n^- and ϕ_n^+, satisfying the properties $\phi_n^-\{p_n(x)\} = p_{n-1}(x)$ and $\phi_n^+\{p_n(x)\} = p_{n+1}(x)$, are called decreasing and increasing operators, respectively. Obtaining the decreasing and increasing operators for a given sequence of polynomials give rise to differential equation such as

$$\phi_{n+1}^- \phi_n^+ \{p_n(x)\} = p_n(x). \tag{34}$$

Currently, we obtain the raising and lowering operators for q-TEP $E_{n,q}(x)$. Using equation (24) for $a = 1/q$, we have

$$D_{q,x}E_{n,q}\left(\frac{1}{q}x\right) = \frac{1}{q}E_{n-1,q}(x), \quad n \geq 1, \quad 0 < q < 1.$$

Using (25) in the left hand side of above equation, we find

$$D_{q,x}T_x^{-1}E_{n,q}(x) = \frac{1}{q}E_{n-1,q}(x) \quad n \geq 1, \quad 0 < q < 1,$$

which gives the lowering operator as

$${}_x\phi_{n,q}^- = qD_{q,x}T_x^{-1}. \tag{35}$$

Also, in view of (31), we have

$$[n+1]_q E_{n+1,q}(x) = ((1+x)T_x + q[n]_q - qxD_{q,x})E_{n,q}(x),$$

which gives the increasing operator as

$$ {}_x\phi^{+}_{n,q} = \frac{1}{[n+1]_q}((1+x)T_x + q[n]_q - qxD_{q,x}). \tag{36} $$

Using (35) and (36) in equation (34), we get the following q-differential equation for the q-TEP $E_{n,q}(x)$ as

$$ q^3 x D^2_{q,x} T_x^{-1} E_{n,q}(x) - (q(1+x)T_x + [n]_q q^2 - q^2) D_{q,x} T_x^{-1} E_{n,q}(x) - q^2 E_{n,q}(x) + [n+1] E_{n,q}(x) = 0. $$

Remark 2. *In view of* (24) *for* $a = 1$, *we have*

$$ E_{n,q}(qx) = D_{q,x}(E_{n+1,q}(x)), \quad x \in \mathbb{C}, \quad 0 < q < 1. $$

Taking q-integration of both sides of above equation from m to r, we get

$$ \int_m^r E_{n,q}(qx) d_q x = \int_m^r D_{q,x}(E_{n+1,q}(x))) d_q x. $$

The q-definite integral of the q-derivative of a function $f(x)$ is given (see [18]) as

$$ \int_r^m D_{q,x} f(x) d_q x = f(m) - f(r). $$

This gives the following integral representations for q-TEP $E_{n,q}(x)$:

$$ \int_m^r E_{n,q}(qx) d_q x = (E_{n+1,q}(r) - E_{n+1,q}(m)), \quad 0 < q < 1, \quad m, r \in \mathbb{R}. \tag{37} $$

Since, in view of (21), $E_{n,q}(0) = 1$, *therefore, for $m = 0$ and $r = x$, equation* (37) *gives*

$$ \int_0^x E_{n,q}(qz) d_q z = E_{n+1,q}(x) - 1, \quad x \in \mathbb{C}, \quad 0 < q < 1. $$

2 The higher order q-truncated exponential polynomials

In this section, we introduce the associated q-TEP $E^{(\alpha)}_{n,q}(x)$, higher order q-TEP ${}_{[2]}E_{n,q}(x)$, ${}_{[2]}E^{*}{}_{n,q}(x)$, ${}_{[m]}E_{n,q}(x)$, ${}_{[m]}E^{*}{}_{n,q}(x)$ and mth order associated q-TEP ${}_{[m]}E^{(\alpha)}_{n,q}(x)$, ${}_{[m]}E^{*}{}^{(\alpha)}_{n,q}(x)$ by means of their integral forms. Also, we obtain their generating functions and series definitions.

Currently, we introduce the associated q-TEP (Aq-TEP for short) $E^{(\alpha)}_{n,q}(x)$ by means of the following integral representation

$$ E^{(\alpha)}_{n,q}(x) = \frac{1}{[n]_q!} \int_0^{\frac{1}{1-q}} E_q(-q\zeta) \zeta^{\alpha} (\zeta + x)^n_q d_q \zeta, \tag{38} $$

which on using (1) gives

$$ E^{(\alpha)}_{n,q}(x) = \frac{1}{[n]_q!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} x^k \int_0^{\frac{1}{1-q}} E_q(-q\zeta) \zeta^{n-k+\alpha} d_q \zeta. $$

Using (11) in aforementioned equation, we get

$$ E^{(\alpha)}_{n,q}(x) = \sum_{k=0}^{n} q^{\binom{k}{2}} \frac{x^k \Gamma_q(n-k+\alpha+1)}{[k]_q! [n-k]_q!}, \quad x \in \mathbb{C}, \quad 0 < q < 1, \quad 0 \leq k \leq n. \tag{39} $$

In view of equation (38), we have the following integral form

$$E^{*(\alpha)}_{n,q}(x) = \frac{1}{[n]_q!}\int_0^{\frac{1}{1-q}} E_q(-q\zeta)\zeta^\alpha (x+\zeta)^n_q d_q\zeta \tag{40}$$

and series definition

$$E^{*(\alpha)}_{n,q}(x) = \sum_{k=0}^{n} q^{\binom{n-k}{2}} \frac{x^k \Gamma_q(n-k+\alpha+1)}{[k]_q![n-k]_q!}. \tag{41}$$

Since $\binom{n-k}{2} = \binom{n}{2} + \binom{k}{2} - k(n-1)$, therefore in view of equations (39) and (41), we have

$$E^{*(\alpha)}_{n,q}(x) = q^{\binom{n}{2}} E^{(\alpha)}_{n,q}\Big(\frac{x}{q^{n-1}}\Big). \tag{42}$$

For $x=0$, the above equation gives the following initial condition $E^{*(\alpha)}_{n,q}(0) = q^{\binom{n}{2}}$. In view of equations (40) and (41) for $\alpha = 0$, we deduce for q-TEP $E^*_{n,q}(x)$ the following integral form

$$E^{*}_{n,q}(x) = \frac{1}{[n]_q!}\int_0^{\frac{1}{1-q}} E_q(-q\zeta)(x+\zeta)^n_q d_q\zeta$$

and series definition

$$E^{*}_{n,q}(x) = \sum_{k=0}^{n} q^{\binom{n-k}{2}} \frac{x^k}{[k]_q!}.$$

Also, for $\alpha = 0$, equation (42) gives

$$E^{*}_{n,q}(x) = q^{\binom{n}{2}} E_{n,q}\Big(\frac{x}{q^{n-1}}\Big).$$

Now, we obtain the following result for generating function of Aq-TEP $E^{(\alpha)}_{n,q}(x)$.

Theorem 4. *The Aq-TEP $E^{(\alpha)}_{n,q}(x)$ have the following generating function*

$$\frac{\Gamma_q(\alpha+1)}{(1-t)^{\alpha+1}_q} E_q(xt) = \sum_{n=0}^{\infty} E^{(\alpha)}_{n,q}(x)t^n, \quad x,\alpha \in \mathbb{C}, \quad |t| < \frac{1}{1-q}, \quad 0<q<1. \tag{43}$$

Proof. In view of (39), we have

$$\sum_{n=0}^{\infty} E^{(\alpha)}_{n,q}(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} q^{\binom{k}{2}} \frac{x^k}{[k]_q![n-k]_q!}\Gamma_q(\alpha+n-k+1)t^n,$$

which on using equation (23) gives

$$\sum_{n=0}^{\infty} E^{(\alpha)}_{n,q}(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{x^k}{[k]_q!}\frac{\Gamma_q(\alpha+n+1)}{[n]_q!}t^{n+k}.$$

Multiplying right-hand side of aforementioned formula by $\Gamma_q(\alpha+1)/\Gamma_q(\alpha+1)$ and then using (10), we get

$$\sum_{n=0}^{\infty} E^{(\alpha)}_{n,q}(x)t^n = \sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{x^k}{[k]_q!}t^k \Gamma_q(\alpha+1)\sum_{n=0}^{\infty} \begin{bmatrix} \alpha+n \\ n \end{bmatrix}_q t^n,$$

which on using (3) and (9) in the right hand side of aforementioned equation gives assertion (43). The proof of Theorem 4 is completed. □

Consider the 2nd order q-TEP by means of the following integral integral forms

$$ {}_{[2]}E_{n,q}(x) = \frac{1}{[n]_q!} \int_0^{\frac{1}{1-q}} E_q(-q\zeta) H_{n,q}(x,\zeta) d_q\zeta \tag{44}$$

and

$$ {}_{[2]}E^*_{n,q}(x) = \frac{1}{[n]_q!} \int_0^{\frac{1}{1-q}} E_q(-q\zeta) H_{n,q}(\zeta,x) d_q\zeta. \tag{45}$$

Using (14) in equations (44) and (45), then using (11) in the right hand sides of resultant equations and comparing the equal powers of t from both the sides, we get the following series definitions of 2nd order q-TEP ${}_{[2]}E_{n,q}(x)$ and ${}_{[2]}E^*_{n,q}(x)$:

$$ {}_{[2]}E_{n,q}(x) = \sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{[n-2k]_q!}, \quad |x| < \frac{1}{1-q}, \quad 0 < q < 1, \quad 0 \le k \le \frac{n}{2}, \tag{46}$$

and

$$ {}_{[2]}E^*_{n,q}(x) = \sum_{k=0}^{[n/2]} \frac{x^k}{[k]_q!}, \quad |x| < \frac{1}{1-q}, \quad 0 < q < 1, \tag{47}$$

respectively.

Now, we obtain the following result for generating functions of ${}_{[2]}E_{n,q}(x)$ and ${}_{[2]}E^*_{n,q}(x)$.

Theorem 5. *The* 2*nd order q-TEP* ${}_{[2]}E_{n,q}(x)$ *and* ${}_{[2]}E^*_{n,q}(x)$ *have the following generating functions:*

$$ \sum_{n=0}^{\infty} {}_{[2]}E_{n,q}(x) t^n = \frac{e_q(xt)}{1-t^2}, \quad |x|,|t| < \frac{1}{1-q}, \quad 0 < q < 1, \tag{48}$$

and

$$ \sum_{n=0}^{\infty} {}_{[2]}E^*_{n,q}(x) t^n = \frac{e_q(xt^2)}{1-t}, \quad |x|,|t| < \frac{1}{1-q}, \quad 0 < q < 1, \tag{49}$$

respectively.

Proof. In view of equation (46), we have

$$ \sum_{n=0}^{\infty} {}_{[2]}E_{n,q}(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{[n-2k]_q!} t^n, $$

which on using the following series rearrangement technique [1]

$$ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A(k,n-2k) $$

gives

$$ \sum_{n=0}^{\infty} {}_{[2]}E_{n,q}(x) t^n = \sum_{k=0}^{\infty} t^{2k} \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} t^n, $$

which on using (9) for $m = 1$ and (2) gives assertion (48).

Similarly, following the same steps involved in the proof of assertion (48), equation (47) gives assertion (49). The proof of Theorem 5 is completed. □

As a generalization of 2-variable Hermite polynomials can be considered the Gould-Hopper polynomials [14], which satisfy the generalized heat equation [7]. In view of (13), we introduce the q-Gould-Hopper polynomials (q-GHP for short) $H_{n,q}^{(m)}(x,y)$ by means of the following generating function

$$e_q(xt)e_q(yt^m) = \sum_{k=0}^{\infty} H_{n,q}^{(m)}(x,y)\frac{t^n}{[n]_q!}, \tag{50}$$

which on simplifying by using (2) in the left hand side and then comparing the equal powers of t from both sides of the resultant equation, gives the following series definition of q-GHP $H_{n,q}^{(m)}(x,y)$

$$H_{n,q}^{(m)}(x,y) = [n]_q! \sum_{k=0}^{[n/m]} \frac{y^k x^{n-mk}}{[k]_q![n-mk]_q!}, \quad 0 \le k \le \frac{n}{m}. \tag{51}$$

For $m = 2$, equations (50) and (51) reduce to (13) and (14), respectively. Therefore, for $m = 2$, q-GHP $H_{n,q}^{(m)}(x,y)$ reduces to the 2-variable q-Hermite polynomial $H_{n,q}(x,y)$.

Now, in view of equations (17) and (19), we introduce the mth order q-TEP ${}_{[m]}E_{n,q}(x)$ and ${}_{[m]}E^*_{n,q}(x)$ by means of the following integral forms

$${}_{[m]}E_{n,q}(x) = \frac{1}{[n]_q!}\int_0^{\frac{1}{1-q}} E_q(-q\zeta)H_{n,q}^{(m)}(x,\zeta)d_q\zeta \tag{52}$$

and

$${}_{[m]}E^*_{n,q}(x) = \frac{1}{[n]_q!}\int_0^{\frac{1}{1-q}} E_q(-q\zeta)H_{n,q}^{(m)}(\zeta,x)d_q\zeta, \tag{53}$$

respectively. Using (51) in equations (52) and (53), then using (11) in the right hand sides of the resultant equations and comparing the equal powers of t from both sides, we get the following series definitions of mth order q-TEP ${}_{[m]}E_{n,q}(x)$ and ${}_{[m]}E^*_{n,q}(x)$:

$${}_{[m]}E_{n,q}(x) = \sum_{k=0}^{[n/m]} \frac{x^{n-mk}}{[n-mk]_q!}, \quad |x| < \frac{1}{1-q}, \quad 0<q<1, \quad 0 \le k \le \frac{n}{m}, \tag{54}$$

and

$${}_{[m]}E^*_{n,q}(x) = \sum_{k=0}^{[n/m]} \frac{x^k}{[k]_q!}, \quad |x| < \frac{1}{1-q}, \quad 0<q<1, \tag{55}$$

respectively.

We obtain the following result for generating functions of ${}_{[m]}E_{n,q}(x)$ and ${}_{[m]}E^*_{n,q}(x)$.

Theorem 6. *The mth order q-TEP ${}_{[m]}E_{n,q}(x)$ and ${}_{[m]}E^*_{n,q}(x)$ have the following generating functions:*

$$\sum_{n=0}^{\infty} {}_{[m]}E_{n,q}(x)t^n = \frac{e_q(xt)}{1-t^m}, \quad |x|,|t| < \frac{1}{1-q}, \quad 0<q<1, \tag{56}$$

and

$$\sum_{n=0}^{\infty} {}_{[m]}E^*_{n,q}(x)t^n = \frac{e_q(xt^m)}{1-t}, \quad |x|,|t| < \frac{1}{1-q}, \quad 0<q<1, \tag{57}$$

respectively.

Proof. In view of (54), we have

$$\sum_{n=0}^{\infty} {}_{[m]}E_{n,q}(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]} \frac{x^{n-mk}}{[n-mk]_q!}t^n,$$

which on using the following series rearrangement technique [1]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]} A(k,n-mk) \tag{58}$$

gives

$$\sum_{n=0}^{\infty} {}_{[m]}E_{n,q}(x)t^n = \sum_{k=0}^{\infty} t^{mk} \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}t^n,$$

which on using (2) and (9), gives assertion (56).

Similarly, following the same steps involved in the proof of assertion (56), equation (55) gives assertion (57). The proof of Theorem 6 is completed. □

Currently, in view of equations (18), (52) and (53), we introduce the mth order associated q-TEP ${}_{[m]}E^{(\alpha)}_{n,q}(x)$ and ${}_{[m]}E^{*(\alpha)}_{n,q}(x)$ by means of the following integral forms:

$${}_{[m]}E^{(\alpha)}_{n,q}(x) = \frac{1}{[n]_q!}\int_0^{\frac{1}{1-q}} E_q(-q\zeta)\zeta^{\alpha} H^{(m)}_{n,q}(x,\zeta)d_q\zeta \tag{59}$$

and

$${}_{[m]}E^{*(\alpha)}_{n,q}(x) = \frac{1}{[n]_q!}\int_0^{\frac{1}{1-q}} E_q(-q\zeta)\zeta^{\alpha} H^{(m)}_{n,q}(\zeta,x)d_q\zeta, \tag{60}$$

respectively, which on using (51) gives

$${}_{[m]}E^{(\alpha)}_{n,q}(x) = \sum_{k=0}^{[n/m]} \frac{x^{n-mk}}{[k]_q![n-mk]_q!}\int_0^{\frac{1}{1-q}} E_q(-q\zeta)\zeta^{k+\alpha}d_q\zeta \tag{61}$$

and

$${}_{[m]}E^{*(\alpha)}_{n,q}(x) = \sum_{k=0}^{[n/m]} \frac{x^{k}}{[k]_q![n-mk]_q!}\int_0^{\frac{1}{1-q}} E_q(-q\zeta)\zeta^{n-mk+\alpha}d_q\zeta. \tag{62}$$

Using (11) in the right hand sides of equations (61) and (62), we get the following series definitions of ${}_{[m]}E^{(\alpha)}_{n,q}(x)$ and ${}_{[m]}E^{*(\alpha)}_{n,q}(x)$:

$${}_{[m]}E^{(\alpha)}_{n,q}(x) = \sum_{k=0}^{[n/m]} \frac{\Gamma_q(\alpha+k+1)}{[k]_q![n-mk]_q!}x^{n-mk}, \tag{63}$$

$${}_{[m]}E^{*(\alpha)}_{n,q}(x) = \sum_{k=0}^{[n/m]} \frac{\Gamma_q(n-mk+\alpha+1)}{[k]_q![n-mk]_q!}x^{k}, \tag{64}$$

respectively, where $\alpha \in \mathbb{C}$, $|x| < 1/(1-q)$, $0 < q < 1$, $0 \le k \le n/m$.

We obtain the following result for generating functions of ${}_{[m]}E^{(\alpha)}_{n,q}(x)$ and ${}_{[m]}E^{*(\alpha)}_{n,q}(x)$.

Theorem 7. *The mth order associated q-TEP ${}_{[m]}E^{(\alpha)}_{n,q}(x)$ and ${}_{[m]}E^{*(\alpha)}_{n,q}(x)$ have the following generating functions:*

$$\sum_{n=0}^{\infty} {}_{[m]}E^{(\alpha)}_{n,q}(x)t^n = \frac{\Gamma_q(\alpha+1)}{(1-t^m)_q^{\alpha+1}} e_q(xt), \tag{65}$$

$$\sum_{n=0}^{\infty} {}_{[m]}E^{*(\alpha)}_{n,q}(x)t^n = \frac{\Gamma_q(\alpha+1)}{(1-t)_q^{\alpha+1}} e_q(xt^m), \tag{66}$$

respectively, where $|x|, |t| < 1/(1-q)$, $0 < q < 1$.

Proof. In view of equation (63), we have

$$\sum_{n=0}^{\infty} {}_{[m]}E^{(\alpha)}_{n,q}(x)t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} \frac{x^{n-mk}\Gamma_q(k+\alpha+1)}{[k]_q![n-mk]_q!} t^n.$$

Using (58), we get

$$\sum_{n=0}^{\infty} {}_{[m]}E^{(\alpha)}_{n,q}(x)t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^n\Gamma_q(\alpha+k+1)}{[n]_q![k]_q!} t^{n+mk}.$$

Multiplying right-hand side of aforementioned formula by $\Gamma_q(\alpha+1)/\Gamma_q(\alpha+1)$ and then using equation (10), we obtain

$$\sum_{n=0}^{\infty} {}_{[m]}E^{(\alpha)}_{n,q}(x)t^n = \Gamma_q(\alpha+1) \sum_{k=0}^{\infty} \begin{bmatrix} \alpha+k \\ k \end{bmatrix}_q t^{mk} \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} t^n,$$

which on using (2) and (9) for $m = \alpha+1$ gives assertion (65).

Similarly, following the same steps involved in the proof of assertion (65), equation (64) gives assertion (66). The proof of Theorem 7 is completed. □

Remark 3. *In view of equations* (13) *and* (50)*, for $m = 2$, the q-Gould Hopper polynomials $H^{(m)}_{n,q}(x,y)$ reduce to the 2-variable q-Hermite polynomials $H_{n,q}(x,y)$. Therefore, for $m = 2$ the mth order associated q-TEP ${}_{[m]}E^{(\alpha)}_{n,q}(x)$ and ${}_{[m]}E^{*(\alpha)}_{n,q}(x)$ reduce to the 2nd order associated q-TEP ${}_{[2]}E^{(\alpha)}_{n,q}(x)$ and ${}_{[2]}E^{*(\alpha)}_{n,q}(x)$, which we introduce in view of* (59) *and* (60) *as*

$${}_{[2]}E^{(\alpha)}_{n,q}(x) = \frac{1}{[n]_q!} \int_0^{\frac{1}{1-q}} E_q(-q\zeta)\zeta^\alpha H_{n,q}(x,\zeta) d_q\zeta,$$

$${}_{[2]}E^{*(\alpha)}_{n,q}(x) = \frac{1}{[n]_q!} \int_0^{\frac{1}{1-q}} E_q(-q\zeta)\zeta^\alpha H_{n,q}(\zeta,x) d_q\zeta,$$

respectively.

The other properties of ${}_{[2]}E^{(\alpha)}_{n,q}(x)$ and ${}_{[2]}E^{(*\alpha)}_{n,q}(x)$, listed in the Table 1, can be obtained by substituting $m = 2$ in equations (63)–(66).

S. No.	Polynomials	Series definitions	Generating functions
I.	${}_{[2]}E^{(\alpha)}_{n,q}(x)$	${}_{[2]}E^{(\alpha)}_{n,q}(x)=\sum\limits_{k=0}^{[n/2]}\frac{x^{n-2k}\Gamma_q(\alpha+k+1)}{[k]_q![n-2k]_q!}$	$\sum\limits_{n=0}^{\infty}{}_{[2]}E^{(\alpha)}_{n,q}(x)t^n=\frac{\Gamma_q(\alpha+1)}{(1-t^2)_q^{\alpha+1}}e_q(xt)$
II.	${}_{[2]}E^{*(\alpha)}_{n,q}(x)$	${}_{[2]}E^{*(\alpha)}_{n,q}(x)=\sum\limits_{k=0}^{[n/2]}\frac{x^{k}\Gamma_q(n-2k+\alpha+1)}{[k]_q![n-2k]_q!}$	$\sum\limits_{n=0}^{\infty}{}_{[2]}E^{*(\alpha)}_{n,q}(x)t^n=\frac{\Gamma_q(\alpha+1)}{(1-t)_q^{\alpha+1}}e_q(xt^2)$

Table 1. Series definitions and generating functions of ${}_{[2]}E^{(\alpha)}_{n,q}(x)$ and ${}_{[2]}E^{*(\alpha)}_{n,q}(x)$

Remark 4. *In view of equations* (19) *and* (38)*, we observe that for* $\alpha=0$*, Aq-TEP* $E^{(\alpha)}_{n,q}(x)$ *reduce to the q-TEP* $E_{n,q}(x)$*. Also, in view of equations* (52)*,* (53)*,* (59)*, and* (60)*, we observe that, for* $\alpha=0$*,* m*th order Aq-TEP* ${}_{[m]}E^{(\alpha)}_{n,q}(x)$ *and* ${}_{[m]}E^{*(\alpha)}_{n,q}(x)$ *reduce to the* m*th order q-TEP* ${}_{[m]}E_{n,q}(x)$ *and* ${}_{[m]}E^*{}_{n,q}(x)$*, respectively.*

Further in view of Remark 3, for $\alpha=0$*, the* 2*nd order Aq-TEP* ${}_{[2]}E^{(\alpha)}_{n,q}(x)$ *and* ${}_{[2]}E^{*(\alpha)}_{n,q}(x)$ *reduce to the* 2*nd order q-TEP* ${}_{[2]}E_{n,q}(x)$ *and* ${}_{[2]}E^*_{n,q}(x)$*, respectively. Also, since, for* $q\to 1^-$*, the results in quantum calculus reduce to the results of ordinary calculus. Therefore, for* $q\to 1^-$*, the results involving these q-TEP* $E_{n,q}(x)$*,* $E^{(\alpha)}_{n,q}(x)$*,* ${}_{[2]}E^{(\alpha)}_{n,q}(x)$*,* ${}_{[2]}E^{*(\alpha)}_{n,q}(x)$*,* ${}_{[m]}E^{(\alpha)}_{n,q}(x)$*,* ${}_{[m]}E^{*(\alpha)}_{n,q}(x)$*,* ${}_{[2]}E_{n,q}(x)$*,* ${}_{[2]}E^*_{n,q}(x)$*,* ${}_{[m]}E_{n,q}(x)$*, and* ${}_{[m]}E^*_{n,q}(x)$ *reduce to the corresponding results for* $e_n(x)$*,* $e^{(\alpha)}_n(x)$*,* ${}_{[2]}e^{(\alpha)}_n(x)$*,* ${}_{[2]}e^{*(\alpha)}_n(x)$*,* ${}_{[m]}e^{(\alpha)}_n(x)$*,* ${}_{[m]}e^{*(\alpha)}_n(x)$*,* ${}_{[2]}e_n(x)$*,* ${}_{[2]}e^*_n(x)$*,* ${}_{[m]}e_n(x)$*, and* ${}_{[m]}e^*_n(x)$*, respectively.*

Remark 5. *Taking* k *times q-partial derivatives with respect to* x *of both the sides of equation* (43) *by using* (8) *and taking* k *times q-partial derivative of both the sides of equations* (65) *and* (66) *by using* (7) *and then again using equations* (43)*,* (65)*, and* (66) *in the respective resultant equations, we get*

$$D^k_{q,x}E^{(\alpha)}_{n,q}(x)=q^{\binom{k}{2}}E^{(\alpha)}_{n-k,q}(q^kx),\quad 0\le k\le n, \tag{67}$$

$$D^k_{q,x[m]}E^{(\alpha)}_{n,q}(x)={}_{[m]}E^{(\alpha)}_{n-mk,q}(x),\quad 0\le k\le n/m, \tag{68}$$

$$D^k_{q,x[m]}E^{*(\alpha)}_{n,q}(x)={}_{[m]}E^{*(\alpha)}_{n-mk,q}(x),\quad 0\le k\le n/m. \tag{69}$$

In view of Remark 4, for $\alpha=0$*, equations* (67)–(69) *give the following* k*th derivatives of* $E_{n,q}(x)$*,* ${}_{[m]}E_{n,q}(x)$ *and* ${}_{[m]}E^*_{n,q}(x)$*:*

$$D^k_{q,x}E_{n,q}(x)=q^{\binom{k}{2}}E_{n-k,q}(q^kx),\quad 0\le k\le n,$$

$$D^k_{q,x[m]}E_{n,q}(x)={}_{[m]}E_{n-mk,q}(x),\quad 0\le k\le n/m, \tag{70}$$

$$D^k_{q,x[m]}E^*_{n,q}(x)={}_{[m]}E^*_{n-mk,q}(x),\quad 0\le k\le n/m, \tag{71}$$

respectively. Further, in view of Remark 3, for $m=2$*, equations* (70) *and* (71) *give the following* k*th derivatives of* ${}_{[2]}E_{n,q}(x)$ *and* ${}_{[2]}E^*_{n,q}(x)$*:*

$$D^k_{q,x[2]}E_{n,q}(x)={}_{[2]}E_{n-2k,q}(x),\qquad D^k_{q,x[2]}E^*_{n,q}(x)={}_{[2]}E^*_{n-2k,q}(x),\quad 0\le k\le n/2.$$

3 Summation and operational formulas

In this section, we obtain some summation and operational formulas for $E_{n,q}(x)$, $E^{(\alpha)}_{n,q}(x)$, ${}_{[2]}E_{n,q}(x)$, ${}_{[2]}E^*_{n,q}(x)$, ${}_{[m]}E_{n,q}(x)$, ${}_{[m]}E^*_{n,q}(x)$, ${}_{[m]}E^{(\alpha)}_{n,q}(x)$, and ${}_{[m]}E^{*(\alpha)}_{n,q}(x)$. First, we obtain the following summation formula for Aq-TEP $E^{(\alpha)}_{n,q}(x)$.

Theorem 8. *The following summation formula for $E^{(\alpha)}_{n,q}(x)$*

$$\sum_{k=0}^{n}\frac{(-x)^k}{[k]_q!}E^{(\alpha)}_{n-k,q}(x)=\frac{\Gamma_q(\alpha+n+1)}{[n]_q!},\qquad \alpha,x\in\mathbb{C},\quad 0<q<1,\quad 0\le k\le n, \tag{72}$$

holds true.

Proof. In view of equation (4), we have

$$e_q(-xt)\left(\frac{\Gamma_q(\alpha+1)}{(1-t)_q^{\alpha+1}}E_q(xt)\right)=\frac{\Gamma_q(\alpha+1)}{(1-t)_q^{\alpha+1}},$$

which on using (2), (9) and (43), gives

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}E^{(\alpha)}_{n,q}(x)\frac{(-x)^k t^{n+k}}{[k]_q!}=\sum_{n=0}^{\infty}\begin{bmatrix}\alpha+n\\ n\end{bmatrix}_q\Gamma_q(\alpha+1)t^n.$$

Using equation (23) in the left hand side and using (10) in the right hand side of the above equation, we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-x)^k}{[k]_q!}E^{(\alpha)}_{n-k,q}(x)t^n=\sum_{n=0}^{\infty}\frac{\Gamma_q(\alpha+n+1)}{[n]_q!}t^n,$$

which on comparing the equal powers of t from both sides, gives assertion (72). The proof of Theorem 8 is completed. □

In view of Remark 4 and Theorem 8, for $\alpha=0$, we deduce the following result.

Corollary 1. *The following summation formula for $E_{n,q}(x)$*

$$\sum_{k=0}^{n}\frac{(-x)^k}{[k]_q!}E_{n-k,q}(x)=1,\quad 0\le k\le n,$$

holds true.

Next, we obtain some summation formulas for ${}_{[m]}E^{(\alpha)}_{n,q}(x)$ and ${}_{[m]}E^{*(\alpha)}_{n,q}(x)$.

Theorem 9. *The following summation formulas for ${}_{[m]}E^{(\alpha)}_{n,q}(x)$ and ${}_{[m]}E^{*(\alpha)}_{n,q}(x)$*

$$\sum_{k=0}^{n}q^{\binom{k}{2}}\frac{(-x)^k}{[k]_q!}{}_{[m]}E^{(\alpha)}_{n-k,q}(x)=\begin{cases}0, & (n/m)\notin\mathbb{N}\cup\{0\},\\ \Gamma_q(\alpha+n+1)/[n]_q!, & (n/m)\in\mathbb{N}\cup\{0\},\end{cases}\quad 0\le k\le n, \tag{73}$$

and

$$\sum_{k=0}^{[n/m]}q^{\binom{k}{2}}\frac{(-x)^k}{[k]_q!}{}_{[m]}E^{*(\alpha)}_{n-mk,q}(x)=\frac{\Gamma_q(\alpha+n+1)}{[n]_q!},\quad 0\le k\le\frac{n}{m}, \tag{74}$$

hold true, respectively.

Proof. In view of equation (4), we have

$$\frac{\Gamma_q(\alpha+1)}{(1-t^m)_q^{\alpha+1}}e_q(xt)E_q(-xt)=\frac{\Gamma_q(\alpha+1)}{(1-t^m)_q^{\alpha+1}},$$

which on using (3), (9) and (65), gives

$$\sum_{n=0}^{\infty} {}_{[m]}E_{n,q}^{(\alpha)}(x)t^n \sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{(-x)^k t^k}{[k]_q!} = \sum_{n=0}^{\infty} \begin{bmatrix} \alpha+n \\ n \end{bmatrix}_q \Gamma_q(\alpha+1)t^{nm}.$$

Using (23) in the left hand side and (10) in the right hand side of the above equation and then comparing the equal powers of t from both sides, we get assertion (73).

Again, in view of equation (4), we have

$$\frac{\Gamma_q(\alpha+1)}{(1-t)_q^{\alpha+1}} e_q(xt^m)E_q(-xt^m) = \frac{\Gamma_q(\alpha+1)}{(1-t)_q^{\alpha+1}},$$

which on using (3), (9) and (66), gives

$$\sum_{n=0}^{\infty} {}_{[m]}E_{n,q}^{*(\alpha)}(x)t^n \sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{(-x)^k t^{mk}}{[k]_q!} = \sum_{n=0}^{\infty} \begin{bmatrix} \alpha+n \\ n \end{bmatrix}_q \Gamma_q(\alpha+1)t^{n}.$$

Using (58) in the left hand side and (10) in the right hand side of the above equation and then comparing the equal powers of t from both sides, we get assertion (74). The proof of Theorem 9 is completed. □

In view of Remark 4 and Theorem 9, for $\alpha = 0$, we deduce the following summation formulas for ${}_{[m]}E_{n,q}(x)$ and ${}_{[m]}E_{n,q}^{*}(x)$.

Corollary 2. *The following summation formulas for* ${}_{[m]}E_{n,q}(x)$ *and* ${}_{[m]}E_{n,q}^{*}(x)$

$$\sum_{k=0}^{n} q^{\binom{k}{2}} \frac{(-x)^k}{[k]_q!} {}_{[m]}E_{n-k,q}(x) = \begin{cases} 0, & (n/m) \notin \mathbb{N} \cup \{0\}, \\ 1, & (n/m) \in \mathbb{N} \cup \{0\}, \end{cases} \quad 0 \le k \le n,$$

and

$$\sum_{k=0}^{[n/m]} q^{\binom{k}{2}} \frac{(-x)^k}{[k]_q!} {}_{[m]}E_{n-mk,q}^{*}(x) = 1, \quad 0 \le k \le \frac{n}{m},$$

hold true, respectively.

Example 3. *Applying formulas* (72)*,* (73) *and* (74)*, we have the following:*

$$\sum_{k=0}^{3} \frac{(-x)^k}{[k]_{2/3}!} E_{3-k,2/3}^{(1/3)}(x) = \frac{\Gamma_{2/3}(13/3)}{[3]_{2/3}!},$$

$$\sum_{k=0}^{4} (3/4)^{\binom{k}{2}} \frac{(-x)^k}{[k]_{3/4}!} {}_{[2]}E_{4-k,3/4}^{(1/3)}(x) = \begin{cases} 0, \\ \Gamma_q(16/3)/[4]_{3/4}!, \end{cases}$$

and

$$\sum_{k=0}^{3} (5/6)^{\binom{k}{2}} \frac{(-x)^k}{[k]_{5/6}!} {}_{[2]}E_{6-2k,5/6}^{*(1/3)}(x) = \frac{\Gamma_{5/6}(22/3)}{[6]_{5/6}!},$$

respectively.

Theorem 10. *The following operational rules for q-TEP $E_{n,q}^{(\alpha)}(x)$ and ${}_{[m]}E_{n,q}^{*(\alpha)}(x)$*

$$E_q(-X_{q,x})E_{n,q}^{(\alpha)}(x) = \frac{\Gamma_q(\alpha+n+1)}{[n]_q!}, \quad x,\alpha \in \mathbb{C}, \quad 0<q<1, \tag{75}$$

$$E_q(-X_{q,x}^{*})_{[m]}E_{n,q}^{*(\alpha)}(x) = \frac{\Gamma_q(\alpha+n+1)}{[n]_q!}, \quad x,\alpha \in \mathbb{C}, \quad 0<q<1, \tag{76}$$

hold true, respectively, where the kth power of q-operators $X_{q,x}$ and $X_{q,x}^{*}$ are defined by

$$X_{q,x}^{k} := \frac{x^k}{q^{k(k-1)}}T_x^{-k}D_{q,x}^{k}, \qquad X_{q,x}^{*k} := x^k D_{q,x}^{k}, \quad k \in \mathbb{N}, \tag{77}$$

respectively.

Proof. In view of (25), equation (72) can be rewritten as

$$\sum_{k=0}^{n}\frac{(-x)^k}{q^{\binom{k}{2}}[k]_q!}T_x^{-k}q^{\binom{k}{2}}E_{n-k,q}^{(\alpha)}(q^k x) = \frac{\Gamma_q(\alpha+n+1)}{[n]_q!}, \qquad 0\le k\le n. \tag{78}$$

If the q-operator $X_{q,x}^{k}$ is defined by (77), then using equations (3) and (67) in the left hand side of (78), we obtain assertion (75).

Again, in view of (69), equation (74) becomes

$$\sum_{k=0}^{[n/m]}\frac{(-x)^k}{[k]_q!}q^{\binom{k}{2}}D_{q,x}^{k}{}_{[m]}E_{n,q}^{*(\alpha)}(x) = \frac{\Gamma_q(\alpha+n+1)}{[n]_q!}.$$

If the q-operator $X_{q,x}^{*k}$ is defined by (77), then using equation (3) in the left hand side of above equation, we obtain assertion (76). The proof of Theorem 10 is completed. □

Corollary 3. *The following operational rules for $E_{n,q}(x)$ and ${}_{[m]}E_{n,q}^{*}(x)$*

$$E_q(-X_{q,x})E_{n,q}(x) = 1, \qquad E_q(-X_{q,x}^{*})_{[m]}E_{n,q}^{*}(x) = 1, \quad x,\alpha \in \mathbb{C}, \quad 0<q<1,$$

hold true, respectively.

4 Conclusions

Many professionals related to special functions have an affinity for investigating quantum calculus, which is an effective tool that has been frequently used in several applications like modelling quantum computing, non-commutative probability, combinatorics, functional analysis, mathematical physics, approximation theory and other fields. The interest in the properties of classical TEP and their families is manifold and they appear indeed in many problems in optics and quantum mechanics. Also, they played a crucial role in the evaluation of integrals that involve products of special functions. This led to the introduction of q-TEP and the presentation of their features using the integral representation of q-Gamma function and a few q-calculus identities. In previous sections, we explored ways to enrich the situation with various types of q-TEP. The integral representation, generating function and series definition of the q-TEP $E_{n,q}(x)$ are introduced and it was the most important part of all of these tasks. Certain properties of these polynomials like series definition, recurrence relations, differential equations are established in Section 1. Also, in Section 2, we introduce the associated q-TEP $E^{(\alpha)}_{n,q}(x)$, higher order q-TEP ${}_{[2]}E_{n,q}(x)$, ${}_{[2]}E^{*}_{n,q}(x)$, ${}_{[m]}E_{n,q}(x)$, ${}_{[m]}E^{*}_{n,q}(x)$ as well as higher order associated q-TEP ${}_{[m]}E^{(\alpha)}_{n,q}(x)$ and ${}_{[m]}E^{*(\alpha)}_{n,q}(x)$. Further, we derive their integral forms, generating functions, series definitions. In Section 3, summation and operational formulas for $E_{n,q}(x)$, $E^{(\alpha)}_{n,q}(x)$, ${}_{[2]}E_{n,q}(x)$, ${}_{[2]}E^{*}_{n,q}(x)$, ${}_{[m]}E_{n,q}(x)$, ${}_{[m]}E^{*}_{n,q}(x)$, ${}_{[m]}E^{(\alpha)}_{n,q}(x)$, and ${}_{[m]}E^{*(\alpha)}_{n,q}(x)$ are also established. We have provided multiple examples to demonstrate the efficacy of the proposed technique. We offer further study avenues, beginning with the work presented. The findings reported in this research suggest many options for characterization. Furthermore, the previous sections findings can be applied to different polynomial families to draw additional conclusions.

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Раза Н., Фадель М., Чезарано К. *Про q-усічені експоненціальні поліноми* // Карпатські матем. публ. — 2024. — Т.16, №1. — С. 128–147.

У цій статті введено q-усічені експоненціальні поліноми за допомогою інтегральної форми. Отримано певні властивості q-усічених експоненційних поліномів, таких як визначення ряду, рекурентні співвідношення, q-диференціальні рівняння та інтегральні представлення. Крім того, представлено асоційовані q-усічені експоненціальні поліноми, q-усічені експоненціальні поліноми вищого порядку та асоційовані q-усічені експоненціальні поліноми вищого порядку. Отримано їхні інтегральні форми, породжуючі функції, визначення рядів, підсумовування та операційні формули.

Ключові слова і фрази: квантове числення, усічені експоненціальні поліноми, рекурентні співвідношення, підсумовування та інтегральні формули, q-поліноми Ерміта, оператор q-дилатації.