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Research article

On certain properties of three parametric kinds of Apostol-type unified Bernoulli-Euler polynomials

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Abstract: In this paper, we define the three parametric types of Apostol-type unified Bernoulli-Euler polynomials. We present fundamental properties of these polynomials through the utilization of their generating functions. Furthermore, we derive the partial derivatives of these polynomials. Subsequently, we introduce bivariate polynomials and determine their zeros, graphical representations, and approximation values for specific parameters.

Keywords: unified Bernoulli-Euler polynomials; Apostol-type polynomials; partial derivatives; generating functions
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1. Introduction and preliminaries

In recent years, a number of scholars have made significant contributions to the development of generating functions for newly discovered families of special polynomials, such as Bernoulli, Euler, and Genocchi polynomials. These researchers have successfully established the essential properties of these polynomials and have derived a variety of identities and relationships connecting trigonometric functions with two types of special polynomials using generating functions. Additionally, by applying the partial derivative operator to these generating functions, several derivative formulae and finite combinatorial sums involving the aforementioned polynomials and numbers have been obtained. Let

$\mathbb{N}, \mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ indicate the set of positive integers, the set of integers, the set of real numbers, and the set of complex numbers, respectively. Let $\alpha \in \mathbb{N}$ $x \in \mathbb{R}$, and $\lambda \in \mathbb{C}$ (or $\mathbb{R}$). The two classical polynomials, specifically the Bernoulli polynomials (BP) denoted as $\mathcal{B}_n(x)$ and the Euler polynomials (EP) represented as $\mathcal{E}_n(x)$, have a rich history dating back centuries and have found extensive applications across diverse mathematical domains. Notably, they have played a pivotal role in finite difference calculus and number theory, as substantiated by references [1, 2, 4, 5, 7, 13, 27, 28]. It is worth emphasizing that these polynomials are characterized by the following exponential generating functions:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x)\frac{t^n}{n!}, \ |t| < 2\pi$$

and

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_n(x)\frac{t^n}{n!}, \ |t| < \pi.$$

Because of their importance, numerous extensions for these polynomials and others that share similar structures have been extensively investigated, leading to some fascinating results [3, 9–11, 15, 20, 21]. For instance, one can consider the generalized Bernoulli polynomials denoted by $\mathcal{B}_n^{(\alpha)}(x)$ and generalized Euler polynomials denoted by $\mathcal{E}_n^{(\alpha)}(x)$, where the parameter α specifies their order

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x)\frac{t^n}{n!}, \quad |t| < 2\pi$$

and

$$\left(\frac{2}{e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x)\frac{t^n}{n!}, \ |t| < \pi.$$

Referentially, please see [14, 16]. Srivastava and Luo studied the Apostol-Bernoulli polynomials, $\mathcal{B}_n^{(\alpha)}(x;\lambda)$, and the Apostol-Euler polynomials (AEP), $\mathcal{E}_n^{(\alpha)}(x;\lambda)$, of order α in their work cited as [17, p. 917, Eq (1)], and [23, p. 395, Eq (1.18)]. The Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order α are defined by means of the following exponential generating function.

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} = \left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt}. \tag{1.1}$$

Note that $\mathcal{B}_n^{(\alpha)}(x;1) = \mathcal{B}_n^{(\alpha)}(x)$ denotes the Bernoulli polynomials of order α, and $\mathcal{B}_n^{(\alpha)}(0;\lambda) = \mathcal{B}_n^{(\alpha)}(\lambda)$ denote the Apostol-Bernoulli numbers of order α, respectively. Setting $\alpha = 1$ into (1.1), we obtain $\mathcal{B}_n^{(1)}(\lambda) = \mathcal{B}_n(\lambda)$ which are the so-called Apostol-Bernoulli numbers. The Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order α are defined by means of the following exponential generating function.

$$\sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} = \left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt}. \tag{1.2}$$

By virtue of (1.2), we have $\mathcal{E}_n^{(\alpha)}(x;1) = \mathcal{E}_n^{(\alpha)}(x)$ denot the Euler polynomials of order α and $\mathcal{E}_n^{(\alpha)}(0;\lambda) = \mathcal{E}_n^{(\alpha)}(\lambda)$ denot the Apostol-Euler numbers of order α, respectively. Setting $\alpha = 1$ into (1.2), we obtain $\mathcal{E}_n^{(1)}(\lambda) = \mathcal{E}_n(\lambda)$ which are the so-called Apostol-Euler numbers.

Srivastava et al. [24, 25] utilized both trigonometric generating functions and exponential generating functions to define two parameter special cases of the Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi polynomials. Additionally, they presented the fundamental properties of these types of polynomials, which can also be found in other papers, such as [18, 19, 26]. These polynomials are defined as follows:

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{(c,\alpha)}(x, y; \lambda)\frac{t^n}{n!} = \left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} \cos(yt), \tag{1.3}$$

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{(s,\alpha)}(x, y; \lambda)\frac{t^n}{n!} = \left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} \sin(yt), \tag{1.4}$$

$$\sum_{n=0}^{\infty} \mathcal{E}_n^{(c,\alpha)}(x, y; \lambda)\frac{t^n}{n!} = \left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt} \cos(yt), \tag{1.5}$$

and

$$\sum_{n=0}^{\infty} \mathcal{E}_n^{(s,\alpha)}(x, y; \lambda)\frac{t^n}{n!} = \left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt} \sin(yt). \tag{1.6}$$

The symbols c and s appearing in the superscripts on the left-hand sides of these aforementioned Eqs (1.3)–(1.6) denote the presence of the trigonometric cosine and the trigonometric sine functions, respectively, in the generating functions on the corresponding right-hand sides.

Recently, in the paper [8], a class of polynomials denoted as Unified Bernoulli-Euler Polynomials of Apostol-type (UBEPA), represented as $\mathcal{U}_n(x; \lambda; \mu)$, was introduced and their properties were systematically examined by Belbachir et al. These UBEPA are defined through the following power series:

$$\frac{2 - \mu + \frac{\mu}{2}t}{\lambda e^t + (1 - \mu)} e^{xt} = \sum_{n=0}^{\infty} \mathcal{U}_n(x; \lambda; \mu)\frac{t^n}{n!}, \tag{1.7}$$

where

$$\left|\ln\left(\frac{\lambda}{1-\mu}\right) + t\right| < \pi, \quad 0 \le \mu < 1$$

and

$$\left|\ln\left(\frac{\lambda}{\mu-1}\right) + t\right| < 2\pi, \quad \text{otherwise.}$$

It is worth noting that by choosing specific values for the parameters μ and λ in Eq (1.7), we can get the well-known Bernoulli, Euler, Apostol-Bernoulli, and Apostol-Euler polynomials. However, this formulation does not encompass the unified polynomials of order α, nor does it take into account the Frobenius-Euler Polynomials (FEP), denoted as $H_n(x; u)$, which are defined using the following generating function:

$$\frac{1-u}{e^t - u} e^{xt} = \sum_{n=0}^{\infty} H_n(x; u)\frac{t^n}{n!}, \; |t| < \left|\log \frac{1}{u}\right|.$$

For more detail about Frobenius-Euler polynomials, please see [22] and [6, p. 2, Def. 1]. For real parameters y and z, the Taylor series representations of the following functions in $t = 0$ are given by:

$$G_{cc}(t; y; z) = \cos(yt)\cos(zt) = \sum_{n=0}^{\infty} C_n^{cc}(y, z)\frac{t^n}{n!}, \tag{1.8}$$

$$G_{ss}(t;y;z) = \sin(yt)\sin(zt) = \sum_{n=0}^{\infty} S_n^{ss}(x,y)\frac{z^n}{n!}, \tag{1.9}$$

$$G_{cs}(t;y;z) = \cos(yt)\sin(zt) = \sum_{n=0}^{\infty} C_n^{cs}(y,z)\frac{t^n}{n!}, \tag{1.10}$$

$$G_{sc}(t;y;z) = \sin(yt)\cos(zt) = \sum_{n=0}^{\infty} S_n^{sc}(y,z)\frac{t^n}{n!}, \tag{1.11}$$

where the expressions $C_n^{cc}(y,z)$, $S_n^{ss}(y,z)$, $C_n^{cs}(y,z)$, and $S_n^{sc}(y,z)$ are given by:

$$C_n^{cc}(y,z) = \sum_{k=0}^{[\frac{n}{2}]}(-1)^n\binom{2n}{2k}z^{2n-2k}y^{2k},$$

$$S_n^{cc}(y,z) = \sum_{k=0}^{[\frac{n-1}{2}]}(-1)^n\frac{\binom{2n+1}{2k}}{2k+1}z^{2n-2k+1}y^{2k+1},$$

$$C_n^{cs}(y,z) = \sum_{k=0}^{[\frac{n-1}{2}]}(-1)^n\binom{2n+1}{2k}z^{2n-2k+1}y^{2k},$$

$$S_n^{sc}(y,z) = \sum_{k=0}^{[\frac{n-1}{2}]}(2n+1)(-1)^n\frac{\binom{2n}{2k}}{2k+1}z^{2n-2k}y^{2k}.$$

Motivated by the above-cited recent papers, we define the three parametric kinds of Apostol-type unified Bernoulli-Euler polynomials. Utilizing the generating functions with their functional equations, some properties of these polynomials are given. Then we give the partial derivatives of these newly established polynomials. As a special cases of these types of polynomials, we define two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials and give some properties of these polynomials. Moreover, by using a computer program, we obtain certain zeros of two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials $\mathcal{U}_n^{C_y}(x;y;\lambda;\mu)$ and $\mathcal{U}_n^{S_y}(x;y;\lambda;\mu)$ and beautifully graphical representations of them.

2. Three parametric kinds of Apostol-type unified Bernoulli-Euler polynomials $\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)$, $\mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)$, $\mathcal{U}_n^{C_yS_z}(x;y;z;\lambda;\mu)$ and $\mathcal{U}_n^{S_yC_z}(x;y;z;\lambda;u)$

In this section, by virtue of the above Eqs (1.7)–(1.11), we define the three parametric kinds of Apostol-type unified Bernoulli-Euler polynomials.

Definition 2.1. *For $\lambda,\mu \in \mathbb{C}$, three parametric kinds of Apostol-type unified Bernoulli-Euler polynomials, are defined through the following generating function:*

$$\mathcal{F}_{cc}(t;x;y;z;\lambda;\mu) = \frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{xt}\cos(yt)\cos(zt) = \sum_{n=0}^{\infty}\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!}, \tag{2.1}$$

$$\mathcal{F}_{ss}(t;x;y;z;\lambda;\mu)=\frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{xt}\sin(yt)\sin(zt)=\sum_{n=0}^{\infty}\mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!}, \tag{2.2}$$

$$\mathcal{F}_{cs}(t;x;y;z;\lambda;\mu)=\frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{xt}\cos(yt)\sin(zt)=\sum_{n=0}^{\infty}\mathcal{U}_n^{C_yS_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!}, \tag{2.3}$$

$$\mathcal{F}_{sc}(t;x;y;z;\lambda;\mu)=\frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{xt}\sin(yt)\cos(zt)=\sum_{n=0}^{\infty}\mathcal{U}_n^{S_yC_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!}, \tag{2.4}$$

where

$$\left|\ln\left(\frac{\lambda}{1-\mu}\right)+t\right|<\pi,\quad 0\le\mu<1$$

and

$$\left|\ln\left(\frac{\lambda}{\mu-1}\right)+t\right|<2\pi,\quad \textit{otherwise.}$$

We now give the following items and examples of some special polynomials related to these extensions.

- For $\mu=0$ and $\lambda=1$, Eqs (2.1) and (2.4) become the three variables of Euler polynomials.
- For $\mu=z=0$ and $\lambda=1$, Eqs (2.1) and (2.4) become the two variables of Euler polynomials.
- For $\mu=2$ and $\lambda=1$, Eqs (2.1) and (2.4) become the three variables of Bernoulli polynomials.
- For $\mu=2$, $\lambda=1$, and $z=0$, Eqs (2.1) and (2.4) become the two variables of Bernoulli polynomials.
- For $\mu=2$, Eqs (2.1) and (2.4) become the three variables of Apostol-Bernoulli polynomials.
- For $\mu=2$ and $z=0$, Eqs (2.1) and (2.4) become the two variables of Apostol-Bernoulli polynomials.
- For $\mu=0$, Eqs (2.1) and (2.4) become the three variables of Apostol-Euler polynomials.
- For $\mu=z=0$, Eqs (2.1) and (2.4) become the two variables of Apostol-Euler polynomials.

Example 2.1. *The first three terms of $\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)$ polynomials in the variable x , y and z, are as follows:*

$$\begin{aligned}
\mathcal{U}_0^{C_yC_z}(x;y;z;\lambda;\mu)&=\frac{-2+\mu}{-1-\lambda+\mu},\\
\mathcal{U}_1^{C_yC_z}(x;y;z;\lambda;\mu)&=-\frac{2\lambda}{(1+\lambda-\mu)^2}+\frac{2x}{1+\lambda-\mu}+\frac{\lambda\mu}{(1+\lambda-\mu)^2}+\frac{\mu}{2(1+\lambda-\mu)}-\frac{x\mu}{1+\lambda-\mu},\\
\mathcal{U}_2^{C_yC_z}(x;y;z;\lambda;\mu)&=-\frac{2\lambda}{(1+\lambda-\mu)^3}+\frac{2\lambda^2}{(1+\lambda-\mu)^3}-\frac{4x\lambda}{(1+\lambda-\mu)^2}+\frac{2x^2}{1+\lambda-\mu}-\frac{2y^2}{1+\lambda-\mu}\\
&\quad-\frac{2z^2}{1+\lambda-\mu}+\frac{3\lambda\mu}{(1+\lambda-\mu)^3}-\frac{\lambda^2\mu}{(1+\lambda-\mu)^3}-\frac{\lambda\mu}{(1+\lambda-\mu)^2}+\frac{2x\lambda\mu}{(1+\lambda-\mu)^2}\\
&\quad+\frac{x\mu}{1+\lambda-\mu}-\frac{x^2\mu}{1+\lambda-\mu}+\frac{y^2\mu}{1+\lambda-\mu}+\frac{z^2\mu}{1+\lambda-\mu}-\frac{\lambda\mu^2}{(1+\lambda-\mu)^3}.
\end{aligned}$$

Example 2.2. *The first four terms of $\mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)$ polynomials in the variables x , y, and z are as follows:*

$$\mathcal{U}_0^{S_yS_z}(x;y;z;\lambda;\mu)=0,$$
$$\mathcal{U}_1^{S_yS_z}(x;y;z;\lambda;\mu)=0,$$
$$\mathcal{U}_2^{S_yS_z}(x;y;z;\lambda;\mu)=-\frac{4yz}{-1-\lambda+\mu}+\frac{2yz\mu}{-1-\lambda+\mu},$$
$$\mathcal{U}_3^{S_yS_z}(x;y;z;\lambda;\mu)=-\frac{12yz\lambda}{(1+\lambda-\mu)^2}+\frac{12xyz}{1+\lambda-\mu}+\frac{6yz\lambda\mu}{(1+\lambda-\mu)^2}+\frac{3yz\mu}{1+\lambda-\mu}-\frac{6xyz\mu}{1+\lambda-\mu},$$
$$\begin{aligned}\mathcal{U}_4^{S_yS_z}(x;y;z;\lambda;\mu)=&-\frac{24yz\lambda}{(1+\lambda-\mu)^3}+\frac{24yz\lambda^2}{(1+\lambda-\mu)^3}-\frac{48xyz\lambda}{(1+\lambda-\mu)^2}+\frac{24x^2yz}{1+\lambda-\mu}-\frac{(8y^3z}{1+\lambda-\mu}\\&-\frac{8yz^3}{1+\lambda-\mu}+\frac{36yz\lambda\mu}{(1+\lambda-\mu)^3}-\frac{12yz\lambda^2\mu}{(1+\lambda-\mu)^3}-\frac{12yz\lambda\mu}{(1+\lambda-\mu)^2}+\frac{24xyz\lambda\mu}{(1+\lambda-\mu)^2}\\&+\frac{12xyz\mu}{1+\lambda-\mu}-\frac{12x^2yz\mu}{1+\lambda-\mu}+\frac{4y^3z\mu}{1+\lambda-\mu}+\frac{4yz^3\mu}{1+\lambda-\mu}-\frac{12yz\lambda\mu^2}{(1+\lambda-\mu)^3}.\end{aligned}$$

Example 2.3. *The first three terms of* $\mathcal{U}_n^{C_yS_z}(x;y;z;\lambda;\mu)$ *polynomials in the variables x , y, and z are as follows:*

$$\mathcal{U}_0^{C_yS_z}(x;y;z;\lambda;\mu)=0,$$
$$\mathcal{U}_1^{C_yS_z}(x;y;z;\lambda;\mu)=-\frac{2z}{-1-\lambda+\mu}+\frac{z\mu}{-1-\lambda+\mu},$$
$$\mathcal{U}_2^{C_yS_z}(x;y;z;\lambda;\mu)=-\frac{4z\lambda}{(1+\lambda-\mu)^2}+\frac{4xz}{1+\lambda-\mu}+\frac{2z\lambda\mu}{(1+\lambda-\mu)^2}+\frac{z\mu}{1+\lambda-\mu}-\frac{2xz\mu}{1+\lambda-\mu},$$
$$\begin{aligned}\mathcal{U}_3^{C_yS_z}(x;y;z;\lambda;\mu)=&-\frac{6z\lambda}{(1+\lambda-\mu)^3}+\frac{6z\lambda^2}{(1+\lambda-\mu)^3}-\frac{12xz\lambda}{(1+\lambda-\mu)^2}+\frac{6x^2z}{1+\lambda-\mu}-\frac{6y^2z}{1+\lambda-\mu}\\&-\frac{2z^3}{1+\lambda-\mu}+\frac{9z\lambda\mu}{(1+\lambda-\mu)^3}-\frac{3z\lambda^2\mu}{(1+\lambda-\mu)^3}-\frac{3z\lambda\mu}{(1+\lambda-\mu)^2}+\frac{6xz\lambda\mu}{(1+\lambda-\mu)^2}\\&+\frac{3xz\mu}{1+\lambda-\mu}-\frac{3x^2z\mu}{1+\lambda-\mu}+\frac{3y^2z\mu}{1+\lambda-\mu}+\frac{z^3\mu}{1+\lambda-\mu}-\frac{3z\lambda\mu^2}{(1+\lambda-\mu)^3}.\end{aligned}$$

Example 2.4. *The first three terms of* $\mathcal{U}_n^{S_yC_z}(x;y;z;\lambda;\mu)$ *polynomials in the variables x, y, and z are as follows:*

$$\mathcal{U}_0^{S_yC_z}(x;y;z;\lambda;\mu)=0,$$
$$\mathcal{U}_1^{S_yC_z}(x;y;z;\lambda;\mu)=-\frac{2y}{-1-\lambda+\mu}+\frac{y\mu}{-1-\lambda+\mu},$$
$$\mathcal{U}_2^{S_yC_z}(x;y;z;\lambda;\mu)=-\frac{4y\lambda}{(1+\lambda-\mu)^2}+\frac{4xy}{1+\lambda-\mu}+\frac{2y\lambda\mu}{(1+\lambda-\mu)^2}+\frac{y\mu}{1+\lambda-\mu}-\frac{2xy\mu}{1+\lambda-\mu},$$
$$\begin{aligned}\mathcal{U}_3^{S_yC_z}(x;y;z;\lambda;\mu)=&-\frac{6y\lambda}{(1+\lambda-\mu)^3}+\frac{6y\lambda^2}{(1+\lambda-\mu)^3}-\frac{12xy\lambda}{(1+\lambda-\mu)^2}+\frac{6x^2y}{1+\lambda-\mu}-\frac{6y^2y}{1+\lambda-\mu}\\&-\frac{2y^3}{1+\lambda-\mu}+\frac{9y\lambda\mu}{(1+\lambda-\mu)^3}-\frac{3y\lambda^2\mu}{(1+\lambda-\mu)^3}-\frac{3y\lambda\mu}{(1+\lambda-\mu)^2}+\frac{6xy\lambda\mu}{(1+\lambda-\mu)^2}\\&+\frac{3xy\mu}{1+\lambda-\mu}-\frac{3x^2y\mu}{1+\lambda-\mu}+\frac{3yz^2\mu}{1+\lambda-\mu}+\frac{y^3\mu}{1+\lambda-\mu}-\frac{3y\lambda\mu^2}{(1+\lambda-\mu)^3}.\end{aligned}$$

We now give some properties for these polynomials in the following Theorems.

Theorem 2.1. *Let* $\{\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)\}_{n\geq 0}$, $\{\mathcal{U}_n^{C_yS_z}(x;y;z;\lambda;\mu)\}_{n\geq 0}$, $\{\mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)\}_{n\geq 0}$ *and* $\{\mathcal{U}_n^{S_yC_z}(x;y;z;\lambda;\mu)\}_{n\geq 0}$ *are the three parametric kinds of Apostol-type unified Bernoulli-Euler polynomials. Then, we have:*

$$\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{U}_{n-k}(x;\lambda;u)C_k^{cc}(y,z), \tag{2.5}$$

$$\mathcal{U}_n^{C_yS_z}(x;y;z;\lambda;\mu) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{U}_{n-k}(x;\lambda;u)C_k^{cs}(y,z), \tag{2.6}$$

$$\mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{U}_{n-k}(x;\lambda;u)S_k^{ss}(y,z), \tag{2.7}$$

$$\mathcal{U}_n^{S_yC_z}(x;y;z;\lambda;\mu) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{U}_{n-k}(x;\lambda;u)S_k^{sc}(y,z). \tag{2.8}$$

Proof. For the proof Eq (2.5), using (1.7) and (1.8), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!} &= \frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{xt}\cos(yt)\cos(zt)\\ &= \sum_{n=0}^{\infty}\mathcal{U}_n(x;\lambda;\mu)\frac{t^n}{n!}\sum_{n=0}^{\infty}C_n^{cc}(y,z)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{U}_{n-k}(x;\lambda;\mu)C_k^{cc}(y,z)\frac{t^n}{n!}.\end{aligned}$$

Equations (2.6)–(2.8) can be shown similarly. □

Theorem 2.2. *Let* $\{\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)\}_{n\geq 0}$, $\{\mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)\}_{n\geq 0}$ *be three parametric kinds of Apostol-type unified Bernoulli-Euler polynomials. Then, we obtain:*

$$\mathcal{U}_n(x;\lambda;\mu) = \mathcal{U}_n^{C_yC_z}(x;y;y;\lambda;\mu) + \mathcal{U}_n^{S_yS_z}(x;y;y;\lambda;\mu)$$

and

$$\mathcal{U}_n^{C_yC_z}(x;2y;0;\lambda;\mu) = \mathcal{U}_n^{C_yC_z}(x;y;y;\lambda;\mu) - \mathcal{U}_n^{S_yS_z}(x;y;y;\lambda;\mu).$$

Proof. Substituting $z=y$ in Eqs (2.1) and (2.2), respectively, we have

$$\frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{xt}\cos^2(yt) = \sum_{n=0}^{\infty}\mathcal{U}_n^{C_yC_z}(x;y;y;\lambda;\mu)\frac{t^n}{n!}$$

$$\frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{xt}\sin^2(yt) = \sum_{n=0}^{\infty}\mathcal{U}_n^{S_yS_z}(x;y;y;\lambda;\mu)\frac{t^n}{n!}.$$

Adding the two previous expressions, we have

$$\frac{2-\mu+\frac{\mu}{2}t}{\lambda e^{t}+(1-\mu)}e^{xt}\left[\cos^{2}(yt)+\sin^{2}(yt)\right]=\sum_{n=0}^{\infty}\left[\mathcal{U}_{n}^{C_yC_z}(x;y;y;\lambda;\mu)+\mathcal{U}_{n}^{S_yS_z}(x;y;y;\lambda;\mu)\right]\frac{t^{n}}{n!}.$$

Because of the above equation, we obtain

$$\frac{2-\mu+\frac{\mu}{2}t}{\lambda e^{t}+(1-\mu)}e^{xt}=\sum_{n=0}^{\infty}\left[\mathcal{U}_{n}^{C_yC_z}(x;y;y;\lambda;\mu)+\mathcal{U}_{n}^{S_yS_z}(x;y;y;\lambda;\mu)\right]\frac{t^{n}}{n!}.$$

□

By equating coefficients, the desired result is obtained. Our second assertion in the theorem follows a very similar path, but instead of adding the expressions, we subtract them and also use the fact that, $\cos^2(t)-\sin^2(t)=\cos(2t)$.

Theorem 2.3. *Let $\{\mathcal{U}_{n}^{C_yS_z}(x;y;z;\lambda;\mu)\}_{n\geq0}$, $\{\mathcal{U}_{n}^{S_yC_z}(x;y;z;\lambda;\mu)\}_{n\geq0}$ be two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials. Then, we have:*

$$\mathcal{U}_{n}^{C_yS_z}(x;y;y;\lambda;\mu)=\mathcal{U}_{n}^{S_yC_z}(x;y;y;\lambda;\mu)=\frac{1}{2}\mathcal{U}_{n}^{C_yS_z}(x;0;2y;\lambda;\mu).$$

Proof. Substituting $z=y$ in Eq (2.3), we have

$$\frac{2-\mu+\frac{\mu}{2}t}{\lambda e^{t}+(1-\mu)}e^{xt}\left[2\cos(yt)\sin(yt)\right]=2\sum_{n=0}^{\infty}\mathcal{U}_{n}^{C_yS_z}(x;y;y;\lambda;\mu)\frac{t^{n}}{n!}$$

$$\frac{2-\mu+\frac{\mu}{2}t}{\lambda e^{t}+(1-\mu)}e^{xt}\sin(2yt)=2\sum_{n=0}^{\infty}\mathcal{U}_{n}^{C_yS_z}(x;y;y;\lambda;\mu)\frac{t^{n}}{n!}.$$

Then, we have

$$\sum_{n=0}^{\infty}\mathcal{U}_{n}^{C_yS_z}(x;0;2y;\lambda;\mu)\frac{t^{n}}{n!}=2\sum_{n=0}^{\infty}\mathcal{U}_{n}^{C_yS_z}(x;y;y;\lambda;\mu)\frac{t^{n}}{n!}.$$

By equating coefficients, the proof is completed. □

3. Partial derivatives for three parametric kinds of Apostol-type unified Bernoulli-Euler polynomials $\mathcal{U}_{n}^{C_yC_z}(x;y;z;\lambda;\mu)$, $\mathcal{U}_{n}^{S_yS_z}(x;y;z;\lambda;\mu)$, $\mathcal{U}_{n}^{C_yS_z}(x;y;z;\lambda;\mu)$, and $\mathcal{U}_{n}^{S_yC_z}(x;y;a;\lambda;u)$

In this section, by applying the partial derivative of three parametric kinds of Apostol-type unified Bernoulli-Euler polynomials operator to Eqs (2.1)–(2.4), we will give the following results.

Theorem 3.1. *For $n,m,k\in\mathbb{N}$, let $\{\mathcal{U}_{n}^{C_yC_z}(x;y;z;\lambda;\mu)\}_{n\geq0}$ be the sequence of three parametric kinds of Apostol-type unified Bernoulli-Euler polynomials; then the following statements hold:*

$$\frac{\partial^{k}}{\partial x^{k}}\left\{\mathcal{U}_{n}^{C_yC_z}(x;y;z;\lambda;\mu)\right\}=k!\binom{n}{k}\mathcal{U}_{n-k}^{C_yC_z}(x;y;z;\lambda;\mu), \tag{3.1}$$

$$\frac{\partial^k}{\partial y^k}\left\{\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)\right\} = (-1)^{\lfloor\frac{k}{2}\rfloor}k!\binom{n}{k}\mathcal{U}_{n-k}^{C_yC_z}(x;y;z;\lambda;\mu), \tag{3.2}$$

$$\frac{\partial^k}{\partial y^k}\left\{\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)\right\} = (-1)^{\lfloor\frac{k+1}{2}\rfloor}k!\binom{n}{k}\mathcal{U}_{n-k}^{S_yC_z}(x;y;z;\lambda;\mu), \tag{3.3}$$

$$\frac{\partial^k}{\partial z^k}\left\{\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)\right\} = (-1)^{\lfloor\frac{k}{2}\rfloor}k!\binom{n}{k}\mathcal{U}_{n-k}^{C_yC_z}(x;y;z;\lambda;\mu), \tag{3.4}$$

$$\frac{\partial^k}{\partial z^k}\left\{\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)\right\} = (-1)^{\lfloor\frac{k+1}{2}\rfloor}k!\binom{n}{k}\mathcal{U}_{n-k}^{C_yS_z}(x;y;z;\lambda;\mu),$$

where $\lfloor * \rfloor$ *denotes the integer part of* $*$.

Proof. If we take the partial derivative of both sides with respect to x in Eq (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\partial^k}{\partial x^k}\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!} &= \frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}\cos(yt)\cos(zt)\frac{\partial^k}{\partial x^k}e^{xt}\\ &= \frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}\cos(yt)\cos(zt)t^k e^{xt}.\end{aligned}$$

So, we obtain

$$\sum_{n=0}^{\infty}\frac{\partial^k}{\partial x^k}\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)\frac{t^{n+k}}{n!}.$$

Comparing the coefficients of t^n in both sides of the above equation, we obtain our assertion (3.1).

In Eq (2.1), we take the partial derivative of both sides with respect to y and use the following fact,

$$\frac{\partial^k}{\partial y^k}(\cos(yt)) = \begin{cases} t^k\cos(yt) & \text{if } k\equiv 0 \pmod 4,\\ -t^k\sin(yt) & \text{if } k\equiv 1 \pmod 4,\\ -t^k\cos(yt) & \text{if } k\equiv 2 \pmod 4,\\ t^k\sin(yt) & \text{if } k\equiv 3 \pmod 4.\end{cases}$$

So, we achieve

$$\frac{\partial^k}{\partial y^k}\mathcal{F}_{cc}(t;x;y;z;\lambda;\mu) = \frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{xt}\cdot(-1)^{\lfloor k/2\rfloor}t^k\cos(yt)\cos(zt),$$

$$\frac{\partial^k}{\partial y^k}\mathcal{F}_{cc}(t;x;y;z;\lambda;\mu) = \frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{xt}\cdot(-1)^{\lfloor (k+1)/2\rfloor}t^k\sin(yt)\cos(zt).$$

By virtue of above identities, we have

$$\sum_{n=0}^{\infty}\frac{\partial^k}{\partial y^k}\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!} = (-1)^{\lfloor k/2\rfloor}\cdot\sum_{n=0}^{\infty}\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)\frac{t^{n+k}}{n!},$$

$$\sum_{n=0}^{\infty}\frac{\partial^k}{\partial y^k}\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!}=(-1)^{\lfloor\frac{k+1}{2}\rfloor}\cdot\sum_{n=0}^{\infty}\mathcal{U}_n^{S_yC_z}(x;y;z;\lambda;\mu)\frac{t^{n+k}}{n!}.$$

Comparing the coefficients of t^n on both sides of the above equation, we obtain our assertions (3.2) and (3.3).

In Eq (2.1), we take the partial derivative of both sides with respect to z and use the following fact,

$$\frac{\partial^k}{\partial z^k}\left(\cos(zt)\right)=\begin{cases}t^k\cos(zt) & \text{if } k\equiv 0 \pmod 4,\\ -t^k\sin(zt) & \text{if } k\equiv 1 \pmod 4,\\ -t^k\cos(zt) & \text{if } k\equiv 2 \pmod 4,\\ t^k\sin(zt) & \text{if } k\equiv 3 \pmod 4.\end{cases}$$

Thus, we have

$$\frac{\partial^k}{\partial z^k}\mathcal{F}_{cc}(t;x;y;z;\lambda;\mu)=\frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{xt}(-1)^{\lfloor k/2\rfloor}t^k\cos(yt)\cos(zt),$$

$$\frac{\partial^k}{\partial z^k}\mathcal{F}_{cc}(t;x;y;z;\lambda;\mu)=\frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{xt}(-1)^{\lfloor (k+1)/2\rfloor}t^k\sin(yt)\cos(zt).$$

Using the above identities, we have

$$\sum_{n=0}^{\infty}\frac{\partial^k}{\partial z^k}\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!}=(-1)^{\lfloor k/2\rfloor}\sum_{n=0}^{\infty}\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)\frac{t^{n+k}}{n!},$$

$$\sum_{n=0}^{\infty}\frac{\partial^k}{\partial z^k}\mathcal{U}_n^{C_yC_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!}=(-1)^{\lfloor\frac{k+1}{2}\rfloor}\sum_{n=0}^{\infty}\mathcal{U}_n^{C_yS_z}(x;y;z;\lambda;\mu)\frac{t^{n+k}}{n!}.$$

Comparing the coefficients of t^n on both sides of the above equation, we get our assertions (3.3) and (3.4). □

Theorem 3.2. *For $n,m,k\in\mathbb{N}$, let $\{\mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)\}_{n\geq 0}$ be the three parametric kinds of Apostol-type unified Bernoulli-Euler polynomials. Then the following identities hold:*

$$\frac{\partial^k}{\partial x^k}\left\{\mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)\right\}=k!\binom{n}{k}\mathcal{U}_{n-k}^{S_yS_z}(x;y;z;\lambda;\mu), \tag{3.5}$$

$$\frac{\partial^k}{\partial y^k}\left\{\mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)\right\}=(-1)^{\lfloor\frac{k}{2}\rfloor}k!\binom{n}{k}\mathcal{U}_{n-k}^{S_yS_z}(x;y;z;\lambda;\mu), \tag{3.6}$$

$$\frac{\partial^k}{\partial y^k}\left\{\mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)\right\}=(-1)^{\lfloor\frac{(k-1)}{2}\rfloor}k!\binom{n}{k}\mathcal{U}_{n-k}^{C_yS_z}(x;y;z;\lambda;\mu), \tag{3.7}$$

$$\frac{\partial^k}{\partial z^k}\left\{\mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)\right\}=(-1)^{\lfloor\frac{k}{2}\rfloor}k!\binom{n}{k}\mathcal{U}_{n-k}^{S_yS_z}(x;y;z;\lambda;\mu), \tag{3.8}$$

$$\frac{\partial^k}{\partial z^k}\left\{\mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)\right\}=(-1)^{\lfloor\frac{(k-1)}{2}\rfloor}k!\binom{n}{k}\mathcal{U}_{n-k}^{S_yC_z}(x;y;z;\lambda;\mu), \tag{3.9}$$

where $\lfloor * \rfloor$ denotes the integer part of $*$.

Proof. If we take the partial derivative of both sides with respect to x in Equation (2.2), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\partial^k}{\partial x^k}\mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!} &= \frac{\partial^k}{\partial x^k}\left[\frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{xt}\sin(yt)\sin(zt)\right]\\ &= \frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}\sin(yt)\sin(zt)\frac{\partial^k}{\partial x^k}e^{xt}\\ &= \frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}\sin(yt)\sin(zt)t^k e^{xt}.\end{aligned}$$

So, we have

$$\sum_{n=0}^{\infty}\frac{\partial^k}{\partial x^k}\mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)\frac{t^{n+k}}{n!}.$$

Comparing the coefficients of t^n on both sides of the above equation, we obtain our assertion (3.5).

In Eq (2.2), we take the partial derivative of both sides with respect to y and use the following fact

$$\frac{\partial^k}{\partial y^k}\sin(yt) = \begin{cases} t^k\sin(yt) & \text{if } k\equiv 0 \pmod 4,\\ t^k\cos(yt) & \text{if } k\equiv 1 \pmod 4,\\ -t^k\sin(yt) & \text{if } k\equiv 2 \pmod 4,\\ -t^k\cos(yt) & \text{if } k\equiv 3 \pmod 4.\end{cases}$$

So, we obtain

$$\frac{\partial^k}{\partial y^k}\mathcal{F}_{ss}(t;x;y;z;\lambda;\mu) = \frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{xt}\times\left(\begin{cases} t^k\sin(yt)\sin(zt) & \text{if } k\equiv 0 \pmod 4,\\ t^k\cos(yt)\sin(zt) & \text{if } k\equiv 1 \pmod 4,\\ -t^k\sin(yt)\sin(zt) & \text{if } k\equiv 2 \pmod 4,\\ -t^k\cos(yt)\sin(zt) & \text{if } k\equiv 3 \pmod 4.\end{cases}\right).$$

Namely, we have

$$\frac{\partial^k}{\partial y^k}\mathcal{F}_{ss}(t;x;y;z;\lambda;\mu) = \frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{xt}(-1)^{\lfloor k/2\rfloor}t^k\sin(yt)\sin(zt),$$

$$\frac{\partial^k}{\partial y^k}\mathcal{F}_{ss}(t;x;y;z;\lambda;\mu) = \frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{xt}(-1)^{\lfloor (k-1)/2\rfloor}t^k\cos(yt)\sin(zt).$$

Using the above identities, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\partial^k}{\partial y^k}\mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!} &= (-1)^{\lfloor k/2\rfloor}\sum_{n=0}^{\infty}\mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)\frac{t^{n+k}}{n!},\\ \sum_{n=0}^{\infty}\frac{\partial^k}{\partial y^k}\mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!} &= (-1)^{\lfloor \frac{k-1}{2}\rfloor}\sum_{n=0}^{\infty}\mathcal{U}_n^{C_yS_z}(x;y;z;\lambda;\mu)\frac{t^{n+k}}{n!}.\end{aligned}$$

Comparing the coefficients of t^n on both sides of the above Eqs (3.6) and (3.7), we obtain our assertions.

In Eq (2.2), we take the partial derivative of both sides with respect to z and use the following fact,

$$\frac{\partial^k}{\partial z^k}(\sin(zt)) = \begin{cases} t^k \sin(zt) & \text{if } k \equiv 0 \pmod 4, \\ t^k \cos(zt) & \text{if } k \equiv 1 \pmod 4, \\ -t^k \sin(zt) & \text{if } k \equiv 2 \pmod 4, \\ -t^k \cos(zt) & \text{if } k \equiv 3 \pmod 4. \end{cases}$$

So, we find that

$$\frac{\partial^k}{\partial z^k}\mathcal{F}_{ss}(t;x;y;z;\lambda;\mu) = \frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t + (1-\mu)} e^{xt}(-1)^{\lfloor k/2 \rfloor} t^k \sin(yt)\sin(zt),$$

$$\frac{\partial^k}{\partial z^k}\mathcal{F}_{ss}(t;x;y;z;\lambda;\mu) = \frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t + (1-\mu)} e^{xt}(-1)^{\lfloor \frac{k-1}{2} \rfloor} t^k \sin(yt)\cos(zt).$$

By using the above identities, we have

$$\sum_{n=0}^{\infty} \frac{\partial^k}{\partial z^k} U_n^{S_yS_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!} = (-1)^{\lfloor \frac{k}{2} \rfloor} \sum_{n=0}^{\infty} \mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)\frac{t^{n+k}}{n!},$$

$$\sum_{n=0}^{\infty} \frac{\partial^k}{\partial z^k} \mathcal{U}_n^{S_yS_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!} = (-1)^{\lfloor \frac{(k-1)}{2} \rfloor} \sum_{n=0}^{\infty} \mathcal{U}_n^{S_yC_z}(x;y;z;\lambda;\mu)\frac{t^{n+k}}{n!}.$$

Comparing the coefficients of t^n on both sides of the above Eqs (3.8) and (3.9), we obtain our assertions. □

Theorem 3.3. *For $n, m, k \in \mathbb{N}$, let $\{\mathcal{U}_n^{C_yS_z}(x;y;z;\lambda;\mu)\}_{n\geq 0}$ be the three parametric kinds of Apostol-type unified Bernoulli-Euler polynomials. Then the following identities hold:*

$$\begin{aligned}
\frac{\partial^k}{\partial x^k}\left\{\mathcal{U}_n^{C_yS_z}(x;y;z;\lambda;\mu)\right\} &= k!\binom{n}{k}\mathcal{U}_{n-k}^{C_yS_z}(x;y;z;\lambda;\mu), \\
\frac{\partial^k}{\partial y^k}\left\{\mathcal{U}_n^{C_yS_z}(x;y;z;\lambda;\mu)\right\} &= (-1)^{\lfloor \frac{k}{2} \rfloor}k!\binom{n}{k}\mathcal{U}_{n-k}^{C_yS_z}(x;y;z;\lambda;\mu), \\
\frac{\partial^k}{\partial y^k}\left\{\mathcal{U}_n^{C_yS_z}(x;y;z;\lambda;\mu)\right\} &= (-1)^{\lfloor \frac{k+1}{2} \rfloor}k!\binom{n}{k}\mathcal{U}_{n-k}^{S_yS_z}(x;y;z;\lambda;\mu), \\
\frac{\partial^k}{\partial z^k}\left\{\mathcal{U}_n^{C_yS_z}(x;y;z;\lambda;\mu)\right\} &= (-1)^{\lfloor \frac{k}{2} \rfloor}k!\binom{n}{k}\mathcal{U}_{n-k}^{C_yS_z}(x;y;z;\lambda;\mu), \\
\frac{\partial^k}{\partial z^k}\left\{\mathcal{U}_n^{C_yS_z}(x;y;z;\lambda;\mu)\right\} &= (-1)^{\lfloor \frac{k+1}{2} \rfloor}k!\binom{n}{k}\mathcal{U}_{n-k}^{S_yS_z}(x;y;z;\lambda;\mu),
\end{aligned}$$

where $\lfloor * \rfloor$ denotes the integer part of $*$.

Proof. Using Eq (2.3), we have the following result.

$$\sum_{n=0}^{\infty} \frac{\partial^k}{\partial x^k}\mathcal{U}_n^{C_yS_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!} = \frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t + (1-\mu)}\cos(yt)\sin(zt)\frac{\partial^k}{\partial x^k}e^{xt} = \frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t + (1-\mu)}\cos(yt)\sin(zt)t^k e^{xt}.$$

So, we get

$$\sum_{n=0}^{\infty}\frac{\partial^k}{\partial x^k}\mathcal{U}_n^{C_yS_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\mathcal{U}_n^{C_yS_z}(x;y;z;\lambda;\mu)\frac{t^{n+k}}{n!}.$$

Comparing the coefficients of t^n on both sides of the above equation, we get our first statement; the other statements follow a similar path. □

Theorem 3.4. *For $n,m,k\in\mathbb{N}$, let $\{\mathcal{U}_n^{S_yC_z}(x;y;z;\lambda;\mu)\}_{n\geq 0}$ are the three parametric kinds of Apostol-type unified Bernoulli-Euler polynomials. Then, we find that:*

$$\begin{aligned}
\frac{\partial^k}{\partial x^k}\left\{\mathcal{U}_n^{S_yC_z}(x;y;z;\lambda;\mu)\right\} &= k!\binom{n}{k}\mathcal{U}_{n-k}^{S_yC_z}(x;y;z;\lambda;\mu),\\
\frac{\partial^k}{\partial y^k}\left\{\mathcal{U}_n^{S_yC_z}(x;y;z;\lambda;\mu)\right\} &= (-1)^{\lfloor\frac{k}{2}\rfloor}k!\binom{n}{k}\mathcal{U}_{n-k}^{S_yC_z}(x;y;z;\lambda;\mu),\\
\frac{\partial^k}{\partial y^k}\left\{\mathcal{U}_n^{S_yC_z}(x;y;z;\lambda;\mu)\right\} &= (-1)^{\lfloor\frac{k-1}{2}\rfloor}k!\binom{n}{k}\mathcal{U}_{n-k}^{C_yC_z}(x;y;z;\lambda;\mu),\\
\frac{\partial^k}{\partial z^k}\left\{\mathcal{U}_n^{S_yC_z}(x;y;z;\lambda;\mu)\right\} &= (-1)^{\lfloor\frac{k}{2}\rfloor}k!\binom{n}{k}\mathcal{U}_{n-k}^{S_yC_z}(x;y;z;\lambda;\mu),\\
\frac{\partial^k}{\partial z^k}\left\{\mathcal{U}_n^{S_yC_z}(x;y;z;\lambda;\mu)\right\} &= (-1)^{\lfloor\frac{k-1}{2}\rfloor}k!\binom{n}{k}\mathcal{U}_{n-k}^{S_yS_z}(x;y;z;\lambda;\mu),
\end{aligned}$$

where $\lfloor * \rfloor$ denotes the integer part of $*$.

Proof. Using Eq (2.4), we have the following result.

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{\partial^k}{\partial x^k}\mathcal{U}_n^{S_yC_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!} &= \frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}\sin(yt)\cos(zt)\frac{\partial^k}{\partial x^k}e^{xt}\\
&= \frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}\sin(yt)\cos(zt)t^k e^{xt}.
\end{aligned}$$

So, we obtain

$$\sum_{n=0}^{\infty}\frac{\partial^k}{\partial x^k}\mathcal{U}_n^{S_yC_z}(x;y;z;\lambda;\mu)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\mathcal{U}_n^{S_yC_z}(x;y;z;\lambda;\mu)\frac{t^{n+k}}{n!}.$$

By comparing the coefficients of t^n on both sides of the equation, we arrive at our first statement. The other statements are derived through a similar reasoning process. □

4. Two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials $\mathcal{U}_n^{C_y}(x;y;\lambda;\mu)$ and $\mathcal{U}_n^{S_y}(x;y;\lambda;\mu)$

In this section, we define the two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials. Substituting $z=0$ in Eqs (2.1) and (2.4), we can give the following definition.

Definition 4.1. *For $\lambda,\mu\in\mathbb{C}$, the two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials, are defined through the following generating function:*

$$\mathcal{F}_c(t;x;y;\lambda;\mu)=\frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{xt}\cos(yt)=\sum_{n=0}^{\infty}\mathcal{U}_n^{C_y}(x;y;\lambda;\mu)\frac{t^n}{n!}, \tag{4.1}$$
$$\mathcal{F}_s(t;x;y;\lambda;\mu)=\frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{xt}\sin(yt)=\sum_{n=0}^{\infty}\mathcal{U}_n^{S_y}(x;y;\lambda;\mu)\frac{t^n}{n!}.$$

Theorem 4.1. *For $n,m,k\in\mathbb{N}$, let $\{\mathcal{U}_n^{C_y}(x;y;\lambda;\mu)\}_{n\geq 0}$ and $\{\mathcal{U}_n^{S_y}(x;y;z;\lambda;\mu)\}_{n\geq 0}$ be the sequences of two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials; then the following identities hold:*

$$\mathcal{U}_n^{C_y}(x;y;\lambda;\mu)=\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{2k}\mathcal{U}_{n-2k}(x;\lambda;\mu)y^{2k}$$

and

$$\mathcal{U}_n^{S_y}(x;y;\lambda;\mu)=\sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k\binom{n}{2k+1}\mathcal{U}_{n-1-2k}(x;\lambda;\mu)y^{2k+1}. \tag{4.2}$$

Proof. By using Eq (1.7) and the complex series of the cosine, we have

$$\sum_{n=0}^{\infty}\mathcal{U}_n^{C_y}(x;y;\lambda;\mu)\frac{t^n}{n!}=\frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{xt}\cos(yt)$$
$$\sum_{n=0}^{\infty}\mathcal{U}_n^{C_y}(x;y;\lambda;\mu)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\mathcal{U}_n(x;\lambda;\mu)\frac{t^n}{n!}\sum_{n=0}^{\infty}\frac{(-1)^n y^{2n}t^{2n}}{2n!}.$$

Applying the Cauchy series product, we have

$$\sum_{n=0}^{\infty}\mathcal{U}_n^{C_y}(x;y;\lambda;\mu)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{n=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{2k}\mathcal{U}_{n-2k}(x;\lambda;\mu)y^{2k}\frac{t^n}{n!}.$$

By equating coefficients, the desired result is obtained. The proof of (4.2) follows a similar approach, where we employ (1.7) and the complex series of the sine. □

Theorem 4.2. *The following identities hold true:*

$$\mathcal{U}_n^{C_y}(x+r;y;\lambda;\mu)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{U}_k^{C_y}(x;y;\lambda;\mu)r^{n-k} \tag{4.3}$$

and

$$\mathcal{U}_n^{S_y}(x+r;y;\lambda;\mu)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{U}_k^{S_y}(x;y;\lambda;\mu)r^{n-k}. \tag{4.4}$$

Proof. By using the Eq (4.1), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{U}_n^{C_y}(x+r;y;\lambda;\mu)\frac{t^n}{n!} &= \frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}e^{(x+r)t}\cos(yt)=\sum_{n=0}^{\infty}\mathcal{U}_n^{C_y}(x;y;\lambda;\mu)\frac{t^n}{n!}\sum_{n=0}^{\infty}\frac{(rt)^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\mathcal{U}_k^{C_y}(x;y;\lambda;\mu)r^{n-k}\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of t^n on both sides of this last equation, we have

$$\mathcal{U}_n^{C_y}(x+r;y;\lambda;\mu)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{U}_k^{C_y}(x;y;\lambda;\mu)r^{n-k},$$

which proves the result (4.4). The assertion (4.3) can be proved similarly. □

5. Approximate roots for two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials and their applications

In this section, certain zeros of the two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials $\mathcal{U}_n^{C_y}(x;y;\lambda;\mu)$ and beautifully graphical representations are shown.

A few of them are

$$\begin{aligned}
\mathcal{U}_0^{C_y}(x;y;\lambda;\mu) &= \frac{-2+\mu}{-1-\lambda+\mu},\\
\mathcal{U}_1^{C_y}(x;y;\lambda;\mu) &= -\frac{2\lambda}{(1+\lambda-\mu)^2}+\frac{2x}{1+\lambda-\mu}+\frac{\lambda\mu}{(1+\lambda-\mu)^2}+\frac{\mu}{2(1+\lambda-\mu)}-\frac{x\mu}{1+\lambda-\mu},\\
\mathcal{U}_2^{C_y}(x;y;\lambda;\mu) &= -\frac{2\lambda}{(1+\lambda-\mu)^3}+\frac{2\lambda^2}{(1+\lambda-\mu)^3}-\frac{4x\lambda}{(1+\lambda-\mu)^2}+\frac{2x^2}{1+\lambda-\mu}-\frac{2y^2}{1+\lambda-\mu}\\
&\quad+\frac{3\lambda\mu}{(1+\lambda-\mu)^3}-\frac{\lambda^2\mu}{(1+\lambda-\mu)^3}-\frac{\lambda\mu}{(1+\lambda-\mu)^2}+\frac{2x\lambda\mu}{(1+\lambda-\mu)^2}\\
&\quad+\frac{x\mu}{1+\lambda-\mu}-\frac{x^2\mu}{1+\lambda-\mu}+\frac{y^2\mu}{1+\lambda-\mu}-\frac{\lambda\mu^2}{(1+\lambda-\mu)^3},\\
\mathcal{U}_3^{C_y}(x;y;\lambda;\mu) &= -\frac{2\lambda}{(1+\lambda-\mu)^4}+\frac{8\lambda^2}{(1+\lambda-\mu)^4}-\frac{2\lambda^3}{(1+\lambda-\mu)^4}-\frac{6x\lambda}{(1+\lambda-\mu)^3}+\frac{6x\lambda^2}{(1+\lambda-\mu)^3}\\
&\quad-\frac{6x^2\lambda}{(1+\lambda-\mu)^2}+\frac{6y^2\lambda}{(1+\lambda-\mu)^2}+\frac{2x^3}{1+\lambda-\mu}-\frac{6xy^2}{1+\lambda-\mu}+\frac{5\lambda\mu}{(1+\lambda-\mu)^4}\\
&\quad-\frac{12\lambda^2\mu}{(1+\lambda-\mu)^4}+\frac{\lambda^3\mu}{(1+\lambda-\mu)^4}-\frac{3\lambda\mu}{2(1+\lambda-\mu)^3}+\frac{9x\lambda\mu}{(1+\lambda-\mu)^3}+\frac{3\lambda^2\mu}{2(1+\lambda-\mu)^3}\\
&\quad-\frac{3x\lambda^2\mu}{(1+\lambda-\mu)^3}-\frac{3x\lambda\mu}{(1+\lambda-\mu)^2}+\frac{3x^2\lambda\mu}{(1+\lambda-\mu)^2}-\frac{3y^2\lambda\mu}{(1+\lambda-\mu)^2}+\frac{3x^2\mu}{2(1+\lambda-\mu)}\\
&\quad-\frac{x^3\mu}{1+\lambda-\mu}-\frac{3y^2\mu}{2(1+\lambda-\mu)}+\frac{3xy^2\mu}{1+\lambda-\mu}-\frac{4\lambda\mu^2}{(1+\lambda-\mu)^4}+\frac{4\lambda^2\mu^2}{(1+\lambda-\mu)^4}\\
&\quad+\frac{3\lambda\mu^2}{2(1+\lambda-\mu)^3}-\frac{3x\lambda\mu^2}{(1+\lambda-\mu)^3}+\frac{\lambda\mu^3}{(1+\lambda-\mu)^4}.
\end{aligned}$$

We investigate the beautiful zeros of the two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials $\mathcal{U}_n^{C_y}(x;y;\lambda;\mu)=0$ by using a computer. We plot the zeros of two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials $\mathcal{U}_n^{C_y}(x;y;\lambda;\mu)=0$ for $n=40$ (Figure 1).

In Figure 1(top-left), we choose $\lambda=2,\mu=5$, and $y=\pi$. In Figure 1(top-right), we choose $\lambda=2,\mu=5$, and $y=\frac{\pi}{2}$. In Figure 1(bottom-left), we choose $\lambda=2,\mu=5$, and $y=\frac{\pi}{3}$. In Figure 1(bottom-right), we choose $\lambda=2,\mu=5$, and $y=\frac{\pi}{4}$.

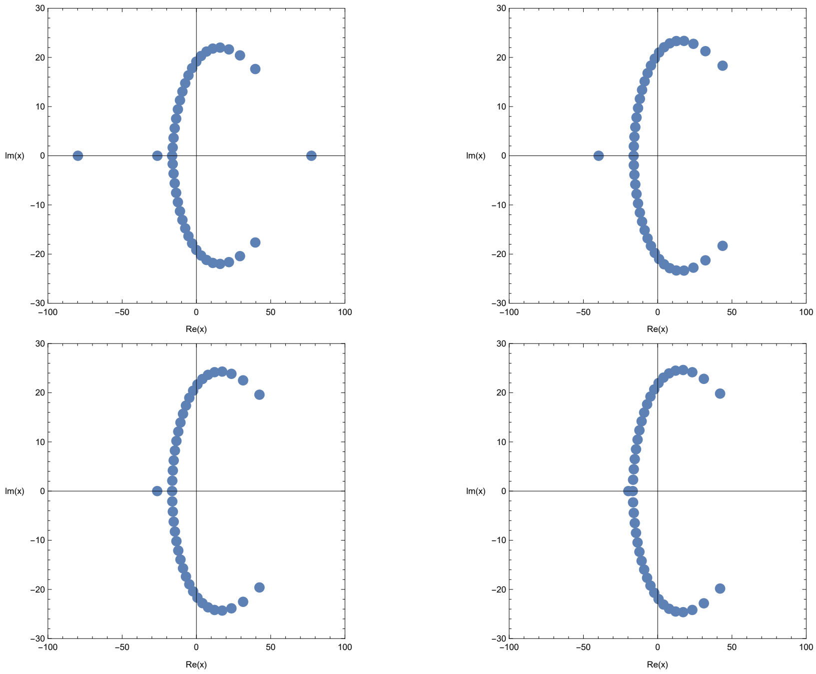

Figure 1. Zeros of $\mathcal{U}_n^{C_y}(x; y; \lambda; \mu) = 0$.

Stacks of zeros of the two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials type $\mathcal{U}_n^{C_y}(x; y; \lambda; \mu) = 0$ for $1 \leq \varepsilon \leq 40$, forming a 3D structure, are presented (Figure 2). In Figure 2 (top-left), we choose $\lambda = 2, \mu = 5$, and $y = \pi$. In Figure 2 (top-right), we choose $\lambda = 2, \mu = 5$, and $y = \frac{\pi}{2}$. In Figure 2 (bottom-left), we choose $\lambda = 2, \mu = 5$, and $y = \frac{\pi}{3}$. In Figure 2 (bottom-right), we choose $\lambda = 2, \mu = 5$, and $y = \frac{\pi}{4}$.

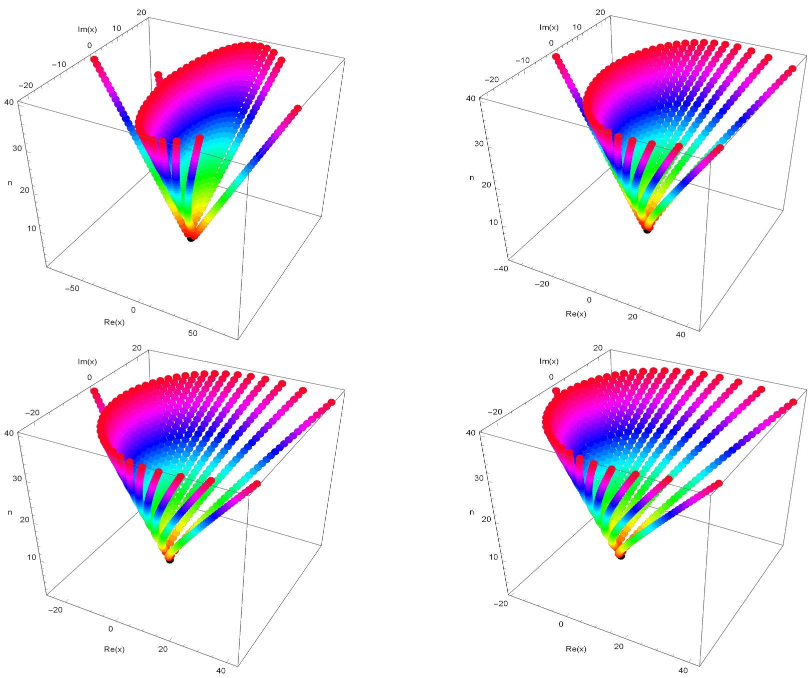

Figure 2. Zeros of $\mathcal{U}_n^{C_y}(x; y; \lambda; \mu) = 0$.

Plots of real zeros of the two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials $\mathcal{U}_n^{C_y}(x;y;\lambda;\mu) = 0$ for $1 \leq n \leq 40$ are presented (Figure 3). In Figure 3 (top-left), we choose $\lambda = 2, \mu = 5$, and $y = \pi$. In Figure 3 (top-right), we choose $\lambda = 2, \mu = 5$, and $y = \frac{\pi}{2}$. In Figure 3 (bottom-left), we choose $\lambda = 2, \mu = 5$, and $y = \frac{\pi}{3}$. In Figure 3 (bottom-right),we choose $\lambda = 2, \mu = 5$, and $y = \frac{\pi}{4}$.

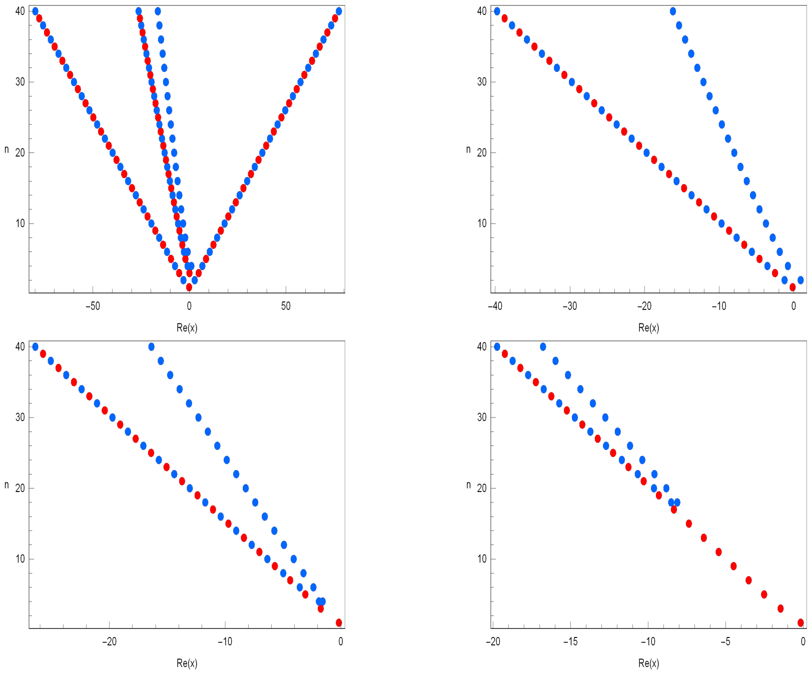

Figure 3. Real zeros of $\mathcal{U}_n^{C_y}(x;y;\lambda;\mu) = 0$.

Table 1. Approximate solutions of $\mathcal{U}_n^{C_y}(x;y;\lambda;\mu) = 0$.

degree n	x
1	-0.16667
2	-3.0931, 2.7598
3	-5.3271, 0.022080, 4.8051
4	-7.4559, -0.97012, 1.0296, 6.7297
5	-9.5327, -2.0616, 1.06441 - 0.49134 i , 1.06441 + 0.49134i, 8.6321
6	-11.584, -2.8515, -0.87769, 1.8890 + 1.4399 i ,1.8890 - 1.4399i , 10.536
7	-13.621, -3.6918, -0.88781 - 0.68377i , -0.88781 + 0.68377 i, 2.7381 - 2.2167i , 2.7381 + 2.2167 i, 12.446
8	-15.649, -4.4325, -2.1661, -0.3586 - 1.5978 i, -0.3586 + 1.5978 i , 3.6336 - 2.9432 i , 3.6336 + 2.9432 i , 14.364
9	-17.671, -5.1878, -2.2437 - 0.7223 i , -2.2437 + 0.7223 i, 0.2105 + 2.4649 i, 0.2105 - 2.4649 i, 4.5674 - 3.6322 i , 4.5674 + 3.6322 i, 16.291
10	-19.689, -5.9057, -3.2565, -1.8925 + 1.6512 i, -1.8925 - 1.6512 i , 0.8405 - 3.2947i, 0.8405 + 3.2947 i, 5.5322 - 4.2897 i, 5.5322 + 4.2897 i, 18.224

Next, we calculated an approximate solution satisfying the two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials $\mathcal{U}_n^{C_y}(x;y;\lambda;\mu) = 0$ for $\lambda = 2, \mu = 5$, and $y = \pi$. The results are given in Table 1.

We investigate the beautiful zeros of the two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials $\mathcal{U}_n^{S_y}(x;y;\lambda;\mu) = 0$ by using a computer. We plot the zeros of two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials $\mathcal{U}_n^{S_y}(x;y;\lambda;\mu) = 0$ for $n = 40$ (Figure 4). In Figure 4 (top-left), we choose $\lambda = 1, \mu = 3$, and $y = \pi$.

In Figure 4 (top-right), we choose $\lambda = 1, \mu = 3$, and $y = \frac{\pi}{2}$. In Figure 4 (bottom-left), we choose $\lambda = 1, \mu = 3$, and $y = \frac{\pi}{3}$. In Figure 4 (bottom-right), we choose $\lambda = 1, \mu = 3$, and $y = \frac{\pi}{4}$.

Stacks of zeros of the two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials $\mathcal{U}_n^{S_y}(x;y;\lambda;\mu) = 0$ for $2 \leq \varepsilon \leq 40$, forming a 3D structure, are presented (Figure 5).

In Figure 5(top-left), we choose $\lambda = 1, \mu = 3$, and $y = \pi$. In Figure 5(top-right), we choose $\lambda = 1, \mu = 3$, and $y = \frac{\pi}{2}$. In Figure 5(bottom-left), we choose $\lambda = 1, \mu = 3$, and $y = \frac{\pi}{3}$. In Figure 5(bottom-right), we choose $\lambda = 1, \mu = 3$, and $y = \frac{\pi}{4}$.

Plots of real zeros of the two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials $\mathcal{U}_n^{S_y}(x;y;\lambda;\mu) = 0$ for $2 \leq n \leq 40$ are presented (Figure 6).

In Figure 6 (top-left), we choose $\lambda = 1, \mu = 3$, and $y = \pi$. In Figure 6 (top-right), we choose $\lambda = 1, \mu = 3$, and $y = \frac{\pi}{2}$. In Figure 6 (bottom-left), we choose $\lambda = 1, \mu = 3$, and $y = \frac{\pi}{3}$. In Figure 6 (bottom-right), we choose $\lambda = 1, \mu = 3$, and $y = \frac{\pi}{4}$.

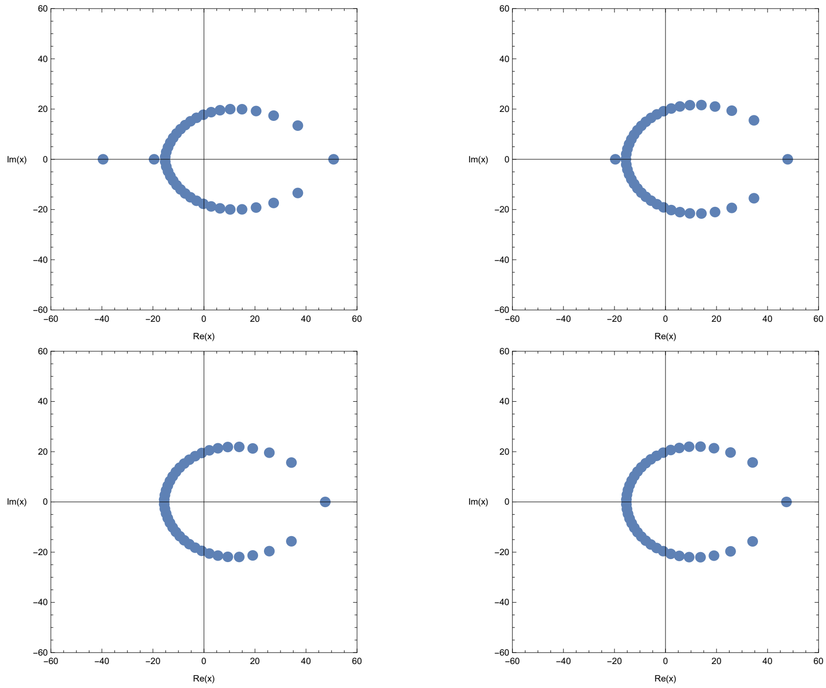

Figure 4. Zeros of $\mathcal{U}_n^{S_y}(x;y;\lambda;\mu) = 0$.

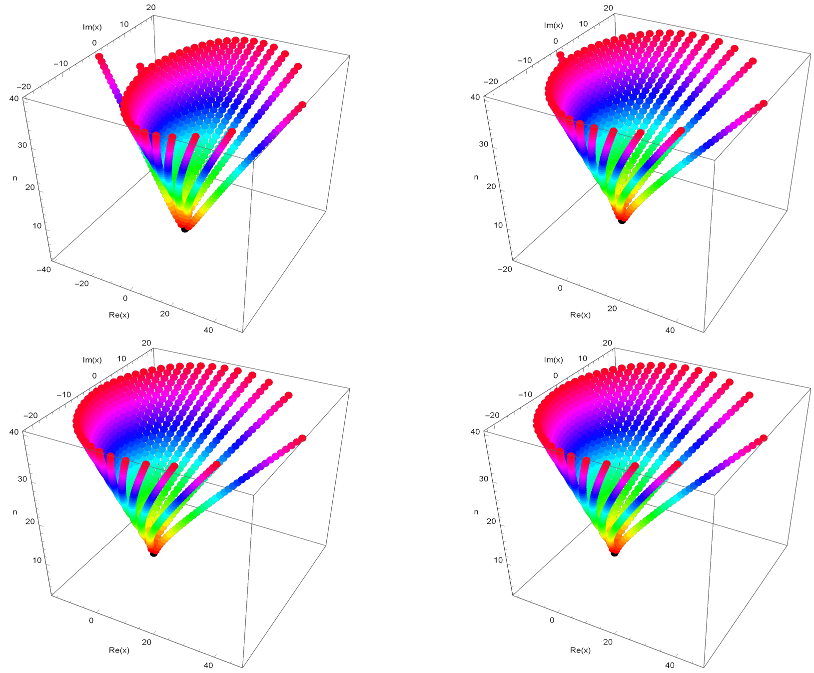

Figure 5. Zeros of $\mathcal{U}_n^{S_y}(x; y; \lambda; \mu) = 0$.

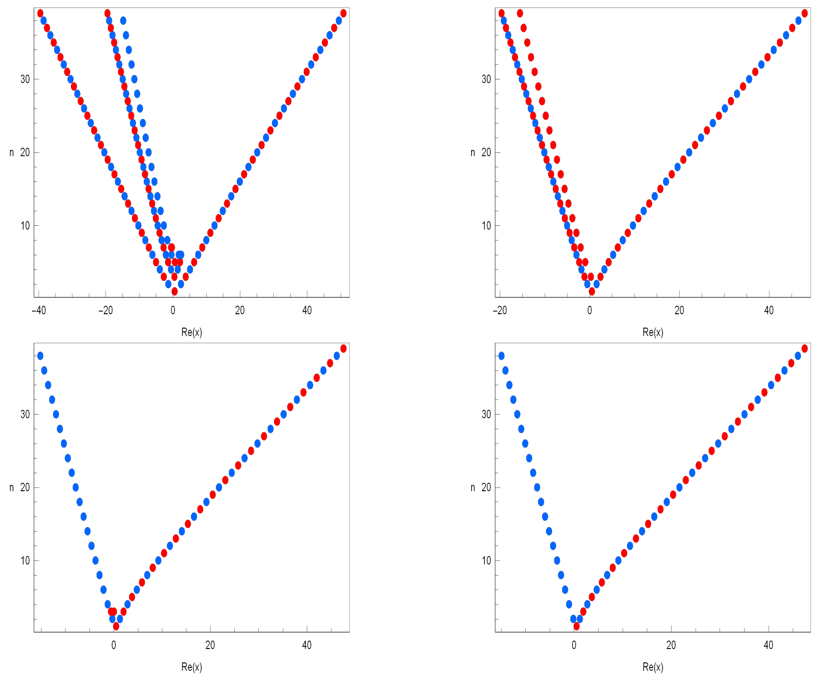

Figure 6. Real zeros of $\mathcal{U}_n^{S_y}(x; y; \lambda; \mu) = 0$.

Next, we calculated an approximate solution satisfying the two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials $\mathcal{U}_n^{S_y}(x; y; \lambda; \mu) = 0$ for $\lambda = 1, \mu = 3$, and $y = \pi$. The results are given in Table 2.

Table 2. Approximate solutions of $\mathcal{U}_n^{S_y}(x;y;\lambda;\mu)=0$.

degree n	x
2	0.50000
3	-1.3815, 2.3815
4	-2.7229, 0.42934, 3.7935
5	-3.9186, -0.58148, 1.4253, 5.0748
6	-5.0425, -1.4358, 0.55667, 2.1148 , 6.3069
7	-6.1297, -2.1173, -0.42912, 1.7145 , 2.4412, 7.5205
8	-7.1938, -2.7930, -0.55400, -0.30241 , 2.8069 + 0.8621i, 2.8069 - 0.8621i, 8.7294
9	-8.2433, -3.3840, -1.6647, 0.09379 - 1.04386 i, 0.09379 + 1.04386i , 3.5819 + 1.4587 i, 3.5819 - 1.4587 i, 9.9406
10	-9.2825, -3.9907, -1.7136 + 0.5208 i, -1.7136 - 0.5208 i , 0.6149 - 1.7860 i , 0.6149 + 1.7860 i , 4.4064 - 2.0008 i , 4.4064 + 2.0008 i, 11.158
11	-10.314, -4.5450, -2.7181, -1.3669 + 1.3569 i, -1.3669 - 1.3569 i, 1.1914 - 2.5026i, 1.1914 + 2.5026 i, 5.2728 - 2.5151 i, 5.2728 + 2.5151 i, 12.383

6. Conclusions

The application of special polynomials is extensive and varied in scientific fields, encompassing areas such as signal processing, geoscience, engineering, and quantum mechanics. These polynomials play a pivotal role in numerical analysis and computational techniques, enabling the resolution of intricate issues across various scientific domains. Researchers in the field of applied mathematics have employed generating functions and function equations of special polynomials in numerous studies to investigate various topics. The results of these investigations have been documented in multiple research papers. In this paper, we have conducted an investigation into the two and three parametric kinds of Apostol-type unified Bernoulli-Euler polynomials, thus broadening the scope of certain special polynomial families that may or may not be present in the literature. Our research has yielded several essential properties of these newly established polynomials. Additionally, we have supplied zeroes and graphical illustrations for the two parametric kinds of Apostol-type unified Bernoulli-Euler polynomials. In [12], the authors constructed a new operator based on Hermite polynomials. Using this paper, researchers can obtain operators for the polynomials mentioned in this paper and study the approximation properties of these operators.

Author contributions

All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Clemente Cesarano and William Ramírez are the Guest Editors of special issue "Orthogonal polynomials and related applications" for AIMS Mathematics. Clemente Cesarano and William Ramírez were not involved in the editorial review and the decision to publish this article.

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