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Solitary Wave Solution of a Generalized Fractional–Stochastic Nonlinear Wave Equation for a Liquid with Gas Bubbles

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Abstract: In the sense of a conformable fractional operator, we consider a generalized fractional–stochastic nonlinear wave equation (GFSNWE). This equation may be used to depict several nonlinear physical phenomena occurring in a liquid containing gas bubbles. The analytical solutions of the GFSNWE are obtained by using the F-expansion and the Jacobi elliptic function methods with the Riccati equation. Due to the presence of noise and the conformable derivative, some solutions that were achieved are shown together with their physical interpretations.

Keywords: fractional generalized nonlinear wave equation; wave; Wiener process; $\mathcal{F}$-expansion method

MSC: 35A20; 60H15; 26A33; 34A08; 34A34; 83C15; 35Q51; 60H10

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1. Introduction

In the fields of finance, engineering, biology, physics, control theory, systems identification, and signal processing, fractional differential equations (FDEs) have received much interest [1–5]. They are also used in social sciences as well as in dietary supplements, finance, climate, and economics. In contrast, stochastic partial differential equations (SPDEs) are employed in the analysis chemical, biological, and physical systems that are affected by random factors. It has been emphasized how important it is to take random impacts into account when modeling complex systems. SPDEs are being increasingly used in information systems, condensed matter physics, finance, biophysics, mechanical and electrical engineering, materials sciences, and climate system modeling to create mathematical models of complicated processes [6,7].

As a result, finding exact solutions to fractional or stochastic differential equations is crucial. For the purpose of solving these equations, several analytical and numerical techniques, such as the modified F-expansion method [8], extended tanh–coth method [9], Riccati–Bernoulli sub-ODE [10], mapping method [11], (G'/G)-expansion method [12], etc., have been developed.

Rayleigh [13] provided the initial study of the bubble dynamics problem. Since liquids containing gas bubbles are widespread in many areas, including medical science and engineering, researchers have studied bubbly liquids. According to some research, the linear partial differential equation of the fourth order may be used to describe the propagation of linear acoustic waves in isothermal bubbly liquids with bubbles that are uniform in radius. The following generalized (3 + 1)-dimensional nonlinear wave equation is one such model, and it is used to describe a liquid containing gas bubbles:

$$(\mathcal{U}_t+\gamma_1\mathcal{U}_x + \gamma_2\mathcal{U}_{xxx} + \gamma_3\mathcal{U}\mathcal{U}_x)_x + \gamma_4\mathcal{U}_{zz} + \gamma_5\mathcal{U}_{yy} = 0, \tag{1}$$

where $\mathcal{U}(x, y, z, t)$ is the wave amplitude, γ_1 is the bubble–liquid viscosity, γ_2 is the bubble–liquid dispersion, γ_3 is the bubble–liquid nonlinearity, γ_4 is the z-transverse perturbation, and γ_5 is the y-transverse perturbation.

Equation (1) may be used to explain various nonlinear physical phenomena in a liquid containing gas bubbles. Therefore, many researchers have studied Equation (1); for instance, Shen et al. [14] used the idea of linear superposition to obtain solutions for N-soliton waves, Wang et al. [15] created both lump-stripe solitons and rogue wave-stripe, Guo and Chen [16] found the lump, periodic solutions, and multi-soliton, Tu et al. [17] constructed the bilinear equation, the Bäcklund transformation, and the N-soliton solution with specific formula for the provided model, Wang et al. [18] used the Hirota bilinear approach to achieve soliton solutions, representing a new generalized exponential rational function [19], Akbulut et al. [20] used the modified Kudryashov and the Nnucci's reduction to acquire information about solitary waves, while the fractional derivative and stochastic term were not earlier considered in (1).

In this study, we look at the generalized fractional–stochastic nonlinear wave equation (GFSNWE) as follows:

$$\mathcal{D}_x^\alpha(\mathcal{U}_t+\gamma_1\mathcal{D}_x^\alpha\mathcal{U} + \gamma_2\mathcal{D}_{xxx}^\alpha\mathcal{U} + \gamma_3\mathcal{U}\mathcal{D}_x^\alpha\mathcal{U}) + \gamma_4\mathcal{D}_{zz}^\alpha\mathcal{U} + \gamma_5\mathcal{D}_{yy}^\alpha = \rho(\mathcal{D}_x^\alpha\mathcal{U})\mathcal{W}_t, \tag{2}$$

where $\mathcal{D}_x^\alpha$ is the conformable derivative (CD) for $\alpha \in (0,1]$ [21], which will be defined in the next section, $\mathcal{W}$ is the standard Wiener process (SWP), and ρ is the noise intensity.

Our contribution in this paper is to analytically determining the fractional–stochastic solutions of GFSNWE (2). The solutions presented in this paper are the first of their kind. These solutions are found using F-expansion and Jacobi elliptical functions methods. Since (2) contains a stochastic term and fractional derivative, physics researchers would find the solution very useful for defining several major physical phenomena. The solutions of GFSNWE (2) are additionally investigated using MATLAB by introducing many graphs to illustrate the effect of noise and fractional derivatives.

This article is summarized as follows: In Section 2, we define the conformable derivative (CD) and the standard Wiener process (SWP), and we explore some of its features. In Section 3, We obtain the wave equation of GFSNWE (2). In Section 4, the Jacobi elliptic functions and F-expansion methods are employed to obtain the exact solutions of the GFSNWE. The impact of noise and the fractional derivative on the acquired solutions of GFSNWE is analyzed in Section 5. Finally, the conclusions of this paper are presented.

2. CD and SWP

Different forms of fractional derivatives have been presented by several mathematicians. The best-known are the ones proposed by Riesz, Marchaud, Kober, Riemann–Liouville, Erdelyi, Hadamard, Grunwald–Letnikov, and Caputo [2,22–24]. The majority of the various fractional derivatives do not adhere to the traditional derivative formulae, such as the product rule, quotient rule, and chain rule. Recently, Khalil et al. [21] developed a novel fractional derivative identified as the conformable derivative, which is dependent on a limit form similar to the standard derivative. In the following, we define the conformable fractional derivative and discuss some of its key characteristics.

Definition 1 ([21])**.** *For $\alpha \in (0,1]$, the CD of $\mathcal{U} : \mathbb{R}^+ \to \mathbb{R}$ is defined as*

$$\mathcal{D}_y^\alpha\mathcal{U}(y) = \lim_{h\to 0}\frac{\mathcal{U}(y + hy^{1-\alpha}) - \mathcal{U}(y)}{h}.$$

Let $\mathcal{U}, \Theta : \mathbb{R}^+ \to \mathbb{R}$ be differentiable, and α also a differentiable function; then, the following characteristics of the CD are satisfied for any real constants c_1, c_2:

1. $\mathcal{D}_y^\alpha[c_1\mathcal{U}(y) + c_2\Theta(y)] = c_1\mathcal{D}_y^\alpha\mathcal{U}(y) + c_2\mathcal{D}_y^\alpha\Theta(y)$,
2. $\mathcal{D}_y^\alpha[c_1] = 0$,
3. $\mathcal{D}_y^\alpha(\mathcal{U} \circ \Theta)(y) = y^{1-\alpha}\Theta'(y)\mathcal{U}(\Theta(y))$,
4. $\mathcal{D}_y^\alpha[y^n] = ny^{n-\alpha}$,
5. $\mathcal{D}_y^\alpha\mathcal{U}(y) = y^{1-\alpha}\frac{d\mathcal{U}}{dy}$,

Moreover, the SWP $\mathcal{W}$ is defined as follows [25]:

Definition 2. *The SWP $\{\mathcal{W}(\tau)\}_{\tau\geq 0}$ is a stochastic process and fulfills:*

1. *$\mathcal{W}(t)$ is continuous for $t \geq 0$,*
2. *$\mathcal{W}(t_2) - \mathcal{W}(t_1)$ has a normal distribution $N(0, t_2 - t_1)$.*
3. *$\mathcal{W}(0) = 0$,*
4. *$\mathcal{W}(t_2) - \mathcal{W}(t_1)$ is independent for $t_1 < t_2$,*

The next lemma is required:

Lemma 1 ([25])**.** $\mathbb{E}(e^{\rho\mathcal{W}(t)}) = e^{\frac{1}{2}\rho^2 t}$ *for* $\rho \geq 0$.

3. Wave Equation for GFSNWE

Applying the following wave transformation:

$$\mathcal{U}(x, y, z, t) = \mathcal{G}(\mu)e^{(\rho\mathcal{W}(t) - \frac{1}{2}\rho^2 t)}, \tag{3}$$

where $\mathcal{G}$ is the function deterministic, and

$$\mu = \frac{\mu_1}{\alpha}x^\alpha + \frac{\mu_2}{\alpha}y^\alpha + \frac{\mu_3}{\alpha}z^\alpha + \mu_4 t, \tag{4}$$

with $\mu_1,\ \mu_2,\ \mu_3$ and μ_4 are unknown constants. We note that

$$\begin{aligned} \mathcal{U}_t &= (\mu_4\mathcal{G}' + \rho\mathcal{G}\mathcal{W}_t + \frac{1}{2}\rho^2\mathcal{G} - \frac{1}{2}\rho^2\mathcal{G})e^{(\rho\mathcal{W}(t) - \frac{1}{2}\rho^2 t)} \\ &= (\mu_4\mathcal{G}' + \rho\mathcal{G}\mathcal{W}_t)e^{(\rho\mathcal{W}(t) - \frac{1}{2}\rho^2 t)}, \end{aligned}$$

$$\begin{aligned} \mathcal{D}_x^\alpha\mathcal{U} &= \mu_1\mathcal{G}'e^{(\rho\mathcal{W}(t) - \frac{1}{2}\rho^2 t)},\ \mathcal{D}_x^\alpha\mathcal{U}_t = (\mu_1\mu_4\mathcal{G}'' + \rho\mu_1\mathcal{G}'\mathcal{W}_t)e^{(\rho\mathcal{W}(t) - \frac{1}{2}\rho^2 t)}, \\ \mathcal{D}_{xx}^\alpha\mathcal{U} &= \mu_1^2\mathcal{G}^{(2)}e^{(\rho\mathcal{W}(t) - \frac{1}{2}\rho^2 t)},\ \mathcal{D}_{xxxx}^\alpha\mathcal{U} = \mu_1^4\mathcal{G}^{(4)}e^{(\rho\mathcal{W}(t) - \frac{1}{2}\rho^2 t)}, \\ \mathcal{D}_{yy}^\alpha\mathcal{U} &= \mu_2^2\mathcal{G}''e^{(\rho\mathcal{W}(t) - \frac{1}{2}\rho^2 t)},\ \mathcal{D}_{zz}^\alpha\mathcal{U} = \mu_3^2\mathcal{G}^{(2)}e^{(\rho\mathcal{W}(t) - \frac{1}{2}\rho^2 t)}. \end{aligned} \tag{5}$$

Inserting Equation (5) into Equation (2) yields

$$\gamma_2\mu_1^4\mathcal{G}^{(4)} + (\mu_1\mu_4 + \gamma_1\mu_1^2 + \gamma_4\mu_3^2 + \gamma_5\mu_2^2)\mathcal{G}^{(2)} + \gamma_3\mu_1^2(\mathcal{G}\mathcal{G}')'e^{(\rho\mathcal{W}(t) - \frac{1}{2}\rho^2 t)} = 0. \tag{6}$$

Based on expectations from both sides, we achieve

$$\gamma_2\mu_1^4\mathcal{G}^{(4)} + (\mu_1\mu_4 + \gamma_3\mu_1^2 + \gamma_4\mu_3^2 + \gamma_5\mu_2^2)\mathcal{G}^{(2)} + \gamma_3\mu_1^2(\mathcal{G}\mathcal{G}')'e^{(-\frac{1}{2}\rho^2 t)}\mathbb{E}e^{\rho\mathcal{W}(t)} = 0. \tag{7}$$

Using Lemma 1, Equation (7) turns into

$$\gamma_2\mu_1^4\mathcal{G}^{(4)} + (\mu_1\mu_4 + \gamma_3\mu_1^2 + \gamma_4\mu_2^2 + \gamma_5\mu_3^2)\mathcal{G}^{(2)} + \gamma_1\mu_1^2(\mathcal{G}\mathcal{G}')' = 0. \tag{8}$$

If we integrate twice without considering the integral constant, we obtain

$$\mathcal{G}^{(2)} + \hbar_1 \mathcal{G} + \hbar_2 \mathcal{G}^2 = 0, \tag{9}$$

where

$$\hbar_1 = \frac{\mu_1\mu_4 + \gamma_3\mu_1^2 + \gamma_4\mu_3^2 + \gamma_5\mu_2^2}{\gamma_2\mu_1^4} \text{ and } \hbar_2 = \frac{\gamma_1}{2\gamma_2\mu_1^2}.$$

4. Exact Solutions of GFSNWE

Using the F-expansion and Jacobi elliptical function (JEF) methods, the solutions to the wave Equation (9) are discovered. After that, the solutions to GFSNWE (2) can be acquired.

4.1. $\mathcal{F}$-Expansion Method

Let the solution $\mathcal{G}$ of Equation (9) be

$$\mathcal{G}(\mu) = a_0 + \sum_{k=1}^{M} (a_k \mathcal{F}^k + \frac{b_k}{\mathcal{F}^k}), \tag{10}$$

where $\mathcal{F}$ solves the Riccati equation:

$$\mathcal{F}' = \mathcal{F}^2 + \Omega. \tag{11}$$

Calculating M requires balancing $\mathcal{G}''$ with $\mathcal{G}^2$ in Equation (9), as follows

$$M + 2 = 2M \Rightarrow M = 2.$$

Equation (10) becomes

$$\mathcal{G}(\mu) = a_0 + a_1\mathcal{F} + a_2\mathcal{F}^2 + \frac{b_1}{\mathcal{F}} + \frac{b_2}{\mathcal{F}^2}. \tag{12}$$

Equation (11) has the following solutions:

$$\mathcal{F}(\mu) = \sqrt{\Omega}\tan(\sqrt{\Omega}\mu) \text{ or } \mathcal{F}(\mu) = -\sqrt{\Omega}\cot(\sqrt{\Omega}\mu), \tag{13}$$

If $\Omega > 0$, or

$$\mathcal{F}(\mu) = -\sqrt{-\Omega}\tanh(\sqrt{-\Omega}\mu) \text{ or } \mathcal{F}(\mu) = -\sqrt{-\Omega}\coth(\sqrt{-\Omega}\mu), \tag{14}$$

If $\Omega < 0$, or

$$\mathcal{F}(\mu) = \frac{-1}{\mu}, \tag{15}$$

If $\Omega = 0$.
Now, putting Equation (12) into Equation (9), we obtain

$$\begin{aligned}&(6a_2 + \hbar_2 a_2^2)\mathcal{F}^4 + (2a_1 + 2\hbar_2 a_1 a_2)\mathcal{F}^3 + (8\Omega a_2 + 2a_0 a_2 \hbar_2 + a_1^2\hbar_2 + \hbar_1 a_2)\mathcal{F}^2 \\ &(2\Omega a_1 + \hbar_1 a_1 + 2\hbar_2 a_0 a_1 + 2a_2 b_1)\mathcal{F} + (2\Omega^2 a_2 + 2b_2 + \hbar_1 a_0 + \hbar_2 a_0^2 + 2\hbar_2 a_1 b_1 \\ &+2\hbar_2 a_2 b_2) + (2\Omega b_1 + 2\hbar_2 a_0 b_1 + 2\hbar_2 a_1 b_2 + \hbar_1 b_1)\mathcal{F}^{-1} + (8\Omega b_2 + 2a_0 b_2 \hbar_2 \\ &+b_1^2\hbar_2 + \hbar_1 b_2)\mathcal{F}^{-2} + (2b_1\Omega^2 + 2\hbar_2 b_1 b_2)\mathcal{F}^{-3} + (6\Omega^2 b_2 + \hbar_2 b_2^2)\mathcal{F}^{-4} = 0\end{aligned}$$

Equating the coefficients of each power of $\mathcal{F}$ to zero:

$$6a_2 + \hbar_2 a_2^2 = 0,$$

$$2a_1 + 2\hbar_2 a_1 a_2 = 0,$$

$$8\Omega a_2 + 2a_0a_2\hbar_2 + a_1^2\hbar_2 + \hbar_1 a_2 = 0,$$
$$2\Omega a_1 + \hbar_1 a_1 + 2\hbar_2 a_0 a_1 + 2a_2 b_1 = 0,$$
$$2\Omega^2 a_2 + 2b_2 + \hbar_1 a_0 + \hbar_2 a_0^2 + 2\hbar_2 a_1 b_1 + 2\hbar_2 a_2 b_2 = 0,$$
$$2\Omega b_1 + 2\hbar_2 a_0 b_1 + 2\hbar_2 a_1 b_2 + \hbar_1 b_1 = 0,$$
$$8\Omega b_2 + 2a_0 b_2 \hbar_2 + b_1^2\hbar_2 + \hbar_1 b_2 = 0,$$
$$2b_1\Omega^2 + 2\hbar_2 b_1 b_2 = 0$$

and

$$6\Omega^2 b_2 + \hbar_2 b_2^2 = 0.$$

We obtain the following four families of solutions by solving these equations:
First family:

$$a_0 = \tfrac{-6\Omega}{\hbar_2},\ a_1 = 0,\ a_2 = \tfrac{-6}{\hbar_2},\ b_1 = b_2 = 0,$$
$$\mu_4 = \tfrac{1}{\mu_1}(4\Omega\gamma_2\mu_1^4 - \gamma_3\mu_1^2 - \gamma_4\mu_3^2 - \gamma_5\mu_2^2), \tag{16}$$

Second family:

$$a_0 = \tfrac{-2\Omega}{\hbar_2},\ a_1 = 0,\ a_2 = \tfrac{-6}{\hbar_2},\ b_1 = b_2 = 0,$$
$$\mu_4 = \tfrac{1}{\mu_1}(-4\Omega\gamma_2\mu_1^4 - \gamma_3\mu_1^2 - \gamma_4\mu_3^2 - \gamma_5\mu_2^2), \tag{17}$$

Third family:

$$a_0 = \tfrac{-12\Omega}{\hbar_2},\ a_1 = b_1 = 0,\ a_2 = \tfrac{-6}{\hbar_2},\ b_2 = \tfrac{-6\Omega^2}{\hbar_2},$$
$$\mu_4 = \tfrac{1}{\mu_1}(16\Omega\gamma_2\mu_1^4 - \gamma_3\mu_1^2 - \gamma_4\mu_3^2 - \gamma_5\mu_2^2), \tag{18}$$

Fourth family:

$$a_0 = \tfrac{8\Omega}{\hbar_2},\ a_1 = b_1 = 0,\ a_2 = \tfrac{-6}{\hbar_2},\ b_2 = \tfrac{-6\Omega^2}{\hbar_2},$$
$$\mu_4 = \tfrac{-1}{\mu_1}(14\Omega\gamma_2\mu_1^4 + \gamma_3\mu_1^2 + \gamma_4\mu_3^2 + \gamma_5\mu_2^2). \tag{19}$$

First family: Equation (9) has the following solution:

$$\mathcal{G}(\mu) = \frac{-6\Omega}{\hbar_2} - \frac{6}{\hbar_2}\mathcal{F}^2(\mu).$$

For $\mathcal{F}(\mu)$, there are three cases:
Case 1: If $\Omega > 0$, then, with (13), we have

$$\mathcal{G}(\mu) = \frac{-6\Omega}{\hbar_2} - \frac{6\Omega}{\hbar_2}\tan^2(\sqrt{\Omega}\mu) = -\frac{6\Omega}{\hbar_2}\sec^2(\sqrt{\Omega}\mu),$$

and

$$\mathcal{G}(\mu) = \frac{-6\Omega}{\hbar_2} - \frac{6\Omega}{\hbar_2}\cot^2(\sqrt{\Omega}\mu) = \frac{-6\Omega}{\hbar_2}\csc^2(\sqrt{\Omega}\mu).$$

Therefore, the solution of GFSNWE (2) is

$$\mathcal{U}(x,y,z,t) = -\frac{6\Omega}{\hbar_2}\sec^2(\sqrt{\Omega}\mu)e^{(\rho\mathcal{W}(t) - \frac{1}{2}\rho^2 t)}, \tag{20}$$

and

$$\mathcal{U}(x,y,z,t) = \frac{-6\Omega}{\hbar_2}\csc^2(\sqrt{\Omega}\mu)e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}, \tag{21}$$

where $\mu = \frac{1}{\alpha}(\mu_1 x^\alpha + \mu_2 y^\alpha + \mu_3 z^\alpha) + \frac{1}{\mu_1}(4\Omega\gamma_2\mu_1^4 - \gamma_3\mu_1^2 - \gamma_4\mu_3^2 - \gamma_5\mu_2^2)t$.

Case 2: If $\Omega < 0$, then, using (14), we obtain

$$\mathcal{G}(\mu) = \frac{-6\Omega}{\hbar_2} + \frac{6\Omega}{\hbar_2}\tanh^2(\sqrt{-\Omega}\mu) = \frac{-6\Omega}{\hbar_2}\text{sech}^2(\sqrt{-\Omega}\mu),$$

and

$$\mathcal{G}(\mu) = \frac{-6\Omega}{\hbar_2} + \frac{6\Omega}{\hbar_2}\coth^2(\sqrt{-\Omega}\mu) = \frac{6\Omega}{\hbar_2}\text{csch}^2(\sqrt{-\Omega}\mu).$$

Therefore, the solution of GFSNWE (2) is

$$\mathcal{U}(x,y,z,t) = \frac{-6\Omega}{\hbar_2}\text{sech}^2(\sqrt{-\Omega}\mu)e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}, \tag{22}$$

and

$$\mathcal{U}(x,y,z,t) = \frac{6\Omega}{\hbar_2}\text{csch}^2(\sqrt{-\Omega}\mu)e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}. \tag{23}$$

Case 3: If $\Omega = 0$, then, we obtain, using (15)

$$\mathcal{G}(\mu) = \frac{6}{\hbar_2}\frac{1}{\mu^2}.$$

Therefore, the solution of GFSNWE (2) is

$$\mathcal{U}(x,y,z,t) = [-\frac{6}{\hbar_2}\frac{1}{\mu^2}]e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}, \tag{24}$$

where $\mu = \frac{1}{\alpha}(\mu_1 x^\alpha + \mu_2 y^\alpha + \mu_3 z^\alpha) + \frac{1}{\mu_1}(4\Omega\gamma_2\mu_1^4 - \gamma_3\mu_1^2 - \gamma_4\mu_3^2 - \gamma_5\mu_2^2)t$.

Second family: Equation (9) has the solution

$$\mathcal{G}(\mu) = \frac{-2\Omega}{\hbar_2} - \frac{6}{\hbar_2}\mathcal{F}^2(\mu).$$

For $\mathcal{F}(\mu)$, there are three cases:

Case 1: If $\Omega > 0$, then, we obtain, using (13)

$$\mathcal{G}(\mu) = \frac{-2\Omega}{\hbar_2} - \frac{6\Omega}{\hbar_2}\tan^2(\sqrt{\Omega}\mu),$$

and

$$\mathcal{G}(\mu) = \frac{-2\Omega}{\hbar_2} - \frac{6\Omega}{\hbar_2}\cot^2(\sqrt{\Omega}\mu).$$

Therefore, the solution of GFSNWE (2) is

$$\mathcal{U}(x,y,z,t) = [\frac{-2\Omega}{\hbar_2} - \frac{6\Omega}{\hbar_2}\tan^2(\sqrt{\Omega}\mu)]e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}, \tag{25}$$

and

$$\mathcal{U}(x,y,z,t) = [\frac{-2\Omega}{\hbar_2} - \frac{6\Omega}{\hbar_2}\cot^2(\sqrt{\Omega}\mu)]e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}, \tag{26}$$

where $\mu = \frac{1}{\alpha}(\mu_1 x^\alpha + \mu_2 y^\alpha + \mu_3 z^\alpha) - \frac{1}{\mu_1}(4\Omega\gamma_2\mu_1^4 + \gamma_3\mu_1^2 + \gamma_4\mu_3^2 + \gamma_5\mu_2^2)t$.

Case 2: If $\Omega < 0$, then, we obtain, using (14)

$$\mathcal{G}(\mu) = \frac{-2\Omega}{\hbar_2} + \frac{6\Omega}{\hbar_2}\tanh^2(\sqrt{-\Omega}\mu),$$

and

$$\mathcal{G}(\mu) = \frac{-2\Omega}{\hbar_2} + \frac{6\Omega}{\hbar_2}\coth^2(\sqrt{-\Omega}\mu).$$

Therefore, the solution of GFSNWE (2) is

$$\mathcal{U}(x,y,z,t) = [\frac{-2\Omega}{\hbar_2} + \frac{6\Omega}{\hbar_2}\tanh^2(\sqrt{-\Omega}\mu)]e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}, \tag{27}$$

and

$$\mathcal{U}(x,y,z,t) = [\frac{-2\Omega}{\hbar_2} + \frac{6\Omega}{\hbar_2}\coth^2(\sqrt{-\Omega}\mu)]e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}. \tag{28}$$

Case 3: If $\Omega = 0$, then, we obtain, using (15)

$$\mathcal{G}(\mu) = \frac{6}{\hbar_2}\frac{1}{\mu^2}.$$

Therefore, the solution of GFSNWE (2) is

$$\mathcal{U}(x,y,z,t) = \frac{6}{\hbar_2}\frac{1}{\mu^2}e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}, \tag{29}$$

where $\mu = \frac{1}{\alpha}(\mu_1 x^\alpha + \mu_2 y^\alpha + \mu_3 z^\alpha) - \frac{1}{\mu_1}(4\Omega\gamma_2\mu_1^4 + \gamma_3\mu_1^2 + \gamma_4\mu_3^2 + \gamma_5\mu_2^2)t$.

Third family: Equation (9) has the solution

$$\mathcal{G}(\mu) = \frac{-12\Omega}{\hbar_2} - \frac{6}{\hbar_2}\mathcal{F}^2(\mu) - \frac{6\Omega^2}{\hbar_2}\mathcal{F}^{-2}(\mu).$$

For $\mathcal{F}(\mu)$, there are three cases:

Case 1: If $\Omega > 0$, then, using (13), we obtain

$$\begin{aligned}\mathcal{G}(\mu) &= \frac{-12\Omega}{\hbar_2} - \frac{6\Omega}{\hbar_2}\tan^2(\sqrt{\Omega}\mu) - \frac{6\Omega}{\hbar_2}\cot^2(\sqrt{\Omega}\mu)\\ &= -\frac{6\Omega}{\hbar_2}[\sec^2(\sqrt{\Omega}\mu) + \csc^2(\sqrt{\Omega}\mu)].\end{aligned}$$

Therefore, the solution of GFSNWE (2) is

$$\mathcal{U}(x,y,z,t) = -\frac{6\Omega}{\hbar_2}[\sec^2(\sqrt{\Omega}\mu) + \csc^2(\sqrt{\Omega}\mu)]e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}, \tag{30}$$

where $\mu = \frac{1}{\alpha}(\mu_1 x^\alpha + \mu_2 y^\alpha + \mu_3 z^\alpha) + \frac{1}{\mu_1}(16\Omega\gamma_2\mu_1^4 - \gamma_3\mu_1^2 - \gamma_4\mu_3^2 - \gamma_5\mu_2^2)t$.

Case 2: If $\Omega < 0$, then, using (14), we obtain

$$\begin{aligned}\mathcal{G}(\mu) &= \frac{-12\Omega}{\hbar_2} + \frac{6\Omega}{\hbar_2}\tanh^2(\sqrt{-\Omega}\mu) + \frac{6\Omega}{\hbar_2}\coth^2(\sqrt{-\Omega}\mu)\\ &= \frac{-6\Omega}{\hbar_2}[\operatorname{sech}^2(\sqrt{-\Omega}\mu) - \operatorname{csch}^2(\sqrt{-\Omega}\mu)].\end{aligned}$$

Therefore, the solution of GFSNWE (2) is

$$\mathcal{U}(x,y,z,t) = \frac{-6\Omega}{\hbar_2}[\operatorname{sech}^2(\sqrt{-\Omega}\mu) - \operatorname{csch}^2(\sqrt{-\Omega}\mu)]e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}. \tag{31}$$

Case 3: If $\Omega = 0$, then, using (15), we obtain

$$\mathcal{G}(\mu) = \frac{6}{\hbar_2}\frac{1}{\mu^2} + \frac{6}{\hbar_2}\mu^2.$$

Therefore, the solution of GFSNWE (2) is

$$\mathcal{U}(x,y,z,t) = \frac{6}{\hbar_2}[\frac{1}{\mu^2} + \mu^2]e^{(\rho\mathcal{W}(t) - \frac{1}{2}\rho^2 t)}, \tag{32}$$

where $\mu = \frac{1}{\alpha}(\mu_1 x^\alpha + \mu_2 y^\alpha + \mu_3 z^\alpha) + \frac{1}{\mu_1}(16\Omega\gamma_2\mu_1^4 - \gamma_3\mu_1^2 - \gamma_4\mu_3^2 - \gamma_5\mu_2^2)t$.

Fourth family: Equation (9) has the solution

$$\mathcal{G}(\mu) = \frac{8\Omega}{\hbar_2} - \frac{6}{\hbar_2}\mathcal{F}^2(\mu) - \frac{6\Omega^2}{\hbar_2}\mathcal{F}^{-2}(\mu).$$

For $\mathcal{F}(\mu)$, there are three cases:

Case 1: If $\Omega > 0$, then, using (13), we obtain

$$\mathcal{G}(\mu) = \frac{8\Omega}{\hbar_2} - \frac{6\Omega}{\hbar_2}\tan^2(\sqrt{\Omega}\mu) - \frac{6\Omega}{\hbar_2}\cot^2(\sqrt{\Omega}\mu).$$

Therefore, the solution of GFSNWE (2) is

$$\mathcal{U}(x,y,z,t) = [\frac{8\Omega}{\hbar_2} - \frac{6\Omega}{\hbar_2}\tan^2(\sqrt{\Omega}\mu) - \frac{6\Omega}{\hbar_2}\cot^2(\sqrt{\Omega}\mu)]e^{(\rho\mathcal{W}(t) - \frac{1}{2}\rho^2 t)}, \tag{33}$$

where $\mu = \frac{1}{\alpha}(\mu_1 x^\alpha + \mu_2 y^\alpha + \mu_3 z^\alpha) - \frac{1}{\mu_1}(14\Omega\gamma_2\mu_1^4 + \gamma_3\mu_1^2 + \gamma_4\mu_3^2 + \gamma_5\mu_2^2)t$.

Case 2: If $\Omega < 0$, then, using (14), we obtain

$$\mathcal{G}(\mu) = \frac{8\Omega}{\hbar_2} + \frac{6\Omega}{\hbar_2}\tanh^2(\sqrt{-\Omega}\mu) + \frac{6\Omega}{\hbar_2}\coth^2(\sqrt{-\Omega}\mu).$$

Therefore, the solution of GFSNWE (2) is

$$\mathcal{U}(x,y,z,t) = [\frac{8\Omega}{\hbar_2} + \frac{6\Omega}{\hbar_2}\tanh^2(\sqrt{-\Omega}\mu) + \frac{6\Omega}{\hbar_2}\coth^2(\sqrt{-\Omega}\mu)]e^{(\rho\mathcal{W}(t) - \frac{1}{2}\rho^2 t)}. \tag{34}$$

Case 3: If $\Omega = 0$, then, using (15), we obtain

$$\mathcal{G}(\mu) = \frac{6}{\hbar_2}\frac{1}{\mu^2} + \frac{6}{\hbar_2}\mu^2.$$

Thus, the GFSNWE (2) has the solution

$$\mathcal{U}(x,y,z,t) = \frac{6}{\hbar_2}[\frac{1}{\mu^2} + \mu^2]e^{(\rho\mathcal{W}(t) - \frac{1}{2}\rho^2 t)}, \tag{35}$$

where $\mu = \frac{1}{\alpha}(\mu_1 x^\alpha + \mu_2 y^\alpha + \mu_3 z^\alpha) - \frac{1}{\mu_1}(14\Omega\gamma_2\mu_1^4 + \gamma_3\mu_1^2 + \gamma_4\mu_3^2 + \gamma_5\mu_2^2)t$.

Remark 1. *Setting $\rho = 0$, and $\alpha = 1$ in Equations (20)–(34), then we obtain the same solutions as in [19].*

Remark 2. *Setting $\rho = 0$, and $\alpha = 1$ in Equations (22) and (23), then we obtain the same solutions (51) and (52) as in [20].*

4.2. JEF Method

We employ here the JEF method defined in [26]. Assuming the solutions of Equation (9) (with $M = 2$) as follows:

$$\mathcal{G}(\mu) = \ell_0 + \ell_1\varphi(\mu) + \ell_2\varphi^2(\mu), \tag{36}$$

where $\varphi(\mu) = cn(\mu, m)$, for $0 < m < 1$, is a Jacobi elliptic cosine function and ℓ_0, ℓ_1, and ℓ_2 are unknown constants. Differentiating Equation (36) twice

$$\mathcal{G}''(\mu) = 2\ell_2(1-m^2) + \ell_1(2m^2-1)\varphi + 4\ell_2(2m^2-1)\varphi^2 - 2\ell_1 m^2\varphi^3 - 6\ell_2 m^2\varphi^4. \quad (37)$$

Plugging Equations (36) and (37) into Equation (9), we have

$$(\hbar_2\ell_2^2 - 6\ell_2 m^2)\varphi^4 + (2\hbar_2\ell_1\ell_2 - 2\ell_1 m^2)\varphi^3 + (2\hbar_2\ell_0\ell_2 + \hbar_2\ell_1^2 + \hbar_1\ell_2 + 4\ell_2(2m^2-1))\varphi^2$$
$$+[\ell_1(2m^2-1) + 2\hbar_2\ell_0\ell_1 + \hbar_1\ell_1]\varphi + (2\ell_2(1-m^2) + \ell_0\hbar_1 + \hbar_2\ell_0^2) = 0.$$

Balancing the coefficient of $\varphi^k (k = 4,3,2,1,0)$ to 0, we have

$$\hbar_2\ell_2^2 - 6\ell_2 m^2 = 0,$$
$$2\hbar_2\ell_1\ell_2 - 2\ell_1 m^2 = 0,$$
$$2\hbar_2\ell_0\ell_2 + \hbar_2\ell_1^2 + \hbar_1\ell_2 + 4\ell_2(2m^2-1) = 0,$$
$$\ell_1(2m^2-1) + 2\hbar_2\ell_0\ell_1 + \hbar_1\ell_1 = 0$$

and

$$2\ell_2(1-m^2) + \ell_0\hbar_1 + \hbar_2\ell_0^2 = 0.$$

The following are the two cases we obtain from solving these equations.

First case:

$$\ell_0 = \frac{-2(2m^2-1) - \sqrt{6m^4 - 6m^2 + 4}}{\hbar_2},\ \ell_1 = 0,\ \ell_2 = \frac{6m^2}{\hbar_2},$$

where $\mu = \frac{1}{\alpha}(\mu_1 x^\alpha + \mu_2 y^\alpha + \mu_3 z^\alpha) + \frac{1}{\mu_1}(\gamma_2\mu_1^4\sqrt{6m^4 - 6m^2 + 4} - \gamma_3\mu_1^2 - \gamma_4\mu_3^2 - \gamma_5\mu_2^2)t$.

Second case:

$$\ell_0 = \frac{-2(2m^2-1) + \sqrt{6m^4 - 6m^2 + 4}}{\hbar_2},\ \ell_1 = 0,\ \ell_2 = \frac{6m^2}{\hbar_2},$$

where $\mu = \frac{1}{\alpha}(\mu_1 x^\alpha + \mu_2 y^\alpha + \mu_3 z^\alpha) + \frac{1}{\mu_1}(\gamma_2\mu_1^4\sqrt{6m^4 - 6m^2 + 4} - \gamma_3\mu_1^2 - \gamma_4\mu_3^2 - \gamma_5\mu_2^2)t$.

First case: The solution of Equation (9) is

$$\mathcal{G}(\mu) = \frac{-2(2m^2-1) - \sqrt{6m^4 - 6m^2 + 4}}{\hbar_2} + \frac{6m^2}{\hbar_2}cn^2(\mu, m).$$

Therefore, the solution of GFSNWE (2) is

$$\mathcal{U}(x,y,z,t) = [\frac{-2(2m^2-1) - \sqrt{6m^4 - 6m^2 + 4}}{\hbar_2} + \frac{6m^2}{\hbar_2}cn^2(\mu, m)]e^{[\rho\mathcal{W}(t) - \frac{1}{2}\rho^2 t]}. \quad (38)$$

If $m \to 1$, then Equation (38) is transferred:

$$\mathcal{U}(x,y,z,t) = [\frac{-4}{\hbar_2} + \frac{6}{\hbar_2}\text{sech}^2(\mu)]e^{[\rho\mathcal{W}(t) - \frac{1}{2}\rho^2 t]}. \quad (39)$$

Second case: The solution of Equation (9) is

$$\mathcal{G}(\mu) = \frac{-2(2m^2-1) + \sqrt{6m^4 - 6m^2 + 4}}{\hbar_2} + \frac{6m^2}{\hbar_2}cn^2(\mu, m).$$

Therefore, the solution of GFSNWE (2) is

$$\mathcal{U}(x,y,z,t) = [\frac{-2(2m^2-1) + \sqrt{6m^4 - 6m^2 + 4}}{\hbar_2} + \frac{6m^2}{\hbar_2}cn^2(\mu, m)]e^{[\rho\mathcal{W}(t) - \frac{1}{2}\rho^2 t]}. \quad (40)$$

If $m \to 1$, then Equation (38) is transferred:

$$\mathcal{U}(x,y,z,t) = [\frac{6}{\hbar_2}\text{sech}^2(\mu)]e^{[\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t]}. \quad (41)$$

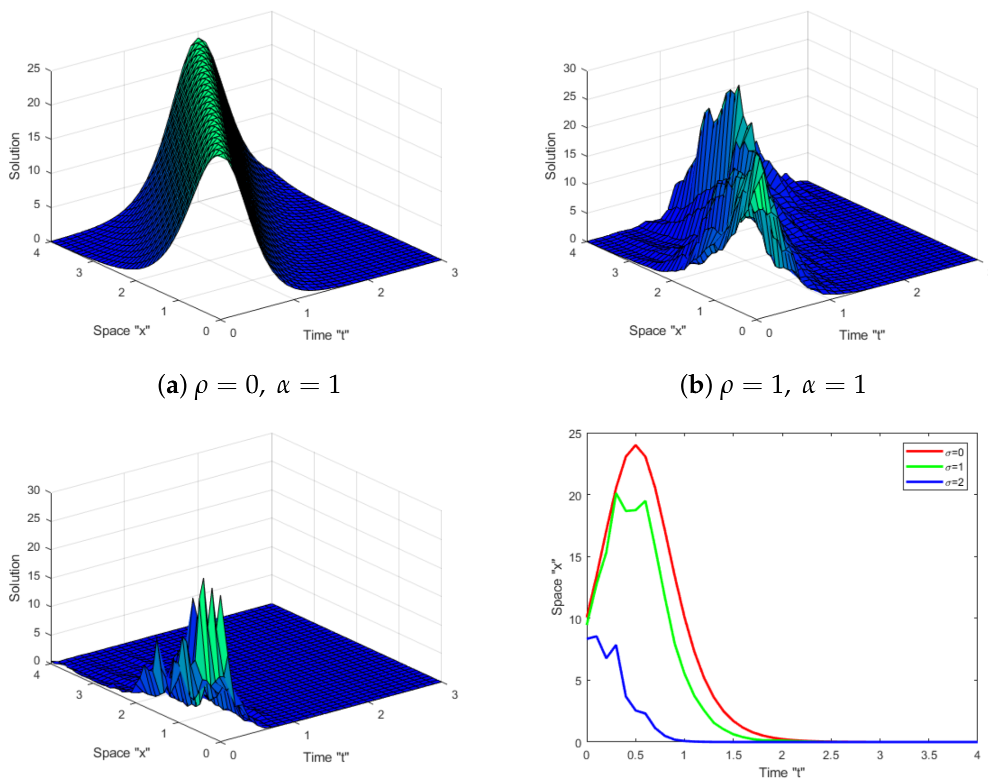
In a similar way, we can replace cn in (36) with the Jacobi elliptic sine function $sn(\mu,m)$ or the Jacobi elliptic delta amplitude $dn(\mu,m)$ in order to obtain other different solutions for GFSNWE (2).

5. Impacts of SWP and the Fractional Derivative

The next step is to look at how SWP and the fractional derivative affect the exact solution of the GFSNWE (2). To explain the status of these solutions, we examine various graphs. Let us set the parameters $\mu_1 = 1$, $\mu_2 = \mu_3 = 1$, $\gamma_1 = \gamma_2 = \gamma_4 = 1$, $\gamma_3 = 2$, $\gamma_5 = 3$, $y = z = 0$, $x \in [0,4]$, and $t \in [0,4]$, for the specific solutions that have been acquired, for example, (22) with $\Omega = -1$ and (38) with $m = 0.5$, so that we may simulate these graphs.

First, the noise impacts: In the following figures, we show the effect of noise:

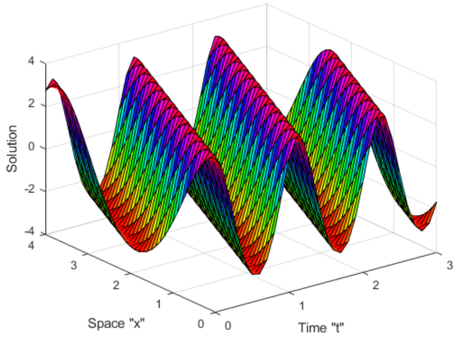
Based on Figures 1 and 2, we may infer that there are different kind of solutions, such as periodic, dark, bright, and others, when the noise is ignored (i.e., at $\rho = 0$). After a few minor transits, the surface becomes much flatter when noise is introduced, and its strength is raised. Thus, it appears that SWP stabilizes GFSNWE solutions.



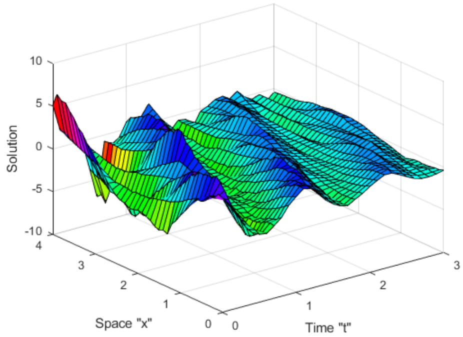
(**a**) $\rho = 0,\ \alpha = 1$ (**b**) $\rho = 1,\ \alpha = 1$

(**c**) $\rho = 2,\ \alpha = 1$ (**d**) $\alpha = 1, \rho = 0,\ 1,\ 2$

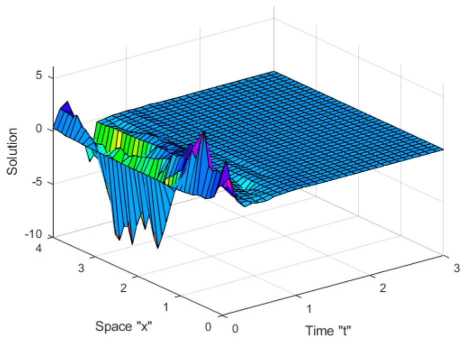
Figure 1. The (**a**–**c**) 3D and (**d**) 2D shapes of the solution given in Equation (22) for various values of $\rho = 0,\ 1,\ 2$.



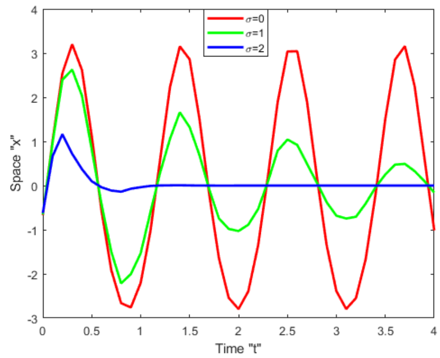
(**a**) $\rho = 0,\ \alpha = 1$



(**b**) $\rho = 1,\ \alpha = 1$



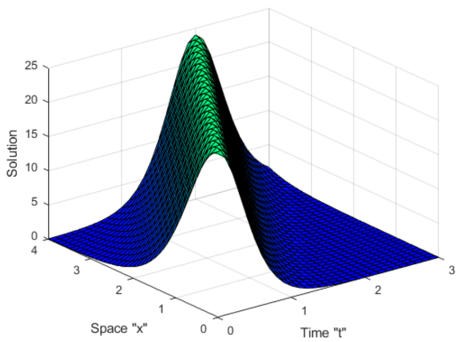
(**c**) $\rho = 2,\ \alpha = 1$



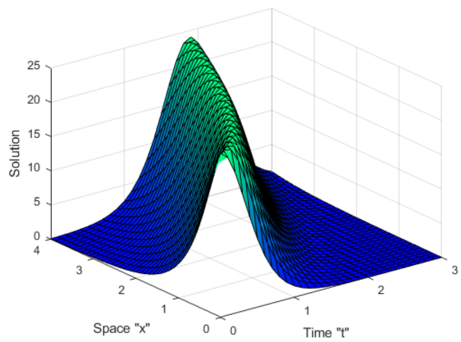
(**d**) $\alpha = 1, \rho = 0,\ 1,\ 2$

Figure 2. The (**a**–**c**) 3D and (**d**) 2D shapes of the solution given in Equation (38) for various values of $\rho = 0,\ 1,\ 2$.

Second, the fractional derivative impacts: In Figures 3 and 4 if $\rho = 0$, we can see that the graph's shape is compressed as the value of α decreases:

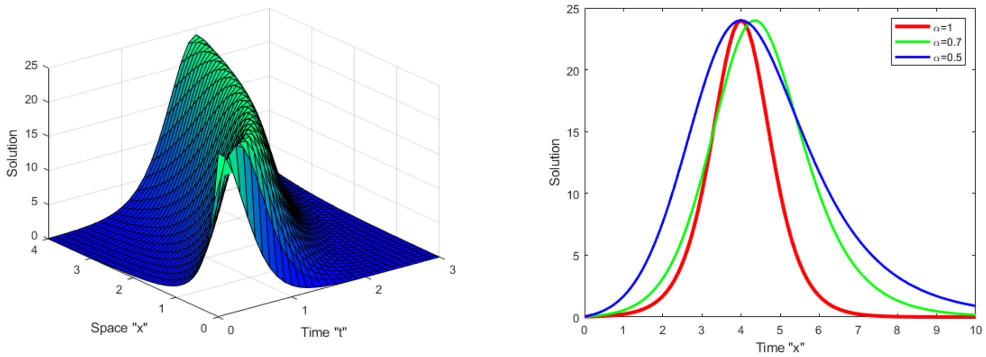


(**a**) $\rho = 0,\ \alpha = 1$



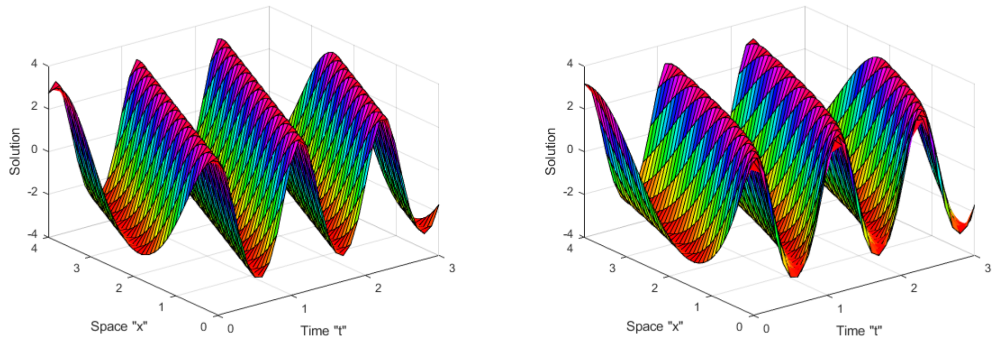
(**b**) $\rho = 0,\ \alpha = 0.7$

Figure 3. *Cont.*

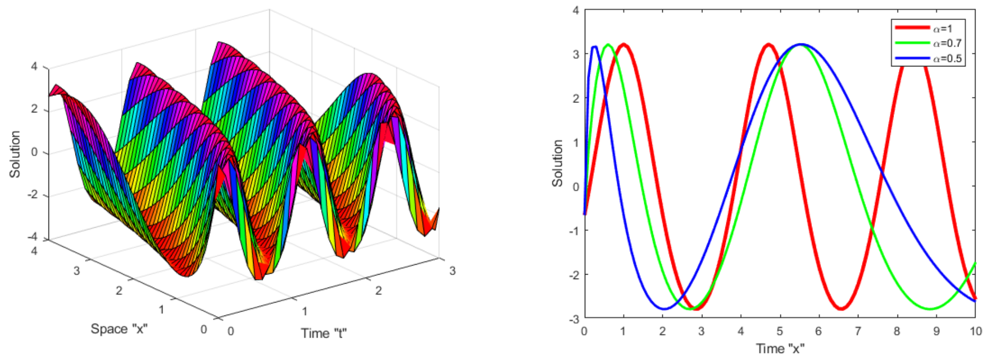


(**c**) $\rho = 0,\ \alpha = 0.5$ (**d**) $\rho = 0,\ \alpha = 1,\ 0.7,\ 0.5$

Figure 3. The (**a**–**c**) 3D and (**d**) 2D profiles of Equation (22) with $\rho = 0$ and different values of $\alpha = 1,\ 0.7,\ 0.5$.



(**a**) $\rho = 0,\ \alpha = 1$ (**b**) $\rho = 0,\ \alpha = 0.7$



(**c**) $\rho = 0,\ \alpha = 0.5$ (**d**) $\rho = 0,\ \alpha = 1,\ 0.7,\ 0.5$

Figure 4. The (**a**–**c**) 3D and (**d**) 2D profiles of Equation (38) with $\rho = 0$ and different values of $\alpha = 1,\ 0.7,\ 0.5$.

From these two Figures 3 and 4, we were able to infer that as the order of the fractional derivative goes down, the surface grows bigger.

6. Conclusions

In this paper, the generalized fractional–stochastic nonlinear wave equation (GFSNWE) was considered in the Itô sense. This equation can characterize several nonlinear physical phenomena in a liquid with gas bubbles. Using the Jacobi elliptic function and the $\mathcal{F}$-expansion methods, exact stochastic–fractional solutions for GFSNWE were discovered. The methods we used are very efficient and strong in their ability to discover several solutions of GFSNWE. In this study, we obtained new solutions using a strategy that has never been explored before. These solutions are necessary to understand a range of fascinating and difficult physical phenomena. Using the MATLAB software, the impact of the Wiener process and conformable derivative on the acquired solutions of GFSNWE (2) is discussed. We deduced that the standard Wiener process stabilizes the solutions around 0. Additionally, we deduced that the derivative order decreased and the surface is extended.

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