A Research Announcement on Generalized Discrete U–Bernoulli–Korobov Polynomials

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Abstract

The aim of this work is to introduce a new family of generalized discrete U–Bernoulli–Korobov–type polynomials. We provide several explicit representations of this class, together with connections to other well–known families of special polynomials. In addition, we establish properties involving the forward and backward difference operators Δ and ∇. Finally, we examine the orthogonality structure satisfied by these polynomials and derive the corresponding three-term recurrence relation.

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1. Introduction

The study of generating functions and their various extensions leads to polynomials and numbers known for their exceptional and valuable properties, which have applications in some branches of mathematics, probability, engineering, and other scientific disciplines. Many mathematical physics issues can be solved analytically thanks to the recent developments in generating functions theory [12; 13; 20**?**]. We can find certain results related to the generating functions for the Bernoulli polynomials and degenerate Bernoulli polynomials. Also, in the literature, we find various kinds of versions of the Euler, Genocchi, and Dahee degenerate polynomials, see for example [2; 3; 8; 10; 11; 17–19]. On the other hand, in recent years the investigations of discrete orthogonal polynomials have gained high attention for their applications on functional equations and differential and their use to establish various analytic number theory properties (cf. [4–6; 9]).

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The motivation of this work is to introduce a new family of generalized discrete U–Bernoulli–Korobov–kind polynomials equipped with a parameter that outlines the advantages of techniques associated with the generating functions we have in mind to give some representative properties, and we show that these polynomials are orthogonal to $\mathbb{N}$ with respect to the inner product that will be studied. Here $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{N}$ will be denoted the sets of the numbers complex, real, positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{P}$ is the space of all polynomials in one variable with real coefficients, and $\log(z)$ denotes the principal value of the multi-valued logarithm function.

The outline of this paper is as follows: In Section 2, we provide well-known basic formulas and definitions that we shall need to use for the rest of the work. In Section 3, a new class of discrete polynomials is introduced using their generating function. We derive certain properties and explicit formulas for these polynomials. Also, we study relations with the Korobov polynomials, the Stirling numbers of the first kind, and the Dahee and Cauchy numbers. Moreover, in section 4, we establish that these new polynomials satisfy an orthogonality relationship. Finally, we study that they satisfy to three-term recurrence relation.

2. Background and preliminary results

The classical Bernoulli polynomials $B_n(x)$, are defined by employing the following generating function (see [16; 17]):

$$\left(\frac{z}{e^z - 1}\right) e^{zx} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}, \quad (|z| < 2\pi). \tag{1}$$

For $x = 0$, we find from (1) the classical Bernoulli numbers B_n given by $B_n := B_n(0) = B_n^{(0)}$, $(n \in \mathbb{N}_0)$.

The Bernoulli polynomials of the second kind $b_n(x)$, are defined as below (see [3, p. 167, Eq(1.2)]):

$$\frac{z}{\log(1+z)}(1+z)^x = \sum_{n=0}^{\infty} b_n(x)\frac{z^n}{n!}, \quad (|z| < 1). \tag{2}$$

For $x = 0$ in (2), $b_n := b_n(0)$, $(n \in \mathbb{N}_0)$ is called Bernoulli numbers of the second kind. (cf. [8; 9]). The Bernoulli polynomials of the second kind are called also, Korobov polynomials of the first kind.

The Daehee polynomials $D_n(x)$ are defined by employing the generating function (see [7; 8; 10]):

$$\frac{\log(1+z)}{z}(1+z)^x = \sum_{n=0}^{\infty} D_n(x)\frac{z^n}{n!}, \quad (|z| < 1). \tag{3}$$

If $x = 0$, in (3) $D_n := D_n(0)$ denotes the so called Daehee numbers.

The falling factorial x of order n; $\langle x\rangle$, is (see [10]):

$$\langle x\rangle_n = x(x-1)\cdots(x-n+1), \quad n\geq 1;\ \langle x\rangle_0 = 1. \tag{4}$$

The Cauchy numbers of the first kind C_n, are given by (see [8]):

$$\frac{z}{\log(1+z)} = \sum_{n=0}^{\infty} C_n \frac{z^n}{n!}, \quad (|z|<1). \tag{5}$$

The Stirling numbers of the first kind $s(n,k)$, appear as the coefficients in the following generating function (see [17]):

$$\frac{(\log(1+z))^k}{k!} = \sum_{n=k}^{\infty} s(n,k)\frac{z^n}{n!}, \quad (|z|<1).\ (k\in\mathbb{N}_0). \tag{6}$$

These numbers can also be given as (see [10; 17]).

$$\langle x\rangle_n = \sum_{k=0}^{n} s(n,k)x^n. \tag{7}$$

Of the classical exponential function, is received

$$e^{-\alpha z}-1 = \sum_{m=0}^{\infty}\frac{(-\alpha)^{m+1}z^{m+1}}{(m+1)!}. \tag{8}$$

Let f be some function of real variable x, the backward and forward difference operators Δ and ∇ respectively are defined as (see [14]):

$$\nabla f(x) := f(x) - f(x-1), \tag{9}$$
$$\Delta f(x) := f(x+1) - f(x). \tag{10}$$

Further, for any real number a we have

$$\Delta_a f(x) := f(x+a) - f(x). \tag{11}$$

If $a=1$ in (11), we obtain (10).

It is also satisfied (see [14]).

$$\nabla f(x) = \Delta f(x) - \Delta\nabla f(x). \tag{12}$$
$$\nabla(f(x)g(x)) = f(x)\nabla g(x) + g(x-1)\nabla f(x). \tag{13}$$

For two arbitrary sequences $\{c_k\}_{k\geq 0}$ and $\{d_k\}_{k\geq 0}$, if $d_{-1}=0$ then applying summation by parts there holds (see [14]):

$$\sum_{k=0}^{\infty}(\Delta c_k)d_k = -\sum_{k=0}^{\infty} c_k\nabla d_k \tag{14}$$

3. New Family of generalized discrete $U-$Bernoulli$-$Korobov$-$kind polynomials

In this section, a new class of discrete polynomials is introduced which we denote by $\mathscr{P}_n(x;\alpha)$ and will we call generalized discrete $U-$Bernoulli$-$Korobov$-$kind polynomials, and study certain properties and explicit formulas that satisfy these new polynomials.

Definition 3.1. *The new family of generalized discrete $U-$Bernoulli$-$Korobov$-$kind polynomials $\mathscr{P}_n(x;\alpha)$ of degree n in the variable x and parameter $\alpha \in \mathbb{R}-\{0\}$ are defined through the generating function*

$$L(x,z;\alpha) = \left(\frac{z}{e^{-z\alpha}-1}\right)(1+z)^x = \sum_{n=0}^{\infty} \mathscr{P}_n(x;\alpha)\frac{z^n}{n!}, \quad \left(|z| < \frac{2\pi}{|\alpha|}\right). \tag{15}$$

By using (15) we can compute the first generalized discrete $U-$Bernoulli$-$Korobov$-$kind polynomials $\mathscr{P}_n(x;\alpha)$, as follows:

$$\begin{aligned}
\mathscr{P}_0(x;\alpha) &= -\frac{1}{\alpha},\\
\mathscr{P}_1(x;\alpha) &= -\frac{x}{\alpha} - \frac{1}{2},\\
\mathscr{P}_2(x;\alpha) &= -\frac{x^2}{\alpha} + \left(\frac{1-\alpha}{\alpha}\right)x - \frac{\alpha}{6},\\
\mathscr{P}_3(x;\alpha) &= \left(\frac{-1}{\alpha}\right)x^3 + \frac{3(2-\alpha)}{2\alpha}x^2 + \frac{(3\alpha-\alpha^2-4)}{2\alpha}x,\\
\mathscr{P}_4(x;\alpha) &= \left(\frac{-1}{\alpha}\right)x^4 + \left(\frac{6-2\alpha}{\alpha}\right)x^3 + \left(\frac{-\alpha^2+6\alpha-11}{\alpha}\right)x^2 + \left(\frac{\alpha^2-4\alpha+6}{\alpha}\right)x + \frac{\alpha^3}{30},\\
\mathscr{P}_5(x;\alpha) &= \left(\frac{-1}{\alpha}\right)x^5 + \left(\frac{20-5\alpha}{2\alpha}\right)x^4 + \left(\frac{45\alpha-105-5\alpha^2}{3\alpha}\right)x^3 + \left(\frac{10\alpha^2-55\alpha+100}{2\alpha}\right)x^2\\
&\quad + \left(\frac{\alpha^4+20\alpha^2+90\alpha-144}{6\alpha}\right)x.
\end{aligned}$$

For $x=0$, in (15) corresponds to the generating function of the generalized $U-$Bernoulli$-$Korobov$-$kind numbers, $\mathscr{P}_n(\alpha) = \mathscr{P}_n := \mathscr{P}(0;\alpha)$ given by

$$\frac{z}{e^{-z\alpha}-1} = \sum_{n=0}^{\infty} \mathscr{P}_n(\alpha)\frac{z^n}{n!}, \quad \left(|z| < \frac{2\pi}{|\alpha|}\right). \tag{16}$$

From (16), we get some of these numbers as below:

$$\mathscr{P}_0(\alpha) = -\frac{1}{\alpha}; \quad \mathscr{P}_1(\alpha) = -\frac{1}{2}; \quad \mathscr{P}_2(\alpha) = -\frac{\alpha}{6}; \quad \mathscr{P}_3(\alpha) = 0; \quad \mathscr{P}_4(\alpha) = \frac{\alpha^3}{30}; \quad \mathscr{P}_5(\alpha) = 0.$$

Proposition 3.1. *Let $\alpha \in \mathbb{R}-\{0\}$, and $\{\mathscr{P}_n(\alpha)\}_{n\geq 0}$ be a sequence of generalized $U-$Bernoulli–Korobov–kind numbers. Then, the following relationship is fulfilled.*

$$\sum_{k=0}^{n} \frac{(-\alpha)^{k+1}}{(k+1)} \binom{n}{k} \frac{\mathscr{P}_{n-k}(\alpha)}{n!} = \begin{cases} 1, & si\ n = 0, \\ 0, & si\ n \neq 0. \end{cases}$$

Proof. By using (16) we have

$$\frac{z}{e^{-\alpha z}-1} = \sum_{n=0}^{\infty} \mathscr{P}_n(\alpha)\frac{z^n}{n!}, \qquad |z| < \frac{2\pi}{|\alpha|}.$$

$$z = (e^{-\alpha z}-1)\sum_{n=0}^{\infty} \mathscr{P}_n(\alpha)\frac{z^n}{n!}. \tag{17}$$

From (8) in (17) it follows that

$$z = \alpha z \left(\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\alpha^n}{(n+1)!} z^n\right)\left(\sum_{n=0}^{\infty} \mathscr{P}_n(\alpha)\frac{z^n}{n!}\right). \tag{18}$$

In (18), we obtain

$$1 = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-\alpha)^{k+1}}{(k+1)} \binom{n}{k} \mathscr{P}_{n-k}(\alpha)\frac{z^n}{n!}. \tag{19}$$

Comparing the coefficients in (19) completes the proof.

Proposition 3.2. *Let $\alpha \in \mathbb{R}-\{0\}$, and $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$ be a sequence of generalized discrete $U-$Bernoulli–Korobov–kind polynomials. Then, the following relations hold:*

$$\mathscr{P}_n(x;\alpha) = \sum_{k=0}^{\infty} \langle x\rangle_k \binom{n}{k} \mathscr{P}_{n-k}(\alpha), \tag{20}$$

$$\mathscr{P}_n(x;\alpha) - \mathscr{P}_n(\alpha) = \sum_{k=0}^{n-1} \frac{n}{k+1}\binom{n-1}{k} \langle x\rangle_{k+1} \mathscr{P}_{n-1-k}(\alpha), \tag{21}$$

with $\langle x\rangle_k$ given in (4).

Proof. From (15) and (16), we can write

$$\sum_{k=0}^{n} \mathscr{P}_n(x;\alpha)\frac{z^n}{n!} = \left(\sum_{n=0}^{\infty} \mathscr{P}_n(\alpha)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\binom{x}{n} z^n\right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{x}{k}\binom{n}{k} k!\mathscr{P}_{n-k}(\alpha)\frac{z^n}{n!}. \tag{22}$$

As a result of (22), we obtain (20).

The assertion (21) follows by utilizing (15), (16), and the Cauchy product rule, which finally yields

$$\sum_{n=0}^{\infty}[\mathscr{P}_n(x;\alpha)-\mathscr{P}_n(\alpha)]\frac{z^n}{n!} = \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}\binom{x}{k+1}\binom{n-1}{k}nk!\mathscr{P}_{n-1-k}(\alpha)\right)\frac{z^n}{n!}, \quad (23)$$

by comparing coefficients in (23), we obtain (21).

Proposition 3.3. *The following summation formulae for the generalized discrete U−Bernoulli–Korobov−kind polynomials $\mathscr{P}_n(x;\alpha)$ and $\mathscr{P}_n(y;\beta)$ with $\alpha,\beta\in\mathbb{R}-\{0\}$ and $n\in\mathbb{N}$ hold true:*

$$\mathscr{P}_n(x+y;\alpha) = \sum_{k=0}^{n}\binom{n}{k}\langle x+y\rangle_k\,\mathscr{P}_{n-k}(\alpha), \quad (24)$$

$$\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_n(x+y;\alpha)\mathscr{P}_{n-k}(\beta) = \sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}(x;\alpha)\mathscr{P}_k(y;\beta), \quad (25)$$

$$\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_n(x+y;\alpha)\mathscr{P}_{n-k}(\alpha) = \sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}(x;\alpha)\mathscr{P}_k(y;\alpha), \quad (26)$$

$$\mathscr{P}_n(x;\alpha) = \sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_k(x;\alpha)\alpha^{n-k}+\sum_{k=0}^{n-1}n\,\langle x\rangle_k\binom{n-1}{k}\alpha^{n-k-1}. \quad (27)$$

Proof. The representation (24), follows from (4) and (15). On the other hand, because of (15) for α, β, and $x,y\in\mathbb{Z}^+$, we have

$$\left(\frac{z}{e^{-\alpha z}-1}\right)(1+z)^x = \sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}, \quad (28)$$

$$\left(\frac{z}{e^{-\beta z}-1}\right)(1+z)^y = \sum_{n=0}^{\infty}\mathscr{P}_n(y;\beta)\frac{z^n}{n!}. \quad (29)$$

Multiplying member by member to (28) and (29), we deduce

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}(\beta)\mathscr{P}_k(x+y;\alpha)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}(x;\alpha)\mathscr{P}_k(y;\beta)\frac{z^n}{n!}. \quad (30)$$

Therefore, of (30) we derive (25). Similarly, we can obtain (26).

To prove of (27), multiplying (15) by $e^{\alpha z}$ leads to

$$z\left(\sum_{n=0}^{\infty}\frac{\alpha^n z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\binom{x}{n}z^n\right) = \sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}-\left(\sum_{n=0}^{\infty}\frac{\alpha^n z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}\right).$$

Hence,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x!}{(x-k)!k!}\frac{\alpha^{n-k}}{(n-k)!}\frac{n!z^{n+1}}{n!}=\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}-\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\mathscr{P}_k(x;\alpha)}{k!}\frac{\alpha^{n-k}}{(n-k)!}\frac{n!z^n}{n!}. \tag{31}$$

From (4) and (31), we derive

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\langle x\rangle_k\binom{n-1}{k}\alpha^{n-k-1}n\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left[\mathscr{P}_n(x;\alpha)-\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_k(x;\alpha)\alpha^{n-k}\right]\frac{z^n}{n!}, \tag{32}$$

whence the formula (27) follows of (32).

Theorem 3.1. *For every $n\in\mathbb{N}$ and $\alpha\in\mathbb{R}-\{0\}$, the generalized discrete U–Bernoulli–Korobov–kind polynomials satisfy.*

$$\begin{aligned}(n-1)\mathscr{P}_n(x;\alpha)-n\,x\,\mathscr{P}_{n-1}(x-1;\alpha)\\ =\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l(\alpha)^{l+1}\mathscr{P}_{n-j-l}(\alpha)\mathscr{P}_j(x;\alpha).\end{aligned} \tag{33}$$

PROOF. By differentiating both sides of (15) with respect to z, we get

$$\frac{(1+z)^x}{(e^{-\alpha z}-1)}+\frac{x\,z\,(z+1)^{x-1}}{e^{-\alpha z}-1}+\frac{\alpha z e^{-\alpha z}(1+z)^x}{(e^{-\alpha z}-1)^2} = \sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)n\frac{z^{n-1}}{n!}.$$

So,

$$\begin{aligned}\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)n\frac{z^n}{n!}-\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}-\sum_{n=0}^{\infty}nx\mathscr{P}_{n-1}(x-1;\alpha)\frac{z^n}{n!}\\ =\alpha\left(\sum_{n=0}^{\infty}(-\alpha)^n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(\alpha)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}\right),\end{aligned}$$

therefore

$$\begin{aligned}\sum_{n=0}^{\infty}[n\mathscr{P}_n(x;\alpha)-\mathscr{P}_n(x;\alpha)-nx\mathscr{P}_{n-1}(x-1;\alpha)]\frac{z^n}{n!}&=\alpha\left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}(-\alpha)^l\mathscr{P}_{n-l}(\alpha)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}\right)\\ &=\alpha\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-\alpha)^l\mathscr{P}_{n-j-l}(\alpha)\mathscr{P}_j(x;\alpha)\right)\frac{z^n}{n!}.\end{aligned}$$

As a result of the above expression (33), follows.

Theorem 3.2. *The following relations hold for the generalized discrete U–Bernoulli–Korobov–kind polynomials defined in* (15).

$$\frac{\partial \mathscr{P}_n(x;\alpha)}{\partial x} = \sum_{k=0}^{n-1}(-1)^k\, n\binom{n-1}{k}\frac{k!}{k+1}\mathscr{P}_{n-k-1}(x;\alpha), \quad (n\in\mathbb{N}), \tag{34}$$

$$(n-1)\mathscr{P}_n(x;\alpha) - n\,\gamma(x,z)\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x;\alpha) - n\,\psi(z;\alpha)\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x;\alpha) = 0, \tag{35}$$

where $\alpha\in\mathbb{R}-\{0\}$, $z\in\mathbb{C}-\{0,-1\}$ *and* $n\in\mathbb{N}$, *with*

$$\gamma(x,z) = \frac{x}{(1+z)\log(1+z)}, \ \ \textit{and}\ \psi(z;\alpha) = \frac{\alpha\, e^{-\alpha z}}{(e^{-\alpha z}-1)\log(1+z)}.$$

Proof. By differentiating (15) with respect to x, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\partial \mathscr{P}_n(x;\alpha)}{\partial x}\frac{z^n}{n!} &= \left(\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}z^{n+1}\right)\\ &= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\mathscr{P}_{n-1-k}(x;\alpha)(-1)^k\binom{n-1}{k}\frac{k!}{(k+1)(n-1)!}z^n.\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty}\frac{\partial \mathscr{P}_n(x;\alpha)}{\partial x}\frac{z^n}{n!} = \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{n\,k!}{(k+1)}\mathscr{P}_{n-1-k}(x;\alpha)\frac{z^n}{n!}.$$

As a result of these computations, we obtain (34).

To prove (35), we differentiate (15) concerning z as follows:

$$\frac{\partial}{\partial z}L(x,z;\alpha) = \sum_{n=1}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^{n-1}}{(n-1)!}, \tag{36}$$

and

$$\frac{\partial}{\partial z}L(x,z;\alpha) = \frac{(1+z)^x}{(e^{-\alpha z}-1)} + \left[\frac{z\,(1+z)^x}{(e^{-\alpha z}-1)}\right]\left[\frac{x}{(1+z)}\right] + \left[\frac{z\,(1+z)^x}{(e^{-\alpha z}-1)}\right]\left[\frac{\alpha\, e^{-\alpha z}}{(e^{-\alpha z}-1)}\right]. \tag{37}$$

Furthermore, differentiating (15) concerning x, we have

$$\frac{\partial}{\partial x}L(x,z;\alpha) = \sum_{n=0}^{\infty}\frac{\partial}{\partial x}\mathscr{P}_n(x,\alpha)\frac{z^n}{n!}, \tag{38}$$

$$\frac{\partial}{\partial x}L(x,z;\alpha) = \frac{z\,(1+z)^x\log(1+z)}{(e^{-\alpha z}-1)}. \tag{39}$$

Combining (37) with (38) and (39), we can be written

$$\frac{\partial}{\partial z}L(x,z;\alpha)-\left[\frac{x}{(1+z)\log(1+z)}+\frac{\alpha\, e^{-\alpha z}}{(e^{-\alpha z}-1)\log(1+z)}\right]\frac{\partial}{\partial x}L(x,z;\alpha)-\frac{(1+z)^{x}}{(e^{-\alpha z}-1)}=0. \quad (40)$$

Thus, from (40) we have

$$z\frac{\partial}{\partial z}L(x,z;\alpha)-\left[\frac{zx}{(1+z)\log(1+z)}+\frac{z\alpha e^{-\alpha z}}{(e^{-\alpha z}-1)\log(1+z)}\right]\frac{\partial}{\partial x}L(x,z;\alpha)-L(x,z;\alpha)=0. \quad (41)$$

Hence, from (38) and (41), and after simplifying, we can get

$$\begin{aligned} \sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\; n\,\frac{z^n}{n!}-\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\;\frac{z^n}{n!}- \\ \sum_{n=0}^{\infty}\left[\frac{x}{(1+z)\log(1+z)}+\frac{\alpha\, e^{-\alpha z}}{(e^{-\alpha z}-1)\log(1+z)}\right]\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x;\alpha)\frac{nz^n}{n!} &= 0, \end{aligned}$$

and consequently

$$\begin{aligned} n\mathscr{P}_n(x;\alpha)-\mathscr{P}_n(x;\alpha)-\left[\frac{n\,x}{(1+z)\log(1+z)}\right]\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x;\alpha) \\ -\left[\frac{n\,\alpha\, e^{-\alpha z}}{(e^{-\alpha z}-1)\log(1+z)}\right]\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x;\alpha)=0. \end{aligned} \quad (42)$$

In (42) doing $\gamma(x,z)=\dfrac{x}{(1+z)\log(1+z)}$, $\psi(z;\alpha)=\dfrac{\alpha\, e^{-\alpha z}}{(e^{-\alpha z}-1)(\log(1+z))}$ follows (35). Theorem 3.2 is proved.

Theorem 3.3. *Given $\alpha\in\mathbb{R}-\{0\}$, and let $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$ be a sequence of generalized discrete U–Bernoulli–Korobov–kind polynomials. Then, the following assertions hold:*

$$\mathscr{P}_n(x;\alpha)=\sum_{k=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}x^k\mathscr{P}_{n-j}(\alpha)s(j,k), \quad \text{with } s(n,k) \text{ given in (6).} \quad (43)$$

$$\mathscr{P}_n(x;\alpha)=\sum_{q=0}^{n}\sum_{l=0}^{q}\sum_{j=0}^{n-q}\sum_{k=0}^{\infty}\binom{n}{q}\binom{q}{l}\binom{n-q}{l}x^k\mathscr{P}_{q-l}(\alpha)s(l,k)b_{n-q-j}D_j, \quad (44)$$

where b_n and D_n are given in (2) and (3), respectively.

$$\mathscr{P}_n(x;\alpha)=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\binom{n}{k}b_{n-k}(x)\mathscr{P}_{k-j}(\alpha)D_j, \quad (45)$$

where $b_n(x)$ is defined in (2).

$$\mathscr{P}_n(x;\alpha) = \sum_{l=0}^{n}\sum_{j=0}^{l}\sum_{q=0}^{n-l}\sum_{k=0}^{\infty}\binom{n}{l}\binom{l}{j}\binom{n-l}{q}C_jD_{l-j}s(q,k)\mathscr{P}_{n-l-q}(\alpha)x^k, \tag{46}$$

where C_n is defined in (5).

Proof. The statement (43) follows from (6) and (15).
By using (2), (3), (6), and (15), we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!} &= \left(\sum_{n=0}^{\infty}\mathscr{P}_n(\alpha)\frac{z^n}{n!}\right)\left(e^{x\log(1+z)}\right)\\
&= \left(z\sum_{n=0}^{\infty}\mathscr{P}_n(\alpha)\frac{z^n}{n!}\right)\left(\sum_{k=0}^{\infty}\frac{x^k}{z}\frac{[\log(1+z)]^k}{k!}\right)\\
&= \left(\sum_{n=0}^{\infty}\mathscr{P}_n(\alpha)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}s(n,k)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}b_n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}D_n\frac{z^n}{n!}\right)\sum_{k=0}^{\infty}x^k\\
&= \sum_{n=0}^{\infty}\left[\sum_{q=0}^{n}\binom{n}{q}\binom{q}{l}\binom{n-q}{j}\sum_{l=0}^{q}\sum_{j=0}^{n-q}\sum_{k=0}^{\infty}x^k\mathscr{P}_{q-l}(\alpha)s(l,k)b_{n-q-j}D_j\right]\frac{z^n}{n!},
\end{aligned}$$

from which (44) follow. Taking (2), (3) into account, and, by (15) we can find (45).

To prove (46), we use (5) as well as (6), and (15). Thus, we deduce

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!} &= \left(\sum_{n=0}^{\infty}C_n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}D_n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(\alpha)\frac{z^n}{n!}\right)\left(\sum_{k=0}^{\infty}x^k\frac{[\log(1+z)]^k}{k!}\right)\\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{j=0}^{l}\sum_{q=0}^{n-l}\sum_{k=0}^{\infty}\binom{n}{q}\binom{l}{j}\binom{n-q}{q}C_jD_{l-j}s(q,k)\mathscr{P}_{n-l-q}(\alpha)x^k\frac{z^n}{n!},
\end{aligned}$$

from which assertion (46) follows. This completes the proof.

Theorem 3.4. *Let $\alpha\in\mathbb{R}-\{0\}$ and $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$ be a sequence of generalized discrete $U-$Bernoulli$-$Korobov$-$kind polynomials. Then, the following relations hold:*

$$\begin{aligned}
\Delta_a\mathscr{P}_n(x;\alpha) &= \sum_{k=0}^{n}\binom{n}{k}\langle a\rangle_k\,\mathscr{P}_{n-k}(x;\alpha)-\mathscr{P}_n(x;\alpha), &(47)\\
\Delta\mathscr{P}_n(x;\alpha) &= n\mathscr{P}_{n-1}(x;\alpha), &(48)\\
\nabla\mathscr{P}_n(x;\alpha) &= n\mathscr{P}_{n-1}(x-1;\alpha), &(49)\\
\Delta\mathscr{P}_n(x;\alpha)+n\Delta\mathscr{P}_{n-1}(x;\alpha) &= n\mathscr{P}_{n-1}(x+1;\alpha). &(50)
\end{aligned}$$

with ∇ and Δ_a the operators given in (9) *and* (11)*, respectively.*

Proof. We see that from (11) and (15) it follows

$$\sum_{n=0}^{\infty} \Delta_a \mathscr{P}(x;\alpha)\frac{z^n}{n!} = \frac{z}{e^{-\alpha z}-1}(1+z)^x(1+z)^a - \frac{z}{e^{-\alpha z}-1}(1+z)^x$$
$$= \left(\sum_{n=0}^{\infty} \mathscr{P}_n(x;\alpha)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\binom{a}{n}z^n\right) - \sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}. \tag{51}$$

Hence, in (51) we have

$$\sum_{n=0}^{\infty} \Delta_a \mathscr{P}(x;\alpha)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{a}{k}\frac{n!}{(n-k)!}\mathscr{P}_{n-k}(x;\alpha) - \mathscr{P}_n(x;\alpha)\right)\frac{z^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\langle a\rangle_k\,\mathscr{P}_{n-k}(x;\alpha) - \mathscr{P}_n(x;\alpha)\right)\frac{z^n}{n!},$$

from which, (47) follows. For the case $a = 1$, we obtain (48).

To prove (49), we see that of (9) and (15), we get

$$\sum_{n=0}^{\infty} \nabla \mathscr{P}_n(x;\alpha)\frac{z^n}{n!} = \left(\frac{z^2}{e^{-\alpha z}-1}\right)(1+z)^x\left(\frac{1}{1+z}\right),$$

and consequently

$$\sum_{n=1}^{\infty} \nabla \mathscr{P}_n(x;\alpha)\frac{z^n}{n!} = \sum_{n=0}^{\infty} \mathscr{P}_n(x-1;\alpha)\frac{z^{n+1}}{n!}$$
$$= \sum_{n=1}^{\infty} \mathscr{P}_{n-1}(x-1;\alpha)\,n\,\frac{z^n}{n!},$$

from which, (49) follows.

Taking (15) into account, as well as using the operator Δ, we get the following expression

$$(1+z)\sum_{n=0}^{\infty} \Delta\mathscr{P}_n(x;\alpha)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\mathscr{P}_n(x+1;\alpha)\frac{z^{n+1}}{n!}$$

thus, we have

$$\sum_{n=1}^{\infty}[\Delta\mathscr{P}_n(x;\alpha) + n\Delta\mathscr{P}_{n-1}(x;\alpha) - n\mathscr{P}_{n-1}(x+1;\alpha)]\frac{z^n}{n!} = 0,$$

and, as a consequence, (50) follows. Hence, Theorem 3.4 is proved.

On the other hand, by using (49) and (50), we can see that the polynomials $\mathscr{P}_n(x;\alpha)$ satisfy (12) in such a way that

$$\nabla\mathscr{P}_n(x;\alpha) = \Delta\mathscr{P}_n(x;\alpha) - \Delta\nabla\mathscr{P}_n(x;\alpha). \tag{52}$$

Proposition 3.4. *Let $\alpha\in\mathbb{R}-\{0\}$, $n\in\mathbb{R}$ and $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$ be a sequence of generalized discrete−kind polynomials. Then, the following relations hold:*

$$\Delta(2\mathscr{P}_n(x;\alpha)+n\mathscr{P}_{n-1}(x;\alpha)) - 2\Delta\nabla\mathscr{P}_n(x;\alpha) = 2n\mathscr{P}_{n-1}(x;\alpha)+n(n-1)\mathscr{P}_{n-2}(x;\alpha), \tag{53}$$

$$\Delta\mathscr{P}_n(x;\alpha) - \Delta\nabla\mathscr{P}_n(x;\alpha) = n\mathscr{P}_{n-1}(x;\alpha) - n(n-1)\mathscr{P}_{n-2}(x-1;\alpha). \tag{54}$$

Proof. By using (15) and applying the operator Δ it follows

$$\left(\frac{z}{e^{-\alpha z}-1}\right)(1+z)^{x+1}(1+z) = (1+2z+z^2)\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}$$

$$\sum_{n}^{\infty}[\mathscr{P}_n(x+1;\alpha)-\mathscr{P}_n(x;\alpha)]\frac{z^n}{n!} = 2\sum_{n=1}^{\infty}n\mathscr{P}_{n-1}(x;\alpha)\frac{z^n}{n!} + \sum_{n=2}^{\infty}n(n-1)\mathscr{P}_{n-2}(x;\alpha)\frac{z^n}{n!}$$
$$-\sum_{n=0}^{\infty}\mathscr{P}_n(x+1;\alpha)\frac{z^{n+1}}{n!}$$

$$\sum_{n=1}^{\infty}\Delta\mathscr{P}_n(x;\alpha)\frac{z^n}{n!} = \sum_{n=1}^{\infty}[2n\mathscr{P}_{n-1}(x;\alpha)+n(n-1)\mathscr{P}_{n-2}(x;\alpha)$$
$$-n\mathscr{P}_{n-1}(x+1;\alpha)]\frac{z^n}{n!},$$

whence

$$n\mathscr{P}_{n-1}(x+1;\alpha) = 2n\mathscr{P}_{n-1}(x;\alpha)+n(n-1)\mathscr{P}_{n-2}(x;\alpha) - \Delta\mathscr{P}_n(x;\alpha). \tag{55}$$

Now, taking (50), (55) into account, and by (52), we get

$$2\Delta\mathscr{P}_n(x;\alpha)+n\Delta\mathscr{P}_{n-1}(x;\alpha) = 2n\mathscr{P}_{n-1}(x;\alpha)+n(n-1)\mathscr{P}_{n-2}(x;\alpha),$$

thus

$$2\Delta\mathscr{P}_n(x;\alpha) - 2\Delta\nabla\mathscr{P}_n(x;\alpha)+n\Delta\mathscr{P}_{n-1}(x;\alpha) = 2n\mathscr{P}_{n-1}(x;\alpha)+n(n-1)\mathscr{P}_{n-2}(x;\alpha),$$

and, as a consequence, we obtain (53).

To prove (54), we use (15) and the operator ∇, then

$$(1+z)^2\sum_{n=0}^{\infty}\mathscr{P}_n(x-1;\alpha)\frac{z^n}{n!} = (1+z)\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}$$

$$2n\mathscr{P}_{n-1}(x-1;\alpha) - n\mathscr{P}_{n-1}(x;\alpha)+n(n-1)\mathscr{P}_{n-2}(x-1;\alpha) = \nabla\mathscr{P}_n(x;\alpha),$$

this implies,

$$2n\mathscr{P}_{n-1}(x-1;\alpha) = n\mathscr{P}_{n-1}(x;\alpha) - n(n-1)\mathscr{P}_{n-2}(x-1;\alpha) + \nabla\mathscr{P}_n(x;\alpha). \tag{56}$$

From (49), (56) and by using (50), we have

$$\nabla\mathscr{P}_n(x;\alpha) + n(n-1)\mathscr{P}_{n-2}(x-1;\alpha) - n\mathscr{P}_{n-1}(x;\alpha) = 0,$$

from which, (54) follows. This completes the proof.

4. Orthogonality of the generalized discrete $U-$Bernoulli$-$Korobov$-$kind polynomials

Let ω^α be the discrete weights function

$$\omega^\alpha(x;\beta) = \frac{(-\alpha)^x e^\alpha (1-e^{\alpha\beta})^2}{x!}, \quad x \in \mathbb{N}, \tag{57}$$

with $\alpha < 0$, $z,\ v \in \mathbb{C}$ and $\lambda_1 \in \mathrm{Re}(z)$, $\sigma_1 \in \mathrm{Re}(v)$, $\beta = \lambda_1 = \sigma_1$.

With this weight, we can consider on $\mathbb{P}$, the inner product $\langle f,\ g\rangle_{\omega^\alpha}$

$$\langle f,\ g\rangle_{\omega^\alpha} = \sum_{x=0}^{\infty} f(x)g(x)\omega^\alpha(x;\beta), \tag{58}$$

which has positive weights for every $\alpha < 0$.

The weight function $\omega^\alpha(x;\beta)$ satisfies the Pearson$-$type difference equation

$$\begin{aligned}\nabla\omega^\alpha(x;\beta) &= \omega^\alpha(x;\beta) - \omega^\alpha(x-1;\beta)\\ &= \left(1 - \frac{x}{(-\alpha)}\right)\left(\frac{e^\alpha(-\alpha)^x(1-e^{\alpha\beta})^2}{x!}\right)\\ &= \left(\frac{\alpha+x}{\alpha}\right)\omega^\alpha(x;\beta).\end{aligned} \tag{59}$$

Theorem 4.1. *If $\alpha \in \mathbb{R}$, with $\alpha < 0$ and $m,\ n \in \mathbb{N}$. Then, the generalized discrete $U-$Bernoulli $-$Korobov$-$kind polynomials satisfy the following orthogonality relation:*

$$\sum_{x=0}^{\infty} \mathscr{P}_m(x;\alpha)\mathscr{P}_n(x;\alpha)\,\omega^\alpha(x;\beta) = (-\alpha)^{n-1}\, n^2\,\Gamma(n)\delta_{mn}, \tag{60}$$

with $\omega^\alpha(x;\beta)$ given in (57) and $|z|,\ |v| < \dfrac{2\pi}{|\alpha|}$

Proof. Using (16), the Cauchy product property and taking into account the binomial theorem, we can see that

$$L(x,z;\alpha)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{x}{k}\frac{\mathscr{P}_{n-k}(\alpha)}{(n-k)!}\,z^{n}. \tag{61}$$

Hence,

$$L(x,z;\alpha)=\sum_{n=0}^{\infty}L_{n}(x;\alpha)z^{n}, \tag{62}$$

we note that

$$L_{n}(x;\alpha)=\sum_{k=0}^{n}\binom{x}{k}\frac{\mathscr{P}_{n-k}(\alpha)}{(n-k)!}. \tag{63}$$

Therefore by using (4), it follows that

$$L_{n}(x;\alpha)=\sum_{k=0}^{n}\frac{\langle x\rangle_{k}}{k!}\frac{\mathscr{P}_{n-k}(\alpha)}{(n-k)!}. \tag{64}$$

Likewise, of (15) we see that

$$L(x,v;\alpha)=\sum_{m=0}^{\infty}L_{m}(x;\alpha)\,v^{m}, \tag{65}$$

hence

$$L_{m}(x;\alpha)=\sum_{k=0}^{m}\binom{x}{k}\frac{\mathscr{P}_{m-k}(\alpha)}{(m-k)!}=\sum_{k=0}^{m}\frac{\langle x\rangle_{k}}{k!}\frac{\mathscr{P}_{m-k}(\alpha)}{(m-k)!}. \tag{66}$$

Now, for any k it follows from (61) and (65) that

$$(-\alpha)^{k}L(k,z;\alpha)L(k,v;\alpha)=\left[\frac{zve^{\alpha z+\alpha v}}{(1-e^{\alpha z})(1-e^{\alpha v})}\right][-\alpha(1+z)(1+v)]^{k}, \tag{67}$$

hence

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-\alpha)^{k}L(k,z;\alpha)L(k,v;\alpha)}{k!}&=\frac{zve^{\alpha z+\alpha v}}{(1-e^{\alpha z})(1-e^{\alpha v})}\sum_{k=0}^{\infty}\frac{[-\alpha(1+z)(1+v)]^{k}}{k!}\\&=\left[\frac{zv\,e^{-\alpha}}{(1-e^{\alpha z})(1-e^{\alpha v})}\right]e^{-\alpha zv}\\&=\sum_{n=0}^{\infty}\left[\frac{n\,e^{-\alpha}(-\alpha)^{n-1}}{(1-e^{\alpha z})(1-e^{\alpha v})}\right]\frac{z^{n}\,v^{n}}{n!}.\end{aligned} \tag{68}$$

On the other hand, because of (62) and (65) we also have

$$\sum_{k=0}^{\infty} \frac{(-\alpha)^k L(k,z;\alpha)L(k,v;\alpha)}{k!} = \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{k!} \sum_{n=0}^{\infty} L_n(k;\alpha)\, z^n \sum_{m=0}^{\infty} L_m(k;\alpha)\, v^m$$
$$= \sum_{m,n=0}^{\infty} \sum_{k=0}^{\infty} L_m(k;\alpha), L_n(k;\alpha) \frac{(-\alpha)^k}{k!} z^n v^m. \qquad (69)$$

So, from (68) and (69) follows

$$\sum_{m,n=0}^{\infty} \sum_{k=0}^{\infty} L_m(k;\alpha)L_n(k;\alpha) \frac{(-\alpha)^k}{k!} z^n v^m = \sum_{n=0}^{\infty} \left[\frac{e^{-\alpha}(-\alpha)^{n-1}}{(1-e^{\alpha z})(1-e^{\alpha v})}\right] \frac{n z^n v^n}{n!}. \qquad (70)$$

Comparing the coefficients of $z^n\, v^m$ in (70), we conclude

$$\sum_{k=0}^{\infty} L_m(k;\alpha)L_n(k;\alpha) \frac{(-\alpha)^k}{k!} = \begin{cases} \dfrac{(-\alpha)^{n-1}\, n\, e^{-\alpha}}{n!} \left[\dfrac{1}{(1-e^{\alpha z})(1-e^{\alpha v})}\right], & \text{si } m = n, \\ 0, & \text{si } m \neq n. \end{cases} \qquad (71)$$

If we now consider that $t_1 = e^{\alpha z}$, $t_2 = e^{\alpha v}$ and $z = \lambda_1 + i\lambda_2$, $v = \sigma_1 + i\sigma_2$ then,

$$t_1 = e^{\alpha z} = e^{\alpha\lambda_1} e^{i\,\alpha\lambda_2} \text{ and } t_2 = e^{\alpha v} = e^{\alpha\sigma_1} e^{i\,\alpha\sigma_2},$$

so $|t_1| = e^{\alpha\lambda_1}$, $|t_2| = e^{\alpha\sigma_1}$ with $|\lambda_1|$, $|\lambda_2|$, $|\sigma_1|$, $|\sigma_2| < \dfrac{2\pi}{|\alpha|}$.
Thus,

$$(1-e^{\alpha z})(1-e^{\alpha v}) = (1-|t_1|e^{i\,\alpha\lambda_2})(1-|t_2|e^{i\,\alpha\sigma_2}). \qquad (72)$$

If now we establish $\lambda_2, \sigma_2 \longrightarrow 0$ and $\beta = \lambda_1 = \sigma_1$, then we can be written (72) as

$$(1-e^{\alpha z})(1-e^{\alpha v}) = (1-e^{\alpha\beta})^2.$$

Therefore in (71), we find

$$\sum_{x=0}^{\infty} \frac{\mathscr{P}_m(x;\alpha)}{m!} \frac{\mathscr{P}_n(x;\alpha)}{n!} \omega^{\alpha}(x;\beta) = \frac{n\,(-\alpha)^{n-1}}{n!} \delta_{mn}, \quad \text{where } \delta_{mn} \text{ is the Kronecker delta.}$$

And, as a consequence (60) follows. Theorem 4.1 is proved.

Due to Theorem 4.1, we obtain a three-term recurrence relation that the sequence $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$ satisfies.

Theorem 4.2. *Let* $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$ *be the sequence generalized discrete* U*−Bernoulli−Korobov −kind polynomials which are orthogonal on* $\mathbb{N}$ *for the inner product* (58)*. Then, they satisfy the following three-term recurrence relation:*

$$x\mathscr{P}_{n-1}(x;\alpha) = \gamma_n \mathscr{P}_n(x;\alpha) + \xi_n \mathscr{P}_{n-1}(x;\alpha) + \lambda_n \mathscr{P}_{n-2}(x;\alpha), \quad n > 2, \tag{73}$$

with

$$\begin{aligned} \gamma_n &= \frac{n\,\alpha}{2}, \\ \xi_n &= \left[(s(n-1,n-2) - s(n,n-1)) - \frac{\alpha(2n+1)}{6} \right], \\ \lambda_n &= \frac{(-\alpha)(n-1)^3}{n(n-3)^2} \frac{\Gamma(n-1)}{\Gamma(n-2)}, \end{aligned} \tag{74}$$

and $s(n,k)$ *given in* (6)*.*

Proof. To prove (73), we first expand the polynomial $x\mathscr{P}_{n-1}(x;\alpha)$, which is of degree n in terms of $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$:

$$x\mathscr{P}_{n-1}(x;\alpha) = \sum_{k=0}^{n} \frac{a(k,n-1)}{g_k(\alpha)} \mathscr{P}_n(x;\alpha), \tag{75}$$

with $\alpha < 0$ is a fixed parameter, $n \in \mathbb{N}$ and $x \in \mathbb{N}_0$. From the orthogonality of $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$, we obtain

$$\begin{aligned} \frac{a(k,n-1)}{g_k(\alpha)} &= \frac{\langle x\mathscr{P}_{n-1}(x;\alpha), \mathscr{P}_k(x;\alpha)\rangle_{\omega^\alpha}}{\langle \mathscr{P}_k(x;\alpha), \mathscr{P}_k(x;\alpha)\rangle_{\omega^\alpha}} \\ &= \frac{\langle \mathscr{P}_{n-1}(x;\alpha), x\mathscr{P}_k(x;\alpha)\rangle_{\omega^\alpha}}{\langle \mathscr{P}_k(x;\alpha), \mathscr{P}_k(x;\alpha)\rangle_{\omega^\alpha}}. \end{aligned}$$

As $x\mathscr{P}_k(x;\alpha)$ is a polynomial of degree $k+1$, by orthogonality $a(k,n-1) = 0$ for $k < n-2$ and therefore (75) can be written in the form

$$\begin{aligned} x\mathscr{P}_{n-1}(x;\alpha) = \frac{a(n,n-1)}{g_n(\alpha)} \mathscr{P}_n(x;\alpha) + \frac{a(n-1,n-1)}{g_{n-1}(\alpha)} \mathscr{P}_{n-1}(x;\alpha) \\ + \frac{a(n-2,n-1)}{g_{n-2}(\alpha)} \mathscr{P}_{n-2}(x;\alpha). \end{aligned} \tag{76}$$

On the other hand, taking (6), (20) into account, and (76), we can obtain:

$$\begin{aligned} \mathscr{P}_n(x;\alpha) = \mathscr{P}_0(\alpha)x^n + \left(\mathscr{P}_0(\alpha)s(n,n-1) + \mathscr{P}_1(\alpha)\binom{n}{n-1} \right) x^{n-1} \\ + \left(\mathscr{P}_0(\alpha)s(n,n-2) + \mathscr{P}_1(\alpha)\binom{n}{n-1} s(n,n-2) + \mathscr{P}_2(\alpha)\binom{n}{n-2} \right) x^{n-2} + \cdots, \end{aligned} \tag{77}$$

$$\begin{aligned}\mathscr{P}_{n-1}(x;\alpha) = \mathscr{P}_1(\alpha)\binom{n}{n-1}x^{n-1} + \Bigg(\mathscr{P}_1(\alpha)s(n-1,n-2)\binom{n}{n-1}\\ +\mathscr{P}_2(\alpha)\binom{n}{n-2}\Bigg)x^{n-2} + \cdots,\end{aligned} \tag{78}$$

$$\begin{aligned}\mathscr{P}_{n-2}(x;\alpha) = \mathscr{P}_1(\alpha)\binom{n-1}{n-2}x^{n-2} + \Bigg(\mathscr{P}_1(\alpha)s(n-2,n-3)\binom{n-1}{n-2}\\ +\mathscr{P}_2(\alpha)\binom{n-1}{n-3}\Bigg)x^{n-3} + \cdots,\end{aligned} \tag{79}$$

$$\begin{aligned}x\mathscr{P}_{n-1}(x;\alpha) = \mathscr{P}_1(\alpha)\binom{n}{n-1}x^{n} + \Bigg(\mathscr{P}_1(\alpha)s(n-1,n-2)\binom{n}{n-1}\\ +\mathscr{P}_2(\alpha)\binom{n}{n-2}\Bigg)x^{n-1} + \cdots,\end{aligned} \tag{80}$$

moreover

$$\begin{aligned}x\mathscr{P}_{n-2}(x;\alpha) = \mathscr{P}_1(\alpha)\binom{n-1}{n-2}x^{n-1} + \Bigg(\mathscr{P}_1(\alpha)s(n-2,n-3)\binom{n-1}{n-2}\\ +\mathscr{P}_2(\alpha)\binom{n-1}{n-3}\Bigg)x^{n-2} + \cdots,\end{aligned} \tag{81}$$

also, we can write $x\mathscr{P}_{n-2}(x;\alpha)$ in terms of $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$, we have

$$\begin{aligned}x\mathscr{P}_{n-2}(x;\alpha) &= \sum_{k=0}^{n-1}\frac{a(k,n-2)}{g_k(\alpha)}\mathscr{P}_k(x;\alpha)\\ &= \frac{a(n-1,n-2)}{g_{n-1}(\alpha)}\mathscr{P}_{n-1}(x;\alpha) + \frac{a(n-2,n-2)}{g_{n-2}(\alpha)}\mathscr{P}_{n-2}(x;\alpha)\\ &\quad + \frac{a(n-3,n-2)}{g_{n-3}(\alpha)}\mathscr{P}_{n-3}(x;\alpha).\end{aligned} \tag{82}$$

Now, from (79) and (82) we deduce:

$$\frac{a(n-1,n-2)}{g_{n-1}(\alpha)} = \left(\frac{n-1}{n}\right). \tag{83}$$

So,

$$x\mathscr{P}_{n-2}(x;\alpha) = \left(\frac{n-1}{n}\right)\mathscr{P}_{n-1}(x;\alpha) + P(x). \tag{84}$$

From (76) and (84) it is seen that

$$\begin{aligned}\frac{a(n-2,n-1)}{g_{n-2}(\alpha)} &= \frac{\langle \mathscr{P}_{n-1}(x;\alpha), x\mathscr{P}_{n-2}(x;\alpha)\rangle_{\omega^\alpha}}{g_{n-2}(\alpha)}\\ &= \frac{(n-1)g_{n-1}(\alpha)}{n\, g_{n-2}(\alpha)}.\end{aligned} \tag{85}$$

Also, using (76), (77) on (78), we obtain

$$\frac{a(n,n-1)}{g_n(\alpha)} = \frac{\mathscr{P}_1(\alpha)}{\mathscr{P}_0(\alpha)}\binom{n}{n-1}. \tag{86}$$

Substitution of (85) and (86) into (76) gives

$$\begin{aligned} x\mathscr{P}_n(x;\alpha) = {} & \frac{\mathscr{P}_1(\alpha)}{\mathscr{P}_0(\alpha)}\binom{n}{n-1}\mathscr{P}_n(x;\alpha) + \frac{a(n-1,n-2)}{g_{n-1}(\alpha)}\mathscr{P}_{n-1}(x;\alpha) \\ & + \frac{(n-1)g_{n-1}(\alpha)}{n\, g_{n-2}(\alpha)}\mathscr{P}_{n-2}(x;\alpha). \end{aligned} \tag{87}$$

Comparing the coefficients of the highest terms on the left-hand and right-hand sides of (87), we have

$$\frac{a(n-1,n-2)}{g_{n-1}(\alpha)} = (s(n-1,n-2) - s(n,n-1)) - \frac{\alpha(2n+1)}{6}. \tag{88}$$

Because of Theorem 4.1, it follows

$$\frac{a(n-2,n-1)}{g_{n-2}(\alpha)} = \frac{(-\alpha)(n-1)^3}{n(n-3)^2}\frac{\Gamma(n-1)}{\Gamma(n-2)} \tag{89}$$

and from (86)

$$\frac{a(n,n-1)}{g_n(\alpha)} = \frac{n\alpha}{2}. \tag{90}$$

Theorem 4.2 is proved.

By using the orthogonality of the polynomials $\mathscr{P}_n(x;\alpha)$, we give the following relation.

Proposition 4.1. *The generalized discrete U–Bernoulli–Korobov–kind polynomials, which are orthogonal with respect to the inner product* (58)*, fulfill the relation*

$$\Delta\mathscr{P}_n(x;\alpha) = J^{\alpha}_{k,n}\mathscr{P}_{n-1}(x;\alpha), \tag{91}$$

where $J^{\alpha}_{k,n}$ *are the Fourier coefficients.*

Proof. If we write the polynomial $\Delta\mathscr{P}_n(x;\alpha)$ in terms of $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$, we have

$$\mathscr{P}_n(x+1;\alpha) - \mathscr{P}_n(x;\alpha) = \sum_{k=0}^{n-1} J^{\alpha}_{k,n}\mathscr{P}_k(x;\alpha),$$

besides, for $0 \leq k \leq n-1$

$$J^{\alpha}_{k,n} = \frac{\langle \Delta\mathscr{P}_n, \mathscr{P}_k\rangle_{\omega^\alpha}}{\langle \mathscr{P}_k, \mathscr{P}_k\rangle_{\omega^\alpha}}.$$

Hence, by (4) and (13), we have

$$\begin{aligned}\langle \mathscr{P}_k, \mathscr{P}_k \rangle_{\omega^\alpha} J^\alpha_{k,n} &= \langle \Delta \mathscr{P}_n, \mathscr{P}_k \rangle_{\omega^\alpha} \\ &= \sum_{g=0}^{\infty} \Delta \mathscr{P}_n(g;\alpha) \mathscr{P}_k(g;\alpha) \omega^\alpha(g;\beta) \\ &= -\sum_{g=0}^{\infty} \mathscr{P}_n(g;\alpha) \nabla (\omega^\alpha(g;\beta) \mathscr{P}_k(g;\alpha)) \\ &= -\sum_{g=0}^{\infty} \mathscr{P}_n(g;\alpha) \omega^\alpha(g;\beta) \nabla \mathscr{P}_k(g;\alpha) - \sum_{g=0}^{\infty} \mathscr{P}_n(g;\alpha) \mathscr{P}_k(g-1;\alpha) \nabla \omega^\alpha(g;\beta),\end{aligned}$$

from which, by orthogonality, the first sum is zero since $\nabla \mathscr{P}_k$ is degree $k < n + 1$. For the second sum let us consider (59)

$$\langle \mathscr{P}_k, \mathscr{P}_k \rangle_{\omega^\alpha} J^\alpha_{k,n} = -\frac{1}{\alpha} \sum_{g=0}^{\infty} \mathscr{P}_n(g;\alpha) \mathscr{P}_k(g-1;\alpha)(\alpha + g) \omega^\alpha(g;\beta),$$

if we use orthogonality again, only $J^\alpha_{n-1,n}$ can be non zero, and as a consequence (91), follows. Proposition 4.1 is proved.

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