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New Oscillation Criteria for Sturm–Liouville Dynamic Equations with Deviating Arguments

Taher S. Hassan [1,2,3,*], Clemente Cesarano [3], Loredana Florentina Iambor [4,*], Amir Abdel Menaem [5], Naveed Iqbal [1] and Akbar Ali [1]

1 Department of Mathematics, College of Science, University of Ha'il, Ha'il 2440, Saudi Arabia
2 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
3 Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy; c.cesarano@uninettunouniversity.net
4 Department of Mathematics and Computer Science, University of Oradea, Univeritatii nr.1, 410087 Oradea, Romania
5 Department of Automated Electrical Systems, Ural Power Engineering Institute, Ural Federal University, 620002 Yekaterinburg, Russia
* Correspondence: tshassan@mans.edu.eg (T.S.H.); iambor.loredana@uoradea.ro (L.F.I.)

Abstract: The aim of this study is to refine the known Riccati transformation technique to provide new oscillation criteria for solutions to second-order dynamic equations over time. It is important to note that the convergence or divergence of some improper integrals on time scales depends not only on the integration function but also on the integration time scale. Therefore, there has been a motivation to find new oscillation criteria that can be applicable regardless of whether $\int_{\zeta_0}^{\infty} \frac{\Delta\xi}{a(\xi)}$ is convergent or divergent, in contrast to what has been followed in most previous works in the literature. We have provided an example to illustrate the significance of the obtained results.

Keywords: oscillation behavior; second-order; linear; dynamic equations; time scales

MSC: 34K11; 34N05; 39A21; 39A99; 34C10

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1. Introduction

Oscillation phenomena are present in several models derived from real-world applications; see the papers [1,2] for mathematical biology models in which oscillation and/or delay actions might be depicted using cross-diffusion terms. This work discusses dynamic equations on time scales because they are relevant to many practical problems, such as non-Newtonian fluid theory and the turbulent flow of a polytrophic gas in a porous media (for further details, see, [3–7]). Therefore, we are interested in the oscillatory behavior of second-order Sturm–Liouville dynamic equations in the form

$$\left(ay^{\Delta}\right)^{\Delta}(\zeta) + p(\zeta)y(\tau(\zeta)) = 0 \tag{1}$$

on an arbitrary unbounded above time scale $\mathbb{T}$, where $\zeta \in [\zeta_0, \infty)_{\mathbb{T}} := [\zeta_0, \infty) \cap \mathbb{T}$, $\zeta_0 \geq 0$, $\zeta_0 \in \mathbb{T}$; a, p are positive rd-continuous functions on $\mathbb{T}$; and $\tau : \mathbb{T} \to \mathbb{T}$ is a nondecreasing rd-continuous function satisfying $\tau(\zeta) \leq \sigma(\zeta)$ on $[\zeta_0, \infty)_{\mathbb{T}}$ and $\lim_{\zeta\to\infty} \tau(\zeta) = \infty$.

We presume the reader is already acquainted with the fundamentals of time scales and time scale notations. By a solution of Equation (1) we mean a nontrivial real-valued function $y \in C^1_{\mathrm{rd}}[\zeta_y, \infty)_{\mathbb{T}}$, $\zeta_y \in [\zeta_0, \infty)_{\mathbb{T}}$ such that $ay^{\Delta} \in C^1_{\mathrm{rd}}[\zeta_y, \infty)_{\mathbb{T}}$ and y satisfies (1) on $[\zeta_y, \infty)_{\mathbb{T}}$, where C_{rd} is the set of right-dense continuous functions. According to Trench [8], we state that (1) is in noncanonical form if

$$\int_{\zeta_0}^{\infty} \frac{\Delta\xi}{a(\xi)} < \infty, \tag{2}$$

and we state that (1) is in canonical form if

$$\int_{\zeta_0}^{\infty} \frac{\Delta\xi}{a(\xi)} = \infty. \tag{3}$$

We refer to a solution y of Equation (1) as nonoscillatory if it is either eventually positive or negative; otherwise, it is considered oscillatory. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. Equation (1) is said to oscillate if all of its solutions oscillate. For nonoscillatory solutions of (1), we define

$$\mathcal{N}_1 := \left\{ y(\zeta) : \; y(\zeta)\, y^{\Delta}(\zeta) < 0 \qquad \text{eventually} \right\}$$

and

$$\mathcal{N}_2 := \left\{ y(\zeta) : \; y(\zeta)\, y^{\Delta}(\zeta) > 0 \qquad \text{eventually} \right\}.$$

Stefan Hilger [9] proposed the theory of dynamic equations on time scales in order to establish a unified framework for analyzing both continuous and discrete systems. A time scale $\mathbb{T}$ is a nonempty, closed subset of the reals, and the cases when this time scale is real or the integers represent the classical theories of differential and of difference equations. Many applications use different time scales. The new theory of the so-called "dynamic equations" includes classical theories for differential and difference equations and instances in between. The q-difference equations, which have quantum theory implications (refer to [10]), can be investigated at different time scales. The time scales include $\mathbb{T}=q^{\mathbb{N}_0} := \{q^{\lambda} : \lambda \in \mathbb{N}_0 \text{ for } q > 1\}$, as well as $\mathbb{T}=h\mathbb{N}$, $\mathbb{T}= \mathbb{N}^2$, and $\mathbb{T} = \mathbb{T}_n$, where $\mathbb{T}_n$ represents the set of harmonic numbers, see [11–13], for an introduction to the study of calculus on time scales. Note that if $\mathbb{T} = \mathbb{R}$, then

$$\sigma(\zeta) = \zeta,\; \mu(\zeta) = 0,\; y^{\Delta}(\zeta) = y'(\zeta),\; \int_{\alpha}^{\beta} y(\xi)\Delta\xi = \int_{\alpha}^{\beta} y(\xi)\mathrm{d}\xi,$$

and (1) becomes the linear Sturm–Liouville delay differential equation

$$\big(a\, y'\big)'(\zeta) + p(\zeta)y(\tau(\zeta)) = 0. \tag{4}$$

The oscillatory characteristics of particular cases of Equation (4) are examined by Fite [14] and showed that if

$$\int_{\zeta_0}^{\infty} p(\xi)\mathrm{d}\xi = \infty, \tag{5}$$

then every solution of the differential equation

$$y''(\zeta) + p(\zeta)y(\zeta) = 0 \tag{6}$$

oscillates. Hille [15] improved condition (5) and proved that if

$$\liminf_{\zeta\to\infty} \zeta \int_{\zeta}^{\infty} p(\xi)\mathrm{d}\xi > \frac{1}{4}, \tag{7}$$

then every solution of Equation (6) oscillates. If $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(\zeta) = \zeta + 1,\; \mu(\zeta) = 1,\; y^{\Delta}(\zeta) = \Delta y(\zeta),\; \int_{\alpha}^{\beta} y(\xi)\Delta\xi = \sum_{\xi=\alpha}^{\beta-1} y(\xi),$$

and (1) obtains the linear Sturm–Liouville difference equation

$$\Delta(a\, \Delta y)(\zeta) + p(\zeta)y(\tau(\zeta)) = 0. \tag{8}$$

Thandapani et al. [16] studied the oscillation behavior of Equation (8) when $a(\zeta) = 1$ and $\tau(\zeta) = \zeta$, and it was proven that every solution of Equation (8) oscillates if

$$\sum_{\zeta=\zeta_0}^{\infty} p(\zeta) = \infty. \tag{9}$$

We will show that our findings not only unify some differential and difference equation oscillation results but can also be extended to determine oscillatory behavior in other cases. If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, thus

$$\sigma(\zeta) = \zeta + h,\ \mu(\zeta) = h,\ y^{\Delta}(\zeta) = \Delta_h y(\zeta) := \frac{y(\zeta+h) - y(\zeta)}{h},\ \int_{\alpha}^{\beta} y(\xi)\Delta\xi = \sum_{k=0}^{\frac{\beta-\alpha-h}{h}} y(\alpha + kh)h,$$

and (1) converts the linear Sturm–Liouville difference equation

$$\Delta_h(a\,\Delta_h y)(\zeta) + p(\zeta)y(\tau(\zeta)) = 0. \tag{10}$$

If $\mathbb{T} = \{\zeta : \zeta = q^k,\ k \in \mathbb{N}_0,\ q > 1\}$, then

$$\sigma(\zeta) = q\zeta,\ \mu(\zeta) = (q-1)\zeta,\ y^{\Delta}(\zeta) = \Delta_q y(\zeta) = \frac{y(q\,\zeta) - y(\zeta)}{(q-1)\,\zeta},\ \int_{\zeta_0}^{\infty} y(\xi)\Delta\xi = \sum_{k=n_0}^{\infty} y(q^k)\mu(q^k),$$

where $\zeta_0 = q^{n_0}$, and (1) becomes the linear Sturm–Liouville q-difference equation

$$\Delta_q\big(a\,\Delta_q y\big)(\zeta) + p(\zeta)y(\tau(\zeta)) = 0. \tag{11}$$

If $\mathbb{T} = \mathbb{N}_0^2 := \{n^2 : n \in \mathbb{N}_0\}$, then

$$\sigma(\zeta) = (\sqrt{\zeta}+1)^2,\ \mu(\zeta) = 1 + 2\sqrt{\zeta},\ \Delta_N y(\zeta) = \frac{y((\sqrt{\zeta}+1)^2) - y(\zeta)}{1 + 2\sqrt{\zeta}},$$

and (1) obtains the linear Sturm–Liouville difference equation

$$\Delta_N(a\,\Delta_N y)(\zeta) + p(\zeta)y(\tau(\zeta)) = 0. \tag{12}$$

If $\mathbb{T} = \{H_n : n \in \mathbb{N}\}$ where H_n is the n-th harmonic number defined by $H_0 = 0$, $H_n = \sum_{k=1}^{n} \frac{1}{k}$, $n \in \mathbb{N}_0$, then

$$\sigma(H_n) = H_{n+1},\ \mu(H_n) = \frac{1}{n+1},\ y^{\Delta}(\zeta) = \Delta_{H_n} y(H_n) = (n+1)\Delta y(H_n)$$

and (1) converts the linear Sturm–Liouville difference equation

$$\Delta_{H_n}(a\,\Delta_{H_n} y)(H_n) + p(H_n)y(\tau(H_n)) = 0. \tag{13}$$

Recall that in the case of a discrete time scale,

$$\int_{\alpha}^{\beta} y(\xi)\Delta\xi = \sum_{\xi \in [\alpha,\beta)_{\mathbb{T}}} y(\xi)\mu(\xi).$$

Regarding dynamic equations, there have been a large number of papers devoted to studying the oscillatory behavior of solutions to second-order dynamic equations on time scales. As an illustration, Agarwal et al. [17] established some sufficient conditions for the oscillation of the delay dynamic equation

$$y^{\Delta\Delta}(\zeta) + p(\zeta)y(\tau(\zeta)) = 0, \tag{14}$$

where $\tau(\zeta) \leq \zeta$ on $[\zeta_0, \infty)_{\mathbb{T}}$. Zhang et al. [18] examined the oscillation of the dynamic equation

$$y^{\Delta\Delta}(\zeta) + p(\zeta)y(\sigma(\zeta)) = 0 \tag{15}$$

and obtained some oscillation criteria and comparison theorems for (15). By utilizing the Riccati transformation method, Sahiner [19] was able to derive sufficient conditions for the oscillation of the delay dynamic equation

$$y^{\Delta\Delta}(\zeta) + p(\zeta)y(\tau(\zeta)) = 0. \tag{16}$$

Erbe et al. [20] extended Sahiner's result to the delay dynamic Equation (1), where $\tau(\zeta) \leq \zeta$ on $[\zeta_0, \infty)_{\mathbb{T}}$ and (3) holds. Erbe et al. [21] established Hille–Kneser type nonoscillation necessary and sufficient criteria for the pair of dynamic equations

$$\left(ay^{\Delta}\right)^{\Delta}(\zeta) + p(\zeta)y(\zeta) = 0 \tag{17}$$

and

$$\left(ay^{\Delta}\right)^{\Delta}(\zeta) + p(\zeta)y(\sigma(\zeta)) = 0,$$

where (3) holds. Erbe et al. [22] studied the canonical form of Equation (1), i.e., (3) holds, and established the following results:

Theorem 1 (see [22] (Theorems 2.1 and 2.2))**.** *Let* (3) *hold. Then Equation* (1) *oscillates if there exists a function* $\rho \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^+)$ *such that*

$$\limsup_{\zeta\to\infty} \left[\int_{\zeta_0}^{\zeta} \left(\rho(\xi)p(\xi)\frac{A(\tau(\xi))}{A^{\sigma}(\xi)} - \frac{\left(\rho^{\Delta}(\xi)\right)^2 a(\xi)}{4\rho(\xi)}\right)\Delta\xi\right] = \infty, \tag{18}$$

where

$$A(\xi) := \int_{\zeta_0}^{\xi} \frac{\Delta s}{a(s)}.$$

For further results, see articles [23–42] and the references indicated therein. It is worth noting here that most of the works are concerned with obtaining sufficient conditions for oscillation when (2) holds, while others do so when (3) holds.

Here, it is important to highlight a property that is not expected in the usual calculus of integrals and sums but has been achieved for some time scales, which is that the convergence of the improper integral $\int_{\zeta_0}^{\infty} \frac{\Delta\xi}{\xi^{\alpha}}$ does not depend only on α but also on the time scale, such as for the unbounded above time scales

$$\mathbb{T} = \left\{\xi_k : \xi_k = 2^{\beta^k},\ \beta > 1,\ k \in \mathbb{N}_0\right\},$$

such that

$$\int_{\zeta_0}^{\infty} \frac{\Delta\xi}{\xi^{\alpha}} = \sum_{k=0}^{\infty} \frac{\mu(\xi_k)}{\xi_k^{\alpha}}$$

is divergent if $\alpha \leq \beta$ and convergent if $\alpha > \beta$; see [13] (Examples 5.63, 5.66, and Theorems 5.65, 5.68) for more details. This means that the results obtained when condition (2) is satisfied cannot be applied to all time scales.

Therefore, it was important to present new oscillation criteria that improve the existing criteria in the literature and can be applied to either the noncanonical or canonical form. Moreover, these results will be applicable in the case of improper integrals whose convergence and divergence depend on the time scale.

2. Main Results

The first two theorems are established for non-existence criteria for nonoscillatory solutions in class $\mathcal{N}_1$, and the other two are for class $\mathcal{N}_2$.

Theorem 2. *If there exists a function $\delta \in C^1_{rd}(\mathbb{T}, \mathbb{R}^+)$ such that*

$$\limsup_{\zeta\to\infty}\left[\delta(\zeta)\int_{\zeta}^{\infty}\frac{\Delta\xi}{a(\xi)}+\int_{\zeta_0}^{\zeta}\left(\frac{\delta(\xi)}{a(\xi)}-\frac{\left(\delta^{\Delta}(\xi)\right)^2}{4\delta(\xi)p(\xi)}\right)\Delta\xi\right]=\infty, \tag{19}$$

then $\mathcal{N}_1 = \emptyset$.

Proof. Assume (1) has a nonoscillatory solution $y(\zeta) \in \mathcal{N}_1$ such that $y(\zeta) > 0$ and $y(\tau(\zeta)) > 0$ for $\zeta \in [\zeta_0, \infty)_{\mathbb{T}}$. Then,

$$y^{\Delta}(\zeta) < 0 \text{ and } \left(ay^{\Delta}\right)^{\Delta}(\zeta) = -p(\zeta)y(\tau(\zeta)) < 0 \quad \text{for } \zeta \in [\zeta_0, \infty)_{\mathbb{T}}. \tag{20}$$

Define

$$w_1(\zeta) := -\delta(\zeta)\frac{y(\zeta)}{a(\zeta)y^{\Delta}(\zeta)}. \tag{21}$$

We see by the product and quotient rules that

$$\begin{aligned}
w_1^{\Delta}(\zeta) &= \delta^{\Delta}(\zeta)\left(\frac{-y(\zeta)}{a(\zeta)y^{\Delta}(\zeta)}\right)^{\sigma}+\delta(\zeta)\left(\frac{-y(\zeta)}{a(\zeta)y^{\Delta}(\zeta)}\right)^{\Delta}\\
&= \delta^{\Delta}(\zeta)\left(\frac{-y(\zeta)}{a(\zeta)y^{\Delta}(\zeta)}\right)^{\sigma}\\
&\quad +\delta(\zeta)\left(-y^{\Delta}(\zeta)\left(\frac{1}{a(\zeta)y^{\Delta}(\zeta)}\right)-y^{\sigma}(\zeta)\left(\frac{1}{a(\zeta)y^{\Delta}(\zeta)}\right)^{\Delta}\right)\\
&= \delta^{\Delta}(\zeta)\left(\frac{-y(\zeta)}{a(\zeta)y^{\Delta}(\zeta)}\right)^{\sigma}\\
&\quad +\delta(\zeta)\left(-\frac{1}{a(\zeta)}+\frac{\left(a(\zeta)y^{\Delta}(\zeta)\right)^{\Delta}}{a(\zeta)y^{\Delta}(\zeta)}\left(\frac{y(\zeta)}{a(\zeta)y^{\Delta}(\zeta)}\right)^{\sigma}\right).
\end{aligned}$$

Thanks to the facts that $\left[a(\zeta)y^{\Delta}(\zeta)\right]^{\Delta} < 0$ and $y^{\Delta}(\zeta) < 0$ on $[\zeta_0, \infty)_{\mathbb{T}}$, (1), and (21), we have

$$\begin{aligned}
w_1^{\Delta}(\zeta) &= -\frac{\delta(\zeta)}{a(\zeta)}+\delta^{\Delta}(\zeta)\left(\frac{w_1(\zeta)}{\delta(\zeta)}\right)^{\sigma}+\delta(\zeta)p(\zeta)\frac{y(\tau(\zeta))}{a(\zeta)y^{\Delta}(\zeta)}\left(\frac{w_1(\zeta)}{\delta(\zeta)}\right)^{\sigma}\\
&\leq -\frac{\delta(\zeta)}{a(\zeta)}+\delta^{\Delta}(\zeta)\left(\frac{w_1(\zeta)}{\delta(\zeta)}\right)^{\sigma}+\delta(\zeta)p(\zeta)\left(\frac{y(\zeta)}{a(\zeta)y^{\Delta}(\zeta)}\right)^{\sigma}\left(\frac{w_1(\zeta)}{\delta(\zeta)}\right)^{\sigma}\\
&= -\frac{\delta(\zeta)}{a(\zeta)}+\delta^{\Delta}(\zeta)\left(\frac{w_1(\zeta)}{\delta(\zeta)}\right)^{\sigma}-\delta(\zeta)p(\zeta)\left[\left(\frac{w_1(\zeta)}{\delta(\zeta)}\right)^{\sigma}\right]^2.
\end{aligned} \tag{22}$$

Using the inequality

$$Au - Bu^2 \leq \frac{A^2}{4B}, \quad B > 0, \tag{23}$$

we obtain

$$\delta^{\Delta}(\zeta)\left(\frac{w_1(\zeta)}{\delta(\zeta)}\right)^{\sigma}-\delta(\zeta)p(\zeta)\left[\left(\frac{w_1(\zeta)}{\delta(\zeta)}\right)^{\sigma}\right]^2 \leq \frac{\left(\delta^{\Delta}(\zeta)\right)^2}{4\delta(\zeta)p(\zeta)}.$$

By the latter inequality and (22), we obtain

$$\frac{\delta(\zeta)}{a(\zeta)} - \frac{\left(\delta^{\Delta}(\zeta)\right)^2}{4\delta(\zeta)p(\zeta)} \leq -w_1^{\Delta}(\zeta). \tag{24}$$

By integrating (24) from ζ_0 to ζ, it follows that

$$\int_{\zeta_0}^{\zeta} \left(\frac{\delta(\xi)}{a(\xi)} - \frac{\left(\delta^{\Delta}(\xi)\right)^2}{4\delta(\xi)p(\xi)} \right) \Delta\xi \leq -w_1(\zeta) + w_1(\zeta_0).$$

By the facts that $y^{\Delta}(\zeta)$ and $a(\zeta)y^{\Delta}(\zeta)$ are decreasing on $[\zeta_1, \infty)_{\mathbb{T}}$, we obtain

$$-y(\zeta) \leq \int_{\zeta}^{\infty} \frac{a(\xi)y^{\Delta}(\xi)}{a(\xi)} \Delta\xi \leq a(\zeta)y^{\Delta}(\zeta) \int_{\zeta}^{\infty} \frac{\Delta\xi}{a(\xi)}, \tag{25}$$

which implies

$$w_1(\zeta) \geq \delta(\zeta) \int_{\zeta}^{\infty} \frac{\Delta\xi}{a(\xi)}.$$

Therefore,

$$\delta(\zeta) \int_{\zeta}^{\infty} \frac{\Delta\xi}{a(\xi)} + \int_{\zeta_0}^{\zeta} \left(\frac{\delta(\xi)}{a(\xi)} - \frac{\left(\delta^{\Delta}(\xi)\right)^2}{4\delta(\xi)p(\xi)} \right) \Delta\xi \leq w_1(\zeta_0),$$

which leads to a discrepancy with (19). □

Now, we are prepared to state and demonstrate the Philos-type criterion for Equation (1).

Theorem 3. *If there exist functions $\delta \in C^1_{rd}(\mathbb{T}, \mathbb{R}^+)$ and $R, r \in C_{rd}(\mathbb{D}, \mathbb{R})$, where*

$$\mathbb{D} \equiv \{(\zeta, \xi) : \zeta, \xi \in \mathbb{T},\ \zeta \geq \xi \geq \zeta_0\}$$

such that

$$R(\zeta, \zeta) = 0, \quad \zeta \geq \zeta_0, \quad R(\zeta, \xi) > 0, \quad \zeta > \xi \geq \zeta_0, \tag{26}$$

and suppose R has a nonpositive continuous Δ-partial derivative $R^{\Delta_\xi}(\zeta, \xi)$ that satisfies

$$R^{\Delta_\xi}(\zeta, \xi) + \frac{\delta^{\Delta}(\xi)}{\delta^{\sigma}(\xi)} R(\zeta, \xi) = -\frac{r(\zeta, \xi)}{\delta^{\sigma}(\xi)} \sqrt{R(\zeta, \xi)} \tag{27}$$

and

$$\limsup_{\zeta \to \infty} \frac{1}{R(\zeta, \zeta_0)} \int_{\zeta_0}^{\zeta} \left[\frac{\delta(\xi)}{a(\xi)} R(\zeta, \xi) - \frac{r^2(\zeta, \xi)}{4\delta(\xi)p(\xi)} \right] \Delta\xi = \infty, \tag{28}$$

then $\mathcal{N}_1 = \emptyset$.

Proof. Assume (1) has a nonoscillatory solution $y(\zeta) \in \mathcal{N}_1$ such that $y(\zeta) > 0$ and $y(\tau(\zeta)) > 0$ for $\zeta \in [\zeta_0, \infty)_{\mathbb{T}}$. Hence, (20) holds. As shown in the proof of Theorem 2, we have

$$\frac{\delta(\zeta)}{a(\zeta)} \leq -w_1^{\Delta}(\zeta) + \delta^{\Delta}(\zeta) \left(\frac{w_1(\zeta)}{\delta(\zeta)} \right)^{\sigma} - \delta(\zeta)p(\zeta) \left[\left(\frac{w_1(\zeta)}{\delta(\zeta)} \right)^{\sigma} \right]^2. \tag{29}$$

Replace ζ by ξ, multiply by $R(\zeta,\xi)$, and integrate with regard to ξ from ζ_0 to $\zeta \geq \zeta_0$ to obtain

$$\begin{aligned}\int_{\zeta_0}^{\zeta} \frac{\delta(\xi)}{a(\xi)} R(\zeta,\xi)\Delta\xi \quad \leq \quad & -\int_{\zeta_0}^{\zeta} R(\zeta,\xi) w_1^{\Delta}(\xi)\Delta\xi \\ & + \int_{\zeta_0}^{\zeta} \delta^{\Delta}(\xi) R(\zeta,\xi)\left(\frac{w_1(\xi)}{\delta(\xi)}\right)^{\sigma}\Delta\xi \\ & - \int_{\zeta_0}^{\zeta} \delta(\xi)p(\xi)R(\zeta,\xi)\left[\left(\frac{w_1(\xi)}{\delta(\xi)}\right)^{\sigma}\right]^2\Delta\xi.\end{aligned}$$

Integrating by parts and from (26) and (27), we obtain

$$\begin{aligned}\int_{\zeta_0}^{\zeta} \frac{\delta(\xi)}{a(\xi)} R(\zeta,\xi)\Delta\xi \quad \leq \quad & R(\zeta,\zeta_0)w_1(\zeta_0) + \int_{\zeta_0}^{\zeta} R^{\Delta_\xi}(\zeta,\xi) w_1^{\sigma}(\xi)\Delta\xi \\ & + \int_{\zeta_0}^{\zeta} \frac{\delta^{\Delta}(\xi)}{\delta^{\sigma}(\xi)} R(\zeta,\xi) w_1^{\sigma}(\xi)\Delta\xi \\ & - \int_{\zeta_0}^{\zeta} \delta(\xi)p(\xi)R(\zeta,\xi)\left\{\left(\frac{w_1(\xi)}{\delta(\xi)}\right)^{\sigma}\right\}^2\Delta\xi \\ = \quad & R(\zeta,\zeta_0)w_1(\zeta_0) + \int_{\zeta_0}^{\zeta}\Bigg[-r(\zeta,\xi)\sqrt{R(\zeta,\xi)}\left(\frac{w_1(\xi)}{\delta(\xi)}\right)^{\sigma} \\ & -\delta(\xi)p(\xi)R(\zeta,\xi)\left\{\left(\frac{w_1(\xi)}{\delta(\xi)}\right)^{\sigma}\right\}^2\Bigg]\Delta\xi. \end{aligned} \tag{30}$$

It is easy to check that

$$-r(\zeta,\xi)\sqrt{R(\zeta,\xi)}\left(\frac{w_1(\xi)}{\delta(\xi)}\right)^{\sigma} - \delta(\xi)p(\xi)R(\zeta,\xi)\left\{\left(\frac{w_1(\xi)}{\delta(\xi)}\right)^{\sigma}\right\}^2 \leq \frac{r^2(\zeta,\xi)}{4\delta(\xi)p(\xi)}. \tag{31}$$

From (30) and (31), we obtain

$$\frac{1}{R(\zeta,\zeta_0)}\int_{\zeta_0}^{\zeta}\left[\frac{\delta(\xi)}{a(\xi)}R(\zeta,\xi) - \frac{r^2(\zeta,\xi)}{4\delta(\xi)p(\xi)}\right]\Delta\xi \leq w(\zeta_0),$$

which is a discrepancy with assumption (28). □

Theorem 4. *If there exists a function* $\rho \in C^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^+)$ *such that*

$$\limsup_{\zeta\to\infty}\left[\rho(\zeta)\int_{\zeta}^{\infty} P(\xi)\Delta\xi + \int_{\zeta_0}^{\zeta}\left(\rho(\xi)P(\xi) - \frac{\left(\rho^{\Delta}(\xi)\right)^2 a(\xi)}{4\rho(\xi)}\right)\Delta\xi\right] = \infty, \tag{32}$$

where

$$P(\xi) := p(\xi)\frac{A(\tau(\xi))}{A^{\sigma}(\xi)} \text{ with } A(\xi) := \int_{\zeta_0}^{\xi}\frac{\Delta s}{a(s)},$$

then $\mathcal{N}_2 = \emptyset$.

Proof. Assume (1) has a nonoscillatory solution $y(\zeta) \in \mathcal{N}_2$ such that $y(\zeta) > 0$ and $y(\tau(\zeta)) > 0$ for $\zeta \in [\zeta_0,\infty)_{\mathbb{T}}$. Then,

$$y^{\Delta}(\zeta) > 0 \text{ and } \left(ay^{\Delta}\right)^{\Delta}(\zeta) = -p(\zeta)y(\tau(\zeta)) < 0 \quad \text{for } \zeta \in [\zeta_0,\infty)_{\mathbb{T}}. \tag{33}$$

Define

$$w_2(\zeta) := \rho(\zeta)\frac{a(\zeta)y^{\Delta}(\zeta)}{y(\zeta)}.$$

In a manner analogous to the proof of Theorem 2, we find that

$$w_2^{\Delta}(\zeta) \leq -\rho(\zeta)p(\zeta)\frac{y(\tau(\zeta))}{y^{\sigma}(\zeta)} + \rho^{\Delta}(\zeta)\left(\frac{w_2(\zeta)}{\rho(\zeta)}\right)^{\sigma} - \frac{\rho(\zeta)}{a(\zeta)}\left[\left(\frac{w_2(\zeta)}{\rho(\zeta)}\right)^{\sigma}\right]^2.$$

By virtue of [12] (Theorem 1), we have

$$\left(\frac{y(\zeta)}{A(\zeta)}\right)^{\Delta} < 0. \tag{34}$$

Therefore,

$$\begin{aligned} w_2^{\Delta}(\zeta) &\leq -\rho(\zeta)p(\zeta)\frac{A(\tau(\zeta))}{A^{\sigma}(\zeta)} + \rho^{\Delta}(\zeta)\left(\frac{w_2(\zeta)}{\rho(\zeta)}\right)^{\sigma} - \frac{\rho(\zeta)}{a(\zeta)}\left[\left(\frac{w_2(\zeta)}{\rho(\zeta)}\right)^{\sigma}\right]^2 \\ &= -\rho(\zeta)P(\zeta) + \rho^{\Delta}(\zeta)\left(\frac{w_2(\zeta)}{\rho(\zeta)}\right)^{\sigma} - \frac{\rho(\zeta)}{a(\zeta)}\left[\left(\frac{w_2(\zeta)}{\rho(\zeta)}\right)^{\sigma}\right]^2. \end{aligned}$$

Using the inequality (23), we obtain

$$\rho^{\Delta}(\zeta)\left(\frac{w_2(\zeta)}{\rho(\zeta)}\right)^{\sigma} - \frac{\rho(\zeta)}{a(\zeta)}\left[\left(\frac{w_2(\zeta)}{\rho(\zeta)}\right)^{\sigma}\right]^2 \leq \frac{\left(\rho^{\Delta}(\zeta)\right)^2 a(\zeta)}{4\rho(\zeta)}.$$

Therefore,

$$w_2^{\Delta}(\zeta) \leq -\rho(\zeta)P(\zeta) + \frac{\left(\rho^{\Delta}(\zeta)\right)^2 a(\zeta)}{4\rho(\zeta)}. \tag{35}$$

By integrating (35) from ζ_0 to ζ, we obtain

$$\int_{\zeta_0}^{\zeta}\left(\rho(\xi)P(\xi) - \frac{\left(\rho^{\Delta}(\xi)\right)^2 a(\xi)}{4\rho(\xi)}\right)\Delta\xi \leq -w_2(\zeta) + w_2(\zeta_0).$$

From (1), (33), and (34), we have

$$a(\zeta)y^{\Delta}(\zeta) \geq \int_{\zeta}^{\infty} p(\xi)y(\tau(\xi))\Delta\xi \geq y(\zeta)\int_{\zeta}^{\infty} p(\xi)\frac{A(\tau(\xi))}{A^{\sigma}(\xi)}\Delta\xi$$

which implies

$$w_2(\zeta) \geq \rho(\zeta)\int_{\zeta}^{\infty} p(\xi)\frac{A(\tau(\xi))}{A^{\sigma}(\xi)}\Delta\xi.$$

Hence,

$$\rho(\zeta)\int_{\zeta}^{\infty} P(\xi)\Delta\xi + \int_{\zeta_0}^{\zeta}\left(\rho(\xi)P(\xi) - \frac{\left(\rho^{\Delta}(\xi)\right)^2 a(\xi)}{4\rho(\xi)}\right)\Delta\xi \leq w_2(\zeta_0),$$

which leads to a discrepancy with (32). This completes the proof. □

Theorem 5. *If there exist functions $\rho \in C^1_{rd}(\mathbb{T}, \mathbb{R}^+)$ and $R, r \in C_{rd}(\mathbb{D}, \mathbb{R})$ such that*

$$R^{\Delta_{\xi}}(\zeta,\xi) + \frac{\rho^{\Delta}(\xi)}{\rho^{\sigma}(\xi)}R(\zeta,\xi) = -\frac{r(\zeta,\xi)}{\rho^{\sigma}(\xi)}\sqrt{R(\zeta,\xi)}$$

and

$$\limsup_{\zeta\to\infty}\frac{1}{R(\zeta,\zeta_0)}\int_{\zeta_0}^{\zeta}\left[\rho(\xi)p(\xi)\frac{A(\tau(\xi))}{A^{\sigma}(\xi)}R(\zeta,\xi) - \frac{r^2(\zeta,\xi)a(\xi)}{4\rho(\xi)}\right]\Delta\xi = \infty, \tag{36}$$

where R, r are defined as in Theorem 3, then $\mathcal{N}_2 = \emptyset$.

Proof. Assume (1) has a nonoscillatory solution $y(\zeta) \in \mathcal{N}_2$ such that $y(\zeta) > 0$ and $y(\tau(\zeta)) > 0$ for $\zeta \in [\zeta_0, \infty)_{\mathbb{T}}$. Hence, (33) holds. As shown in the proof of Theorem 4, we have

$$\rho(\zeta)p(\zeta)\frac{A(\tau(\zeta))}{A^{\sigma}(\zeta)} \leq -w_2^{\Delta}(\zeta) + \rho^{\Delta}(\zeta)\left(\frac{w_2(\zeta)}{\rho(\zeta)}\right)^{\sigma} - \frac{\rho(\zeta)}{a(\zeta)}\left[\left(\frac{w_2(\zeta)}{\rho(\zeta)}\right)^{\sigma}\right]^2.$$

Replacing ζ by ξ, multiplying by $R(\zeta,\xi)$, and integrating with regard to ξ from ζ_0 to $\zeta \geq \zeta_0$, we obtain

$$\begin{aligned}\int_{\zeta_0}^{\zeta} \rho(\xi)p(\xi)\frac{A(\tau(\xi))}{A^{\sigma}(\xi)}R(\zeta,\xi)\Delta\xi \leq & -\int_{\zeta_0}^{\zeta} R(\zeta,\xi)w_2^{\Delta}(\xi)\Delta\xi \\ & + \int_{\zeta_0}^{\zeta} \rho^{\Delta}(\xi)R(\zeta,\xi)\left(\frac{w_2(\xi)}{\rho(\xi)}\right)^{\sigma}\Delta\xi \\ & - \int_{\zeta_0}^{\zeta} \frac{\rho(\xi)}{a(\xi)}R(\zeta,\xi)\left\{\left(\frac{w_2(\xi)}{\rho(\xi)}\right)^{\sigma}\right\}^2\Delta\xi.\end{aligned}$$

In a manner similar to the proof of Theorem 3, we find a discrepancy with assumption (36). □

Next, by combining the results of previous theorems, we set new oscillation criteria for Equation (1).

Theorem 6. *If conditions (19) or (28) and (32) or (36) are satisfied, then Equation (1) oscillates.*

Example 1. *Consider the dynamic equation of second order*

$$\left(\zeta^2 y^{\Delta}\right)^{\Delta}(\zeta) + \gamma\frac{A^{\sigma}(\zeta)}{A(\tau(\zeta))}y(\tau(\zeta)) = 0, \tag{37}$$

where $\gamma > 0$, $a(\zeta) = \zeta^2$, and $p(\zeta) = \dfrac{A^{\sigma}(\zeta)}{A(\tau(\zeta))}$. By choosing $\delta(\zeta) = \zeta$, we have

$$\begin{aligned}&\limsup_{\zeta\to\infty}\left[\delta(\zeta)\int_{\zeta}^{\infty}\frac{\Delta\xi}{a(\xi)} + \int_{\zeta_0}^{\zeta}\left(\frac{\delta(\xi)}{a(\xi)} - \frac{(\delta^{\Delta}(\xi))^2}{4\delta(\xi)p(\xi)}\right)\Delta\xi\right] \\ &= \limsup_{\zeta\to\infty}\left[\zeta\int_{\zeta}^{\infty}\frac{\Delta\xi}{\xi^2} + \int_{\zeta_0}^{\zeta}\left(\frac{1}{\xi} - \frac{1}{4\gamma\xi}\frac{A(\tau(\xi))}{A^{\sigma}(\xi)}\right)\Delta\xi\right] \\ &\geq \limsup_{\zeta\to\infty}\left[\zeta\int_{\zeta}^{\infty}\left(\frac{-1}{\xi}\right)^{\Delta}\Delta\xi + \left(1 - \frac{1}{4\gamma}\right)\int_{\zeta_0}^{\zeta}\frac{1}{\xi}\Delta\xi\right] = \infty.\end{aligned}$$

If $\gamma > \dfrac{1}{4}$ and by choosing $\rho(\zeta) = 1$, (32) holds. As a result of Theorem 6, then Equation (37) oscillates if $\gamma > \dfrac{1}{4}$. It is very important here to note that with the time scale

$$\mathbb{T} = \{\zeta : \zeta = 2^{\beta^k},\ \beta > 1,\ k \in \mathbb{N}_0\},$$

we obtain

$$\int_{\zeta_0}^{\infty}\frac{\Delta\xi}{a(\xi)} = \int_{\zeta_0}^{\infty}\frac{\Delta\xi}{\xi^2}\begin{cases} = \infty & \text{if} \quad \beta \geq 2, \\ < \infty & \text{if} \quad \beta < 2.\end{cases}$$

For more details, see [13] (Example 5.63 and Theorem 5.65). Therefore, all previous results in the literature fail to apply to this Equation (37) on any time scale.

In a particular case, we note that if (3) holds, then $\mathcal{N}_1 = \emptyset$; see [22] (Lemma 2.1). Together with Theorems 4 and 5, we get further oscillation criteria for Equation (1).

Corollary 1. *If conditions (3) and (32) or (36) are satisfied, then Equation (1) oscillates.*

3. Discussion and Conclusions

(1) The results obtained in this paper are applicable to all time scales without restrictive conditions, such as $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, etc. (see [11]).

(2) Novel and enhanced criteria have been developed for the oscillation of the solutions of Equation (1) without relying on convergence and divergence of the improper integral $\int_{\zeta_0}^{\infty} \frac{\Delta\xi}{a(\xi)}$. Compared to previous works in the literature, this approach is more appropriate and applicable to all time scales.

(3) By virtue of

$$\rho(\zeta) \int_{\zeta}^{\infty} P(\xi)\Delta\xi + \int_{\zeta_0}^{\zeta} \left(\rho(\xi)P(\xi) - \frac{\left(\rho^{\Delta}(\xi)\right)^2 a(\xi)}{4\rho(\xi)} \right) \Delta\xi \geq \int_{\zeta_0}^{\zeta} \left(\rho(\xi)P(\xi) - \frac{\left(\rho^{\Delta}(\xi)\right)^2 a(\xi)}{4\rho(\xi)} \right) \Delta\xi,$$

condition (32) improves (18). Therefore, Corollary 1 improves Theorem 1.

(4) It would be interesting to find such conditions for the half-linear second order dynamic equations of the form

$$\left(a(\zeta) \left| y^{\Delta}(\zeta) \right|^{\alpha-1} y^{\Delta}(\zeta) \right)^{\Delta} + p(\zeta) |y(\tau(\zeta))|^{\alpha-1} y(\tau(\zeta)) = 0,$$

where $\alpha > 0$ is a constant.

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