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# Oscillation Test for Second-Order Differential Equations with Several Delays

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**Abstract:** In this paper, the oscillatory properties of certain second-order differential equations of neutral type are investigated. We obtain new oscillation criteria, which guarantee that every solution of these equations oscillates. Further, we get conditions of an iterative nature. These results complement and extend some beforehand results obtained in the literature. In order to illustrate the results we present an example.

**Keywords:** oscillatory behavior; neutral differential equations; several delay arguments

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## 1. Introduction

In this work, we suppose a second-order neutral delay differential equation (NDDE) with several delays

$$\left(r(\varrho)\left(y'(\varrho)\right)^{\gamma}\right)' + \sum_{i=1}^{l} g_i(\varrho)\varkappa^{\beta}(\vartheta_i(\varrho)) = 0, \tag{1}$$

where $\varrho \geq \varrho_0$ and $y(\varrho) = \varkappa(\varrho) + p\varkappa(\tau(\varrho))$. Throughout this study, we suppose

**G**$_1$ $\gamma, \beta \in \{u/v : u, b \in \mathbb{Z}^+ \textit{are odd}\}$ and $l$ is a positive integer;

**G**$_2$ $r \in C([\varrho_o, \infty), (0, \infty))$, $p \geq 0$ is a constant, $g_i(\varrho)$ is not congruently zero, eventually and

$$\mu_{\varrho_0}(\varrho) = \int_{\varrho_0}^{\varrho} r^{-1/\gamma}(\xi)\mathrm{d}\xi = \infty; \tag{2}$$

**G**$_3$ $\tau, \vartheta_i \in c([\varrho_o, \infty), \mathbb{R})$, $\tau(\varrho) \leq \varrho$, $\vartheta_i(\varrho) \leq \varrho$, $\lim_{\varrho\to\infty} \tau(\varrho) = \infty$ and $\lim_{\varrho\to\infty} \vartheta_i(\varrho) = \infty$ for all $i = 1, 2, ..., l$.

By a solution of (2), we mean a nontrivial real-valued function $\varkappa \in C^1([\varrho_\varkappa, \infty))$ for all $\varrho_\varkappa \geq \varrho_0$, which has the property $r(y')^{\gamma} \in C^1([\varrho_\varkappa, \infty))$ and satisfies (1) on $[\varrho_\varkappa, \infty)$. We will consider only those solutions $\varkappa$ of (1) that satisfy the condition

$$\sup\{|\varkappa(\varrho)| : \varrho \geq \varrho_\varkappa\} > 0, \text{ for all } \varrho > \varrho_\varkappa.$$

A solution is said to be oscillatory if it is distinguished that it is neither positive nor negative eventually. A differential equation whose solutions all oscillate is called an oscillatory equation.

NDDEs appear in a variety of situations, including issues with electric networks using lossless transmission lines (as in high-speed computers where such lines are used to interconnect switching circuits). Additional applications in population dynamics, automatic control, mixing liquids, and vibrating masses attached to an elastic bar are available, see

Hale [1,2]. Dynamical systems with several delays have been intensively studied in control theory; see [3,4]. To encourage special interest in the oscillatory behavior of solutions to second-order NDDEs through their applications in the natural sciences and engineering, There is a continuing need to discover new necessary conditions for the oscillation or nonoscillation of solutions varietal type equations; see, e.g., papers [5–11].

Baculikova and Dzurina [12] studied the NDDE

$$\left(r(\varrho)\left((\varkappa(\varrho)+p(\varrho)\varkappa(\tau(\varrho)))'\right)^{\gamma}\right)'+g(\varrho)\varkappa^{\beta}(\vartheta(\varrho))=0, \tag{3}$$

They presented new oscillation criteria for the case under the condition

$$0\le p(\varrho)\le p_0<\infty \text{ and } \tau\circ\vartheta=\vartheta\circ\tau.$$

Dong [13], Liu and Bai [14] and Xu and Meng [15] studied the oscillation of (3), where $0\le p(\varrho)<1$.

Bohner et al. [16] and Agarwal et al. [6] studied the oscillation of the equation

$$\left(r(\varrho)\left((\varkappa(\varrho)+p(\varrho)\varkappa(\tau(\varrho)))'\right)^{\gamma}\right)'+g(\varrho)\varkappa^{\gamma}(\vartheta(\varrho))=0,\ \varrho\ge\varrho_0>0,$$

where

$$\int_{\varrho_o}^{\infty} r^{-1/\gamma}(\xi)\mathrm{d}\xi<\infty.$$

Recently, Moaaz [17] created conditions for the oscillation of NDDEs

$$\left(r(\varrho)\left((\varkappa(\varrho)+p(\varrho)\varkappa(\tau(\varrho)))'\right)^{\gamma}\right)'+f(\varrho,\varkappa(\vartheta(\varrho)))=0,$$

under condition (2).

In this paper we will use some important lemmas and notation

$$B(\varrho)=\begin{cases} c_1^{\beta-\gamma} & if\ \ \gamma\le\beta \\ c_2\mu_{\varrho_0}^{\beta-\gamma}(\varrho) & if\ \ \gamma>\ \beta,\end{cases}$$

where $c_1$ and $c_2$ are positive constants. We will denote by the symbol $\varkappa^+$ the class of all eventually positive solutions of (1).

**Lemma 1.** *[18] Let $\varkappa\in\varkappa^+$.Then,*

$$y>0,\, y'>0 \text{ and } \left(r(y')^{\gamma}\right)'\le 0, \tag{4}$$

*for $\varrho\ge\varrho_1$, where $\varrho_1$ is sufficiently large.*

**Lemma 2.** *[19] If $\varkappa\in\varkappa^+$, then $y^{\beta-\gamma}(\varrho)\ge B(\varrho)$, eventually.*

There is no doubt that the concept of symmetry is of great importance as it appears in many natural phenomena and has many applications. The approach adopted in our paper is based on exploiting the symmetry between positive and negative solutions in studying only positive solutions.

The aim of this work is to find new NDDE oscillation criteria (1). We establish more effective criteria by considering the equation in two cases: $p<1$ and $p>1$. To create more efficient criteria, we take into account the influence of the delay argument $\tau(\varrho)$, and we abandon some of the constraints that are usually imposed on the coefficients of the equation in the case $p>1$. When $\gamma=\beta$ and $p<1$, we also utilize an iterative method to obtain the oscillation criterion of (1).

## 2. Main Results

For convenience, we write the functions without the independent variable, such as $f(\varrho) = f$ and $f(q(\varrho)) = f(q)$. In addition, we suppose that $\tau^0 = \varrho$, $\tau^m = \tau \circ \tau^{m-1}$, $\tau^{-m-1} = \tau^{-1} \circ \tau^{-m}$ for $m = 1, 2, ....$ We use some notations in this paper:

$$g(\varrho) = \min\{g_i(\varrho) : i = 1, 2, ..., l\},$$

$$\vartheta(\varrho) = \max\{\vartheta_i(\varrho) : i = 1, 2, ..., l\}.$$

$$g = g(1 - p(\vartheta))^{\beta},$$

$$\widetilde{\chi}_{\varrho_0}(\varrho) = \mu_{\varrho_0}(\varrho) + \frac{1}{\gamma}\int_{\varrho_0}^{\varrho} \mu_{\varrho_0}(\xi) g(\xi) \sum_{i=1}^{l} B(\vartheta_i(\xi)) \mu_{\varrho_o}^{\gamma}(\vartheta_i(\xi)) \mathrm{d}\xi,$$

$$\widehat{\chi}_{\varrho_0}(\varrho) = \exp\left(-\gamma \int_{\vartheta(\varrho)}^{\varrho} \frac{1}{r^{1/\gamma}(\xi)\widetilde{\chi}_{\varrho_0}(\xi)} \mathrm{d}\xi\right),$$

and

$$\varphi_k(\varrho) = \int_{\varrho}^{\infty} \widehat{\chi}_{\varrho_0}(\varsigma) g(\varsigma) \sum_{i=1}^{l} B(\vartheta_i(\varsigma)) \mathrm{d}\varsigma, \, k = 0, 1.$$

First, we will establish new criteria for the oscillation of solution (1) using the Riccati technique.

**Lemma 3.** *Let $\varkappa \in \varkappa^+$, $p > 1$ and there is an even integer $n > 0$ such that*

$$\widetilde{p} = \sum_{m=1}^{n/2} \frac{1}{p^{2m-1}} \left(1 - \frac{1}{p} \frac{\mu_{\varrho_2}\left(\tau^{-2m}\right)}{\mu_{\varrho_2}\left(\tau^{-(2m-1)}\right)}\right) > 0. \tag{5}$$

*Then*

$$\varkappa(\varrho) \geq \widetilde{p}(\varrho) y(\varrho). \tag{6}$$

**Proof.** We assume that $\varkappa \in \varkappa^+$.Thus, $\varkappa(\varrho)$, $\varkappa(\tau(\varrho))$ and $\varkappa(\vartheta_i(\varrho))$ are positive for all $\varrho \geq \varrho_1$, where $\varrho_1$ is large enough and $1 \leq i \leq l$. From Lemma 1, we have that (4) holds. Since $\left(r^{1/\gamma} y'\right)' \leq 0$, we obtain that

$$y(\varrho) = y(\varrho_1) + \int_{\varrho_1}^{\varrho} \frac{1}{r^{1/\gamma}(\xi)} r^{1/\gamma}(\xi) y'(\xi) \mathrm{d}\xi,$$

so

$$\begin{aligned} y(\varrho) &> \int_{\varrho_1}^{\varrho} \frac{1}{r^{1/\gamma}(\xi)} r^{1/\gamma}(\xi) y'(\xi) \mathrm{d}\xi \\ &> r^{1/\gamma}(\varrho) y'(\varrho) \int_{\varrho_1}^{\varrho} \frac{1}{r^{1/\gamma}(\xi)} \mathrm{d}\xi \\ &> r^{1/\gamma}(\varrho) y'(\varrho) \mu_{\varrho_1}(\varrho), \end{aligned} \tag{7}$$

for all $\varrho \geq \varrho_1$. Using the definition of $y(\varrho)$, we find

$$\varkappa = \frac{1}{p}\left(y\left(\tau^{-1}\right) - \varkappa\left(\tau^{-1}\right)\right) = \frac{1}{p}\left(y\left(\tau^{-1}\right) - \frac{1}{p} y\left(\tau^{-2}\right)\right) + \frac{1}{p^2} \varkappa\left(\tau^{-2}\right).$$

By repeating the above step, we can see

$$\begin{aligned} \varkappa &= \sum_{m=1}^{n} \frac{(-1)^{m+1}}{p^m} y(\tau^{-m}) + \frac{1}{p^n} \varkappa(\tau^{-n}) \\ &> \sum_{m=1}^{n/2} \frac{1}{p^{2m-1}} \left( y\left(\tau^{-(2m-1)}\right) - \frac{1}{p} y\left(\tau^{-2m}\right) \right), \end{aligned} \tag{8}$$

for $\varrho \geq \varrho_2 \geq \varrho_1$, where $\varrho_2$ is large enough, and $n \in \mathbb{Z}^+$ is even . Taking (7) and $\tau^{-2m} \geq \tau^{-(2m-1)}$ into account, we get

$$y\left(\tau^{-2m}\right) < y\left(\tau^{-(2m-1)}\right) \frac{\mu_{t_1}(\tau^{-2m})}{\mu_{t_1}(\tau^{-(2m-1)})}, \tag{9}$$

for $m = 1\,, 2\,, ..., n/2$. From (8) and (9), we obtain

$$\begin{aligned} \varkappa &> \sum_{m=1}^{n/2} \frac{1}{p^{2m-1}} \left( 1 - \frac{1}{p_0} \frac{\mu_{t_1}(\tau^{-2m})}{\mu_{t_1}(\tau^{-(2m-1)})} \right) y\left(\tau^{-(2m-1)}\right) \\ &> \widetilde{p} y. \end{aligned}$$

The proof of the lemma is complete. □

**Lemma 4.** *Let $\varkappa \in \varkappa^+$and $p < 1$. Then*

$$\varkappa(\varrho) \geq \widehat{p}(\varrho) y(\varrho), \tag{10}$$

*for any $n \in \mathbb{Z}^+$ is odd , where*

$$\widehat{p} = (1-p) \sum_{m=0}^{(n-1)/2} p^{2m} \frac{\mu_{\varrho_1}(\tau^{2m+1})}{\mu_{\varrho_1}}, \tag{11}$$

*and*

$$\mu_{\varrho_1}(\varrho) = \int_{\varrho_1}^{\varrho} r^{-1/\gamma}(\xi) \mathrm{d}\xi.$$

**Proof.** Let's move forward as in the proof of Lemma 3, we get (7). Using the definition of $y(\varrho)$, we obtain

$$\varkappa = y - p\varkappa(\tau) = y - py(\tau) + p^2 \varkappa\left(\tau^2\right).$$

By repeating the above step, we can see

$$\begin{aligned} \varkappa &= \sum_{m=0}^{n} (-1)^m p^m y(\tau^m) + p^{n+1} \varkappa\left(\tau^{n+1}\right) \\ &\geq \sum_{m=0}^{(n-1)/2} \left( p^{2m} y\left(\tau^{2m}\right) - p^{2m+1} y\left(\tau^{2m+1}\right) \right), \end{aligned} \tag{12}$$

for $\varrho \geq \varrho_2 \geq \varrho_1$, where $\varrho_2$ is enough, and odd $n \in \mathbb{Z}^+$. Since $\tau^{2m+1}(\varrho) \leq \tau^{2m}(\varrho)$, we have that

$$y(\tau^n) \leq \ ... \ \leq y\left(\tau^{2m+1}\right) \leq y\left(\tau^{2m}\right) \leq \ ... \ \leq y,$$

for $m = 0, 2, ..., (n-1)/2$. From (12), we arrive at

$$\varkappa \geq \sum_{m=0}^{(n-1)/2} p^{2m} (1-p) y\left(\tau^{2m+1}\right). \tag{13}$$

From (7), we obtain

$$y\left(\tau^{2m+1}\right) > y\frac{\mu_{\varrho_1}\left(\tau^{2m+1}\right)}{\mu_{\varrho_1}}.$$

Thus, from (13), we find

$$\varkappa \geq (1-p)y\sum_{m=0}^{(n-1)/2} p^{2m}\frac{\mu_{\varrho_1}\left(\tau^{2m+1}\right)}{\mu_{\varrho_1}}.$$

This completes the proof. □

**Theorem 1.** *If*

$$\liminf_{\varrho\to\infty}\frac{\gamma}{\varphi_1}\int_{\varrho}^{\infty} r^{-1/\gamma}(\varsigma)\varphi_1^{(\gamma+1)/\gamma}(\varsigma)\mathrm{d}\varsigma > \frac{\gamma}{(\gamma+1)^{(\gamma+1)/\gamma}}, \tag{14}$$

*then (1) is oscillatory.*

**Proof.** Supposing that the result we want to achieve is incorrect. We suppose the opposite that $\varkappa$ is a non-oscillatory solution of (1). Without losing generalization, we assume that $\varkappa \in \varkappa^+$.Thus, $\varkappa(\varrho)$, $\varkappa(\tau(\varrho))$ and $\varkappa(\vartheta_i(\varrho))$ are positive for all $\varrho \geq \varrho_1$, where $\varrho_1$ is sufficiently large and $1 \leq i \leq l$. We obtain

$$\varkappa \geq y(1-p(\vartheta)),$$

which with (1) gives

$$\left(r(y')^{\gamma}\right)' \leq -g\sum_{i=1}^{l} y^{\beta}(\vartheta_i). \tag{15}$$

Using the chain rule and simple computation, we find

$$\gamma\left(r^{1/\gamma}y'\right)^{\gamma-1}\frac{\mathrm{d}}{\mathrm{d}\varrho}\left(y-\mu_{\varrho_1}r^{1/\gamma}y'\right) = -\gamma\left(r^{1/\gamma}y'\right)^{\gamma-1}\mu_{\varrho_1}\left(r^{1/\gamma}y'\right)' = -\mu_{\varrho_1}\left(r(y')^{\gamma}\right)', \tag{16}$$

from (15) and (16), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varrho}\left(y-\mu_{\varrho_1}r^{1/\gamma}y'\right) &\geq \frac{1}{\gamma}\left(r^{1/\gamma}y'\right)^{1-\gamma}\mu_{\varrho_1}g\sum_{i=1}^{l} y^{\beta}(\vartheta_i)\\ &\geq \frac{1}{\gamma}\left(r^{1/\gamma}y'\right)^{1-\gamma}\mu_{\varrho_1}g\sum_{i=1}^{l} B(\vartheta_i)y^{\gamma}(\vartheta_i).\end{aligned} \tag{17}$$

Integrating (17) from $\varrho_1$to $\varrho$, we obtain

$$y \geq \mu_{\varrho_1}r^{1/\gamma}y' + \frac{1}{\gamma}\int_{\varrho_1}^{\varrho}\left(r^{1/\gamma}(\xi)y'(\xi)\right)^{1-\gamma}\mu_{\varrho_1}(\xi)g(\xi)\sum_{i=1}^{l} B(\vartheta_i(\xi))y^{\gamma}(\vartheta_i(\xi))\mathrm{d}\xi. \tag{18}$$

Since $\left(r(\varrho)(y'(\varrho))^{\gamma}\right)' \leq 0$, we have

$$y(\vartheta_i) \geq \mu_{\varrho_1}(\vartheta_i)r^{1/\gamma}(\vartheta_i)y'(\vartheta_i) \geq \mu_{\varrho_1}(\vartheta_i)r^{1/\gamma}y'.$$

Thus, (18) becomes

$$y \geq \left(\mu_{\varrho_1} + \frac{1}{\gamma}\int_{\varrho_1}^{\varrho}\mu_{\varrho_1}(\xi)g(\xi)\sum_{i=1}^{l} B(\vartheta_i(\xi))\mu_{\varrho_1}^{\gamma}(\vartheta_i)\mathrm{d}\xi\right)r^{1/\gamma}y',$$

that is

$$y \geq \widetilde{\chi}_{\varrho_1}r^{1/\gamma}y'. \tag{19}$$

Integrating $y'/y \leq 1/\widehat{\mu}_{\varrho_1} r^{1/\gamma}$ from $\vartheta(\varrho)$ to $\varrho$, we find

$$\ln \frac{y(\varrho)}{y(\vartheta(\varrho))} \leq \int_{\vartheta(\varrho)}^{\varrho} \frac{1}{r^{1/\gamma}(\xi)\widetilde{\chi}_{\varrho_1}(\xi)} \mathrm{d}\xi,$$

that is

$$y(\vartheta(\varrho)) \geq \exp\left(-\int_{\vartheta(\varrho)}^{\varrho} \frac{1}{r^{1/\gamma}(\xi)\widetilde{\chi}_{\varrho_1}(\xi)} \mathrm{d}\xi\right) y(\varrho). \tag{20}$$

Define the function

$$v = \frac{r(y')^{\gamma}}{y^{\gamma}}. \tag{21}$$

Then $v(\varrho) > 0$ for $\varrho \geq \varrho_1$. From (1) and (21), we obtain

$$v' \leq -g\frac{y^{\gamma}(\vartheta)}{y^{\gamma}} \sum_{i=1}^{l} y^{\beta-\gamma}(\vartheta_i) - \frac{\gamma}{r^{1/\gamma}(\varrho)} v^{(\gamma+1)/\gamma} < 0. \tag{22}$$

By using (20), we obtain

$$v'(\varrho) \leq -g\widehat{\chi} \sum_{i=1}^{l} B(\vartheta_i) - \frac{\gamma}{r^{1/\gamma}(\varrho)} v^{(\gamma+1)/\gamma} < 0. \tag{23}$$

By integrating (23) from $\varrho$ to $\infty$, we conclude that

$$\int_{\varrho}^{\infty} g(\varsigma)\widehat{\chi}(\varsigma) \sum_{i=1}^{l} B(\vartheta_i(\varsigma))\mathrm{d}\varsigma + \gamma \int_{\varrho}^{\infty} r^{-1/\gamma}(\varsigma) v^{(\gamma+1)/\gamma}(\varsigma)\mathrm{d}\varsigma \leq v(\varrho) - v(\infty).$$

Since $v$ is a positive decreasing function, we see that

$$\varphi_1 + \gamma \int_{\varrho}^{\infty} r^{-1/\gamma}(\varsigma) v^{(\gamma+1)/\gamma}(\varsigma)\mathrm{d}\varsigma \leq v.$$

Hence,

$$1 + \frac{\gamma}{\varphi_1} \int_{\varrho}^{\infty} r^{-1/\gamma}(\varsigma)\varphi_1^{(\gamma+1)/\gamma}(\varsigma) \left(\frac{v(\varsigma)}{\varphi_1(\varsigma)}\right)^{(\gamma+1)/\gamma} \mathrm{d}\varsigma \leq \frac{v}{\varphi_1}. \tag{24}$$

Set

$$\kappa = \inf_{\varrho \geq \varrho_1} \frac{v}{\varphi_1}.$$

From (24), $\kappa \geq 1$. Taking (14) and (24) into account, we find

$$1 + \gamma\left(\frac{\kappa}{\gamma+1}\right)^{1+1/\gamma} \leq \kappa$$

or

$$\left(\frac{\kappa}{\gamma+1}\right)^{\gamma+1} \leq \left(\frac{\kappa-1}{\gamma}\right)^{\gamma},$$

which is not possible with the permissible value $\gamma > 0$ and $\kappa \geq 1$. This contradiction completes the proof. $\square$

**Theorem 2.** *Suppose that $p < 1$. If there is a function $\theta \in C^1([\varrho_o, \infty), (0, \infty))$ with*

$$\limsup_{\varrho \to \infty} \int_{\varrho_1}^{\varrho} \left(\theta(\xi)\delta(\xi)g(\xi) \sum_{i=1}^{l} \widehat{p}^{\beta}(\vartheta_i(\xi))\, B(\vartheta_i(\xi)) \;-\; \frac{r(\xi)(\theta'_+(\xi))^{\gamma+1}}{\theta^{\gamma}(\xi)(\gamma+1)^{(\gamma+1)}}\right) \mathrm{d}\xi = \infty, \tag{25}$$

*then (1) is oscillatory, where*

$$\widehat{\mu}_{\varrho_0}(\varrho) = \mu_{\varrho_0}(\varrho) + \frac{1}{\gamma}\int_{\varrho_0}^{\varrho} \mu_{\varrho_0}(\xi) g(\xi) \sum_{i=1}^{l} \widehat{p}^{\beta}(\vartheta_i(\xi))\, B(\vartheta_i(\xi))\mu_{\varrho_o}^{\gamma}(\vartheta_i(\xi))\mathrm{d}\xi$$

*and*

$$\delta(\varrho) = \exp\left(-\gamma \int_{\vartheta(\varrho)}^{\varrho} \frac{1}{r^{1/\gamma}(\xi)\widehat{\mu}_{\varrho_1}(\xi)}\mathrm{d}\xi\right),$$

*where*

$$\widehat{\mu}_{\varrho_1}(\varrho) = \mu_{\varrho_1}(\varrho) + \frac{1}{\gamma}\int_{\varrho_1}^{\varrho} \mu_{\varrho_1}(\xi) g(\xi) \sum_{i=1}^{l} \widehat{p}^{\beta}(\vartheta_i(\xi))\, B(\vartheta_i(\xi))\mu_{\varrho_1}^{\gamma}(\vartheta_i(\xi))\mathrm{d}\xi.$$

**Proof.** Supposing that the result we want to achieve is incorrect. We suppose the opposite that $\varkappa$ is a non-oscillatory solution of (1). Without losing generalization, we assume that $\varkappa \in \varkappa^{+}$. Thus, $\varkappa(\varrho)$, $\varkappa(\tau(\varrho))$ and $\varkappa(\vartheta_i(\varrho))$ are positive for all $\varrho \geq \varrho_1$, where $\varrho_1$ large enough and $1 \leq i \leq l$. From Lemma 4, we obtain that (10) holds. Combining (1) and (10), we arrive at

$$\left(r(y')^{\gamma}\right)' \leq -g \sum_{i=1}^{l} \widehat{p}^{\beta}(\vartheta_i)\, y^{\beta}(\vartheta_i). \tag{26}$$

Using (16) and (26) gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varrho}\left(y - \mu_{\varrho_1} r^{1/\gamma} y'\right) &\geq \frac{1}{\gamma}\left(r^{1/\gamma}y'\right)^{1-\gamma} \mu_{\varrho_1} g \sum_{i=1}^{l} \widehat{p}^{\beta}(\vartheta_i)\, y^{\beta}(\vartheta_i) \\ &\geq \frac{1}{\gamma}\left(r^{1/\gamma}y'\right)^{1-\gamma} \mu_{\varrho_1} g \sum_{i=1}^{l} \widehat{p}^{\beta}(\vartheta_i)\, B(\vartheta_i) y^{\gamma}(\vartheta_i). \end{aligned} \tag{27}$$

Integrating (27) from $\varrho_1$to $\varrho$, we obtain

$$y \geq \mu_{\varrho_1} r^{1/\gamma} y' + \frac{1}{\gamma}\int_{\varrho_1}^{\varrho} \left(r^{1/\gamma}(\xi) y'(\xi)\right)^{1-\gamma} \mu_{\varrho_1}(\xi) g(\xi) \sum_{i=1}^{l} \widehat{p}^{\beta}(\vartheta_i(\xi))\, B(\vartheta_i(\xi)) y^{\gamma}(\vartheta_i(\xi))\mathrm{d}\xi. \tag{28}$$

Since $\left(r(\varrho)(y'(\varrho))^{\gamma}\right)' \leq 0$, we have

$$y(\vartheta_i) \geq \mu_{\varrho_1}(\vartheta_i) r^{1/\gamma}(\vartheta_i) y'(\vartheta_i) \geq \mu_{\varrho_1}(\vartheta_i) r^{1/\gamma} y'.$$

Thus, (28) becomes

$$y \geq \left(\mu_{\varrho_1} + \frac{1}{\gamma}\int_{\varrho_1}^{\varrho} \mu_{\varrho_1}(\xi) g(\xi) \sum_{i=1}^{l} \widehat{p}^{\beta}(\vartheta_i(\xi))\, B(\vartheta_i(\xi)) \mu_{\varrho_1}^{\gamma}(\vartheta_i)\mathrm{d}\xi\right) r^{1/\gamma} y',$$

that is

$$y \geq \widehat{\mu}_{\varrho_1} r^{1/\gamma} y'. \tag{29}$$

Integrating $y'/y \leq 1/\widehat{\mu}_{\varrho_1} r^{1/\gamma}$ from $\vartheta(\varrho)$ to $\varrho$, we find

$$\ln \frac{y(\varrho)}{y(\vartheta(\varrho))} \leq \int_{\vartheta(\varrho)}^{\varrho} \frac{1}{r^{1/\gamma}(\xi)\widehat{\mu}_{\varrho_1}(\xi)}\mathrm{d}\xi,$$

that is

$$y(\vartheta(\varrho)) \geq \exp\left(-\int_{\vartheta(\varrho)}^{\varrho} \frac{1}{r^{1/\gamma}(\xi)\widehat{\mu}_{\varrho_1}(\xi)}\mathrm{d}\xi\right) y(\varrho). \tag{30}$$

Now, we define

$$\Theta = \theta \frac{r(y')^{\gamma}}{y^{\gamma}}.$$

Therefore, $\Theta(\varrho) > 0$ for all $\varrho \geq \varrho_1$ and

$$\Theta' = \frac{\theta'}{\theta}\Theta + \theta \frac{\left(r(y')^{\gamma}\right)'}{y^{\gamma}} - \gamma\theta \frac{r(y')^{\gamma+1}}{y^{\gamma+1}}.$$

From (26) and (30), we find

$$\Theta' \leq \frac{\theta'}{\theta}\Theta - \theta \frac{g \sum_{i=1}^{l} \widehat{p}^{\beta}(\vartheta_i)\, B(\vartheta_i) y^{\gamma}(\vartheta_i)}{y^{\gamma}} - \gamma\theta \frac{r(y')^{\gamma+1}}{y^{\gamma+1}},$$

so

$$\Theta' \leq \frac{\theta'}{\theta}\Theta - \theta g \sum_{i=1}^{l} \widehat{p}^{\beta}(\vartheta_i)\, B(\vartheta_i)\, \delta_i - \gamma\theta \frac{r(y')^{\gamma+1}}{y^{\gamma+1}}.$$

Using the definition of $\Theta$, we conclude that

$$\Theta' \leq \frac{\theta'}{\theta}\Theta - \theta g \sum_{i=1}^{l} \widehat{p}^{\beta}(\vartheta_i)\, B(\vartheta_i)\, \delta_i - \frac{\gamma}{r^{1/\gamma}\theta^{1/\gamma}}\Theta^{1+1/\gamma}.$$

Using the inequality

$$A\phi - B\phi^{(\gamma+1)/\gamma} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{(\gamma+1)}} \frac{A^{\gamma+1}}{B^{\gamma}}, B > 0,$$

with $A = \theta'/\theta$, $B = \gamma/\left(r^{1/\gamma}\theta^{1/\gamma}\right)$ and $\phi = \Theta$, we get

$$\Theta' \leq -\theta\delta g \sum_{i=1}^{l} \widehat{p}^{\beta}(\vartheta_i)\, B(\vartheta_i) + \frac{1}{(\gamma+1)^{(\gamma+1)}} \frac{r(\theta'_{+})^{\gamma+1}}{\theta^{\gamma}}.$$

Integrating the above inequality from $\varrho_1$to $\varrho$, we obtain

$$\int_{\varrho_1}^{\varrho} \left( \theta(\xi)g(\xi)\delta(\xi) \sum_{i=1}^{l} \widehat{p}^{\beta}(\vartheta_i(\xi))\, B(\vartheta_i(\xi)) - \frac{1}{(\gamma+1)^{(\gamma+1)}} \frac{r(\xi)(\theta'_{+}(\xi))^{\gamma+1}}{\theta^{\gamma}(\xi)} \right) \mathrm{d}\xi \leq \Theta(\varrho_1),$$

which contradicts (25). The proof is complete. □

**Theorem 3.** *Assume that (5) holds. If there exists a function $\varpi \in C^1([\varrho_o, \infty), (0, \infty))$ such that*

$$\limsup_{\varrho \to \infty} \int_{\varrho_1}^{\varrho} \left( \varpi(\xi)g(\xi)\widetilde{\delta}(\xi) \sum_{i=1}^{l} \widetilde{p}^{\beta}(\vartheta_i(\xi))\, B(\vartheta_i(\xi)) - \frac{r(\xi)(\varpi'_{+}(\xi))^{\gamma+1}}{\varpi^{\gamma}(\xi)(\gamma+1)^{(\gamma+1)}} \right) \mathrm{d}\xi = \infty,$$

*then (1) is oscillatory, where*

$$\widetilde{\mu}_{\varrho_0}(\varrho) = \mu_{\varrho_0}(\varrho) + \frac{1}{\gamma} \int_{\varrho_0}^{\varrho} \mu_{\varrho_0}(\xi)g(\xi) \sum_{i=1}^{l} \widetilde{p}^{\beta}(\vartheta_i(\xi))\, B(\vartheta_i(\xi))\mu_{\varrho_o}^{\gamma}(\vartheta_i(\xi))\mathrm{d}\xi,$$

*and*

$$\widetilde{\delta}(\varrho) = \exp\left( -\gamma \int_{\vartheta(\varrho)}^{\varrho} \frac{1}{r^{1/\gamma}(\xi)\widetilde{\mu}_{\varrho_1}(\xi)} \mathrm{d}\xi \right).$$

**Proof.** It is enough to use (6) instead of (10) in the proof of Theorem 2 to prove this theorem. To prove this theorem, we use □

Now, we will establish new criteria for oscillation of solution (1) by using an iterative technique.

**Lemma 5.** *Suppose that $\varkappa \in \varkappa^+$, $\gamma = \beta$ and $p < 1$. Then*

$$y(\varrho) = \phi_k(\varrho) r^{1/\gamma}(\varrho) y'(\varrho), \tag{31}$$

*for $k = 0, 1, ...,$ where $\phi_0(\varrho) = \widehat{\mu}_{\varrho_1}(\varrho)$ and*

$$\phi_{k+1}(\varrho) = \int_{\varrho_1}^{\varrho} \left( \frac{1}{r(\varsigma)} \exp\left( \int_{\varsigma}^{\varrho} g(\xi) \sum_{i=1}^{l} \widehat{p}^{\gamma}(\vartheta_i(\xi)) \phi_k^{\gamma}(\vartheta_i(\xi)) \mathrm{d}\xi \right) \right)^{1/\gamma} \mathrm{d}\varsigma. \tag{32}$$

**Proof.** Suppose that $\varkappa \in \varkappa^+$. Thus, $\varkappa(\varrho)$, $\varkappa(\tau(\varrho))$ and $\varkappa(\vartheta_i(\varrho))$ are positive for all $\varrho \geq \varrho_1$, where $\varrho_1$ large enough. Using Lemma 1, we have that (4) holds. Next, by induction, we will prove (31).

For $k = 1$, as in the proof of Theorem 2, we get that (26) and (29) hold. By (29), we find

$$y \geq \widehat{\mu}_{\varrho_1}(\varrho) r^{1/\gamma} y' = \phi_0(\varrho) r^{1/\gamma} y'.$$

Next, we suppose that (31) holds at $k = n$; that is, $y \geq \phi_n r^{1/\gamma} y'$. Thus, since $\left(\left(r^{1/\gamma} y'\right)^{\gamma}\right)' \leq 0$, we find

$$y(\vartheta_i) \geq \phi_n(\vartheta_i) r^{1/\gamma}(\vartheta_i) y'(\vartheta_i) \geq \phi_n(\vartheta_i) r^{1/\gamma} y'.$$

Which, with (26), gives

$$\left(r(y')^{\gamma}\right)' + g \sum_{i=1}^{l} \widehat{p}^{\gamma}(\vartheta_i)\, y^{\gamma}(\vartheta_i) \leq 0,$$

so

$$\left(r(y')^{\gamma}\right)' + g \sum_{i=1}^{l} \widehat{p}^{\gamma}(\vartheta_i) \phi_n^{\gamma}(\vartheta_i) r(y')^{\gamma} \leq 0. \tag{33}$$

Setting $H = r(y')^{\gamma}$, (33) becomes

$$H'(\varrho) + g \sum_{i=1}^{l} \widehat{p}^{\gamma}(\vartheta_i) \phi_n^{\gamma}(\vartheta_i) H(\varrho) \leq 0. \tag{34}$$

Applying the Gronwall inequality in (34), we get

$$H(\varsigma) \geq H(\varrho) \exp\left( \int_{\varsigma}^{\varrho} g(\xi) \sum_{i=1}^{l} \widehat{p}^{\gamma}(\vartheta_i(\xi)) \phi_n^{\gamma}(\vartheta_i(\xi)) \mathrm{d}\xi \right),$$

for $\varrho \geq \varsigma \geq \varrho_1$, and so

$$y'(\varsigma) \geq r^{1/\gamma}(\varrho) y'(\varrho) \left( \frac{1}{r(\varsigma)} \exp\left( \int_{\varsigma}^{\varrho} g(\xi) \sum_{i=1}^{l} \widehat{p}^{\gamma}(\vartheta_i(\xi)) \phi_n^{\gamma}(\vartheta_i(\xi)) \mathrm{d}\xi \right) \right)^{1/\gamma}.$$

Integrating this inequality from $\varrho_1$ to $\varrho$, we obtain

$$\begin{aligned} y(\varrho) &\geq r^{1/\gamma}(\varrho) y'(\varrho) \int_{\varrho_1}^{\varrho} \left( \frac{1}{r(\varsigma)} \exp\left( \int_{\varsigma}^{\varrho} g(\xi) \sum_{i=1}^{l} \widehat{p}^{\gamma}(\vartheta_i(\xi)) \phi_n^{\gamma}(\vartheta_i(\xi)) \mathrm{d}\xi \right) \right)^{1/\gamma} \mathrm{d}\varsigma \\ &= \phi_{n+1}(\varrho) r^{1/\gamma}(\varrho) y'(\varrho). \end{aligned}$$

The proof is complete. □

**Theorem 4.** *Assume that $\gamma = \beta$ and $p < 1$. Then (1) is oscillatory if*

$$\liminf_{\varrho\to\infty} \int_{\vartheta(\varrho)}^{\varrho} g(\xi) \sum_{i=1}^{l} \widehat{p}^{\gamma}(\vartheta_i(\xi))\phi_k^{\gamma}(\vartheta_i(\xi))\mathrm{d}\xi > \frac{1}{\mathrm{e}}, \tag{35}$$

*for some integers $k > 0$, where $\widehat{p}$, $\phi_k$ are defined as in (11) and (32), respectively.*

**Proof.** Supposing that the result we want to achieve is incorrect. We suppose the opposite that $\varkappa$ is a non-oscillatory solution of (1). Without losing generalization, we assume that $\varkappa \in \varkappa^+$. Thus, $\varkappa(\varrho)$, $\varkappa(\tau(\varrho))$ and $\varkappa(\vartheta_i(\varrho))$ are positive for all $\varrho \geq \varrho_1$, where $\varrho_1$ large enough. Using Lemma 5, we arrive at (31) holds. As in the proof of Theorem 2, we get (26). Using (33) and (31), we obtain

$$\left(r(y')^{\gamma}\right)' + g\sum_{i=1}^{l} \widehat{p}^{\gamma}(\vartheta_i)\phi_k^{\gamma}(\vartheta_i)r(\vartheta_i)\left(y'(\vartheta_i)\right)^{\gamma} \leq 0.$$

If we set $w = (r(y'))^{\gamma}$, we have that $w$ is a positive solution of the delay differential inequality

$$w'(\varrho) + g\sum_{i=1}^{l} \widehat{p}^{\gamma}(\vartheta_i)\phi_k^{\gamma}(\vartheta_i)w(\vartheta_i) \leq 0.$$

Using Theorem 1 in [20] the associated delay differential equation

$$w'(\varrho) + g\sum_{i=1}^{l} \widehat{p}^{\gamma}(\vartheta_i)\phi_k^{\gamma}(\vartheta_i)w(\vartheta_i) = 0, \tag{36}$$

also has a positive solution. But, the equation (36) with condition (35) is oscillatory; this is a contradiction. The proof is complete. □

**Theorem 5.** *Suppose that $\gamma = \beta$ and $p < 1$. Then (1) is oscillatory if there is a function $\rho \in C^1([\varrho_o,\infty),(0,\infty))$ with*

$$\limsup_{\varrho\to\infty} \int_{\varrho_1}^{\varrho} \left(\rho(\xi)g(\xi)\widehat{\delta}_k(\xi)\sum_{i=1}^{l} \widehat{p}^{\beta}(\vartheta_i(\xi))\, B(\vartheta_i(\xi)) - \frac{r(\xi)(\rho'_+(\xi))^{\gamma+1}}{\rho^{\gamma}(\xi)(\gamma+1)^{(\gamma+1)}}\right)\mathrm{d}\xi = \infty, \tag{37}$$

*for some integers $k \geq 0$, where*

$$\widehat{\delta}_k(\varrho) = \exp\left(-\gamma\int_{\vartheta(\varrho)}^{\varrho} \frac{1}{r^{1/\gamma}(\xi)\phi_k(\xi)}\mathrm{d}\xi\right),$$

*$\widehat{p}$ and $\phi_k$ are defined as in (11) and (32), respectively.*

**Proof.** Supposing that the result we want to achieve is incorrect. We suppose the opposite that $\varkappa$ is a non-oscillatory solution of (1). Without losing generalization, we assume that $\varkappa \in \varkappa^+$. Thus, there exist $\varrho_1 > \varrho_0$ such that $\varkappa(\varrho)$, $\varkappa(\tau(\varrho))$ and $\varkappa(\vartheta_i(\varrho))$ are positive for all $\varrho \geq \varrho_1$ and $1 \leq i \leq n$. Now, we define $\psi = \rho r(y'/y)^{\gamma}$. Thus, $\psi(\varrho) > 0$ and

$$\psi' = \frac{\rho'}{\rho}\psi + \rho\frac{(r(y')^{\gamma})'}{y^{\gamma}} - \gamma\rho r\left(\frac{y'}{y^{\gamma+1}}\right)^{\gamma+1}.$$

From Lemma 5, we have that (31) holds. By replacing (29) with (31) in the proof of Theorem 2, this part of the proof is similar to that of Theorem 2 and so the proof is obtained. □

**Example 1.** *Consider the NDDE*

$$\left(\left((\varkappa(\varrho)+p\varkappa(\eta\varrho))'\right)^{\gamma}\right)'+\frac{g_0}{\varrho^{\gamma}}\varkappa^{\gamma}(\lambda_2\varrho)+\frac{g_0}{\varrho^{\gamma+1}}\varkappa^{\gamma}(\lambda_1\varrho)=0, \tag{38}$$

*where* $g_0>0$, $\varrho>1$, $\lambda_2<\lambda_1$ *and* $\eta,\lambda_1,\lambda_2\in(0,1)$. *It is easy to verify that*

$$\mu_{\varrho_0}(\varrho)=\varrho,\ \tau^m(\varrho)=\eta^m\varrho,\ \widehat{\mu}_{\varrho_0}(\varrho)=\left(1+\frac{2}{\gamma}\widehat{p}_0^{\gamma}g_0\lambda_1^{\gamma}\right)\varrho \text{ and } \delta(\varrho)=\lambda_1^{\widehat{\gamma}},$$

$$g(\varrho)=\frac{g_0}{\varrho^{\gamma+1}}(1-p)^{\gamma},\ \widetilde{\chi}_{\varrho_0}(\varrho)=A\varrho,\ \widehat{\chi}_{\varrho_0}(\varrho)=\lambda_1^{A}$$

*and*

$$\varphi_1(\varrho)=\frac{1}{\gamma}\lambda_1^{\frac{\gamma}{A}}(1-p)^{\gamma}g_0\frac{1}{\varrho^{\gamma}},$$

*where*

$$\widehat{p}=(1-p)\sum_{m=0}^{(n-1)/2}p^{2m}\eta^{2m+1}=\widehat{p}_0,\ \widehat{\gamma}=\frac{\gamma}{\left(1+\frac{2}{\gamma}\widehat{p}_0^{\gamma}g_0\lambda_1^{\gamma}\right)}$$

*and* $A=1+\frac{2}{\gamma}g_0\lambda_1^{\gamma}(1-p)^{\gamma}$.

*From Theorem 2, we arrive at (38) is oscillatory if* $p<1$ *and*

$$g_0>\frac{\gamma^{\gamma+1}}{2\lambda_1^{\widehat{\gamma}}\widehat{p}_0^{\gamma}(\gamma+1)^{\gamma+1}}. \tag{39}$$

*By Theorem 3, we have that (38) is oscillatory if* $p>1$ *and*

$$g_0>\frac{\gamma^{\gamma+1}}{2\lambda_1^{\widetilde{\gamma}}\widetilde{p}_0^{\gamma}(\gamma+1)^{\gamma+1}}. \tag{40}$$

*Using Theorem 1, we see that (38) is oscillatory if*

$$g_0>\frac{\gamma^{\gamma+1}}{\lambda_1^{\frac{\gamma}{A}}(1-p)^{\gamma}(\gamma+1)^{\gamma+1}}. \tag{41}$$

**Remark 1.** *By comparing the conditions (39) and (41) for different values* $p,\lambda_1$ *and* $\eta$, *we obtain the following table when* $\gamma=1$.

| | (39) | (41) |
|---|---|---|
| $(p,\lambda_1,\eta)$ | $n=5$ | |
| $(2/3,0.1,0.755)$ | 2.6671 | 4.4348 |
| $(0.5,0.5,0.83)$ | 0.44222 | 0.81579 |

**Remark 2.** *In a special case the best-known criteria for oscillation of neutral delay differential Equation (38) at* $\frac{q_0}{\varrho^{\gamma}}\varkappa^{\gamma}(\lambda_2\varrho)=0$ *are*

$$q_0>\frac{\gamma^{\gamma+1}}{(1-p)^{\gamma}\lambda^{\frac{\gamma}{\left(1+\frac{1}{\gamma}(1-p)^{\gamma}g_0\lambda^{\gamma}\right)}}(\gamma+1)^{\gamma+1}} \text{ for } p<1 \text{ see [21] (Example 3)} \tag{42}$$

*and*

$$g_0\lambda^{\gamma}\ln\left(\frac{\mu}{\lambda}\right)>\frac{\mu+p}{\mu e} \text{ for } p>1 \text{ see [18] (Corollary 2)}. \tag{43}$$

Consider the particular case of (38) in the form

$$\left(\left(\left(\varkappa(\varrho)+\frac{2}{3}\varkappa(0.755\varrho)\right)'\right)^{\gamma}\right)'+\frac{g_0}{\varrho^2}\varkappa^{\gamma}(0.1\varrho)=0.$$

The conditions (39) and (42) reduce to $g_0 > 5.2529$ and $g_0 > 5.30610$, respectively. Consider another particular case of (38) in the form

$$\left(\left((\varkappa(\varrho)+4\varkappa(0.5\varrho))'\right)^{\gamma}\right)'+\frac{g_0}{\varrho^2}\varkappa^{\gamma}(0.4\varrho)=0.$$

The conditions (40) and (43) reduce to $g_0 > 2.0411$ and $g_0 > 37.094$, respectively. So, our results improve the related results in [18,21].

**Remark 3.** *An interesting problem for further research could be to study the problem of oscillation for Equation (1) when*

$$\mu_{\varrho_0}(\varrho)=\int_{\varrho_0}^{\varrho} r^{-1/\gamma}(\xi)\mathrm{d}\xi<\infty.$$

## 3. Conclusions

The oscillatory behavior of a class of NDDEs with multiple delays has been studied. The study depends on establishing new criteria by finding an improved relationship between the solution $\varkappa$ and the corresponding function $y$. We also created criteria of an iterative nature that can be applied more than once in case the previous results fail. By comparing our results with previous results in Remark 2, we have illustrated the significance of the new results.

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