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# Investigating Spacelike Polynomial Curves in Minkowski 3-Space Using a Frenet-Like Curve Fram

Ayman Elsharkawy<sup>1\*</sup>, Hoda Elsayied<sup>1\*\*</sup>, Aya Refaat<sup>1\*\*\*</sup>, and Clemente Cesarano<sup>2\*\*\*\*</sup>

(Submitted by H. A. Matevossian)

<sup>1</sup>*Department of Mathematics, Faculty of Science, Tanta University, Tanta, 31511 Egypt*

<sup>2</sup>*Section of Mathematics, International Telematic, University Uninettuno, Roma, Italy*

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**Abstract**—This study investigates the properties of spacelike polynomial curves in Minkowski 3-space using an innovative framework known as the Frenet-like curve frame (Flc-frame). Traditional methods, such as the Frenet frame, depend on the first and second derivatives of a curve, which can be limiting for polynomial curves where higher-order derivatives lack geometric clarity. The Flc-frame addresses these limitations by leveraging intrinsic curve properties to derive Flc-formulae and curvatures, establishing their interrelationships. A concrete example demonstrates the efficacy of the Flc-frame in characterizing such curves. Additionally, we introduce novel integral curves defined via the Flc-frame and its curvatures, extending this framework to Minkowski 3-space. Our findings reveal that the Flc-frame generalizes the Frenet frame, providing a more adaptable tool for curve analysis, especially in scenarios where traditional methods falter. This research enhances the understanding of polynomial curves in Minkowski space and lays the groundwork for future studies on integral curves and their geometric properties.

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## 1. INTRODUCTION

In classical differential geometry, the Frenet frame serves as a cornerstone for curve analysis. However, its utility diminishes at points where curvature vanishes, as the normal and binormal vectors become undefined [19]. To overcome this, alternative frames like the Bishop frames introduced by Bishop in 1975 as a “parallel transport frame” were developed [1]. While effective in certain applications [17], the Bishop frames is non-analytic and has inherent constraints [4]. For further exploration of alternative frames, see [7–12, 14, 15].

Recent advancements by Dede et al. introduced the Flc-frame for polynomial curves in Euclidean 3-space [3, 4]. This frame defines the binormal vector using the cross product of the first and  $n$ th derivatives, offering geometric insights into higher-order derivatives. The Flc-frame outperforms the Frenet and Bishop frames in computational simplicity and robustness, with fewer singular points [2]. Arslan (2020) further applied the Flc-frame to define integral curves in Euclidean space, deriving their curvature and torsion [13].

This paper extends the Flc-frame to spacelike polynomial curves in Minkowski 3-space, a setting pivotal to relativity and Lorentzian geometry. We derive explicit expressions for Flc-curvatures and

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\* E-mail: ayman\_ramadan@science.tanta.edu.eg

\*\* E-mail: hkelsayied1998@yahoo.com

\*\*\* E-mail: aya30899252@science.tanta.edu.eg

\*\*\*\* E-mail: c.cesarano@uninettunouniversity.net

explore their relationships. New integral curves, constructed by integrating Flc-normal and Flc-binormal vectors, are introduced, and their Frenet apparatus (tangent, normal, binormal vectors, curvature, and torsion) is rigorously analyzed. Our results confirm that the Flc-frame generalizes the Frenet frame, proving indispensable in cases where the latter fails, such as vanishing curvature or higher-order derivatives.

The paper is structured as follows: Section 2 outlines foundational concepts, including Minkowski 3-space and curve classification. Section 3 details the Flc-frame and its curvatures for spacelike polynomial curves. Section 4 introduces integral curves and their Frenet apparatus. The conclusion discusses implications and future applications.

## 2. PRELIMINARIES

This section establishes the foundational concepts and notation for our study. We begin with the Lorentz–Minkowski space and its metric structure, followed by the causal character of vectors and curves. Finally, we review the Frenet frame and its limitations, paving the way for the Flc-frame.

### 2.1. Lorentz-Minkowski Space

The Lorentz–Minkowski space,  $\mathbb{E}_1^3$ , is a 3-dimensional space equipped with a metric tensor  $g$  distinct from Euclidean space. Formally,  $\mathbb{E}_1^3$  is defined as  $(E^3, g)$ , where  $E^3$  represents the underlying Euclidean 3-space, and the metric  $g$  is

$$g = -du_1^2 + du_2^2 + du_3^2,$$

for coordinate  $(u_1, u_2, u_3)$ . This pseudo-Riemannian structure is central to studying curves and surfaces in Minkowski space.

### 2.2. Causal Character of Vectors and Curves

In  $\mathbb{E}_1^3$ , the causal character of a vector  $M$  is determined by the sign of its inner product with itself under the metric  $g$ . Specifically:

- A vector  $M$  is *spacelike* (*Sp-like*) if  $g(M, M) > 0$  or  $M = 0$ .
- A vector  $M$  is *timelike* (*T-like*) if  $g(M, M) < 0$ .
- A vector  $M$  is *lightlike* (*null*) if  $g(M, M) = 0$  and  $M \neq 0$ .

The norm of a vector  $M$  is defined as  $\|M\| = \sqrt{|g(M, M)|}$ . If  $\|M\| = 1$ , the vector is said to be unit. Two vectors  $M$  and  $N$  are orthogonal if  $g(M, N) = 0$ .

The vector product (cross product) of two vectors  $M = (m_1, m_2, m_3)$  and  $N = (n_1, n_2, n_3)$  in  $\mathbb{E}_1^3$  is defined as

$$M \wedge N = (m_3n_2 - m_2n_3, m_3n_1 - m_1n_3, m_1n_2 - m_2n_1).$$

### 2.3. Curves in Minkowski Space

A curve in  $\mathbb{E}_1^3$  is a smooth mapping  $\eta(s) : I \rightarrow \mathbb{E}_1^3$ , where  $I$  is an open interval in  $\mathbb{R}$ , and  $s$  is the arc-length parameter. The causal character of a curve  $\eta(s)$  is determined by its tangent vector  $\frac{d\eta(s)}{ds}$ :

- $\eta(s)$  is *spacelike* if  $\frac{d\eta(s)}{ds}$  is spacelike for all  $s \in I$ .
- $\eta(s)$  is *timelike* if  $\frac{d\eta(s)}{ds}$  is timelike for all  $s \in I$ .
- $\eta(s)$  is *lightlike* if  $\frac{d\eta(s)}{ds}$  is lightlike for all  $s \in I$ .

For non-null curves, the Frenet frame  $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$  is commonly used, where

- $\mathbf{T}(s)$  is the unit tangent vector,
- $\mathbf{N}(s)$  is the principal normal vector,
- $\mathbf{B}(s)$  is the binormal vector.

For non-null curves, the Frenet formulae are given by

$$\frac{d}{ds} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 \\ -\epsilon\kappa_1 & 0 & \kappa_2 \\ 0 & \kappa_2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{bmatrix}, \quad (2.1)$$

where  $\epsilon = \pm 1$  determines the causal character of  $\mathbf{N}$ . If  $\epsilon = 1$ , then  $\mathbf{N}$  is spacelike, and if  $\epsilon = -1$ , then  $\mathbf{N}$  is timelike.

For non-arc-length parametrization, the Frenet frame is given by

$$\mathbf{T}(t) = \frac{\eta'(t)}{\|\eta'(t)\|}, \quad \mathbf{N}(t) = \xi \mathbf{T}(t) \times \mathbf{B}(t), \quad \mathbf{B}(t) = \frac{-\xi(\eta'(t) \times \eta''(t))}{\|\eta'(t) \times \eta''(t)\|}, \quad (2.2)$$

where  $\xi = 1$  if  $\eta''(t)$  is spacelike and  $\xi = -1$  if  $\eta''(t)$  is timeline. Also, the curvature  $\kappa$  and torsion  $\tau$  are computed as

$$\kappa = \frac{\|\eta'(t) \times \eta''(t)\|}{\|\eta'(t)\|^3}, \quad \tau = \frac{\xi[g(\eta'''(t), \eta'(t) \times \eta''(t))]}{\|\eta'(t) \times \eta''(t)\|^2}. \quad (2.3)$$

#### 2.4. Limitations of the Frenet Frame

While the Frenet frame is widely used, it fails at points where the curvature  $\kappa$  vanishes, as the normal and binormal vectors become undefined. This limitation motivates the need for alternative frames, such as the Frenet-like curve frame (Flc-frame), which we introduce and analyze in this work. The Flc-frame provides a more robust framework for studying polynomial curves, particularly in Minkowski space, where the causal character of vectors and curves adds additional complexity.

### 3. SPACELIKE POLYNOMIAL CURVES WITH FLC-FRAME

At the beginning of this part, the Sp-like Flc-frame and Sp-like Flc-formulae in  $\mathbb{E}_1^3$  are investigated. Also, the Flc-curvatures of the curve,  $K_1$ ,  $K_2$ , and  $K_3$  are introduced. Moreover, a relationship between the Flc-curvatures is obtained.

#### 3.1. Flc-Frame and Flc-Formulae of Sp-Like Curves in $\mathbb{E}_1^3$

There are two cases in this subsection. First, the curve is Sp-like with a Sp-like Flc-normal vector and Sp-like  $n$ th derivative. Second, the curve is Sp-like with a T-like Flc-normal vector and T-like  $n$ th derivative. From now on, we consider  $\eta(t)$  a polynomial spatial curve,  $\eta(t)$  is regular for all  $t$ , that is,  $\eta'(t) \neq 0$  for all  $t$ .

**Definition 3.1.** Let  $\eta(t)$  be a Sp-like curve in which the Flc-normal  $D_2$  is a Sp-like vector, then the Flc-binormal vector  $D_1(t)$  is the only T-like unit vector parallel to the Sp-like plane  $\{T(t), D_2(t)\}$ , and we can define the Flc-frame consisting of  $\{T(t), D_2(t), D_1(t)\}$  the tangent, Flc-normal, Flc-binormal vectors, respectively, by

$$T(t) = \frac{\eta'(t)}{\|\eta'(t)\|}, \quad D_1(t) = \frac{-[\eta'(t) \times \eta^{(n)}(t)]}{\|\eta'(t) \times \eta^{(n)}(t)\|}, \quad D_2(t) = D_1(t) \times T(t). \quad (3.1)$$

**Theorem 3.1.** *Let  $\eta(t)$  is a Sp-like curve with Flc-normal vector  $D_2$ , has the same causal character of Frenet normal  $N$ , and let the  $n$ th derivative  $\eta^{(n)}(t)$  be a Sp-like vector. Then, the Flc-formulae can be expressed as*

$$\frac{d}{dt} \begin{bmatrix} T(t) \\ D_2(t) \\ D_1(t) \end{bmatrix} = u \begin{bmatrix} 0 & K_1 & -K_2 \\ -K_1 & 0 & K_3 \\ -K_2 & K_3 & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ D_2(t) \\ D_1(t) \end{bmatrix}, \tag{3.2}$$

where  $u = \|\eta'(t)\| = \frac{ds}{dt}$ ,  $K_1 = \kappa \cosh \psi$ ,  $K_2 = \kappa \sinh \psi$ , and  $K_3 = \frac{d\psi}{ds} + \tau u$ .

**Proof.** Define the the Lorentzian angle  $\psi$  between  $N(t)$  and  $D_2(t)$  in which

$$D_2 = \cosh \psi N + \sinh \psi B, \tag{3.3}$$

$$D_1 = \sinh \psi N + \cosh \psi B. \tag{3.4}$$

That is

$$N = \cosh \psi D_2 - \sinh \psi D_1, \quad B = \sinh \psi D_2 + \cosh \psi D_1.$$

By using Eq. (2.1), we can obtain

$$T' = \frac{ds}{dt} \frac{dT}{ds} = u\kappa \cosh \psi D_2 - u\kappa \sinh \psi D_1.$$

Differentiating (3.3) and substituting (2.1) and (3.4), we get

$$D_2' = -(u\kappa \cosh \psi)T + \left(\frac{d\psi}{ds} + \tau u\right) D_1.$$

Similarly, differentiating (3.3) and substituting (2.1) and (3.4), we have

$$D_1' = -(u\kappa \sinh \psi)T + \left(\frac{d\psi}{ds} + \tau u\right) D_2.$$

By taking the substitution

$$K_1 = \kappa \cosh \psi, \quad K_2 = \kappa \sinh \psi, \quad K_3 = \frac{d\psi}{ds} + \tau u.$$

The results are obtained. □

**Corollary 3.1.** *Let  $\eta(t)$  is a Sp-like curve, with a Sp-like Flc-normal vector  $D_2$ , and let the  $n$ th derivative  $\eta^{(n)}(t)$  be a Sp-like vector in the same space. Then, the Flc-curvatures  $K_1, K_2$ , and  $K_3$  are, respectively, defined by*

$$\begin{aligned} K_1 &= \frac{g(T', D_2)}{u} = -\frac{g(D_2', T)}{u}, \\ K_2 &= \frac{g(T', D_1)}{u} = -\frac{g(D_1', T)}{u}, \\ K_3 &= -\frac{g(D_2', D_1)}{u} = \frac{g(D_1', D_2)}{u}. \end{aligned} \tag{3.5}$$

**Definition 3.2.** *Let  $\eta(t)$  be a Sp-like curve in which the Flc normal  $D_2$  is a T-like vector, then the Flc-binormal vector  $D_1(t)$  is the only Sp-like unit vector parallel to the T-like plane  $\{T(t), D_2(t)\}$ , and we can define the Flc-frame by*

$$T(t) = \frac{\eta'(t)}{\|\eta'(t)\|}, \quad D_1(t) = \frac{\eta'(t) \times \eta^{(n)}(t)}{\|\eta'(t) \times \eta^{(n)}(t)\|}, \quad D_2(t) = -D_1(t) \times T(t).$$

**Theorem 3.2.** If  $\eta(t)$  is a Sp-like curve, with a T-like Flc-normal vector  $D_2$  and the  $n$ th derivative  $\eta^{(n)}(t)$  is a T-like vector, then the Flc-formulae can be obtained as

$$\frac{d}{dt} \begin{bmatrix} T(t) \\ D_2(t) \\ D_1(t) \end{bmatrix} = u \begin{bmatrix} 0 & K_1 & -K_2 \\ K_1 & 0 & K_3 \\ K_2 & K_3 & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ D_2(t) \\ D_1(t) \end{bmatrix}, \quad (3.6)$$

where  $u = \|\eta'(t)\|$ ,  $K_1 = \kappa \cosh \psi$ ,  $K_2 = \kappa \sinh \psi$ , and  $K_3 = \frac{d\psi}{ds} + \tau u$ .

**Corollary 3.2.** Let  $\eta(t)$  is a polynomial Sp-like curve, with a T-like Flc-normal vector  $D_2$ , and let the  $n$ th derivative  $\eta^{(n)}$  be a T-like vector, i.e.,  $D_1$  is a Sp-like vector. Then, the Flc-curvatures  $K_1$ ,  $K_2$ , and  $K_3$  are respectively defined by

$$\begin{aligned} K_1 &= -\frac{g(T', D_2)}{u} = \frac{g(D_2', T)}{u}, \\ K_2 &= -\frac{g(T', D_1)}{u} = \frac{g(D_1', T)}{u}, \\ K_3 &= \frac{g(D_2', D_1)}{u} = -\frac{g(D_1', D_2)}{u}. \end{aligned} \quad (3.7)$$

### 3.2. The Relation Between the Flc-Curvatures

In this subsection, we give formulae for determining the Flc-curvatures,  $K_1$ ,  $K_2$ , and  $K_3$ , and introduce the relation between them in two cases. First, the  $\eta(t)$  is Sp-like with a Sp-like Flc-normal vector. Second, the  $\eta(t)$  is Sp-like with a T-like Flc-normal vector.

**Theorem 3.3.** If  $\eta(t)$  is a Sp-like curve, with a Sp-like Flc-normal vector and the  $n$ th derivative  $\eta^{(n)}$  is a Sp-like vector, then the Flc-curvatures  $K_1$ ,  $K_2$ , and  $K_3$  are given by

$$K_1 = -\frac{g(\eta'(t) \times \eta''(t), \eta'(t) \times \eta^{(n)}(t))}{\|\eta'(t)\|^3 \|\eta^{(n)}(t) \times \eta'(t)\|}, \quad (3.8)$$

$$K_2 = -\frac{g(\eta''(t), \eta'(t) \times \eta^{(n)}(t))}{\|\eta'(t)\|^2 \|\eta'(t) \times \eta^{(n)}(t)\|}, \quad (3.9)$$

$$K_3 = -\frac{g(\eta''(t), \eta'(t) \times \eta^{(n)}(t)) g(\eta'(t), \eta^{(n)}(t))}{\|\eta'(t)\|^2 \|\eta'(t) \times \eta^{(n)}(t)\|^2}. \quad (3.10)$$

**Proof.** Let  $\eta(t)$  be a Sp-like curve, with a Sp-like Flc-normal vector, and let the  $n$ th derivative  $\eta^{(n)}$  be a Sp-like vector. By differentiating the unit tangent vector  $T(t)$  in Eq. (3.1) and substituting into Eqs. (3.2) and (3.5), we have

$$K_1 = \frac{g(T', D_2)}{u} = -\frac{[g(\eta''(t), (\eta'(t) \times \eta^{(n)}(t)) \times \eta'(t))]}{\|\eta'(t)\|^3 \|\eta^{(n)}(t) \times \eta'(t)\|}.$$

By using the fact  $(A \times B) \times C = Ag(B, C) - Bg(A, C)$ , where  $A$ ,  $B$ , and  $C$  are vectors in  $\mathbb{E}_1^3$ , we get

$$K_1 = -\frac{[g(\eta''(t), \eta'(t))g(\eta'(t), \eta^{(n)}(t)) - \|\eta'(t)\|^2 g(\eta''(t), \eta^{(n)}(t))]}{\|\eta'(t)\|^3 \|\eta^{(n)}(t) \times \eta'(t)\|}. \quad (3.11)$$

Since  $g(A \times C, B \times D) = g(A, D)g(B, C) - g(A, B)g(C, D)$ , where  $A$ ,  $B$ ,  $C$ , and  $D$  are vectors in  $\mathbb{E}_1^3$ , we get

$$K_1 = -\frac{g(\eta'(t) \times \eta''(t), \eta'(t) \times \eta^{(n)}(t))}{\|\eta'(t)\|^3 \|\eta^{(n)}(t) \times \eta'(t)\|}.$$

Similarly, we can obtain  $K_2$  as follows

$$K_2 = \frac{g(T', D_1)}{u} = -\frac{g(\eta''(t), \eta'(t) \times \eta^{(n)}(t))}{\|\eta'(t)\|^2 \|\eta'(t) \times \eta^{(n)}(t)\|}.$$

Also, from Eqs. (3.1) and (3.2), we can get  $K_3$  as

$$K_3 = -\frac{g(D_2'(t), D_1(t))}{\|\eta'(t)\|} = -\frac{g(D_1'(t) \times T(t), D_1(t))}{\|\eta'(t)\|}.$$

Similarly, we can obtain  $K_3$  as

$$K_3 = -\frac{[g(D_1'(t), \eta'(t))g(\eta'(t), \eta^{(n)}(t)) - g(D_1'(t), \eta^{(n)}(t))g(\eta'(t), \eta'(t))]}{\|\eta'(t)\| \|\eta'(t) \times \eta^{(n)}(t)\|}. \tag{3.12}$$

On the other hand, by differentiating  $D_1$  in Eq. (3.1), we have

$$D_1' = \frac{[(\eta''(t) \times \eta^{(n)}(t) + \eta'(t) \times \eta^{(n+1)}(t))]}{\|\eta'(t) \times \eta^{(n)}(t)\|} - \frac{[(\eta'(t) \times \eta^{(n)}(t))\frac{d}{dt}\|\eta'(t) \times \eta^{(n)}(t)\|]}{\|\eta'(t) \times \eta^{(n)}(t)\|^2}.$$

Since the  $(n + 1)$ th derivative of the curve vanishes, by substituting into Eq. (3.12), we deduce

$$K_3 = -\frac{g(\eta''(t), \eta'(t) \times \eta^{(n)}(t))g(\eta'(t), \eta^{(n)}(t))}{\|\eta'(t)\|^2 \|\eta'(t) \times \eta^{(n)}(t)\|^2}.$$

□

**Theorem 3.4.** *If  $\eta(t)$  is a Sp-like curve, then the relation between Flc-curvatures  $K_1, K_2$ , and  $K_3$  is*

$$uK_1 = \frac{[(K_3/K_2)']}{1 + (K_3/K_2)^2}. \tag{3.13}$$

**Proof.** Let  $\phi(t)$  be the Lorentzian angle between  $D_2(t)$  and the  $n$ th derivative of the curve  $\eta^{(n)}(t)$ , then we have

$$g(\eta^{(n)}(t), D_2(t)) = \|\eta^{(n)}(t)\| \cos \phi.$$

From Eq. (3.1) we see that the vectors  $\eta^{(n)}$  and  $D_1$  are orthogonal and, so,

$$g(\eta^{(n)}(t), T(t)) = \|\eta^{(n)}(t)\| \sin \phi,$$

which implies that

$$g(\eta^{(n)}(t), \eta'(t)) = u\|\eta^{(n)}(t)\| \sin \phi. \tag{3.14}$$

Combining Eqs. (3.9) and (3.10), we have

$$\frac{K_3}{K_2} = \frac{g(\eta'(t), \eta^{(n)}(t))}{\|\eta'(t) \times \eta^{(n)}(t)\|} = \tan \phi.$$

It follows that

$$\phi = \arctan(K_3/K_2). \tag{3.15}$$

Differentiating Eq. (3.15) yields

$$\frac{d\phi}{dt} = \frac{[(K_3/K_2)']}{1 + (K_3/K_2)^2}. \tag{3.16}$$

On the other hand, by using

$$\eta'(t) = uT(t), \quad \eta''(t) = u'T(t) + u(K_1D_2(t) - K_3D_2(t)),$$

we have

$$g(\eta'(t), \eta''(t)) = uu'. \tag{3.17}$$

Differentiating Eq. (3.14) yields

$$g(\eta^{(n)}(t), \eta''(t)) = u' \left\| \eta^{(n)}(t) \right\| \sin \phi + u \left\| \eta^{(n)}(t) \right\| \cos \phi \frac{d\phi}{dt}. \quad (3.18)$$

Substituting Eqs. (3.14), (3.17), and (3.18) into (3.11), we obtain

$$K_1 = \frac{1}{u} \frac{d\phi}{dt}. \quad (3.19)$$

Combining (3.16) and (3.19), we have the desired relation.  $\square$

**Corollary 3.3.** *If  $\eta(t)$  is a Sp-like curve with Sp-like Flc-normal  $D_2(t)$ , and let the  $n$ th derivative  $\eta^{(n)}(t)$  be a Sp-like vector. Then, the  $n$ th derivative of the curve can be expressed as a function of its base  $\{T, D_1, D_2\}$  in the following form*

$$\eta^n(t) = \|\eta^n(t)\| (\sin \phi T(t) + \cos \phi D_2(t)),$$

where  $\phi = \int u K_1 dt$ .

**Theorem 3.5.** *If  $\eta(t)$  is a Sp-like curve, with a T-like Flc-normal vector, and, then, the  $n$ th derivative  $\eta^{(n)}$  is a T-like vector, then the Flc-curvatures  $K_1, K_2$ , and  $K_3$  are given by*

$$\begin{aligned} K_1 &= \frac{g(\eta'(t) \times \eta''(t), \eta'(t) \times \eta^{(n)}(t))}{\|\eta'(t)\|^3 \|\eta^{(n)}(t) \times \eta'(t)\|}, \\ K_2 &= -\frac{g(\eta''(t), \eta'(t) \times \eta^{(n)}(t))}{\|\eta'(t)\|^2 \|\eta'(t) \times \eta^{(n)}(t)\|}, \\ K_3 &= -\frac{g(\eta''(t), \eta'(t) \times \eta^{(n)}(t)) g(\eta'(t), \eta^{(n)}(t))}{\|\eta'(t)\|^2 \|\eta'(t) \times \eta^{(n)}(t)\|^2}. \end{aligned}$$

**Theorem 3.6.** *If  $\eta(t)$  is a Sp-like curve, then the relation between curvatures  $K_1, K_2$ , and  $K_3$  of the Flc-frame is given by*

$$uk_1 = \frac{[(K_3/K_2)']}{1 - (K_3/K_2)^2}. \quad (3.20)$$

**Corollary 3.4.** *If  $\eta(t)$  is a Sp-like curve, and the  $n$ th derivative  $\eta^{(n)}(t)$  is a T-like vector. Then, the  $n$ th derivative of the curve can be expressed as a function of its base  $\{T, D_1, D_2\}$  in the following form*

$$\eta^n(t) = \|\eta^n(t)\| (\sinh \phi T(t) + \cosh \phi D_2(t)),$$

where  $\phi = \int u K_1 dt$ .

**Example 3.1.** Let  $\eta$  be a spatial Sp-like curve of degree 9 with a Sp-like  $D_2$  vector given by

$$\eta = (t^7 - t^3 + 4, t^7 + t^3 - 4, t^9).$$

Using Eq. (3.1) the tangent, Flc-normal and Flc-binormal vector fields of  $\eta(t)$  are as follows

$$\begin{aligned} T &= \left( \frac{t^2(-3 + 7t^4)}{\sqrt{3}\sqrt{t^8(28 + 27t^8)}}, \frac{t^2(3 + 7t^4)}{\sqrt{3}\sqrt{t^8(28 + 27t^8)}}, \frac{3t^8}{\sqrt{\frac{28t^8}{3} + 9t^{16}}} \right), \\ D_2 &= \left( \frac{3t^{18}(7t^4 - 3)}{2\sqrt{7}(-t^8)^{3/2}\sqrt{t^8(27t^8 + 28)}}, \frac{3t^{18}(7t^4 + 3)}{2\sqrt{7}(-t^8)^{3/2}\sqrt{t^8(27t^8 + 28)}}, -\frac{2\sqrt{7}\sqrt{-t^8}}{\sqrt{t^8(27t^8 + 28)}} \right), \\ D_1 &= \left( \frac{7t^6 + 3t^2}{2\sqrt{21}\sqrt{t^8}}, \frac{7t^6 - 3t^2}{2\sqrt{21}\sqrt{t^8}}, 0 \right). \end{aligned}$$

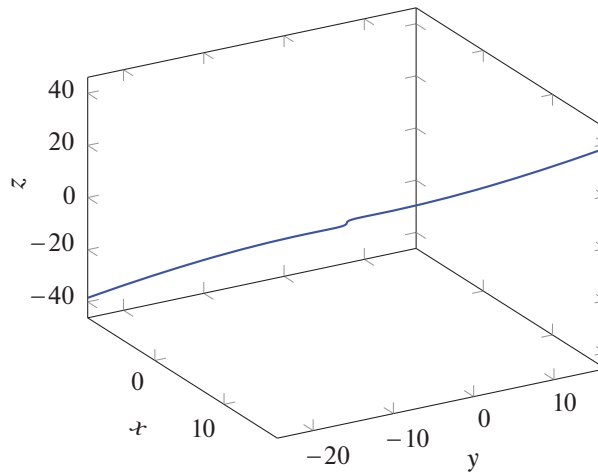
From Eq. (3.5) or using Eqs. (3.8), (3.9), and (3.10) the Flc-curvatures are

$$K_1 = -\frac{24\sqrt{7}t^7\sqrt{-t^8}}{(t^8(27t^8 + 28))^{3/2}}, \quad K_2 = \frac{4\sqrt{\frac{7}{3}}\sqrt{t^8}}{t^9(28 + 27t^8)}, \quad K_3 = \frac{6}{28t + 27t^9}.$$

Also from Eq. (3.13) we get

$$uk_1 = \frac{\frac{d}{dt}(K_3/K_2)}{1 + (K_3/K_2)^2} = \frac{24\sqrt{21}\sqrt{t^8}}{t(27t^8 - 28)}.$$

3D Plot of  $\eta(t) = (t^7 - t^3 + 4, t^7 + t^3 - 4, t^9)$



#### 4. INTEGRAL CURVES

We introduce a new type of integral curves in which the Frenet vectors, curvature, and torsion for these curves are generated by using the Flc-frame and Flc-curvatures.

**Definition 4.1.** If  $\eta(t)$  is a Sp-like curve in which the Flc-normal  $D_2(t)$  is a Sp-like or a T-like vector. The integrated curve  $\gamma(t)$  of  $D_2(t)$  is defined as

$$\gamma(t) = \int D_2(t)dt. \tag{4.1}$$

**Theorem 4.1.** If  $\eta(t)$  is a Sp-like curve in which Flc-normal  $D_2(t)$  is a Sp-like vector and  $\gamma(t)$  is the integrated curve of  $D_2(t)$  with  $K_1, K_3 \neq 0$ , then the Frenet vector fields are

$$T_\gamma(t) = (D_2)_\eta(t),$$

$$N_\gamma(t) = \xi \left( \frac{K_1}{\sqrt{|K_3^2 - K_1^2|}} T(t) - \frac{K_3}{\sqrt{|K_3^2 - K_1^2|}} D_1(t) \right)_\eta,$$

$$B_\gamma(t) = -\xi \left( \frac{K_3}{\sqrt{|K_3^2 - K_1^2|}} T(t) - \frac{K_1}{\sqrt{|K_3^2 - K_1^2|}} D_1(t) \right)_\eta.$$

**Proof.** Let  $\eta(t)$  be a Sp-like curve in which  $D_2(t)$  is a Sp-like vector and  $\gamma(t)$  be the integrated curve of  $D_2(t)$  with  $K_1, K_3 \neq 0$ . By using Eqs. (3.2) and (4.1) and taking the derivatives, we get

$$\gamma'(t) = D_2(t), \tag{4.2}$$

$$\gamma''(t) = D_2'(t) = u(-K_1T(t) + K_3D_1(t)). \tag{4.3}$$

Taking the cross-product of Eqs. (4.2) and (4.3), we get

$$\gamma'(t) \times \gamma''(t) = u(K_3T(t) - K_1D_1(t)). \tag{4.4}$$

Applying Frenet formulae (2.2), we obtain  $T$ ,  $D_1$ , and  $D_2$  as follows

$$T_\gamma(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|} = (D_2)_\eta(t),$$

$$B_\gamma(t) = -\xi \left( \frac{\gamma'(t) \times \gamma''(t)}{\|\gamma'(t) \times \gamma''(t)\|} \right) = -\xi \left( \frac{K_3}{\sqrt{|K_3^2 - K_1^2|}} T(t) - \frac{K_1}{\sqrt{|K_3^2 - K_1^2|}} D_1(t) \right)_\eta,$$

$$N_\gamma(t) = \xi (B_\gamma(t) \times T_\gamma(t)) = \xi \left( \frac{-K_1}{\sqrt{|K_3^2 - K_1^2|}} T(t) + \frac{K_3}{\sqrt{|K_3^2 - K_1^2|}} D_1(t) \right)_\eta.$$

□

**Corollary 4.1.** *If  $\eta(t)$  is a Sp-like curve in which Flc-normal  $D_2(t)$  is a Sp-like vector, if  $\gamma(t)$  is the integrated curve of  $D_2(t)$  of a polynomial Sp-like curve  $\eta(t)$  with  $K_1, K_3 \neq 0$ , then  $\gamma(t)$  is a Sp-like curve in which Frenet-normal  $N_\gamma(t)$  is a Sp-like vector if  $-K_1 < K_3 < K_1$  and, then,  $\xi = 1$  also  $\gamma(t)$  is a Sp-like curve in which Frenet-normal  $N_\gamma(t)$  is a T-like vector if  $K_3 > K_1$  or  $K_3 < -K_1$  and, then,  $\xi = -1$ .*

**Theorem 4.2.** *Let  $\eta(t)$  is a Sp-like curve with a Sp-like Flc-normal  $D_2(t)$  vector, and  $\gamma(t)$  is the integrated curve of  $D_2(t)$  of a polynomial Sp-like curve  $\eta(t)$  with  $K_1, K_3 \neq 0$ . Then, the Frenet curvature and Frenet torsion of  $\gamma(t)$  are, respectively, given by*

$$\kappa_\gamma = \left( u\sqrt{|K_3^2 - K_1^2|} \right)_\eta, \quad \tau_\gamma = \left( \frac{-K_1^2 \frac{d}{dt} (K_3/K_1) + uK_2(K_3^2 - K_1^2)}{|K_3^2 - K_1^2|} \right)_\eta.$$

**Proof.** Let  $\eta(t)$  be a Sp-like curve with a Sp-like Flc-normal  $D_2(t)$  vector, and  $\gamma(t)$  is the integrated curve of  $D_2(t)$  of a polynomial Sp-like curve  $\eta(t)$  with  $K_1, K_3 \neq 0$ . By taking the derivative of Eq. (4.3), we get

$$\gamma'''(t) = (-u'K_1 - uK_1' - u^2K_2K_3) T(t) + u^2 (K_3^2 - K_1^2) D_2(t) + (u'K_3 + uK_3' + u^2K_1K_2) D_1(t).$$

By using Eqs. (2.3), (4.4), and (4.3) the Frenet curvature and Frenet torsion of  $\gamma(t)$  are given by

$$\kappa_\gamma = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3} = \left( u\sqrt{|K_3^2 - K_1^2|} \right)_\eta,$$

$$\tau_\gamma = \frac{\xi (g(\gamma'''(t), \gamma'(t) \times \gamma''(t)))}{\|\gamma'(t) \times \gamma''(t)\|^2} = \xi \left( \frac{-K_1^2 \frac{d}{dt} \left( \frac{\eta K_3}{K_1} \right) + uK_2(K_3^2 - K_1^2)}{|K_3^2 - K_1^2|} \right)_\eta.$$

□

**Theorem 4.3.** *Let  $\eta(t)$  is a Sp-like curve in which Flc-normal  $D_2(t)$  is a T-like vector and  $\gamma(t)$  is the integrated curve of  $D_2(t)$  with  $K_1, K_3 \neq 0$ . Then, the Frenet vector fields are given by*

$$T_\gamma(t) = (D_2)_\eta(t),$$

$$N_\gamma(t) = \left( \frac{K_1}{\sqrt{|K_3^2 + K_1^2|}} T(t) - \frac{K_3}{\sqrt{|K_3^2 + K_1^2|}} D_1(t) \right)_\eta,$$

$$B_\gamma(t) = \left( \frac{K_3}{\sqrt{|K_3^2 + K_1^2|}} T(t) - \frac{K_1}{\sqrt{|K_3^2 + K_1^2|}} D_1(t) \right)_\eta.$$

**Corollary 4.2.** *If  $\eta(t)$  is a polynomial T-like curve with a T-like Flc-normal vector, and if  $\gamma(t)$  is integrated curve of  $D_2(t)$  with  $K_1, K_3 \neq 0$ , then  $\gamma(t)$  is a T-like curve with a Sp-like normal  $N_\gamma(t)$ .*

**Theorem 4.4.** *If  $\eta(t)$  is a polynomial Sp-like curve with a T-like Flc-normal vector. and if  $\gamma(t)$  is integrated curve of  $D_2(t)$  of a polynomial Sp-like curve  $\eta(t)$  with  $K_1, K_3 \neq 0$ , then the curvature and torsion of  $\gamma(t)$  are given, respectively, by*

$$\kappa_\gamma = \left( u\sqrt{|K_3^2 + K_1^2|} \right)_\eta, \quad \tau_\gamma = \left( \frac{K_1^2 \frac{d}{dt} \left( \frac{K_3}{K_1} \right)}{|K_3^2 + K_1^2|} - uK_2 \right)_\eta.$$

**Definition 4.2.** *If  $\eta(t)$  is a polynomial Sp-like curve in which the Flc-binormal  $D_1(t)$  is a Sp-like or a T-like vector. Then, the integrated curve  $\beta(t)$  of  $D_1(t)$  is defined as*

$$\beta(t) = \int D_1(t)dt. \tag{4.5}$$

**Theorem 4.5.** *Let  $\eta(t)$  is a Sp-like curve in which Flc-binormal  $D_1(t)$  is a Sp-like vector and  $\gamma(t)$  is the integrated curve of  $D_1(t)$  with  $K_2, K_3 \neq 0$ . Then, the Frenet vector fields are*

$$\begin{aligned} T_\beta &= (D_1)_\eta(t), \\ N_\beta &= \left( -\frac{K_2}{\sqrt{|K_3^2 - K_2^2|}}T(t) - \frac{K_3}{\sqrt{|K_3^2 - K_2^2|}}D_2(t) \right)_\eta, \\ B_\beta &= \left( -\frac{K_3}{\sqrt{|K_3^2 - K_2^2|}}T(t) - \frac{K_2}{\sqrt{|K_3^2 - K_2^2|}}D_2(t) \right)_\eta. \end{aligned}$$

**Corollary 4.3.** *Let  $\eta(t)$  is a Sp-like curve, in which Flc-binormal  $D_1(t)$  is a Sp-like vector. If  $\beta(t)$  is integrated curve of  $D_1(t)$  of a polynomial Sp-like curve  $\eta(t)$  with  $K_2, K_3 \neq 0$ , then  $\beta(t)$  is a Sp-like curve in which Frenet-normal  $N_\beta(t)$  is a Sp-like vector if  $-K_2 < K_3 < K_2$  and, then,  $\xi = 1$  also  $\beta(t)$  is a Sp-like curve in which Frenet-normal  $N_\beta(t)$  is a T-like vector if  $K_3 > K_2$  or  $K_3 < -K_2$ .*

**Theorem 4.6.** *If  $\eta(t)$  is a Sp-like curve in which Flc-binormal  $D_1(t)$  is a Sp-like vector and  $\beta(t)$  is the integrated curve of  $D_1(t)$  of a polynomial Sp-like curve  $\eta(t)$  with  $K_2, K_3 \neq 0$ , then the Frenet curvature and Frenet torsion of  $\beta(t)$  are respectively given by*

$$\kappa_\beta = \left( u\sqrt{|K_3^2 - K_2^2|} \right)_\eta, \quad \tau_\beta = \left( \frac{-K_2^2 \frac{d}{dt} \left( \frac{K_3}{K_2} \right) + uK_2(K_3^2 - K_2^2)}{|K_3^2 - K_2^2|} \right)_\eta.$$

**Theorem 4.7.** *If  $\eta(t)$  is a Sp-like curve in which Flc-binormal  $D_1(t)$  is a T-like vector and  $\beta(t)$  is the integrated curve of  $D_1(t)$  of a polynomial Sp-like curve  $\eta(t)$  with  $K_2, K_3 \neq 0$ , then the Frenet vector fields are given by*

$$\begin{aligned} T_\beta &= (D_1)_\eta(t), \\ N_\beta &= \left( -\frac{K_2}{\sqrt{|K_3^2 + K_2^2|}}T(t) + \frac{K_3}{\sqrt{|K_3^2 + K_2^2|}}D_2(t) \right)_\eta, \\ B_\beta &= \left( -\frac{K_3}{\sqrt{|K_3^2 + K_2^2|}}T(t) - \frac{K_2}{\sqrt{|K_3^2 + K_2^2|}}D_2(t) \right)_\eta. \end{aligned}$$

**Corollary 4.4.** *If  $\eta(t)$  is a Sp-like curve in which the Flc-binormal  $D_1(t)$  is a T-like vector, and if  $\beta(t)$  is integrated curve of  $D_2(t)$  with  $K_2, K_3 \neq 0$ , then  $\beta(t)$  is a T-like curve with a Sp-like normal  $N_\beta(t)$ .*

**Theorem 4.8.** *If  $\eta(t)$  is a Sp-like curve in which Flc-binormal  $D_1(t)$  is a T-like vector and  $\beta(t)$  is the integrated curve of  $D_1(t)$  of a polynomial Sp-like curve  $\eta(t)$  with  $K_2, K_3 \neq 0$ , then the curvature and torsion are given by*

$$\kappa_\beta = \left( u \sqrt{|K_3^2 + K_2^2|} \right)_\eta, \quad \tau_\beta = \left( \frac{K_2^2 \frac{d}{dt} \left( \frac{K_3}{K_2} \right)}{|K_3^2 + K_2^2|} + u K_2 \right)_\eta.$$

## 5. CONCLUSIONS

This study has investigated the properties of spacelike polynomial curves in Minkowski 3-space using the Frenet-like curve frame (Flc-frame). The Flc-frame provides a powerful alternative to the traditional Frenet frame, overcoming its limitations when dealing with polynomial curves where higher-order derivatives lack clear geometric meaning. By leveraging intrinsic curve properties, we derived the Flc-formulae and established explicit relationships between the Flc-curvatures, offering new insights into the geometric structure of such curves.

A significant contribution of this work is the introduction of novel integral curves defined through the Flc-frame and its associated curvatures. We analyzed these curves in detail, determining their Frenet apparatus and classifying their causal character in Minkowski space. The Flc-frame proves particularly advantageous as it remains well-defined even at points where the Frenet frame fails, such as when curvature vanishes. Moreover, our results demonstrate that the Flc-frame generalizes the Frenet frame, as all classical Frenet frame properties can be recovered by setting  $K_2 = 0$ .

The findings presented here enhance the understanding of polynomial curves in Minkowski space and open new avenues for research in Lorentzian geometry. Potential applications extend to relativistic physics, where the Flc-frame could provide a more robust tool for analyzing particle trajectories and other geometric structures in spacetime. Future work may explore extensions of this framework to timelike and lightlike curves, as well as computational implementations for curve modeling and simulation.

In summary, the Flc-frame offers both theoretical elegance and practical utility, making it a valuable addition to the geometric analysis of curves in Minkowski 3-space. Its adaptability and robustness suggest promising applications in both mathematical research and physics, particularly in scenarios where traditional methods prove insufficient.

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## CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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