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Higher-Order Kinematic Analysis of Particle Motion along Bishop-Framed Curves in Euclidean 3-Space

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Abstract. This paper investigates the higher-order kinematic properties of particle motion along curves equipped with Bishop frames in three-dimensional Euclidean space. We establish comprehensive expressions for jerk and snap vectors, representing the third and fourth derivatives of position with respect to time. Our analysis transcends traditional Frenet-Serret formulations by employing the Bishop frame, which offers enhanced stability and well-defined behavior at points where classical frames may fail. The study presents novel decompositions of these kinematic quantities along Bishop frame components and introduces alternative geometric representations using specialized radial-tangential coordinate systems. These results contribute to differential geometry applications in physics, engineering, and computer graphics where smooth trajectory analysis is essential.

1. Introduction

The geometric analysis of curves in three-dimensional Euclidean space represents a cornerstone of differential geometry with far-reaching applications across physics, engineering, and computer science [6, 20]. Traditional approaches have relied heavily on the Frenet-Serret frame, which, while mathematically elegant, exhibits certain limitations that become problematic in practical applications [2].

The Frenet-Serret frame, consisting of tangent, normal, and binormal vectors, provides a natural orthonormal basis for describing curve properties [18]. However, this framework encounters difficulties when dealing with curves having vanishing curvature or undefined torsion. At such points, the normal vector becomes undefined, leading to discontinuities in the frame orientation [15]. These singularities pose significant challenges in applications requiring smooth, continuous descriptions of curve behavior.

To address these limitations, Bishop [3] introduced an alternative coordinate system, known as the Bishop frame or parallel transport frame. This framework maintains orthonormality while ensuring well-defined behavior across the entire curve domain. The Bishop frame's principal advantage lies in its minimal rotation property, which provides a more intrinsic geometric description and avoids the abrupt orientation changes characteristic of Frenet-Serret frames at singular points [17].

Bishop frame was introduced and studied in Minkowski 3-space in [4, 5, 13, 16].

The study of higher-order kinematic properties, particularly jerk and snap vectors, has gained considerable importance in modern applications [1, 21]. Jerk, defined as the third derivative of position, characterizes the rate of change of acceleration and plays a crucial role in trajectory optimization and control system design [12]. Snap, the fourth derivative, provides additional insights into motion smoothness and is essential for applications requiring precise acceleration profiles [7, 12, 14].

Contemporary research in robotics, computer graphics, and aerospace engineering increasingly demands a sophisticated understanding of these higher-order motion characteristics. Smooth trajectory generation, vibration control, and optimal path planning all benefit from comprehensive jerk and snap analysis. The Bishop frame's stability properties make it particularly suitable for these applications, as it provides consistent geometric references throughout the motion [9].

Recent work by Elsharkawy et al. [10] has explored jerk and snap analysis in non-Euclidean geometries. The comprehensive study by Elsayied et. al [7] on higher-order differential geometry provides additional theoretical foundations for such analyses.

This work establishes a comprehensive framework for analyzing jerk and snap vectors using Bishop frame coordinates. We derive explicit expressions for these kinematic quantities and introduce novel geometric interpretations through specialized radial-tangential decompositions. Our approach extends beyond traditional methodologies by providing alternative representations that maintain geometric significance while offering computational advantages.

The mathematical development proceeds through several key stages. We begin by establishing the fundamental relationships between Bishop and Frenet-Serret frames, including transformation matrices and curvature relationships. Subsequently, we derive detailed expressions for jerk and snap vectors resolved along Bishop frame components. The analysis culminates in the introduction of specialized coordinate systems that provide alternative geometric interpretations of these kinematic quantities.

The significance of this research extends beyond theoretical considerations. The developed framework offers practical tools for trajectory analysis in robotics, computer-aided design, and physical simulation [14]. By providing stable, well-defined expressions for higher-order motion derivatives, this work contributes to the development of more robust algorithms for curve analysis and motion control.

2. Mathematical Preliminaries

We establish our analysis within the context of three-dimensional Euclidean space, denoted $\mathbb{E}^3$, equipped with the standard inner product structure [20]. For arbitrary vectors $\mathbf{P} = (p_1, p_2, p_3)$ and $\mathbf{Q} = (q_1, q_2, q_3)$ in $\mathbb{E}^3$, we define the following fundamental operations:

- Inner product: $\langle \mathbf{P}, \mathbf{Q} \rangle = p_1 q_1 + p_2 q_2 + p_3 q_3$
- Vector norm: $\|\mathbf{Q}\| = \sqrt{\langle \mathbf{Q}, \mathbf{Q} \rangle}$
- Cross product: $\mathbf{P} \times \mathbf{Q} = (p_2 q_3 - p_3 q_2, p_3 q_1 - p_1 q_3, p_1 q_2 - p_2 q_1)$

DEFINITION 1. *A curve $\alpha(s)$ in $\mathbb{E}^3$ is termed regular if its first derivative satisfies $\alpha'(s) \neq 0$ for all parameter values s in the domain [6]. When $\|\alpha'(s)\| = 1$ throughout the domain, the curve is parameterized by arc length.*

For unit-speed curves $\alpha(s)$ with non-vanishing second derivative, the classical Frenet-Serret frame $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is defined as [2, 19]:

- $\mathbf{T}(s) = \alpha'(s)$: unit tangent vector
- $\mathbf{N}(s) = \mathbf{T}'(s)/\|\mathbf{T}'(s)\|$: unit normal vector
- $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$: unit binormal vector

The evolution of this frame along the curve follows the Frenet-Serret equations [18]:

$$\begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{bmatrix}$$

where $\kappa(s) = \|\mathbf{T}'(s)\|$ represents the curvature and $\tau(s) = -\langle \mathbf{B}'(s), \mathbf{N}(s) \rangle$ denotes the torsion.

2.1. Bishop Frame Construction

The Bishop frame provides an alternative orthonormal coordinate system that addresses the limitations of the Frenet-Serret approach [3]. For a regular arc-length parameterized curve $\alpha : I \to \mathbb{E}^3$, the Bishop frame $\{\mathbf{T}(s), \mathbf{M}_1(s), \mathbf{M}_2(s)\}$ is characterized by:

- $\mathbf{T}(s) = \alpha'(s)$: unit tangent vector (identical to Frenet case)
- $\mathbf{M}_1(s), \mathbf{M}_2(s)$: orthonormal vectors spanning the normal plane

The construction of the Bishop frame ensures that the normal plane vectors $\mathbf{M}_1(s)$ and $\mathbf{M}_2(s)$ are obtained by parallel transport along the curve, minimizing their rotation. This property distinguishes the Bishop frame from the Frenet-Serret frame, where the normal and binormal vectors can exhibit rapid rotations.

The Bishop frame evolution equations take the form:

$$\begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{M}_1'(s) \\ \mathbf{M}_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1(s) & \kappa_2(s) \\ -\kappa_1(s) & 0 & 0 \\ -\kappa_2(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{M}_1(s) \\ \mathbf{M}_2(s) \end{bmatrix}$$

The quantities $\kappa_1(s)$ and $\kappa_2(s)$ are the Bishop curvature functions, which relate to classical geometric invariants through [8]:

$$\kappa(s) = \sqrt{\kappa_1^2(s) + \kappa_2^2(s)}$$

$$\tau(s) = \frac{\kappa_1(s)\kappa_2'(s) - \kappa_2(s)\kappa_1'(s)}{\kappa_1^2(s) + \kappa_2^2(s)}$$

provided that $\kappa_1^2(s) + \kappa_2^2(s) \neq 0$ [8, 11].

The transformation between Bishop and Frenet-Serret frames involves a rotation in the normal plane [11]:

$$\begin{bmatrix} \mathbf{T}(s) \\ \mathbf{M}_1(s) \\ \mathbf{M}_2(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(s) & -\sin\theta(s) \\ 0 & \sin\theta(s) & \cos\theta(s) \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{bmatrix}$$

where the rotation angle $\theta(s)$ satisfies $\theta'(s) = \tau(s)$, establishing a direct connection between the geometric frameworks. Recent theoretical developments have extended the Bishop frame concept to higher dimensions and non-Euclidean geometries [10].

The Bishop frame's stability properties are particularly valuable when dealing with curves that exhibit rapid changes in curvature or torsion. Unlike the Frenet-Serret frame, which can become undefined at inflection points, the Bishop frame maintains well-defined orientation throughout the curve domain. This characteristic makes it especially suitable for applications in computer graphics, robotics, and engineering, where smooth, continuous motion is required [14].

3. Main Results

We present our principal findings concerning the kinematic analysis of particle motion along Bishop-framed curves. The development focuses on explicit expressions for jerk and snap vectors, providing both Bishop frame decompositions and alternative geometric representations.

DEFINITION 2. *For a particle undergoing motion along a curve, we define the following kinematic quantities:*

- *Velocity vector:* $\mathbf{v} = \frac{d\mathbf{r}}{dt}$
- *Acceleration vector:* $\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}$
- *Jerk vector:* $\mathbf{j} = \frac{d^3\mathbf{r}}{dt^3}$
- *Snap vector:* $\mathbf{s} = \frac{d^4\mathbf{r}}{dt^4}$

where $\mathbf{r}(t)$ *represents the position vector as a function of time.*

THEOREM 1 (Jerk Vector Decomposition in Bishop Frame). *Consider a particle with constant mass moving along an arc-length parameterized Bishop curve* $\alpha(s)$ *in* $\mathbb{E}^3$*. Assuming the arc-length parameter coincides with time, the jerk vector* $\mathbf{J}$ *admits the decomposition:*

$$\mathbf{J} = J_T\mathbf{T} + J_{M_1}\mathbf{M}_1 + J_{M_2}\mathbf{M}_2$$

where the components are given by:

$$\begin{aligned} J_T &= \frac{d^3s}{dt^3} - (\kappa_1^2(s) + \kappa_2^2(s))\left(\frac{ds}{dt}\right)^3 \\ J_{M_1} &= \left[3\frac{ds}{dt}\frac{d^2s}{dt^2}\sqrt{\kappa_1^2(s)+\kappa_2^2(s)} + \left(\frac{ds}{dt}\right)^3\frac{d}{ds}\sqrt{\kappa_1^2(s)+\kappa_2^2(s)}\right]\cos\theta(s) \\ &\quad - \left[\left(\frac{ds}{dt}\right)^3\sqrt{\kappa_1^2(s)+\kappa_2^2(s)}\theta'\right]\sin\theta(s) \\ J_{M_2} &= \left[3\frac{ds}{dt}\frac{d^2s}{dt^2}\sqrt{\kappa_1^2(s)+\kappa_2^2(s)} + \left(\frac{ds}{dt}\right)^3\frac{d}{ds}\sqrt{\kappa_1^2(s)+\kappa_2^2(s)}\right]\sin\theta(s) \\ &\quad + \left[\left(\frac{ds}{dt}\right)^3\sqrt{\kappa_1^2(s)+\kappa_2^2(s)}\theta'\right]\cos\theta(s) \end{aligned}$$

Proof. Let $\mathscr{P}$ denote a point particle moving along the arc-length parameterized Bishop curve $\alpha(s)$ in $\mathbb{E}^3$. The position vector $\mathbf{X}$ of $\mathscr{P}$ at time t relative to a fixed origin satisfies:

$$\mathbf{T}(s) = \frac{d\mathbf{X}}{ds}$$

From the Bishop frame evolution equations and the chain rule, we obtain:

$$\mathbf{v} = \frac{d\mathbf{X}}{dt} = \left(\frac{ds}{dt}\right)\mathbf{T}$$

$$\mathbf{a} = \left[\frac{d^2s}{dt^2}\right]\mathbf{T} + \left[\sqrt{\kappa_1^2(s)+\kappa_2^2(s)}\left(\frac{ds}{dt}\right)^2\cos\theta(s)\right]\mathbf{M}_1 + \left[\sqrt{\kappa_1^2(s)+\kappa_2^2(s)}\left(\frac{ds}{dt}\right)^2\sin\theta(s)\right]\mathbf{M}_2$$

Differentiating the acceleration vector and applying the Bishop frame evolution equations yields:

$$\mathbf{J} = D_T\mathbf{T} + D_{M_1}\mathbf{M}_1 + D_{M_2}\mathbf{M}_2$$

where:

$$\begin{aligned}
D_T &= \frac{d^3s}{dt^3} - (\kappa_1^2(s) + \kappa_2^2(s))\left(\frac{ds}{dt}\right)^3 \\
D_{M_1} &= \left[3\frac{ds}{dt}\frac{d^2s}{dt^2}\sqrt{\kappa_1^2(s)+\kappa_2^2(s)} + \left(\frac{ds}{dt}\right)^3 \frac{d}{ds}\sqrt{\kappa_1^2(s)+\kappa_2^2(s)}\right]\cos\theta(s) \\
&\quad - \left[\left(\frac{ds}{dt}\right)^3 \sqrt{\kappa_1^2(s)+\kappa_2^2(s)}\theta'\right]\sin\theta(s) \\
D_{M_2} &= \left[3\frac{ds}{dt}\frac{d^2s}{dt^2}\sqrt{\kappa_1^2(s)+\kappa_2^2(s)} + \left(\frac{ds}{dt}\right)^3 \frac{d}{ds}\sqrt{\kappa_1^2(s)+\kappa_2^2(s)}\right]\sin\theta(s) \\
&\quad + \left[\left(\frac{ds}{dt}\right)^3 \sqrt{\kappa_1^2(s)+\kappa_2^2(s)}\theta'\right]\cos\theta(s)
\end{aligned}$$

This completes the proof. □

THEOREM 2 (Snap Vector Decomposition in Bishop Frame). *Under the conditions of Theorem 1, the snap vector* **S** *of the particle admits the decomposition:*

$$\mathbf{S} = S_T\mathbf{T} + S_{M_1}\mathbf{M}_1 + S_{M_2}\mathbf{M}_2$$

where the components involve fourth-order derivatives and higher-order geometric terms.

Proof. The proof follows by successive differentiation of the jerk vector components, applying the Bishop frame evolution equations at each step. Although the detailed expressions are mathematically significant, they are omitted for brevity. □

THEOREM 3 (Jerk Vector in Special Radial-Tangential Frame). *Under the same conditions as Theorem 1, and assuming the components of the angular momentum vector $\mathscr{H}^{\mathcal{O}}$ never vanish, the jerk vector of the particle $\mathscr{P}$ can be expressed as:*

$$\mathbf{J} = J_T\mathbf{T} + J_r\mathbf{e_r} + J_{r^*}\mathbf{e_{r^*}}$$

where

$$\begin{aligned}
J_T &= \frac{d^3s}{dt^3} - (k_1^2 + k_2^2)\left(\frac{ds}{dt}\right)^3 + 3\lambda\frac{ds}{dt}\frac{d^2s}{dt^2}\left(\frac{k_1}{u} - \frac{k_2}{\mu}\right) + \lambda\left(\frac{ds}{dt}\right)^3\left(\frac{k_1'}{u} - \frac{k_2'}{\mu}\right) \\
J_r &= -\frac{3k_1}{ur}\frac{ds}{dt}\frac{d^2s}{dt^2} - \frac{k_1'}{ur}\left(\frac{ds}{dt}\right)^3 \\
J_{r^*} &= \frac{3k_2}{\mu r^*}\frac{ds}{dt}\frac{d^2s}{dt^2} + \frac{k_2'}{\mu r^*}\left(\frac{ds}{dt}\right)^3
\end{aligned}$$

Here, J_T, J_r, and J_{r^*} are the tangential and special radial components of the jerk vector. The special radial components J_r and J_{r^*} lie along lines passing through the

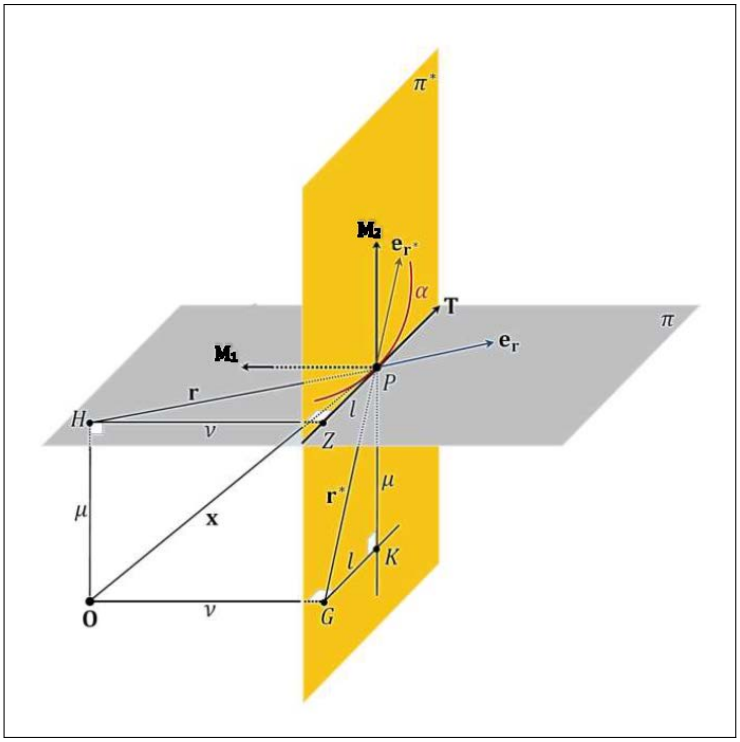

Figure 1: The motion of $\mathscr{P}$ along γ in $\mathbb{E}^3$.

particle P and points H and G, respectively. The tangential component J_T lies along the tangent line T of the curve α at P.

Proof. Let a point particle $\mathscr{P}$ move along an arc-length parameterized Bishop curve $\alpha(s)$ in the space $\mathbb{E}^3$. Then, the point particle has a position vector in terms of the Bishop frame. Assume that the position vector $\mathbf{r}$ of $\mathscr{P}$ is resolved as:

$$\mathbf{r} = \lambda\mathbf{T} - u\mathbf{M}_1 + \mu\mathbf{M}_2, \tag{1}$$

where

$$\lambda = \langle \mathbf{r}, \mathbf{T} \rangle, \quad -u = \langle \mathbf{r}, \mathbf{M}_1 \rangle, \quad \mu = \langle \mathbf{r}, \mathbf{M}_2 \rangle.$$

and

$$M_1 = \frac{1}{u}\left(\lambda T - r e_r\right), \quad M_2 = \frac{1}{\mu}\left(r^* e_{r^*} - \lambda T\right). \tag{2}$$

We note that the vectors $\mathbf{T}, \mathbf{M}_1$ and $\mathbf{M}_2$ are orthogonal. Let us define the vector $\mathbf{r}$ and $\mathbf{r}^*$ as

$$\mathbf{r} = r e_r = \lambda T - u M_1, \quad \mathbf{r}^* = r^* e_{r^*} = \lambda T + \mu M_2 \tag{3}$$

that lie in the planes π and π^* to α at $\mathscr{P}$, respectively. Then, we have

$$\mathbf{r}^2 = \lambda^2 + u^2, \quad \mathbf{r}^{*2} = \lambda^2 + \mu^2. \tag{4}$$

Where r and r^* are the cartesian norms of $\mathbf{r}$ and $\mathbf{r}^*$, respectively. The jerk and snap vectors can be written as

$$\begin{aligned}\mathbf{J} = \mathbf{T}\left[\frac{d^3 s}{dt^3} - \left(k_1^2 + k_2^2\right)\left(\frac{ds}{dt}\right)^3\right] + \frac{1}{u}(\lambda\mathbf{T} - r\mathbf{e}_r)\left[3k_1\frac{ds}{dt}\frac{d^2 s}{dt^2} + k_1'\left(\frac{ds}{dt}\right)^3\right] \\ + \frac{1}{\mu}\left(r^*\mathbf{e}_{r^*} - \lambda\mathbf{T}\right)\left[3k_2\frac{ds}{dt}\frac{d^2 s}{dt^2} + k_2'\left(\frac{ds}{dt}\right)^3\right]\end{aligned} \tag{5}$$

and

$$\begin{aligned}\mathbf{S} = \mathbf{T}\Bigg[\frac{d^4 s}{dt^4} - 6\left(k_1^2 + k_2^2\right)\left(\frac{ds}{dt}\right)^2\frac{d^2 s}{dt^2} - 3\left(k_1k_1' + k_2k_2'\right)\left(\frac{ds}{dt}\right)^4\Bigg] \\ + \frac{1}{u}(\lambda\mathbf{T} - r\mathbf{e}_r)\Bigg[4k_1\frac{ds}{dt}\frac{d^3 s}{dt^3} - k_1\left(k_1^2 + k_2^2\right)\left(\frac{ds}{dt}\right)^4 \\ + 6k_1'\left(\frac{ds}{dt}\right)^2\frac{d^2 s}{dt^2} + 3k_1\left(\frac{d^2 s}{dt^2}\right)^2 + k_1''\left(\frac{ds}{dt}\right)^4\Bigg] \\ + \frac{1}{\mu}\left(r^*\mathbf{e}_{r^*} - \lambda\mathbf{T}\right)\Bigg[4k_2\frac{ds}{dt}\frac{d^3 s}{dt^3} - k_2\left(k_1^2 + k_2^2\right)\left(\frac{ds}{dt}\right)^4 \\ + 6k_2'\left(\frac{ds}{dt}\right)^2\frac{d^2 s}{dt^2} + 3k_2\left(\frac{d^2 s}{dt^2}\right)^2 + k_2''\left(\frac{ds}{dt}\right)^4\Bigg].\end{aligned} \tag{6}$$

respectively. It is well known that the vector $\mathscr{H}^{\mathscr{O}}$ is given by

$$\mathscr{H}^{\mathscr{O}} = \mathscr{X} \times m\mathscr{V}$$

thus, from (1), we get

$$\mathscr{H}^{\mathscr{O}} = -m\mu\left(\frac{ds}{dt}\right)\mathbf{M}_1 + mu\left(\frac{ds}{dt}\right)\mathbf{M}_2 \tag{7}$$

Our goal is to resolve the jerk and snap vectors in (5) and (6) along the vectors $\mathbf{T}$, $\mathbf{e}_r$ and $\mathbf{e}_{r^*}$. To do that, let us write the vectors $\mathbf{M}_1$ and $\mathbf{M}_2$ in terms of $\{r, \mathbf{T}\}$ and $\left\{r^*, \mathbf{T}\right\}$, respectively. By means of (3), we can do this if and only if $\mu \neq 0$ and $u \neq 0$. Through an assumption "the components of the vector $\mathscr{H}^{\mathscr{O}}$ in (7) never vanish", we can guarantee that $\mu \neq 0$ and $u \neq 0$. Thus, we find from (2) that

$$\mathbf{M}_1 = \frac{1}{u}(\lambda\mathbf{T} - \mathbf{r}), \quad \mathbf{M}_2 = \frac{1}{\mu}\left(\mathbf{r}^* - \lambda\mathbf{T}\right) \tag{8}$$

We can also find from (4) that $r \neq 0$ and $r^* \neq 0$. So, we can define the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_{r^*}$ as

$$\mathbf{e}_r = \frac{1}{r}\mathbf{r}, \quad \mathbf{e}_{r^*} = \frac{1}{r^*}\mathbf{r}^*.$$

Thus, (8) becomes

$$\mathbf{M}_1 = \frac{1}{u}(\lambda\mathbf{T} - r\mathbf{e}_r), \quad \mathbf{M}_2 = \frac{1}{\mu}\left(r^*\mathbf{e}_{r^*} - \lambda\mathbf{T}\right) \tag{9}$$

By substituting (9) into (5) and (6), the jerk and snap vectors of the point particle $\mathscr{P}$ are expressed in terms of $\mathbf{T}$, $\mathbf{e}_r$, and $\mathbf{e}_{r*}$ respectively. The components J_T, J_r, J_{r*} follow directly after algebraic simplification. This completes the proof. □

THEOREM 4 (Snap Vector in Special Radial-Tangential Frame). *Under the same conditions as Theorem 3, the snap vector of the particle $\mathscr{P}$ can be expressed as:*

$$\mathbf{S} = S_T\mathbf{T} + S_r\mathbf{e_r} + S_{r^*}\mathbf{e_{r^*}},$$

where S_T, S_r, and S_{r^*} are the tangential and special radial components of the snap vector, respectively. Their expressions are omitted here for brevity but can be derived by differentiating the jerk vector components given in Theorem 3.

Proof. The proof involves differentiating the jerk vector components expressed in the special radial-tangential frame. Details are omitted for brevity. □

These theorems provide a comprehensive description of the higher-order kinematic properties of a particle moving along a Bishop-framed curve in Euclidean 3-space. They offer alternative representations of the jerk and snap vectors, which can be useful in various applications in differential geometry, physics, and engineering.

4. Illustrative Examples

EXAMPLE 1 (Circular Helix). Consider a circular helix parameterized by:

$$\mathbf{r}(t) = (a\cos t, a\sin t, bt)$$

where $a > 0$ is the radius and $b > 0$ determines the pitch.

The kinematic quantities are:

$$\begin{aligned}
\mathbf{v}(t) &= (-a\sin t,\ a\cos t,\ b),\\
\mathbf{a}(t) &= (-a\cos t,\ -a\sin t,\ 0),\\
\mathbf{j}(t) &= (a\sin t,\ -a\cos t,\ 0),\\
\mathbf{s}(t) &= (a\cos t,\ a\sin t,\ 0).
\end{aligned}$$

The Frenet-Serret frame components are:

$$\begin{aligned}
\mathbf{T}(t) &= \frac{1}{\sqrt{a^2+b^2}}(-a\sin t,\ a\cos t,\ b),\\
\mathbf{N}(t) &= (-\cos t,\ -\sin t,\ 0),\\
\mathbf{B}(t) &= \frac{1}{\sqrt{a^2+b^2}}(b\sin t,\ -b\cos t,\ a).
\end{aligned}$$

The curvature and torsion are constant:

$$\kappa = \frac{a}{a^2+b^2}, \quad \tau = \frac{b}{a^2+b^2}$$

For the Bishop frame, we can choose:

$$\begin{aligned} \mathbf{M}_1(t) &= \mathbf{N}(t) = (-\cos t,\ -\sin t,\ 0), \\ \mathbf{M}_2(t) &= \mathbf{B}(t) = \frac{1}{\sqrt{a^2+b^2}}(b\sin t,\ -b\cos t,\ a). \end{aligned}$$

with Bishop curvatures:

$$\kappa_1 = \frac{a}{a^2+b^2}, \quad \kappa_2 = 0$$

EXAMPLE 2 (Trefoil Knot). The trefoil knot can be parameterized as:

$$\mathbf{r}(t) = ((2+\cos(3t))\cos(2t), (2+\cos(3t))\sin(2t), \sin(3t))$$

This provides a more complex example where the Bishop frame offers computational advantages due to its stability properties.

EXAMPLE 3 (Logarithmic Spiral). Consider the logarithmic spiral in 3D space parameterized by:

$$\mathbf{r}(t) = (ae^{bt}\cos t, ae^{bt}\sin t, ce^{bt})$$

where $a > 0$, $b > 0$, and $c \geq 0$ are constants.

Kinematic Quantities

First, we compute the velocity vector:

$$\begin{aligned} \mathbf{v}(t) &= \frac{d\mathbf{r}}{dt} \\ &= \left(ae^{bt}(b\cos t - \sin t),\ ae^{bt}(b\sin t + \cos t),\ cbe^{bt}\right). \end{aligned}$$

The speed is:

$$\|\mathbf{v}(t)\| = e^{bt}\sqrt{a^2(b^2+1)+c^2b^2}$$

The acceleration vector:

$$\begin{aligned} \mathbf{a}(t) &= \frac{d\mathbf{v}}{dt} \\ &= ae^{bt}\left[(b^2-1)\cos t - 2b\sin t,\ (b^2-1)\sin t + 2b\cos t,\ cb^2e^{bt}\right]. \end{aligned}$$

The jerk vector:

$$\begin{aligned}\mathbf{j}(t) &= \frac{d\mathbf{a}}{dt}\\ &= ae^{bt}\Big[(b^3-3b)\cos t-(3b^2-1)\sin t,\\ &\qquad (b^3-3b)\sin t+(3b^2-1)\cos t,\ cb^3e^{bt}\Big].\end{aligned}$$

The snap vector:

$$\begin{aligned}\mathbf{s}(t) &= \frac{d\mathbf{j}}{dt}\\ &= ae^{bt}\Big[(b^4-6b^2+1)\cos t-(4b^3-4b)\sin t,\\ &\qquad (b^4-6b^2+1)\sin t+(4b^3-4b)\cos t,\ cb^4e^{bt}\Big].\end{aligned}$$

Frenet-Serret Frame

The unit tangent vector:

$$\mathbf{T}(t)=\frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}=\frac{1}{\sqrt{a^2(b^2+1)+c^2b^2}}[a(b\cos t-\sin t),a(b\sin t+\cos t),cb]$$

For the normal vector, we compute:

$$\mathbf{T}'(t)=\frac{ae^{bt}}{\sqrt{a^2(b^2+1)+c^2b^2}}[-(b^2+1)\sin t-b\cos t,(b^2+1)\cos t-b\sin t,0]$$

The curvature:

$$\kappa=\frac{\|\mathbf{T}'(t)\|}{\|\mathbf{v}(t)\|}=\frac{a\sqrt{(b^2+1)^2+b^2}}{[a^2(b^2+1)+c^2b^2]^{3/2}}e^{bt}$$

Simplifying:

$$\kappa=\frac{a(b^2+1)}{[a^2(b^2+1)+c^2b^2]^{3/2}}e^{bt}$$

Bishop Frame Analysis

For the Bishop frame, we can choose:

$$\begin{aligned}\mathbf{M}_1(t) &= \frac{1}{\sqrt{(b^2+1)^2+b^2}}\Big[-(b^2+1)\sin t-b\cos t,\ (b^2+1)\cos t-b\sin t,\ 0\Big],\\ \mathbf{M}_2(t) &= \mathbf{T}(t)\times\mathbf{M}_1(t).\end{aligned}$$

The Bishop curvatures are:

$$\begin{aligned} \kappa_1 &= \frac{a(b^2+1)}{\left[a^2(b^2+1)+c^2b^2\right]^{3/2}}e^{bt}, \\ \kappa_2 &= 0. \end{aligned}$$

EXAMPLE 4 (Viviani's Curve). Viviani's curve is the intersection of a sphere and a cylinder, parameterized by:

$$\mathbf{r}(t) = (a(1+\cos t), a\sin t, 2a\sin(t/2))$$

where $a > 0$ is the radius of the sphere.

Kinematic Analysis

The velocity vector:

$$\mathbf{v}(t) = (-a\sin t, a\cos t, a\cos(t/2))$$

The speed:

$$\|\mathbf{v}(t)\| = a\sqrt{\sin^2 t + \cos^2 t + \cos^2(t/2)} = a\sqrt{1+\cos^2(t/2)}$$

The acceleration vector:

$$\mathbf{a}(t) = (-a\cos t, -a\sin t, -\frac{a}{2}\sin(t/2))$$

The jerk vector:

$$\mathbf{j}(t) = (a\sin t, -a\cos t, -\frac{a}{4}\cos(t/2))$$

The snap vector:

$$\mathbf{s}(t) = (a\cos t, a\sin t, \frac{a}{8}\sin(t/2))$$

Geometric Properties

The unit tangent vector:

$$\mathbf{T}(t) = \frac{1}{\sqrt{1+\cos^2(t/2)}}[-\sin t, \cos t, \cos(t/2)]$$

For the curvature calculation:

$$\mathbf{T}'(t) = \frac{1}{\sqrt{1+\cos^2(t/2)}}[-\cos t, -\sin t, -\frac{1}{2}\sin(t/2)] + \text{correction terms}$$

The curvature:

$$\kappa(t) = \frac{\sqrt{1+\frac{1}{4}\sin^2(t/2)}}{(1+\cos^2(t/2))^{3/2}}$$

5. Conclusion

This work has established a comprehensive framework for analyzing higher-order kinematic properties of particle motion along Bishop-framed curves in Euclidean 3-space. The derived expressions for jerk and snap vectors provide both theoretical insights and practical computational tools for applications in differential geometry, physics, and engineering.

The Bishop frame approach offers significant advantages over traditional Frenet-Serret methods, particularly in handling curves with singular points or complex geometric behavior. The alternative radial-tangential decompositions introduce novel geometric perspectives that enhance our understanding of particle motion dynamics.

Future research directions include extensions to higher-dimensional spaces, applications to specific physical systems, and the development of computational algorithms based on these theoretical foundations. The framework presented here provides a solid foundation for such investigations.

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