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Some novel results on Boubaker polynomials leading to an efficient orthogonalization

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Abstract

The orthogonal polynomials have been recognized in literature due to their compatibility in providing robust and accurate solutions of different nonlinear, singular, and complex problems in science and engineering. The Boubaker polynomial (BPs) and its variants to date have been at galore in the recent literature in this context. Although exhaustive applications of the BPs and its variants have been performed, systematic orthogonalization of the polynomials is still a persistent problem. In one way or another, the existing variants compromise in being an efficient orthogonalization due to inclusion of imaginary zeros, rounding off errors of coefficients, the choice of appropriate weight function, and symmetry with the conventional BPs. In this study, we derive a class of weight-functions in a Hilbert space which accommodates an efficient orthogonalization of the BPs for the first time. We also prove theorems on the consequent recurrence, orthogonality and orthonormality relations for the proposed orthogonal BPs (POBPs). The characteristic differential equation and its spectral form have also been derived. The results of this study are a basis for the applicability of POBPs where existing attempts suffered due to lack of efficient orthogonalization.

1 Introduction

Numerical integration and integral equations entail most of the significant mathematical models in the realm of science and engineering. The theory of orthogonalization figures prominently well in terms of identification of special nodal information in the sampling points or in the integrators [1]. When integral equations are embedded with different families of orthogonal polynomials, it works really wonder in the sense of better approximation and elegancy. When a mathematical model contains singularities, solutions can be found efficiently by using orthogonal polynomials [2].

Orthogonal polynomials are mere concise, approximate, fast, and elegant. It has widely gained ground in comparison to the other traditional polynomials because of its accuracy, convergence, and its computational efficiency. One of the interesting properties of such polynomials is that orthogonal basis can be extended to infinite class of differential equations. Not only that, but operational matrices of derivatives are also viable and covering almost all dimensions. Reformation of operational matrices of derivatives and integration is being taken into wider consideration. No doubt, it has yielded interesting results which are making strides in the field of science and engineering. Some classical orthogonal polynomials: Legendre [3], Chebyshev [4] are used in the field of physics, ordinary differential equations and approximation theory [5].

Applications of sequences of polynomials are a galore and considerably prominent by its presence in engineering and natural science. Polynomials are further taken into active consideration and ultimately became instrumental in exploring a new manifestation in the shape of Boubaker polynomials [6]. Dr. Karem Mahmoud Boubaker (1961-2016) is a pioneer of this exploration and named behind his name in 2007, a new dimension of polynomial sequence was highlighted and set the world on the fire and gained international recognition in terms of authenticity and reliability to solve many problems in applied science and engineering. The BPs were considered as ahead of time in comparison to conventional polynomials, and therefore, the polynomial sequences were refined in terms of Boubaker polynomials which culminated from a mathematical model for the

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feasible solution of heat transfer problem.

The efficient orthogonality of BPs is a persistent problem and conspicuous by its absence in the body of existing literature. So, it has been taxing the researchers to define a Hermitian operator in an inner product space with suitable weight function. In this perspective, it is highly appealing and interesting to underline the problem of orthogonality of these polynomials which shed light on consequent applications regarding efficient Gauss quadrature of Boubaker polynomials. In short, the orthogonalization itself is remarkable and a novel contribution. There have been committed various endeavors, but efficient orthogonalization of Boubaker polynomials remains a desirable one. If practical and viable solution is carried out by exploring strong orthogonalization, it not only paves way for seminal development for the researchers, but also it would be a great stride in the domain of exact and analytic solution.

In view of the extensive literature on Boubaker polynomials published in well-known research journals over the last 17 years, it is a well-known fact that Boubaker polynomials were extensively used in the fields of engineering and sciences. So, it has got international recognition in terms of its application and contributions in the real-world phenomena. Hence, soon after the proposal of the conventional Boubaker polynomials CBPs [6], which were not even orthogonal, there emerged numerous case study problems for years which were solved using the BPs, as reviewed in [7]. But, since the CBPs were not orthogonal, some problems demanding orthogonality relation could not be addressed using the CBPs. Hence, the orthogonality has long been desirable one for the BPs. The researchers were trying continuously to prove orthogonality properties for the BPs. In this zeal, there were several attempts and refinements of the CBPs, for example: [8–15]. In one way or other, one can point out that these attempts lacked symmetry with the CBPs and the classical orthogonal polynomial families like Legendre, Leguerre, Chebyshev, etc.

The refined polynomials produced by researchers were blighted by three main problems, which in turn lead to the term efficient orthogonality in case these are adequately addressed. The efficient orthogonality can be considered in terms of three important factors which are instrumental in bringing about any polynomials as far as case of efficient orthogonalization is concerned. Firstly, the coefficients should be integers and user friendly, as it was observed from classical properties of Legendre, Chebyshev, Hermite polynomials, etc. Secondly, all the zeros of polynomials in the interval of definition must be real; rather than complex numbers. Finally, the polynomials should not be deviated from original source of CBPs, i.e. a linear combination of the the Chebyshev polynomials [15]. Unfortunately, most of the attempts at the orthogonalization by the various researchers and writers culminated into failures because these orthogonality-drives were marred by complex number zeros, or non-integer and higher/lower magnitude coefficients, deviation from the CBPs.

In this work, we critically analyze the orthogonality-drive on the BPs with reference to several attempts made so far to redefine the BPs as modified/Boubaker-Turki polynomials [8, 9], 4q-Boubaker polynomials [10], Noah's orthogonal Boubaker polynomials [11], modified orthogonal BPs [12, 13], and orthogonal Boubaker-Turki polynomials [14]. The limitations of these attempts are highlighted, and a way forward towards an efficient orthogonalization is suggested in the context of the Fermat-linked relations of BPs [15]. We also derive a class of weight functions for the suggested orthogonalization and prove their orthogonality and orthonormality properties with respect to the defined weight function in a suitable Hilbert space.

2 Materials and Methods

In this section, we briefly mention the problem of orthogonalization for CBPs and a necessary background on the different attempts that have been made so far in this zeal to provide a final way forward in the forthcoming discussion.

2.1 Conventional Boubaker Polynomials (CBPs)

Conventional Boubaker polynomials [6] have been used extensively in solving mathematical problems in the fields of biology, dynamical systems, nonlinear systems, approximation theory, thermodynamics, mechanics, hydrology, molecular dynamics, complex analysis, matrix analysis, algebra, etc. as reviewed in [7]. The characteristic differential equation of CBPs is:

$$\left(x^2-1\right)\left(3nx^2+n-2\right)\frac{d^2y}{dx^2}+3n\left(nx^2+3n-2\right)\frac{dy}{dx}-n\left(3n^2x^2+n^2-6n+8\right)y=0 \tag{1}$$

The compact form is:

$$B_n(x)=\sum_{p=0}^{\lceil \frac{n}{2}\rceil}\frac{n-4p}{n-p}\binom{n-p}{p}(-1)^p x^{(n-2p)} \tag{2}$$

Where $\lceil . \rceil$ denotes the ceiling integer approximation.

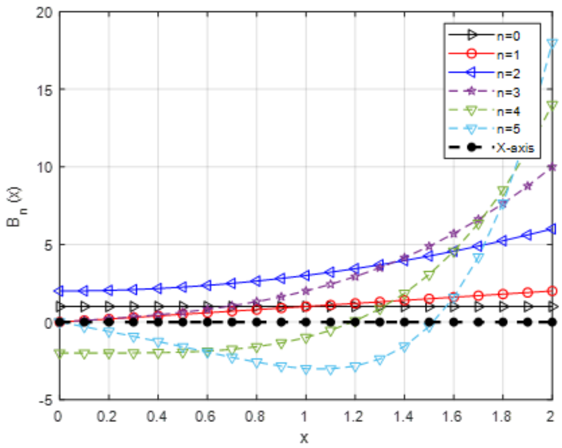

Figure 1: First six CBPs with real zeros intersecting X-axis

The three-term recurrence relation is:

$$B_m(x) = xB_{m-1}(x) - B_{m-2}(x), \quad m > 2 \tag{3}$$

With three initial conditions: $B_0(x) = 1, B_1(x) = x$ and $B_2(x) = x^2 + 2$, one can attain forthcoming polynomials:
$B_3(x) = x^3 + x,$
$B_4(x) = x^4 - 2,$
$B_5(x) = x^5 - x^3 - 3x,$
$B_6(x) = x^6 - 2x^4 - 3x^2 + 2,$
$B_7(x) = x^7 - 3x^5 - x^3 + 5x$, etc.
These can also be generated through the generating function:

$$G(x,t) = \frac{1+3t^2}{1+t(t-x)} \tag{4}$$

Fig. 1 shows first six CBPs in the interval of definition $[0,2]$ and their real zeros as intersecting points with X-axis. The CBPs are not orthogonal. As apparent from Fig. 1, for a particular n, not all zeros of $B_n(x)$ are real; since the number of intersecting points with X-axis is not always n. Therefore, some zeros of these polynomials are also riddled by complex numbers, which is a main hindrance for the orthogonalization and proving and establishing the other classical properties. Besides, unlike other families of orthogonal polynomials, the CBPs requires three initial conditions in the recurrence relation (3) and thus the relation (3) does not produce all polynomials for $m \geq 2$.

2.2 Existing variants of the CBPs and their critical analysis

Here, we review existing variants of CBPs with critical comments on their orthogonality properties focusing the efficiency, if orthogonal, and nearness to the CBPs. The modified Boubaker / Boubaker-Turki polynomials (MBPs) were introduced in [8, 9]. The characteristic differential equation of second order for MBPs is:

$$16\left(1-x^2\right)MBP_n''(x) - 4xMBP_n'(x) + n^2MBP_n(x) = 32(n-1)T_{n-2}(x), n > 2 \tag{5}$$

where $T_n(x)$ are the Chebyshev polynomials of first kind. Also, the relation (6) follows for the MBPs and $T_n(x)$:

$$MBP_n(x) = 2T_n\left(\frac{x}{2}\right) + 8T_{n-2}\left(\frac{x}{2}\right); \quad \text{for } n > 2 \tag{6}$$

Generating function for MBPs is:

$$\sum_{n=0}^{\infty} MBP_n(x)t^n = \frac{1+3t^2}{(1-2xt+t^2)} \tag{7}$$

We can list first few Boubaker-Turki polynomials as:
$MBP_0(x) = 1,$

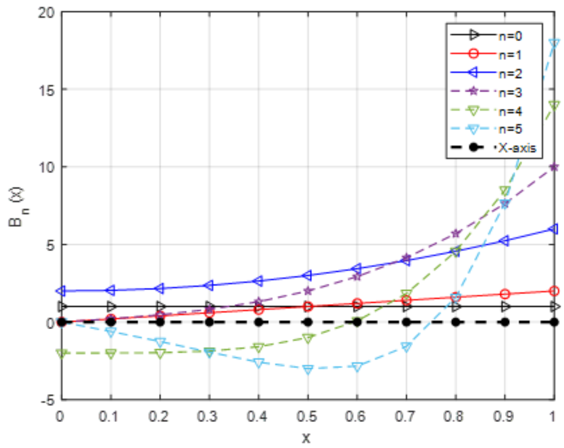

Figure 2: First six MBPs with real zeros intersecting X-axis

$MBP_1(x) = 2x,$
$MBP_2(x) = 4x^2 + 2,$
$MBP_3(x) = 8x^3 + 2x,$
$MBP_4(x) = 16x^4 - 2,$
$MBP_5(x) = 32x^5 - 8x^3 - 6x,$
$MBP_6(x) = 64x^6 - 32x^4 - 12x^2 + 2,$
$MBP_7(x) = 128x^7 - 96x^5 - 16x^3 + 10x$

We can observe clearly that the MBPs, defined in $[0, 1]$, are merely scaled versions of the CBPs in $[0, 2]$, and they are not orthogonal. Another aspect concerns with some of the zeros of these polynomials turning into complex numbers (see Fig. 2), which is again a main impediment to the orthogonalization and proving the other classical properties. Most of the relations are concerned with the Chebyshev polynomials.

The 4q-Boubaker polynomials (4q-BPs) were studied in [10]. The compact form is:

$$B_{4q}(x) = 4\sum_{p=0}^{2q} \frac{(q-p)}{(4q-p)} \binom{4q-p}{p} (-1)^p x^{(2q-p)} \tag{8}$$

The ordinary generating function is:

$$f(x,t) = \frac{1+3t^2}{1+t(t-x)} \tag{9}$$

We can list first few 4q-BPs, which are:
$B_0(x) = 1,$
$B_4(x) = x^4 - 2,$
$B_8(x) = x^8 - 4x^6 + 8x^2 - 2,$
$B_{12}(x) = x^{12} - 8X^{10} + 18x^8 - 35x^4 + 24x^2 - 2,$
$B_{16}(x) = x^{16} - 12x^{14} + 52x^{12} - 88x^{10} + 168x^6 - 168x^4 + 48x^2 - 2$

Zhao et al. [10] considered only those CBPs whose indices were multiples of 4, and these contain effective terms used in the underlying application of counting number of complex zeros of real-coefficient polynomials inside open unit disk [7]. But these carry same limitations as of the classical polynomials in $[0, 2]$. These are highlighted in Fig. 3.

Noah's orthogonal Boubaker polynomials (NOBPs) were studied in [11]. The Gram-Schmidt process (GSP) was applied on the CBPs to find orthogonal Boubaker polynomials denoted by $N_k(t)$. The compact form for these polynomials was not reported in [11]. The interval of orthogonality was $[0, 1]$.

The first few NOBPs are:
$N_0(t) = 1,$
$N_1(t) = \frac{1}{2}(2t - 1),$
$N_2(t) = \frac{1}{6}(6t^2 - 6t + 1),$
$N_3(t) = \frac{1}{20}(20t^3 - 30t^2 + 12t - 1),$
$N_4(t) = \frac{1}{70}(70t^4 - 140t^3 + 90t^2 - 20t + 1),$

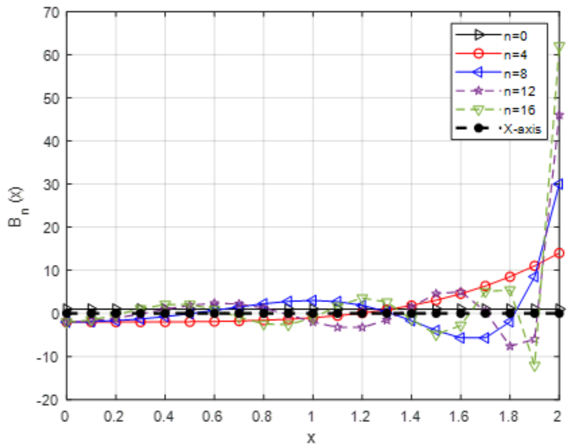

Figure 3: First five 4q-BPs with real zeros intersecting X-axis

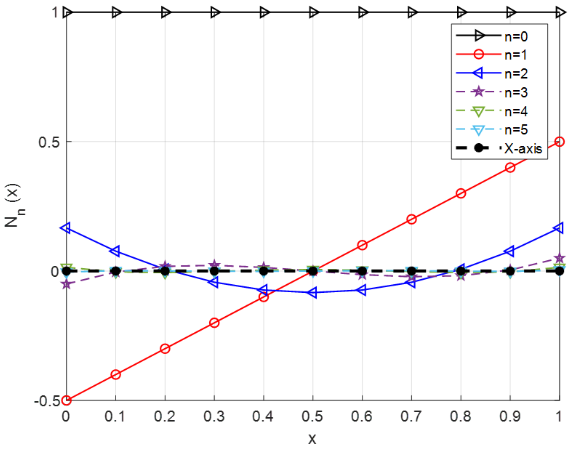

Figure 4: First six NOBPs with real zeros intersecting X-axis

$N_5(t) = \frac{1}{252}(252t^5 - 630t^4 + 560t^3 - 210t^2 + 30t - 1),$

$$N_6(t) = \frac{1}{924}(924t^6 - 2772t^5 + 3150t^4 - 1680t^3 + 420t^2 - 42t + 1) \tag{10}$$

Fig. 4 shows the first six NOBPs in $[0, 1]$, where all zeros are real numbers for every n. Noah [11] discussed only the compact form and the NOBPs in the interval $[0, 1]$. But details on the orthogonality were missing. The orthogonality relation was not discussed. The coefficients of the polynomials in NOBPs are quite higher in magnitude, which are more prone to rounding off errors than other variants in numerical approximations required in application studies.

Modified orthogonal Boubaker polynomials (MOBPs) defined in [0,2] were suggested by Khoso and co-authors in [12, 13]. The Gram-Schmidt process was used to derive modified orthogonal basis for the CBPs, and the new polynomials were referred to as MOBPs. The characteristic differential equation of MOBPs is:

$$\left(x^2 - 2x\right)\frac{d^2y}{dx^2} + 2(x-1)\frac{dy}{dx} - n(n+1)y = 0 \tag{11}$$

The compact form of MOBPs is:

$$P_n(x) = B_n(x) - \sum_{i=0}^{n-1} \frac{< B_n(x), P_i(x) >}{< p_i(x), P_i(x) >} P_i(x) \tag{12}$$

The three-term recurrence relation for MOBPs:

$$P_{n+1}(x) = (x-1)P_n(x) - \frac{n^2}{4n^2 - 1} P_{n-1}(x), n \geq 1 \tag{13}$$

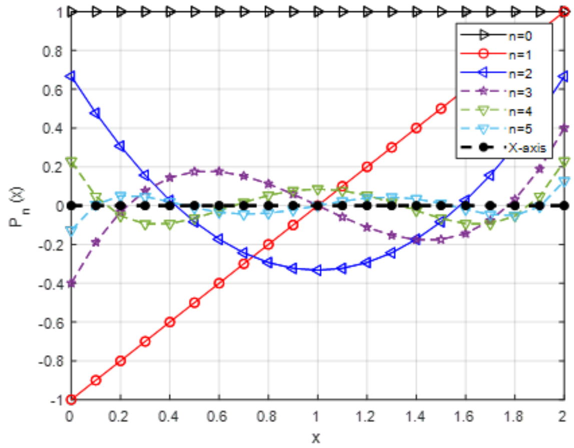

Figure 5: First six MOBPs with real zeros intersecting X-axis

With initial conditions: $P_0(x) = 1$ and $P_1(x) = x - 1$. The generating function for MOBPs was derived to be:

$$G(x,t) = \frac{1}{\sqrt{1-2(x-1)t+t^2}} \tag{14}$$

Rodrigues formula for MOBPs is:

$$P_n(x) = \frac{(n-1)!}{2(2n-1)!}\frac{dy^n}{dx^n}\left(x^2-2x\right)^n, n = 1,2,\ldots \tag{15}$$

The Sturm-Liouville form of the (11) for MOBPs is:

$$\frac{d}{dx}\left[\left(x^2-2x\right)\frac{dy}{dx}\right] + \lambda y = 0 \tag{16}$$

Orthogonality relations was established as:

$$\int_0^2 P_m(x)P_n(x)dx = \left\{\begin{array}{ll} \frac{(n!)^2}{2^{2(n+1)}\left(n+\frac{1}{2}\right)\left[\left(n-\frac{1}{2}\right)!\right]^2} & ,m=n \\ 0, & m \neq n \end{array}\right\} \tag{17}$$

We can list starting MOBPs, which are:
$P_0(x) = 1$,
$P_1(x) = x - 1$,
$P_2(x) = x^2 + 2x - \frac{2}{3}$,
$P_3(x) = x^3 + 3x^2 + \frac{12}{5}x - \frac{2}{5}$,
$P_4(x) = x^4 - 4x^3 + \frac{36}{7}x^2 - \frac{16}{7}x + \frac{8}{35}$,
$P_5(x) = x^5 - 5x^4 + \frac{80}{9}x^3 - \frac{20}{3}x^2 + \frac{40}{21}x - \frac{8}{63}$,
$P_6(x) = x^6 - 6x^5 + \frac{150}{11}x^4 - \frac{160}{11}x^3 + \frac{80}{11}x^2 - \frac{16}{11}x + \frac{16}{231}$

Fig. 5 shows that all the zeros of MOBPs are always real numbers. The MOBPs are orthogonal, and all the classical properties were discussed, but the only problem lies in these being deviated from original source, i.e. CBPs. The coefficients are not integers, and quite higher in magnitude, which are more vulnerable to rounding off errors than the other variants.

Al-Abdali's orthogonal Boubaker-Turki polynomials (OBTPs) defined in [0,1] appeared in a recent study [14]. The characteristic differential equation for OBTPs was reported as:

$$16\left(1-x^2\right)OBT_n'' - 4xOBT_n' + n^2OBT_n = 32(n-1)T_{n-2}, n > 2 \tag{18}$$

The compact form was given by:

$$OBT_n(t) = 2^n t^n - 2^{n-2}(n-4)t^{n-2} + \sum_{p=2}^{\delta(n)}\left[\frac{(n-4p)}{p!}\prod_{j=p+1}^{2p-1}(n-j)\right] \tag{19}$$

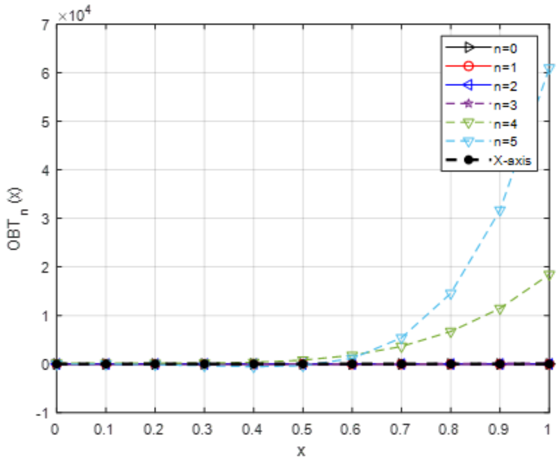

Figure 6: First six OBTPs with real zeros intersecting X-axis

where $OBT_n(t)$ and $T_n(t)$ are Boubaker-Turki polynomials and first kind Chebyshev polynomials respectively. The claimed orthogonality relations were:

$$\int_0^1 OBT_m(t)OBT_n(t)dx = \left\{ \begin{array}{ll} \delta, & m=n \\ 0, & m \neq n \end{array} \right\} \tag{20}$$

where

$$\delta = \frac{2^{2n}}{\binom{2n}{n}(2n+1)} \tag{21}$$

We can list starting polynomials using (19), which are: $OBT_0 = 1$,
$OBT_1 = -2 + 4t$,
$OBT_2 = 23t^2 - 16t + \frac{8}{3}$
$OBT_3 = 384t^3 - 192t^2 + \frac{96}{5}t - \frac{16}{5}$
$OBT_4 = \frac{215040}{7}t^4 - \frac{107520}{7}t^3 + \frac{23040}{7}t^2 - \frac{2560}{7}t + \frac{128}{7}$
$OBT_5 = \frac{7741440}{63}t^5 - \frac{3870720}{63}t^4 + \frac{122880}{63}t^3 - \frac{161280}{63}t^2 + \frac{7680}{63}t - \frac{256}{63}$
$OBT_6 = \frac{681246720}{231}t^6 - \frac{340623360}{231}t^5 + \frac{77414400}{231}t^4 - \frac{10321920}{23}t^3 + \frac{860160}{231}t^2 - \frac{43008}{231}t + \frac{1024}{231}$
These polynomials in [14] were claimed orthogonal on the part of author, but generated polynomials are in fact not orthogonal. One can observe that for any pair of different polynomials, the inner product defined in the claimed orthogonality relation (20) does not vanish for all different polynomials. First six OBTPs are shown in Fig. 6, which also show that the polynomials are not orthogonal as only a few of the zeros, but not all, are real numbers. Besides, the coefficients are much higher in magnitude and rational, which are more blighted by rounding off errors than other variants. The OBTPs are highly flawed with regard to their claimed orthogonality properties, and the rounding off concerns of the coefficients.

The Boubaker-Fermat polynomials (BFPs), defined in [-2,2], were studied in [15] while an attempt to study Fermat-linked relations for the CBPs through Riordan analysis. In collaborative endeavor, they came out with new mathematical expression in which they combined CBPs with Chebyshev polynomials of first kind which was defined as a linear combination:

$$Q_n(2\cos t) = \frac{B_n(2\cos t) - 2T_n(\cos t)}{4} \tag{22}$$

whereas

$$Q_n(\mathrm{x}) = \frac{B_n(x) - 2T_n\left(\frac{x}{2}\right)}{4} \tag{23}$$

where

$$x = 2\cos t \tag{24}$$

A few BFPs in $[-2, 2]$ are:
$Q_0(x) = 1$
$Q_1(x) = x$
$Q_2(x) = x^2 - 1$
$Q_3(x) = x^3 - 2x$

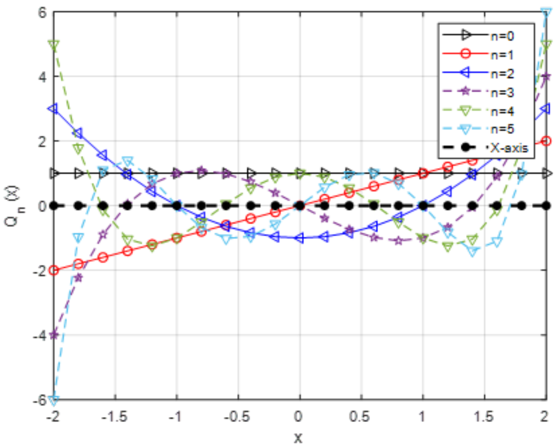

Figure 7: First six BFPs with real zeros intersecting X-axis

$Q_4(x) = x^4 - 3x^2 + 1$
$Q_5(x) = x^5 - 4x^3 + 3x$
Fig. 7 show first six BFPs.
A trigonometric relation was also evident in [15]:

$$Q_n(2\cos t) = \frac{\sin(n+1)t}{\sin t} \tag{25}$$

In form of Riordan matrix, BFPs were written as:

$$\sum_{n>0} Q_n(t)x^n = T\left(1 \mid 1+x^2\right)\left(\frac{1}{1-xt}\right) \tag{26}$$

In fact, we have the Riordan matrix for the BFPs, which is $T(1|1+x^2)$, and can be expressed as:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -3 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & -4 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{27}$$

The BFPs are closer in source to the CBPs and were proposed by the same author Boubaker in collaboration with Zhang in 2012 [15]. The BFPs are redefined, in contrast to the CBPs, using Chebyshev polynomials of first kind (CP-I) in (22) and (23). However, in the next section, we also emphasize on their relation to Chebyshev polynomials of second kind (CP-II). The coefficients of BFPs are always integers and user-friendly. This is an encouraging feature as compared to the other discussed variants. Therefore, the BFPs are more inclined to efficiency with regard to orthogonalization and friendly nature of coefficients in approximation processes.

2.3 Present contributions and a way forward

As per our observation, the BFPs are also orthogonal, and we consider these as an efficient orthogonalization of the CBPs in context of the existing variants of the CBPs and different orthogonalization attempts. This maps back to the three important features outlined for an efficient orthogonalization in the Section 1. The BFPs, $Q_n(x)$, are closer to the CBPs and more inclined to a Chebyshev polynomial class as expected, possess integer coefficients and all real zeros in [-2,2], and also proposed by the same author in 2012 [15]. The orthogonality of the BFPs was never actualized so far. For the BFPs, the classical properties, like weight function, the three-term recurrence relation, generating function, Rodrigues formula, orthogonality relation, orthonormality relation, characteristic differential equation, series solution and compact form, Strum-Liouville (self-adjoint) form; all have not been proved yet in literature. We call the BFPs as proposed orthogonal Boubaker polynomials (POBPs) defined in a Hilbert space in [-2,2], i.e. $H_Q[-2,2]$ for brevity in forthcoming discussions. Here, we establish weight function, orthogonality and orthonormality relations for the POBPs.

We quote following definitions and concepts from [16].

Definition 1. Weight function Two different real-valued functions $f(x)$ and $g(x)$ are orthogonal relative to a real-valued non-negative weight function $w(x)$ on an interval [a,b] if their inner product defined in (28) vanishes.

$$< f,g >= \int_a^b w(x)f(x)g(x)dx \tag{28}$$

Definition 2. Three-term recurrence relation It is a relation among three successive polynomials that allows the systematic determination of the polynomials in the sequence, given two initial polynomials and the relation. The general form of three-term recurrence relation for a polynomial sequence $P_n(x), n = 0,1,2,...$ is:

$$P_{n+1}(x) = (x - a_n)P_n(x) - b_nP_{n-1}(x), \quad n \geq 1 \tag{29}$$

with initial conditions $P_0(x)$ and $P_1(x)$. Here, a_n, b_n are coefficients depending on n, and are defined as: $a_n = \frac{<xP_n(x),P_n(x)>}{<P_n(x),P_n(x)>}$, and $b_n = \frac{<P_n(x),P_n(x)>}{<P_{n-1}(x),P_{n-1}(x)>}$.

Definition 3. Orthogonality relation If a sequence of functions $P_n(x), n = 0,1,2,...$ is orthogonal in an interval [a,b] with respect to a weight function w(x), then the orthogonality relation is defined as:

$$< P_m(x), P_n(x) >= \int_a^b w(x)P_m(x)P_n(x)dx = B(n)\delta_{mn} \tag{30}$$

Where

$$\delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \tag{31}$$

is Kronecker delta and $B(n)$ is a function of n.

Definition 4. Orthonormality relation A sequence of orthogonal functions $\hat{P}_n(x), n = 0,1,2,...$ is orthonormal in an interval [a,b] with respect to the weight function w(x), if the orthonormality relation is satisfied, which is defined as:

$$\hat{P}_m(x), \hat{P}_n(x) >= \int_a^b w(x)\hat{P}_m(x)\hat{P}_n(x)dx = \delta_{mn} \tag{32}$$

The relationship which is used to get the orthonormal polynomials $\hat{P}_k(x)$ from a given family of orthogonal polynomials $P_k(x)$ is: $\hat{P}_k(x) = \frac{P_k(x)}{<P_k(x),P_k(x)>}$

Now, we prove main results on the weight function, orthogonality and orthonormality of POBPs in theorems 1-3, respectively.

Theorem 1. *The weight function of the suggested POBPs is* $w(x) = \frac{1}{2}\sqrt{4-x^2}$.

Proof of Theorem 1. The inner product of two different POBPs in $H_Q[-2,2]$ with general weight function w(x) is:

$$< Q_m(x), Q_n(x) >= \int_{-2}^{2} w(x)Q_m(x)Q_n(x)dx \tag{33}$$

Taking substitution $x = 2cost$ so that: $dx = -2sintdt$, and $x = -2,2$ transform to $t = \pi, 0$, respectively. The equation (33) transforms as:

$$< Q_m(x), Q_n(x) >= \int_{\pi}^{0} w(2\cos t)Q_m(2\cos t)Q_n(2\cos t)(-2\sin t dt) \tag{34}$$

Using (25) in (34) gives:

$$< Q_m(x), Q_n(x) >= \int_{\pi}^{0} \frac{w(t)\sin(m+1)t\sin(n+1)t}{\sin^2 t}(-2\sin t dt) \tag{35}$$

(or)

$$< Q_m(x), Q_n(x) >= -2\int_{\pi}^{0} \left[\frac{w(t)}{\sin t}\right]\sin(m+1)t\sin(n+1)tdt \tag{36}$$

The integral $\int_{\pi}^{0} \sin(m+1)t\sin(n+1)tdt$ vanishes always for each $m \neq n$, so in (36):
$< Q_m(x), Q_n(x) >= 0$ iff $\frac{w(t)}{\sin t} = 1$

$$\Rightarrow w(t) = \sin t \tag{37}$$

As $x = 2\cos t$, so $\sin t = \frac{1}{2}\sqrt{4-x^2}$. Hence

$$w(x) = \frac{1}{2}\sqrt{4-x^2} \tag{38}$$

Q.E.D.

Theorem 2. *The complete orthogonality relation of POBPs in a Hilbert space* $H_Q[-2,2]$ *is:*

$$< Q_m(x), Q_n(x) >= \int_{-2}^{2} \frac{1}{2}\sqrt{4-x^2} Q_n(x) Q_m(x) dx \quad = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases} \tag{39}$$

Proof of Theorem 2.

With the same transformation $x = 2\cos\theta$, the weight function from (38), and the relation (25) from [15], (34) becomes:

$$< Q_m, Q_n >= -\int_0^{\pi} \frac{1}{2}(2\sin\theta) \frac{\sin(n+1)\theta \sin(m+1)\theta(-2\sin\theta) d\theta}{\sin^2\theta} \tag{40}$$

Or,

$$< Q_m, Q_n >= 2\int_0^{\pi} \sin(n+1)\theta \sin(m+1)\theta d\theta \tag{41}$$

If $m = n$ in (41), then:
$< Q_n, Q_n >= 2\int_0^{\pi} \sin^2(n+1)\theta d\theta = 2\int_0^{\pi} \left[\frac{1-\cos 2(n+1)\theta}{2}\right] d\theta$
$< Q_n, Q_n >= \int_0^{\pi} 1.d\theta - \int_0^{\pi} \cos(2n+2)\theta d\theta = \pi - 0 - 0 = \pi$
So, if $m = n$, we have:

$$\int_{-2}^{2} \frac{1}{2}\sqrt{4-x^2} Q_n(x) Q_m(x) dx = \pi \tag{42}$$

If $m \neq n$ in (41), then:

$$< Q_m, Q_n >= \int_0^{\pi} 2\,\mathrm{Sin}(n+1)\theta\, \mathrm{Sin}(m+1)\theta d\theta \tag{43}$$

Since,

$$2\sin\alpha \sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta) \tag{44}$$

$$< Q_m, Q_n >= \int_0^{\pi} \cos(n-m) d\theta - \int_0^{\pi} \cos(n+m+2) d\theta \tag{45}$$

$< Q_m, Q_n >= \left.\frac{\sin(n-m)\theta}{n-m}\right|_0^{\pi} - \left.\frac{\sin(n+m+2)\theta}{n+m+2}\right|_0^{\pi}$
$< Q_m, Q_n >= \left[\frac{\sin(n-m)\pi}{n-m} - \frac{\sin(n-m)(0)}{n-m}\right] - \left[\frac{\sin(n+m+2)\pi}{n+m+2} - \frac{\sin(n+m+2)(0)}{n+m+2}\right] = 0$
So, if $m \neq n$ then

$$\int_{-2}^{2} \frac{1}{2}\sqrt{4-x^2} Q_n(x) Q_m(x) dx = 0 \tag{46}$$

Finally, using both (42) and (46), we reach at the required orthogonality relation (39) in $H_Q[-2,2]$. Q.E.D.

Theorem 3. *The orthonormal POBPs in* $H_Q[-2,2]$ *can be defined as:*
$\hat{Q}_n(x) = \frac{Q_n(x)}{\sqrt{\pi}}$.
The orthonormality relation for the POBPs is:

$$< \widehat{Q}_m(x), \hat{Q}_n(x) = \int_{-2}^{2} \frac{1}{2}\sqrt{4-x^2} \hat{Q}_m(x) \hat{Q}_n(x) dx = \delta_{mn} \tag{47}$$

Proof of Theorem 3.

Let the orthonormal set of polynomials $\hat{Q}_n(x)$ in $H_Q[-2,2]$ corresponding to the POBPs $Q_n(x)$ be defined in general as:

$$\left\{\widehat{Q}_n(x) : \hat{Q}_n(x) = \frac{Q_n(x)}{||Q_n(x)||}, n = 0, 1, 2, \dots\right\} \tag{48}$$

where:

$$||Q_n(x)|| = \left[\int_{-2}^{2} w(x) Q_n^2(x) dx\right]^{\frac{1}{2}} \tag{49}$$

But, $< Q_n, Q_n >= \int_{-2}^{2} \frac{1}{2}\sqrt{4-x^2} Q_n^2 dx = \pi$,
for all n (equation (42)), so,

$$\|Q_n(x)\| = \sqrt{< Q_n, Q_n >} = \sqrt{\pi} \tag{50}$$

Hence,

$$\hat{Q}_n(x) = \frac{Q_n(x)}{\sqrt{\pi}}, n = 0, 1, 2, \ldots \tag{51}$$

The first few orthonormalized POBPs are:
$\hat{Q}_0(x) = \frac{1}{\sqrt{\pi}}$
$\hat{Q}_1(x) = \frac{1}{\sqrt{\pi}} x$
$\hat{Q}_2(x) = \frac{1}{\sqrt{\pi}} \left(x^2 - 1\right)$
$\hat{Q}_3(x) = \frac{1}{\sqrt{\pi}} \left(x^3 - 2x\right)$
$\hat{Q}_4 x) = \frac{1}{\sqrt{\pi}} \left(x^4 - 3x^2 + 1\right)$
Therefore, the final form of the orthonormality relation for the POBPs in $H_Q[-2, 2]$ is:

$$< \widehat{Q}_m(x), \hat{Q}_n(x) = \int_{-2}^{2} \frac{1}{2}\sqrt{4-x^2} \hat{Q}_m(x) \hat{Q}_n(x) dx = \delta_{mn} \tag{52}$$

Q.E.D.

Remark 1. It can be noticed that unlike most of the families of orthogonal polynomials, like Legendre, Laguerre, Chebyshev, etc. where the three-term recurrence relation assumes only two initial conditions for the starting polynomials P_0 and P_1 only to generate the remaining polynomials $P_i, i \geq 2$, the three-term recurrence relation of the CBPs uses three initial conditions. Also, for the case of CBPs, the recurrence relation does not define $B_2(x)$ using B_0 and B_1, instead B_2 is also defined explicitly, and the recurrence relation is used for $m > 2$, i.e. to generate $B_i, i > 2$. But, when the same recurrence relation of the CBPs is used with only two initial conditions B_0 and B_1 which are also same as proposed Q_0 and Q_1, then relaxing the relation for $m \geq 2$ we get the remaining POBPs $Q_i, i \geq 2$. Thus, the three-term recurrence relation for the POBPs is the same as that of CBPs with only two initial conditions and relaxed to generate all forthcoming polynomials for $m \geq 2$ from the same relation as expected conventionally. Thus, the POBPs are orthogonal and nearer in source to the CBPs than other existing variants. The three-term recurrence relation for the POBPs can be written as:

$$Q_m(x) = x Q_{m-1}(x) - Q_{m-2}(x) \quad \text{for } m \geq 2 \tag{53}$$

With initial conditions: $Q_0(x) = 1, Q_1(x) = x$.

Remark 2. It can be easily verified through Table 1 that all zeros of POBPs are real and contained in $[-2, 2]$.

Table 1. Zeros of first few POBPs

Serial number	$Q_n(x)$	Zeroes (exact /closed form)	Zeroes (approximate form)
1	$Q_0(x)$	NA	NA
2	$Q_1(x)$	1	1
3	$Q_2(x)$	± 1	± 1
4	$Q_3(x)$	$0, \pm\sqrt{2}$	0, ± 1.4142135623710
5	$Q_4(x)$	$\pm\frac{1}{2} - \frac{\sqrt{5}}{2}, \pm\frac{1}{2} + \frac{\sqrt{5}}{2}$	± 0.61803398874989, ± 1.61803398874989
6	$Q_5(x)$	$0, \pm 1, \pm\sqrt{3}$	$0, \pm 1, \pm 1.732050808$

The following theorems: theorems 4-6, showcase the Rodrigues formula, characteristic differential equation and Strum-Liouville form for the POBPs, respectively. These are based on the derived weight function in theorem 1.

Theorem 4. *The Rodrigues formula for the POBPs is given by*

$$Q_n(x) = \frac{1}{\sqrt{4-x^2}} \frac{(-1)^n (n+1)!}{(2n+1)!} \frac{d^n}{dx^n} \left[\left(4-x^2\right)^{n+\frac{1}{2}}\right], \text{ where } n = 0, 1, 2, \ldots \tag{54}$$

Proof of Theorem 4. Rodrigues formula is defined [16] generally in a family of orthogonal polynomial as:

$$\alpha_n Q_n(x) = \frac{1}{w(x)} \frac{d^n}{dx^n} [w(x).(A(x))^n] \tag{55}$$

where A(x) is a quadratic polynomial of the form:

$$A(x) = ax^2 + bx + c \tag{56}$$

and

$$w(x) = \frac{1}{2}\sqrt{4-x^2} \tag{57}$$

With $n = 0$ in (55), we have: $\alpha_0 Q_0(x) = \frac{2}{\sqrt{4-x^2}} \frac{d^0}{dx^0}\left[\frac{1}{2}\sqrt{4-x^2}.\left(ax^2+bx+c\right)^0\right]$.
So,

$$\alpha_0 = 1 \tag{58}$$

With $n = 1$ in (55), we have:

$$\alpha_1 Q_1(x) = \frac{2}{\sqrt{4-x^2}} \frac{d}{dx}\left[\frac{1}{2}\sqrt{4-x^2}\cdot\left(ax^2+bx+c\right)^1\right] \tag{59}$$

But, $Q_1(x) = x$
So

$$\alpha_1 \cdot x = \frac{3ax^3 + 2bx^2 + (-8a+c)x - 4b}{(x^2-4)} \tag{60}$$

(or)

$$\alpha_1 \cdot x\left(x^2-4\right) = 3ax^3 + 2bx^2 + (-8a+c)x - 4b \tag{61}$$

By Comparing the coefficients of x^n i.e x^0, x^1, x^2, x^3 on both sides of (61) gives:

$$\alpha_1 = 3a \tag{62}$$

$$-4\alpha_1 = -8a + c \tag{63}$$

$$0 = b \Rightarrow b = 0 \tag{64}$$

$$0 = 4b \Rightarrow b = 0,\ \text{again} \tag{65}$$

Using (62) and (63), we have:

$$4a + c = 0 \tag{66}$$

Going for higher n, results in more equations for remaining unknowns a, c in $A(x)$, but all the equations suggest that the systems for a, c have interdependent solutions.

With $x = 1$ in (61), we have after simplifications:

$$4a - 2b + c = 0 \tag{67}$$

Now, solving (66) and (67) by Crammer's rule for a non-trivial solution of the homogeneous system, we have:
$a = -1, b = 0$ and $c = 4$.
Here b is an independent and a and c are dependent to each other.
Now, it only remains to determine the expression of α_n.
$A(x) = ax^2 + bx + c$, but, $a = -1, b = 0$ and $c = 4$ So, $A(x) = 4 - x^2$
With these in (55), we have:

$$\alpha_n Q_n(x) = \frac{2}{\sqrt{4-x^2}} \frac{d^n}{dx^n}\left[\frac{1}{2}\sqrt{4-x^2}.\left(4-x^2\right)^n\right]$$

(or)

$$\alpha_n Q_n(x) = \frac{1}{\sqrt{4-x^2}} \frac{d^n}{dx^n}\left[\left(4-x^2\right)^{n+\frac{1}{2}}\right] \tag{68}$$

Here, we put $n = 0$ in (68), we get:

$$\alpha_0 Q_0(x) = \left[\ \frac{1}{\sqrt{4-x^2}} \quad \sqrt{4-x^2}\ \right] \tag{69}$$

Where as $Q_0(x) = 1$, so $\alpha_0 = 1$.
Simialrly, with $n = 1, 2, 3, \ldots$ in (68), we get: $\{\alpha_n\} = \{1, -3, 20, -210, 3024, \ldots\}$
The n$^{\text{th}}$ term of $\{\alpha_n\}$ is:

$$\alpha_n = \frac{(-1)^n(2n+1)!}{(n+1)!} \tag{70}$$

Finally, we put this in (68) to have:

$$Q_n(x) = \frac{1}{\sqrt{4-x^2}} \frac{(-1)^n (n+1)!}{(2n+1)!} \frac{d^n}{dx^n} \left[\left(4-x^2\right)^{n+\frac{1}{2}} \right]$$

Q.E.D.

Theorem 5. *The characteristic differential equation for the POBPs is:*

$$\left(4-x^2\right) y'' - 3xy' + n(n+2)y = 0 \tag{71}$$

Proof of Theorem 5. The general form of the characteristic differential equations is [16, 17]:

$$a(x)y''(x) + b(x)y'(x) = \lambda_n y \tag{72}$$

Where $a(x)$ is quadratic polynomial from Rodrigues formula, and:
$a(x) = 4 - x^2$.

$$b(x) = a(x)[D\log(w(x))] + a'(x) \tag{73}$$

$$b(\mathrm{x}) = \left(4-x^2\right) \frac{d}{dx} \left[\log \left\{ \frac{1}{2} \sqrt{4-x^2} \right\} \right] - 2x = -x - 2x = -3x \tag{74}$$

Also

$$\lambda_n = p.n(n-1) + q.n \tag{75}$$

Where p is coefficient of x^2 in $a(x)$, so:

$$p = -1 \tag{76}$$

and q is coefficient of x in $b(x)$, so:

$$q = -3 \tag{77}$$

So,

$$\lambda_n = -1n(n-2) + (-3)n = -n(n+2) \tag{78}$$

Finally, using these in the general form leads to: $\left(4-x^2\right) y'' - 3xy' + n(n+2)y = 0$. Q.E.D.

Theorem 6. *The Strum-Liouville / Spectral / Self-adjoint form of the POBPs is:*

$$\frac{1}{\sqrt{4-x^2}} \frac{d}{dx} \left[\quad \left(4-x^2\right)^{\frac{3}{2}} \frac{dy}{dx} \right] + \lambda_n y = 0 \tag{79}$$

Proof of Theorem 6. The characteristic differential equation of the POBPs is:

$$\left(4-x^2\right) y'' - 3xy' + n(n+2)y = 0 \tag{80}$$

So, $a(x) = 4 - x^2, b(x) = -3x, c(x) = 0$ (homogeneous equation) and $\lambda_n = -n(n+2)$.
For the Strum-Liouville form [16, 17], now, we need to find the polynomials: $p(x), q(x)$ and $r(x)$.

$$p(x) = e^{\int \frac{b(x)}{a(x)} dx} = e^{\int \frac{-3x}{4-x^2} dx} = \left(4-x^2\right)^{\frac{3}{2}} \tag{81}$$

$$q(x) = \frac{c(x)}{a(x)} p(x) = 0 \tag{82}$$

as $c(x) = 0$

$$r(x) = \frac{p(x)}{a(x)} = \sqrt{4-x^2} \tag{83}$$

Finally, the Strum-Liouville form [16], [17] is:

$$\left(p(x)y'\right)' + [q(x) + \lambda r(x)]y = 0 \tag{84}$$

For POBPs, it becomes:

$$\frac{d}{dx} \left[\left(4-x^2\right)^{\frac{3}{2}} \frac{dy}{dx} \right] + n(n+2)\sqrt{4-x^2}\, y = 0 \tag{85}$$

(or)

$$\frac{1}{\sqrt{4-x^2}} \frac{d}{dx} \left[\left(4-x^2\right)^{\frac{3}{2}} \frac{dy}{dx} \right] + n(n+2)y = 0. \quad \text{Q.E.D.}$$

Remark 3. The literature in mathematics motivates to discuss the connections among different families of polynomials that exist. So, exploring connection of a polynomial family with some well-known classical families of polynomials is not at all considered weak or bad, rather it adds to the authenticity of the results on the former. A recent study [18] explores the connections between Chebyshev and Legendre polynomials. In fact, in the late 19th century many polynomial families were introduced and connections between these and Chebyshev polynomials were explored; for example: Lucas, Fibonacci, Vieta and Jacobsthal polynomials [19, 20]. As such many shifted Chebyshev polynomials independently exist to date. By searching above mentioned shifted Chebyshev polynomials, one finds exhaustive literature on their independent applications even in the recent years. So, it adds to authenticity if a polynomial family has connection with a well-known classical family of polynomials, like CP-I and CP-II.

Remark 4. The POBPs, as studied here in detail, were inspired from the work carried out in 2012 by Boubaker and Zhang [15], where these were defined as a linear combination of CBPs and CP-I, as stated in (22) and (23). However, it is apparent from discussions motivated in [15] and through the orthgonality and other properties explored in this work that these are closer to CP-II as expected. In fact, these can be seen as scaled Chebyshev polynomials because the linear combination (22), (23) eliminates the CP-I terms from the CBPs leaving out scaled CP-II. So, we can restate from [15] that:

$$Q_n(x) = U_n(x/2) \tag{86}$$

Remark 5. We note again that the CBPs are diametrically opposite to the Chebyshev polynomials in terms of domain and nodes. Since CBPs originally were not imbued with orthogonality, these were transformed into such polynomials as were distinctly different from Chebyshev polynomials. In this context, pioneer of polynomials (Boubaker himself in [15]) applied Riordan arrays technique which become chiefly instrumental in bringing about such polynomials (POBPs) having real zeros. Problem of complex zeros are resolved which has long been remained main obstacle for efficient orthogonalization of Boubaker polynomials.

3. Final reflections and discussion

The present work highlights an independent effort on the orthogonalization of Boubaker polynomials, searching out all the variants of Boubaker polynomials. Outstanding work left by Boubaker himself [15] was taken as a fundamental source to work further and explore the missing properties of Boubaker polynomials. One of the formidable challenge was to construct knowledge of efficient orthogonalization of Boubaker polynomials because it will pay the way for Gauss quadrature [21] with contributions in integral equations, cubature [22], Riemann-Stieltjes integral [23], modifications of Newton-Cotes quadrature [24] and optimal control theory [14].

Polynomials generated by Boubaker [6] i.e. the CBPs, were not orthogonal. It was main driving force and great motivation for venturing into realm of orthogonalization. Limitations were conspicuous by its absence in the existing literature. In this perspective, these polynomials were massively marred by the zeros of complex numbers which was main stumbling block to the establishment of orthogonalization and the rest of classical properties.

Second variant, i.e. MBPs [8, 9], was essentially not different from the CBPs, rather; it was just scaled CBPs and subjected to the same limitations and imperfections. Most of the relations hinge upon the Chebyshev polynomials.

In [10], polynomials were blighted by the same limitations. They considered only the polynomials whose indices were multiples of 4 , and these contained effective terms used in the underlying application of counting complex zeros inside open unit disk [7]. But these carry same problems as of the classical polynomials.

Noah [11] discussed only compact form and its polynomials in the interval $[0, 1]$. But details on the orthogonality were missing. The orthogonality relation was not discussed. The coefficients were rationals, which are more prone to rounding off errors.

Khoso, Shaikh and Shaikh [12, 13] derived polynomials were systematic and orthogonal, but the original source was not considered; it deviated chiefly from original source. So, it was the least concerned with orthogonalization of conventional Boubaker polynomials rather produced their own polynomials by using Gram-Schmidt process. But nevertheless, the MOBPs were orthogonal, and all the properties were discussed, but away from original source. The coefficients were rational numbers, which are more prone to rounding off errors.

In [14], the suggested polynomials were not orthogonal and are grappled with same challenges. These were claimed orthogonal on the part of author, but generated polynomials were not orthogonal. The coefficients were also quite higher in magnitude, which are more prone to rounding-off errors than other variants.

The POBPs were motivated from the study [15]. We successfully derived the weight function, orthogonality and orthonormality relations. Moreover, the recurrence relation was demonstrated to be in full symmetry with the CBPs while also more authentic than the CBPs. The zeros of POBPs are all reals in [-2,2]. As far as proximity to conventional recurrence relations is concerned, the POBPs is more exact and applicable, because it is laced with interesting feature that is compatible with traditional relations. CBPs are defined with the condition $m > 2$, that is in contravention with the traditional one with condition $m > 1$. Therefore, our derived recurrence relations are exactly same as that of conventional one; it lends authenticity to our POBPs. In summary, no one derived the weight function for the polynomials in [15], which was done in this study through theorem 1. Consequently, orthogonality and orthonormality relations for the POBPs for the first time have also been proved. Of course, the Rodrigues formula and the spectral form quoted are worthwhile for seminal results, while the weight function is basis for all that. Also, the spectral form cannot be obtained unless one knows the characteristic differential equation for the orthogonal polynomials. So, the study accomplished with three more theorems to lend the authenticity, originality and completeness for our study, which are: Rodrigues formula, characteristic differential equation and Sturm-Liouville / spectral form for POBPs.

Finally, it is a positive development that our POBPs is compatible with traditional polynomials which lends its authenticity of POBPs. Therefore, so called Boubaker polynomials do not spring from a figment of imagination rather it is an undeniable fact and concrete reality in the sense of ever-increasing applications of Boubaker polynomials ranging from social science to natural science. After meriting careful consideration and introspection regarding orthogonalization, it is not counter-intuitive for having relative compatibility of POBPs with CP-II in terms of results. So, overlapping the result essentially does not signify that it is tantamount to inconclusive and invalid result and obviates the need for any further investigation into exploring different avenues of Boubaker polynomials.

4. Concluding remarks

There are various case studies springing from science and engineering where applications of Boubaker polynomials are pervading, especially involving either differential equations or integral equations. It is an established fact that orthogonalization is a prominent feature of any polynomials which are essential for Gauss quadrature. In this context, various research scholars tried their hands at the orthogonalization of Boubaker polynomials, but could not attain desirable results. We, in this paper, measured, poked, and investigated their works in terms of mode of approach, strengths, weaknesses, applicability, efficiency, elegancy and proximity to the orthogonalization of Boubaker polynomials. Comprehensive review and analytic observation of findings culminated into a recommendation of promising orthogonalization especially in determining the viable and proper weight function in the Hilbert space. In the broadest sense, many problems stem from science and engineering, and their practical and elegant solutions seem to be almost impossible without orthogonalization of Boubaker polynomials. It is essentially an intellectually demanding and academically useful to have an efficient orthogonalization of conventional Boubaker polynomials as it is predominantly an instrumental in bringing about seminal development in the sense of Gauss quadrature, cubature and Riemann-Stieltjes integrals. In sum, efficient orthogonalization of Boubaker polynomials paves the way for the exploration of new avenues in terms of Gauss quadrature and more effective rules can be derived than traditional Newton-Cotes rules for numerical integration.

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