



# A high statistics lattice calculation of heavy-light meson decay constants

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## Abstract

We present a high statistics study of the  $D$ - and  $B$ -meson decay constants. The results were obtained by using the Clover and Wilson lattice actions at two different values of the lattice spacing  $a$ , corresponding to  $\beta = 6.0$  and  $6.2$ . After a careful analysis of the systematic errors present in the extraction of the physical results, by assuming quite conservative discretization errors, we find  $f_{D_s} = 237 \pm 16$  MeV,  $f_D = 221 \pm 17$  MeV ( $f_{D_s}/f_D = 1.07(4)$ ),  $f_{B_s} = 205 \pm 35$  MeV,  $f_B = 180 \pm 32$  MeV ( $f_{B_s}/f_B = 1.14(8)$ ), in good agreement with previous estimates. © 1997 Elsevier Science B.V.

## 1. Introduction

$f_B$  is a relevant parameter in phenomenological studies of the Standard Model, in determinations of the elements of the Cabibbo–Kobayashi–Maskawa (CKM) matrix and in studies of  $B$ – $\bar{B}$  mixing.

The leptonic decay width of the  $B$  meson is given by

$$\Gamma(B^+ \rightarrow \tau^+ \nu_\tau) = \frac{G_F^2 |V_{ub}|^2}{8\pi} M_B \left(1 - \frac{M_\tau^2}{M_B^2}\right)^2 M_\tau^2 f_B^2. \quad (1)$$

A precise knowledge of the leptonic decay constant  $f_B$ , analogous to  $f_\pi$  in  $\pi \rightarrow \mu\nu\mu$  decays<sup>2</sup>, would then allow an accurate extraction of the CKM matrix

element  $|V_{ub}|$ , which is actually known with a relative error of about 25%.

$f_B$  also enters in phenomenological analyses of the  $B^0$ – $\bar{B}^0$  mixing amplitudes, together with the so-called renormalization group invariant  $B$ -parameter  $\hat{B}_B$ . The square of the mixing parameter  $\xi = f_B \sqrt{\hat{B}_B}$  is in fact related to the matrix element of the renormalized  $\Delta B = 2$  Hamiltonian [1]. All theoretical calculations of  $\hat{B}_B$  tend to give values very close to one for both the  $B_d^0$  and the  $B_s^0$  mesons. Thus the strength of the mixing is essentially regulated by the meson decay constant. It was realized a long time ago [2] that a value of  $\xi \sim 200$  MeV, combined with a large value of the top mass, leads to a large value of  $\sin 2\beta$ , the parameter which controls CP violation in  $B \rightarrow J/\psi K_s$  decays

$$\mathcal{A} = \frac{N(B_d^0 \rightarrow J/\psi K_s)(t) - N(\bar{B}_d^0 \rightarrow J/\psi K_s)(t)}{N(B_d^0 \rightarrow J/\psi K_s)(t) + N(\bar{B}_d^0 \rightarrow J/\psi K_s)(t)} = \sin 2\beta \sin \Delta M_{B_d} t. \quad (2)$$

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<sup>2</sup> We use the normalization convention in which  $f_\pi = 132$  MeV.

Thus a precise theoretical determination of  $f_B$  is very important.

One of the most precise determination of  $f_B$  is presently given by lattice QCD calculations. Leptonic decay constants are usually computed from the matrix element of the lattice axial current:

$$Z_A \langle 0 | A_0 | P(\mathbf{p} = 0) \rangle = i f_P M_P, \quad (3)$$

where  $M_P$  is the mass of the pseudoscalar meson,  $A_0 = \bar{Q}(x) \gamma_0 \gamma_5 q(x)$ , where  $Q$  and  $q$  denote the heavy and light quark fields respectively, is the fourth component of the lattice (local) axial current, and  $Z_A$  is the renormalization constant necessary to relate the lattice operator to the continuum one [3,4]. In the case of  $f_\pi$ ,  $f_K$ , and more recently  $f_{D_s}$ , this is an example of a simple quantity for which a comparison of lattice results with experimental data is possible (we notice *in passing* that  $f_D$  was predicted by lattice calculations long before its measurement [5]).

The major sources of uncertainty in the determination of  $f_P$ , besides the effects due to the use of the quenched approximation, come from the calculation of the constant  $Z_A$  in Eq. (3) and from discretization errors of  $O(a)$ ,  $a$  being the lattice spacing, present in the operator matrix elements. The use of chiral Ward identities allows a non-perturbative determination of  $Z_A$  [4,6], thus eliminating this source of error. Errors of order  $a$  can be reduced by the use of “improved” lattice actions [7,8]. Another method to get rid of  $Z_A$  consists of extracting the decay constants of heavier pseudoscalar mesons by computing the ratio  $R_P = f_P/f_\pi$  and multiplying  $R_P$  by the experimental value of the pion decay constant. Hopefully, by taking the ratio, some of the  $O(a)$  effects are eliminated. These effects are expected to be more important for  $f_D$  than for  $f_{\pi,K}$  since, at current values of  $a$ , the relevant parameter  $m_{\text{charm}} a$  is not very small.

In the last few years, a wide set of results for  $f_P$ , obtained in the quenched approximation by using several lattice quark actions and from numerical simulations on different lattice volumes and at different values of the lattice spacing, have appeared in the literature [9]. Some of these studies tried to extrapolate the decay constant to the continuum limit  $a = 0$  by using values of  $f_P$  obtained at different values of the lattice spacing. If successful, this extrapolation would eliminate the error due to discretization. The extrapolation, how-

ever, is difficult because either one has to use results obtained at large values of  $a$ , where the dynamics is very different from that of the continuum limit, or the range of  $a$  is too small to observe and correct the  $O(a)$  effects with a sufficient precision for the extrapolation. For these reasons, the conclusions of these studies are not very convincing and have changed in time.

In this paper, we present the results of a high statistics study of  $f_P$ , at two different values of the lattice spacing, corresponding to  $\beta = 6.0$  and  $\beta = 6.2$ , obtained with the Wilson and the SW-Clover actions [7]. We have preferred to concentrate our computational effort on two values of  $\beta$  which have been chosen:

- (a) small enough to obtain accurate results on reasonably large physical volumes;
- (b) large enough to avoid the dangerous strong coupling region, which sits at around  $\beta = 5.7$ .

The main parameters of the numerical simulations are given in Table 1. A comparison of the results from two different actions allows us to study the reduction of discretization errors in the improved case and to verify the validity of some recipes that have been proposed to correct  $O(a)$  effects in the Wilson case [12,13]<sup>3</sup>. In the following, we will denote these recipes with the generic name of KLM prescriptions.

In analyzing our results we have been particularly careful to isolate the ground state to avoid higher-mass state contamination. We also studied the dependence of the final results on the extrapolation in the heavy and light quark masses. A poor control of these aspects in the extraction of  $f_P$  can mimic spurious  $O(a)$  effects.

The study of the effects which can fake discretization errors, combined with the high statistics of the numerical runs, leads us to the conclusion that a even higher statistics and a larger spread of  $a$  values are required to uncover satisfactorily the  $O(a)$  dependence of the decay constants. Further studies at smaller values of the lattice spacing (corresponding to  $\beta = 6.4$  and 6.6) with comparable (or smaller) statistical errors and similar (or larger) physical volumes are required to reduce this source of uncertainty.

The main physical results of our study have been given in the abstract. They substantially agree with

<sup>3</sup> A study of the same problem, performed by using the Ward identities to determine the renormalization constants of the axial-vector and vector currents,  $Z_A$  and  $Z_V$ , as a function of the heavy quark mass,  $m_H$ , can be found in Ref. [14].

Table 1  
Summary of the parameters of the runs analyzed in this work

Run	C60	C62	W60	W62a	W62b
$\beta$	6.0	6.2	6.0	6.2	6.2
Action	SW	SW	Wil	Wil	Wil
# Confs	170	250	120	250	110
Volume	$18^3 \times 64$	$24^3 \times 64$	$18^3 \times 64$	$24^3 \times 64$	$24^3 \times 64$
$K_I$	–	0.14144	–	0.1510	–
	0.1425	0.14184	0.1530	0.1515	0.1510
	0.1432	0.14224	0.1540	0.1520	0.1520
	0.1440	0.14264	0.1550	0.1526	0.1526
$K_H$	0.1150	0.1210	0.1255	0.1300	0.1300
	0.1200	0.1250	0.1320	0.1350	0.1350
	0.1250	0.1290	0.1385	0.1400	0.1400
	0.1330	0.1330	0.1420	0.1450	0.1450
	–	–	0.1455	–	0.1500
$t_1-t_2$	15–28	20–28	15–28	20–28	20–28
$a^{-1}(\sigma)$ [10]	1.88(2)	2.55(1)	1.88(2)	2.55(1)	2.55(1)
$a^{-1}(M_\rho)$	1.92(11)	2.56(21)	2.19(9)	2.88(14)	2.92(20)
$a^{-1}(f_\pi)$	2.00(6)	2.73(18)	1.91(6)	2.95(14)	2.96(11)
$a^{-1}(K^*)$ -lp plane [11]	2.00(10)	2.7(1)	2.25(6)	3.0(1)	3.0(1)

recent compilations made in Refs. [9,15]. Preliminary results of this study can be found in Ref. [16].

## 2. Details of the analysis

In this section we describe in detail the extraction of the decay constants from two-point correlation functions and the extrapolation of the results in the heavy and light quark masses to the physical points (including the calibration of the lattice spacing). We discuss systematic errors coming from the time interval of the fit, from the method used to derive the decay constant, from the fit used for the extrapolation in the heavy quark mass or in the light quark mass, and from the choice of the physical scale (from the string tension  $\sigma$ , the mass of the rho  $M_\rho$ , the mass of the  $K^*$   $M_{K^*}$  or from  $f_\pi$ ).

### 2.1. Extraction of the decay constants

Consider the correlation function

$$C_{AP}(t) = \sum_{\mathbf{x}} \langle 0 | A_0(\mathbf{x}, t) P^\dagger(\mathbf{0}, 0) | 0 \rangle, \quad (4)$$

where  $P(x) = \bar{Q}(x) \gamma_5 q(x)$ . We also introduce

$$C_{PP}(t) = \sum_{\mathbf{x}} \langle 0 | P(\mathbf{x}, t) P^\dagger(\mathbf{0}, 0) | 0 \rangle. \quad (5)$$

At large Euclidean times  $C_{AP}(t)$  and  $C_{PP}(t)$  behave as

$$C_{AP}(t) = \frac{Z_{AP}}{M_P} e^{-M_P T/2} \sinh(M_P(T/2 - t)), \quad (6)$$

$$C_{PP}(t) = \frac{Z_{PP}}{M_P} e^{-M_P T/2} \cosh(M_P(T/2 - t)), \quad (7)$$

where  $T$  is the temporal extension of the lattice. We extract the raw lattice value of  $f_P$ ,  $f_P^{\text{latt}}$ , using the usual ratio method

$$f_P^{\text{latt}} = \left\langle \frac{C_{AP}(t)}{C_{PP}(t)} \coth(M_P(T/2 - t)) \right\rangle \frac{\sqrt{Z_{PP}}}{M_P}, \quad (8)$$

where  $\langle \dots \rangle$  is a weighted average over a given time interval  $t_1-t_2$ .  $M_P$  and  $Z_{PP}$  are extracted from a fit of  $C_{PP}(t)$  as a function of  $t$  to the expression given in Eq. (7) in the same interval  $t_1-t_2$  in which we compute  $f_P^{\text{latt}}$ . The values of  $t_1-t_2$  used to extract all the results reported below are given, for the different simulations, in Table 1.

The physical value of  $f_P$  is then simply given by

$$f_P \equiv \frac{\langle 0|A_0|P(\mathbf{p}=0)\rangle}{M_P} = f_P^{\text{latt}} Z_A(a) a^{-1}. \quad (9)$$

Alternatively, we can extract the decay constant of the heavier mesons, by normalizing it to  $f_\pi$  (or to  $f_K$ ), defined as the pseudoscalar decay constant computed in the chiral limit

$$f_P = R_P \times f_\pi^{\text{exp}} = \frac{f_P^{\text{latt}}(M_P)}{f_\pi^{\text{latt}}} \times f_\pi^{\text{exp}}, \quad (10)$$

where  $f_\pi^{\text{latt}} = f_P^{\text{latt}}(M_P = 0)$ . In the following, it will be useful to introduce also  $R_{P_s} = f_{P_s}^{\text{latt}}/f_K^{\text{latt}}$ , where  $f_K^{\text{latt}} = f_P^{\text{latt}}(M_P = M_K)$  and  $f_{P_s}^{\text{latt}}$  is  $f_P^{\text{latt}}$  computed at a value of the light quark mass corresponding to the strange quark,  $m_s$ . We have then

$$f_{P_s} = R_{P_s} \times f_K^{\text{exp}}. \quad (11)$$

We now discuss the choice of the time intervals given in Table 1 and the size of the systematic error due to higher mass states. In all the cases, the range was chosen by demanding that the contamination of the excited states in the parameters of the fits (i.e.  $\langle \frac{C_{AP}}{C_{PP}} \coth \rangle$ ,  $M_P$  and  $Z_{PP}$ ) is at most 20% of the statistical error. We impose this strict criterion to stop the systematic effects of the higher states swamping the  $a$  dependence of  $f_P^{\text{latt}}$ . In particular, we have verified that, due to the contamination of the excited states, the error on the decay constant  $f_P$ , for values of the heavy and light quark masses close to the appropriate ones for the  $D_s$  meson, i.e. without almost any extrapolation in  $m_H$  and  $m_s$ , is of about 2 MeV (assuming  $a^{-1}(\beta = 6.0) = 2$  GeV and  $a^{-1}(\beta = 6.2) = 2.6$  GeV) and that the relative error on  $R_P$  is  $\delta R_P/R_P \sim 0.005$ . This makes us confident that this error is much smaller than the other ones (the statistical error, the error due to the extrapolation in the quark masses and the error coming from the calibration of the lattice spacing).

## 2.2. Extrapolation of the raw results

In order to obtain the physical values of  $f_D$ ,  $f_{D_s}$ , etc., we have to extrapolate  $f_P$  both in the heavy and light quark masses, to fix the value of the lattice spacing in physical units and to determine  $Z_A$  (or to use  $R_P$  to extract the decay constant).

The best method to monitor  $O(a)$  effects is to study  $R_P$ , computed for a fixed physical value of the me-

son mass, using the lattice spacing  $a$  calibrated by the string tension for which, in the quenched approximation, errors are of  $O(a^2)$ . By extracting the decay constant from  $R_P$ , we eliminate the apparent  $a$ -dependence coming from the variation of  $Z_A$  with  $\beta$ .  $Z_A$  is in fact independent of quark masses, except for  $O(a)$  corrections. Since meson masses are not exactly the same for the different simulations, we are forced to make some extrapolation both in the heavy and the light quark masses. In order to minimize the extrapolation, which will be discussed below, and at the same time obtain a physical quantity, we have chosen  $R_{D_s} = f_{D_s}/f_K$ , which is obtained from  $R_{P_s}$ <sup>4</sup>. Indeed, for all the runs listed in Table 1, one of the heavy quark and one of the light quark masses are very close to their physical values ( $m_{\text{charm}}$  and  $m_s$  respectively), and so in this case the effect of the extrapolation is negligible. Only the extrapolation in one of the light quark masses (to the chiral limit) is then relevant in order to get  $R_{D_s}$ . It turns out, see Table 2, that for  $R_{D_s}$  the difference between the value obtained with a linear or a quadratic fit to  $R_{P_s}$  is (at most) 2%, corresponding to an error of about (less) 4 MeV for  $f_{D_s}$ . This is to be contrasted with  $R_D = f_D/f_\pi$  as obtained from a fit to  $R_P$ . In this case, differences between values obtained from a linear or a quadratic fit can be as large as 10%, mostly due to a quadratic dependence of  $f_\pi$  on the light quark masses. The relevance of these differences for the physical results will be discussed in the next section.

As for the mass of the charmed and strange quarks, they have been determined by fixing the  $D$ - and  $K$ -meson masses to their physical values. The choice of the physical quantities used to fix the different quark masses may also affect the effective  $a$ -dependence of the decay constants, since the spectrum also suffers from discretization errors. A different possibility for the method used to fix  $m_s$  will be discussed later on. Our results for  $R_D$ ,  $R_{D_s}$  and  $f_{D_s}/f_D$  are given in Table 2 for the following cases: (i) run C60, (ii) run C62, (iii) run C60 with KLM corrections, (iv) run C62 with KLM corrections, (v) run W60, (vi) run W62 ((a) and (b)), (vii) run W60 with KLM correc-

<sup>4</sup> It may also be that for this quantity the error due to the quenched approximation is smaller, although we have not checked this point yet.

Table 2

Summary of the physical results for  $R_{D_s} = f_{D_s}/f_K$  and  $R_D = f_D/f_\pi$  obtained by extrapolating  $R_P$  and  $R_{P_s}$ . We also give  $f_{D_s}/f_D$ ; “linear” and “quadratic” refer to the fit in the light quark masses

	Method	C60	C62	W60	W62a	W62b
$R_{D_s} = f_{D_s}/f_K$	linear	1.56(3)	1.48(6)	1.11(3)	1.23(4)	1.19(5)
	quadratic	1.57(4)	1.49(7)	1.13(4)	1.25(5)	1.20(5)
	linear KLM	1.59(3)	1.50(6)	1.48(6)	1.52(5)	1.47(6)
	quadratic KLM	1.61(4)	1.51(7)	1.51(5)	1.55(7)	1.47(6)
$R_D = f_D/f_\pi$	linear	1.63(4)	1.58(8)	1.14(4)	1.31(6)	1.25(7)
	quadratic	1.69(8)	1.73(16)	1.19(8)	1.43(11)	1.28(9)
	linear KLM	1.67(4)	1.60(8)	1.52(5)	1.61(7)	1.53(8)
	quadratic KLM	1.72(8)	1.75(16)	1.59(10)	1.77(13)	1.59(11)
$f_{D_s}/f_D$	linear	1.08(1)	1.07(2)	1.06(1)	1.07(1)	1.09(2)
	quadratic	1.09(3)	1.04(4)	1.06(3)	1.06(3)	1.13(3)

tions (KLM-Wilson), (viii) run W62 ((a) and (b)) with KLM corrections.

Since several versions of the KLM prescription exist and, in the Clover case, it must be implemented from  $O(a^2)$  only [14], we now state the recipe used in this work to obtain the results in Table 2. We corrected  $f_P^{\text{latt}}$ , as obtained from Eq. (8), for any given pair of values of the quark masses ( $m_{1,2}$ ), by multiplying it by the factor

$$\mathcal{F}_{\text{KLM}}^{\text{W}} = \sqrt{(1 + am_1)(1 + am_2)}, \quad (12)$$

in the Wilson case and by the factor [14]

$$\mathcal{F}_{\text{KLM}}^{\text{C}} = \frac{\mathcal{F}_{\text{KLM}}^{\text{W}}}{F_1 F_2}, \quad (13)$$

in the Clover case<sup>5</sup>, where

$$F_{1,2} = 1 + \frac{1}{4} [(1 + am_{1,2}) - (1 + am_{1,2})^{-1}].$$

In Table 2, results from linear and quadratic fits in the light quark masses are both given. We do not give results for  $f_{D_s}/f_D$  with KLM corrections because they essentially cancel out for this ratio.

We have the following observations on the results given in Table 2. (a) Without KLM factors the results in the Wilson case are incompatible with those obtained with the Clover action. (b) KLM-Wilson results are compatible (indistinguishable) with the

<sup>5</sup> The factor  $\sqrt{4K_1 K_2}$ , where  $K_{1,2}$  are the hopping parameters corresponding to the masses  $m_{1,2}$ , is always included in our definition of the currents.

Clover results at  $\beta = 6.0$  ( $\beta = 6.2$ ). (c) Within the statistical errors which are of the order of 3% (corresponding to about 6 MeV of error for  $f_{D_s}$ ), KLM-Wilson results do not exhibit any appreciable  $a$ -dependence; (d) Even with such small statistical errors, it is impossible to decide whether there is a systematic shift of the Clover results between  $\beta = 6.0$  and  $\beta = 6.2$  or the difference only comes from a statistical fluctuation. This is due to several reasons: differences between KLM-Wilson and Clover results at  $\beta = 6.0$  are of the same size than those between  $\beta = 6.0$  and  $\beta = 6.2$  in the Clover case; in the Clover case, the KLM corrections to  $f_{D_s}/f_K$  go in the wrong direction (they increase the difference between the results at  $\beta = 6.0$  and  $6.2$ ). If we fit  $R_P$  linearly in the light quark masses, the KLM corrections increase the difference between the results at  $\beta = 6.0$  and  $6.2$  also in the case of  $R_D$ . This is opposite to what happens with a quadratic fit, although within larger errors: in this case the KLM corrections reduce the differences between the results at  $\beta = 6.0$  and  $6.2$ . (e) In the Wilson case, the variations observed between two different runs at  $\beta = 6.2$  are about one half of the differences between the two Clover results  $\beta = 6.0$  and  $\beta = 6.2$ .

From the above considerations, it is clear that any attempt to extrapolate our results to  $a = 0$  in order to reduce discretization error would be fruitless. This is even more true because, as observed in Ref. [11], the results of the extrapolation are extremely sensitive to the choice of the scale, given the small range in  $a$  at disposal. We could have easily done a high statistics,

very accurate calculation at lower values of  $\beta$ , e.g.  $\beta = 5.7$ , to increase the range in  $a$ . We believe, however, that at such large values of  $a$  the behaviour of the lattice dynamics is very different from the continuum one, so that the inclusion of this point could fake completely the final result. This is true not only for our data, which have tiny statistical errors, and have been obtained on (approximatively) the same physical volume, but also for many other analyses which have been recently presented in the literature [9]. With this we do not mean that in the KLM-Wilson and Clover cases there are not sizeable  $O(a)$  effects for  $m_H$  around the charm quark mass<sup>6</sup> (indeed there are indications, from calculations of the dependence of the effective  $Z_A$  on  $m_H$ , that these effects may be large [14]). We want to stress that, in spite of the very good accuracy of our data, it would be illusory to try to correct  $O(a)$  effects by extrapolating to  $a = 0$ . This can be done only by enlarging the range of  $a$  towards even smaller values, corresponding to  $\beta = 6.4$  or even 6.6, while keeping the physical volume, and the statistical accuracy the same as at  $\beta = 6.0$  and 6.2, see also Ref. [17].

### 2.3. Other sources of uncertainty

In this subsection, we discuss some tests which we performed in order to check the stability of the results for the D-meson decay constants (see Table 2) and the B-meson decay constants (presented in Table 3). These calculations consist in extracting the physical results with different assumptions on the parameters and the method chosen for the extrapolation.

To be specific, we consider  $R_{D_s}$ , obtained in the Clover case from a linear fit in the light quark mass and (then) from a fit in the heavy quark mass (at fixed  $m_s$ ) of the form

$$f_{P_s} \sqrt{M_{P_s}} \approx \Phi_s^{\text{inf}} + \frac{\Phi'_s}{M_{P_s}} + \frac{\Phi''_s}{M_{P_s}^2} + \dots, \quad (14)$$

and similarly for  $f_P \sqrt{M_P}$ .  $\Phi_s^{\text{inf}}$ ,  $\Phi'_s$ ,  $\Phi''_s$  are functions which are expected to depend logarithmically on  $m_H$  but have been taken constant in the fit. For  $R_{D_s}$  the inclusion of the quadratic term of Eq. (14) in the fit is

immaterial, since it amounts to a difference of less than about 3 MeV for  $f_{D_s}$ . In the B-case, the difference is of about 5 MeV for  $f_{B_s}$ . Eq. (14) has also been used to extrapolate in  $m_H$  the results of Table 2, discussed in the previous subsection. If not stated otherwise, the fits in the heavy quark mass always include the quadratic term.

We have considered the following cases:

1. *Choice of the scale.* Although  $R_{D_s}$  is a dimensionless quantity, the calibration of the lattice spacing can affect its value because it enters in the determination of the values of the quark masses at which we extrapolate  $R_{P_s}$ . At  $\beta = 6.2$  with the Clover action, using a linear fit in the light quark masses, we obtain  $R_{D_s} = 1.48(6)$  (scale from  $\sigma$ ),  $1.48(6)$  (scale from  $M_\rho$ ),  $1.47(5)$  (scale from  $f_\pi$  with  $Z_A = 1.045$  as determined non-perturbatively with the Ward identities [18]) and  $R_{D_s} = 1.47(6)$  (scale from  $K^*$ , using the lp-plane method of Ref. [11], see below). A similar result is obtained at  $\beta = 6.0$  where we used  $Z_A = 1.06$  [19]. Thus the error due to the calibration of the lattice spacing is negligible for this ratio.
2. *Strange quark mass:* There are different equivalent methods to fix  $m_s$ , from  $M_K$ , from  $M_{K^*}$ , or with the ‘‘lattice physical plane’’ (lp-plane) method which has been used in Refs. [11,20] and that we briefly recall below. All of these methods should give the same value of  $m_s$ , apart from discretization errors and quenching effects. Thus we also computed  $R_{D_s}$  by fixing  $m_s$  with the lp-plane method for comparison. With this method, for the problem at hand, one defines a physical plane  $[M_V a, (M_P a)^2]$ , where  $M_V$  is the mass of the vector meson. In  $[M_V a, (M_P a)^2]$ , assuming that only linear terms are important, one looks for the point where  $M_V/M_P$  coincides with the physical value  $M_{K^*}/M_K$ , and reads off the value of  $m_s$  which is then used to evaluate  $R_{D_s}$ . The results that we obtain in this case are indistinguishable from those given in Table 2.
3. *Extrapolation in the heavy quark mass.* Among the suggestions introduced with the aim of correcting  $O(a)$  effects in the Wilson case, it has been proposed [21], besides rescaling the quark fields according to the KLM prescription, to shift

<sup>6</sup> It is evident that without the KLM factors these effects are very large in the Wilson case.

the mass  $M_P$  by

$$\begin{aligned} M_P a &\rightarrow \tilde{M}_P a = M_P a - \tilde{m}_H + \bar{m}_H, \\ (Q, q) &\rightarrow \mathcal{N}(K_{H,l})(Q, q), \end{aligned} \quad (15)$$

where

$$\begin{aligned} \mathcal{N}(K_H) &= \left( \frac{4K_{\text{crit}}}{K_H} - 3 \right)^{1/2}, \quad \tilde{m}_H = \\ &\log(\mathcal{N}^2(K_H)), \\ \bar{m}_H &= e^{\tilde{m}_H} \sinh \tilde{m}_H / (\sinh \tilde{m}_H + 1). \end{aligned} \quad (16)$$

$K_{\text{crit}}$  is the critical value of the Wilson hopping parameter and  $\mathcal{N}(K_H)$  is simply a different form of the KLM prescription which was used in Eq. (12). We have then fitted the result in  $\tilde{M}_P$  instead than  $M_P$ . The values of  $R_{D_s}$  change as follows:  $R_{D_s} = 1.48(6) \rightarrow 1.44(3)$  (W60),  $1.52(5) \rightarrow 1.51(5)$  (W62a) and  $1.47(6) \rightarrow 1.47(6)$  (W62b). Thus, the results obtained by using the recipe in Eq. (15) are indistinguishable, within the errors from the KLM-Wilson results reported in Table 2. This is not surprising because  $\tilde{M}_P$  is only shifted by a small amount from  $M_P$  and the range of the extrapolation is very small. The effect is only slightly larger after the extrapolation to  $B$ -mesons, and give a small shift, corresponding to an uncertainty of about 5 MeV for  $f_{B_s}$ . We included this error, by combining it in quadrature with other ones, in the final evaluation of  $f_{B_s}$  and  $f_B$ , which can be found in Subsection 3.2.

This concludes the discussion of several minor uncertainties in the calculation of  $R_{D_s}$ ,  $f_{D_s}/f_D$ ,  $R_{B_s}$  and  $f_{B_s}/f_B$ . All of them have little influence in both the  $D$ - and  $B$ -meson cases.

### 3. Physical results

On the basis of the discussion of Section 2, we are now ready to present our final results and errors. We will first give the results for charmed mesons, for which the extrapolation in the heavy quark mass is not a relevant source of systematic uncertainties, and then discuss the  $B$ -meson case.

#### 3.1. Results for $f_{D_s}$ and $f_D$

From Section 2, we learned that an extrapolation to  $a = 0$  it is not possible at this stage. Thus we believe that the best estimate of  $f_{D_s}$  is obtained from the Clover data at  $\beta = 6.2$ , by using  $R_{D_s}$  in Eq. (11) (from a linear fit in the light quark masses, a quadratic fit in  $1/M_P$ , and without any KLM factor). As for the error, we take as a conservative estimate of the discretization error the difference between the results obtained at  $\beta = 6.0$  and 6.2, and combine it in quadrature with the statistical one. This gives  $R_{D_s} = 1.48(10)$  from which, by using  $f_K^{\text{exp}} = 159.8$  MeV, we obtain  $f_{D_s} = 237 \pm 16$  MeV. By using  $R_{D_s} = 1.48(10)$  combined with  $f_{D_s}/f_D = 1.07(4)$  we obtain  $f_D = 221 \pm 17$  MeV. We believe that these results, which have also been given in the abstract, are our “best” results. They are in very good agreement with the compilation of lattice calculations presented in Refs. [9,15]. Since at  $\beta = 6.2$  the ratio  $R_{D_s}$  is essentially identical in the Clover and KLM-Wilson case, the Wilson results do not add much information, besides giving a consistency check.

If we extract  $f_D$  from  $R_D$  by using Eq. (10) instead, we have to take into account the larger differences due to the use of the linear or the quadratic extrapolation in the light quark masses. We also take into account an error on  $f_D$  of about 6 MeV, which comes from the difference in the results obtained with a linear or quadratic fit of  $f_P$  to Eq. (14). Proceeding as before, i.e. taking as discretization error the difference between the result obtained at  $\beta = 6.0$  and 6.2, and combining it in quadrature with the statistical error, we find  $f_D = 209 \pm 16$  MeV (linear) or  $f_D = 228 \pm 22$  MeV (quadratic). Our previous result for  $f_D$  sits in the middle of these two numbers.

To reduce the uncertainties due to the extrapolation in the light quark masses, and which are mainly related to the quadratic dependence of  $f_\pi$  on the quark masses, we can compute directly  $f_D$ , by assuming some value for the calibration of  $a$  and for  $Z_A$ . By taking  $a^{-1}$  as determined from  $M_\rho$ , which in the Clover case is  $a^{-1}(\beta = 6.0) = 1.92(11)$  and  $a^{-1}(\beta = 6.2) = 2.56(21)$ ,  $Z_A(\beta = 6.0) = 1.06$  and  $Z_A(\beta = 6.2) = 1.045$  [18,19], and fixing  $m_s$  from  $M_K$ , we obtain (only linear fits in the light quark masses)  $f_D(\beta = 6.0) = 212 \pm 20$  MeV,  $f_D(\beta = 6.2) = 204 \pm 20$  MeV,  $f_{D_s}(\beta = 6.0) = 219 \pm 17$  MeV and  $f_{D_s}(\beta = 6.2) =$

Table 3

Summary of the physical results for  $f_{B_s}/f_K$ ,  $f_B/f_\pi$  obtained by extrapolating  $R_P$  and  $R_{P_s}$ . We also give  $f_{B_s}/f_B$ . “linear” and “quadratic” refer to the fit in the light quark masses

	Method	C60	C62	W60	W62a	W62b
$f_{B_s}/f_K$	linear	1.48(7)	1.28(9)	0.79(4)	0.83(4)	0.84(5)
	quadratic	1.49(9)	1.27(12)	0.81(5)	0.84(5)	0.84(5)
	linear KLM	1.56(7)	1.33(9)	1.29(4)	1.26(6)	1.16(7)
	quadratic KLM	1.57(12)	1.33(11)	1.33(6)	1.27(8)	1.16(7)
$f_B/f_\pi$	linear	1.53(9)	1.32(13)	0.81(5)	0.86(6)	0.86(6)
	quadratic	1.57(16)	1.37(66)	0.87(9)	0.91(11)	0.86(9)
	linear KLM	1.61(10)	1.38(13)	1.32(6)	1.31(8)	1.19(8)
	quadratic KLM	1.65(32)	1.43(50)	1.41(11)	1.39(15)	1.19(11)
$f_{B_s}/f_B$	linear	1.10(3)	1.14(6)	1.05(2)	1.10(3)	1.12(3)
	quadratic	1.13(7)	1.17(17)	1.03(6)	1.14(6)	1.20(7)

$230 \pm 9$  MeV, which are slightly lower than the previous results, but perfectly consistent with them.

The latter two determinations of  $f_{D_s}$  and  $f_D$  are, however, subject to larger systematic effects, either due to the choice of the fit or to the assumptions for the values of  $a$  and  $Z_A$ , than those extracted from  $R_{D_s}$  and  $f_{D_s}/f_D$ . We therefore prefer these as our best values.

### 3.2. Results for $f_{B_s}$ and $f_B$

In order to obtain  $f_{B_s}$  and  $f_B$ , an extrapolation in the heavy quark mass well outside the range available in our numerical simulations is necessary. Discretization errors can affect the final results in two ways. Not only do they change the actual values of the decay constants, but also deform the dependence of  $f_P$  on  $m_H$ . Moreover, points obtained at the largest values of  $m_H a$  become the most important, since we extrapolate in the direction of even larger values of  $m_H$ .

In Fig. 1, we show the KLM-Wilson and Clover results for  $f_P/f_\pi \sqrt{M_P/\sigma^{1/2}}$  as a function of the dimensionless scale  $\sigma^{1/2}/M_P$ . We note the remarkable agreement between the scaled KLM-Wilson and the Clover data (as was first observed in [22]). In the figure, we do not show the Wilson results without the KLM prescription, because they are inconsistent between each other (at the two values of  $\beta$ ) and with the Clover results.

The results reported in Table 3 were obtained by fitting  $R_P$  ( $R_P$ ) to Eq. (14), including the quadratic term. We notice that all the differences observed in Table 2 for  $D$ -mesons are amplified when we extrapolate

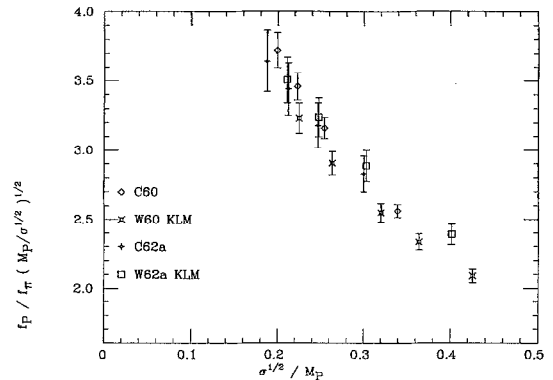


Fig. 1. Dependence of  $f_P/f_\pi (M_P/\sigma^{1/2})^{1/2}$  on  $\sigma^{1/2}/M_P$ . For these points a linear extrapolation in the light quark masses to the chiral limit has been used.

to  $B$ -mesons. Not only is this true for the results of the runs C60 and C62, where we can suspect that the differences are due to discretization errors, but also for the runs W62a and W62b, performed at the same value of  $\beta$ . Thus, the same general considerations made in Section 2 for  $R_{D_s}$  apply here. In order to extract our “best” values for  $f_{B_s}$  and  $f_B$  we proceed exactly as in Subsection 3.1. We obtain  $f_{B_s} = 205 \pm 15 \pm 31$  MeV =  $205 \pm 35$  MeV, where the second error (31 MeV) is the “discretization” error, estimated by comparing the results from C60 and C62. We also obtain  $f_B = 180 \pm 32$  MeV. As in the  $D$ -meson case, these results are in good agreement with the compilation of lattice calculations presented in Refs. [9,15].



#### 4. Conclusion

From several runs obtained using the Wilson and Clover actions at  $\beta = 6.0$  and  $6.2$ , from a careful analysis of all possible effects which can fake discretization errors, we conclude that, using the methods outlined above, we require even higher statistics, or a larger spread of  $a$  values, to uncover satisfactorily the  $O(a)$  dependence of the decay constants. We believe that other studies of the same problem have the same difficulties as us in controlling and correcting discretization errors. Further studies, with comparable (or smaller) statistical errors and physical volume, at smaller values of the lattice spacing, corresponding to  $\beta = 6.4$  and  $6.6$ , are required to reduce this source of uncertainty. The use of the action of Ref. [8] can be of great help in this respect.

By assuming quite conservative discretization errors we found

$$\begin{aligned} f_{D_s} &= 237 \pm 16 \text{ MeV}, & f_D &= 221 \pm 17 \text{ MeV}, \\ f_{B_s} &= 205 \pm 35 \text{ MeV}, & f_B &= 180 \pm 32 \text{ MeV}, \\ \frac{f_{D_s}}{f_D} &= 1.07 \pm 0.04, & \frac{f_{B_s}}{f_B} &= 1.14 \pm 0.08 \end{aligned} \quad (17)$$

in good agreement with previous estimates [9,15].

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