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# An Advanced Approach to Bertrand Curves in 4-Dimensional Minkowski Space

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**Abstract.** This work presents a novel and comprehensive approach to the study of Bertrand curves in 4-dimensional Minkowski space ($\mathbb{R}^4_1$). We introduce a new Frenet frame specifically tailored for analyzing Bertrand curves in the $(1,3)$-normal plane, which allows us to derive significant relationships between the curvature functions $\kappa_1$, $\kappa_2$, and $\kappa_3$. Our analysis provides new formulas and explicit conditions for these curvatures, offering a deeper understanding of their geometric properties in $\mathbb{R}^4_1$.
We investigate four distinct cases of Bertrand curves, each characterized by specific conditions on the curvature functions. For each case, we derive explicit solutions and relationships, demonstrating the versatility of our approach. Furthermore, we establish the existence of a Bertrand mate curve $\zeta^*$ for a given Bertrand curve $\zeta$ and derive the parameter $\lambda$ that defines the mate curve. This parameter is expressed in terms of the curvature functions, providing a clear connection between the original curve and its mate.
To illustrate the practical application of our theoretical results, we provide detailed examples of Bertrand curve pairs in $\mathbb{R}^4_1$. These examples include the explicit construction of the Frenet frames and the computation of the associated curvature functions, showcasing the effectiveness of our methodology.

**Key Words and Phrases**: Bertrand curves, new Frenet frame, curvature functions, Minkowski space

**2010 Mathematics Subject Classifications**: 53A04, 53A35, 53B30

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## 1. Introduction

Curve analysis is a fundamental and fascinating topic in classical differential geometry, with applications spanning various fields such as physics, engineering, and biology. Curves, as geometric objects, play a significant role in understanding the intrinsic properties of spaces, and their study has led to the development of numerous mathematical tools and frameworks. Among the many types of curves, curve pairs—defined as curves that are mutually dependent and share common geometric characteristics—have garnered considerable attention. Examples of such curve pairs include the Bertrand curve, the quaternionic-Bertrand curve, the involute-evolute curve, the Mannheim curve, helices, slant helices, and rectifying curves [1, 2, 3]. These curve pairs are not only of theoretical interest but also have practical applications in areas such as robotics, computer graphics, and the modeling of physical systems.

The concept of Bertrand curves traces its origins to the work of the French mathematician B. Saint-Venant [4], who posed a problem in differential geometry that led to the discovery of these curves. In 1850, Joseph Bertrand [5] provided a comprehensive investigation into this problem, leading to the formal definition of Bertrand curves. Two curves are said to be Bertrand curves if they share a common principal normal vector at every corresponding point. The first curve is referred to as the Bertrand curve, while the second curve is known as its mate curve. The Bertrand curve is also sometimes called the conjugate curve [6], and its study has found applications in classical differential geometry, differential equations, physics, and biology [7, 8]. The geometric properties of Bertrand curves, such as their curvature and torsion, have been extensively studied, and their generalizations to various spaces have led to new insights in differential geometry.

The study of Bertrand curves was significantly advanced by L. R. Pears [9], who extended the concept from Euclidean 3-space ($\mathbb{E}^3$) to Riemannian $n$-space, providing generalized results about these curves. This extension opened the door for further exploration of Bertrand curves in higher-dimensional spaces and more complex geometric settings. Matsuda and Yorozu [10] introduced a novel description of the Bertrand curve, known as the $(1,3)$-Bertrand curve, and conducted an exhaustive analysis of its properties. Their work laid the foundation for subsequent studies on these curves, which have been further explored by various scholars [11, 12, 13, 14, 15, 16, 17]. For instance, Nolasco and Pacheco [18] demonstrated a correlation between planar curves and null curves in $\mathbb{E}^3_1$, while Çöken and Çiftçi [12] defined pseudo-spherical lightlike curves from null Bertrand curves. Additional findings on Bertrand curves can be found in [19, 20, 21, 22, 23, 24, 25, 26], which explore their properties in various contexts, including Minkowski space-time.

In recent years, the study of Bertrand curves has been extended to Minkowski space-time, a pseudo-Riemannian manifold that plays a crucial role in the theory of relativity. Several scholars [13, 14, 15] have engaged in discussions on the Bertrand curve and its partner curve within the framework of Minkowski space-time, leading to new insights into their geometric properties. These studies have highlighted the importance of understanding the curvature functions and the relationships between the original curve and its mate curve in higher-dimensional spaces. Furthermore, the development of new Frenet frames and the exploration of their applications have provided powerful tools for analyzing Bertrand curves in these settings [27].

This study employs an innovative methodology to examine Bertrand curves in a four-dimensional Minkowski space ($\mathbb{R}^4_1$), focusing on their geometric properties and the relationships between their curvature functions. By introducing a new Frenet frame specifically tailored for analyzing Bertrand curves in the $(1,3)$-normal plane, we derive significant relationships between the curvature functions $\kappa_1$, $\kappa_2$, and $\kappa_3$. Our analysis provides new formulas and explicit conditions for these curvatures, offering a deeper understanding of their geometric properties in $\mathbb{R}^4_1$. We investigate four distinct cases of Bertrand curves, each characterized by specific conditions on the curvature functions, and derive explicit solutions and relationships for each case. Additionally, we establish the existence of a Bertrand mate curve $\zeta^*$ for a given Bertrand curve $\zeta$ and derive the parameter $\lambda$ that defines the mate curve. This parameter is expressed in terms of the curvature functions, providing a clear connection between the original curve and its mate.

To illustrate the practical application of our theoretical results, we provide detailed examples of Bertrand curve pairs in $\mathbb{R}^4_1$. These examples include the explicit construction of the Frenet frames and the computation of the associated curvature functions, showcasing the effectiveness of our methodology. The findings of this study not only enhance the theoretical foundation of Bertrand curves in Minkowski space but also provide a framework for further exploration in differential geometry and related fields. Future research directions include the extension of this framework to higher-dimensional Minkowski spaces, the application of these results to physical models, and the development of computational tools for the automated analysis of Bertrand curves and their mate curves.

## 2. Preliminaries

$\mathbb{R}^4_1$ is basically a 4-dimensional Euclidean space, together with an indefinite flat metric $g$ with the signature $(-,+,+,+)$. We define the bilinear metric as follows:

$$g(L,M) = -l_1m_1 + l_2m_2 + l_3m_3 + l_4m_4.$$

For any two vectors $L = \{l_1, l_2, l_3, l_4\}$ and $M\{m_1, m_2, m_3, m_4\}$. Recall a vector $u \in \mathbb{R}^4_1 \setminus \{0\}$ is said to be *spacelike* if $g(u,u) > 0$, *timelike* if $g(u,u) < 0$ or $u = 0$, and null (*lightlike*) if $g(u,u) = 0$ and $u \neq 0$. We can take the norm of any vector, say $u \neq 0$, as $||u|| = \sqrt{|g(u,u)|}$ in $\mathbb{R}^4_1$.

A curve $\zeta : I \subset \mathbb{R} \mapsto \mathbb{R}^4_1$ is considered to be *spacelike* if for $\zeta'(s) \neq 0$ and $\langle\ \zeta'(s), \zeta'(s)\ \rangle > 0$ and *timelike* if $\langle\ \zeta'(s), \zeta'(s)\ \rangle < 0$ and null(*lightlike*) if $\langle\ \zeta'(s), \zeta'(s)\ \rangle = 0$ for all $s \in \mathbb{R}$ according to B O'Neill [22].

Consider a regular curve $\zeta : I \subset \mathbb{R} \mapsto \mathbb{R}^4_1$ with $s$ as an arc length parameter on an open interval under the Frenet frame $\{T, N, B_1, B_2\}$, where $T$ is a tangent vector, $N$ is a principal normal vector, and $B_1,\ B_2$ are first and second binormal vectors that coincide with the standard orientation such that $T \wedge N \wedge B_1 \wedge B_2$. In $\mathbb{R}^4_1$ K. Ilarslan [19] introduced a Frenet serret formula for curves in $\mathbb{R}^4_1$ as:

$$\begin{cases} T' = \epsilon_2 k_1 N, \\ N' = -\epsilon_1 k_1 T + \epsilon_3 k_2 B_1, \\ {B_1}' = -\epsilon_2 k_2 N - \epsilon_1\epsilon_2\epsilon_3 k_3 B_2, \\ {B_2}' = -\epsilon_3 k_3 B_1. \end{cases} \tag{1}$$

Here $\kappa_1,\ \kappa_2$ and $\kappa_3$ represents non-vanishing curvatures functions. Correspondingly, the following conditions hold:

$$g(T,T) = \epsilon_1 = \pm 1, \quad g(N,N) = \epsilon_2 = \pm 1, \quad g(B_1,B_1) = \epsilon_3 = \pm 1, \quad g(B_2 B_2) = \epsilon_4 = \pm 1,$$

also

$$\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = -1.$$

More precisely, the following requirements are satisfied:

$$g(T,N) = g(T,B_1) = g(T,B_2) = g(N,B_1) = g(N,B_2) = g(B_1,B_2) = 0.$$

A new Frenet frame is given by:

$$\begin{cases} T' &= \kappa_1 N, \\ N' &= -\kappa_1 T + \kappa_2 B_1, \\ B_1' &= \kappa_3 N + \kappa_2 B_2, \\ B_2' &= \kappa_3 B_1. \end{cases} \tag{2}$$

Since

$$g(T,T) = g(B_1,B_1) = 1, \quad g(N,N) = g(B_2,B_2) = 0,$$

$g(T,N) = 0,$ when $\kappa_2 = 0$ and $g(T,B_2) = g(N,B_1) = g(T,B_1) = g(B_1,B_2) = 0, \quad g(N,B_2) = -1.$

## 3. Bertrand Curves in 4-Dimensional Minkowski Space with a New Frenet Frame

The Bertrand curve in Minkowski 4-space ($\mathbb{R}_1^4$) using new Frenet frame is examined in this section. The following definition will give you a good idea of what the Bertrand and the Bertrand mate curve are in $\mathbb{R}_1^4$.

**Definition 3.1.** Two curves $\zeta : I \subset \mathbb{R} \mapsto \mathbb{R}_1^4$ and $\zeta^* : I^* \subset \mathbb{R} \mapsto \mathbb{R}_1^4$ are considered a Bertrand curve and its mate curve if and only if $\exists$ a bijection $\varphi : \zeta \mapsto \zeta^*$ with a common principle normal vectors at corresponding points of $\zeta$ and there exist $\lambda$ such that

$$T + \lambda N = T^*,$$

and $T^*$ is a tangent vector of $\zeta^*$.

Suppose $\zeta : I \subset \mathbb{R} \mapsto \mathbb{R}_1^4$ as a Bertrand curve with non-vanishing curvatures and Frenet frame $\{T(s), N(s), B_1(s), B_2(s)\}$ in $(1,3)$-normal plane. Then, we can write the Bertrand mate curve $\zeta^* : I^* = (a,b) \subset \mathbb{R} \mapsto \mathbb{R}_1^4$ with the Frenet frame $\{T^*(s), N^*(s), {B_1}^*(s), {B_2}^*(s)\}$ in the $(1,3)$-normal plan as follows:

$$\zeta^*(s^*) = \zeta(s) + a(s)N(s) + b(s)B_2(s),$$

$\forall s^* \in I^*$ and $a(s),\ b(s)$ are smooth functions.

**Theorem 3.1.** Consider $\zeta : I \subset \mathbb{R} \mapsto \mathbb{R}_1^4$ is a Bertrand curve in the $(1,3)$-normal plan, with the Frenet frame $\{T, N, B_1, B_2\}$ and non-vanishing curvature functions. Then, using the Bertrand-mate curve $\zeta^* : I^* \subset \mathbb{R} \mapsto \mathbb{R}_1^4$, we obtain the following results in the $(1,3)$-normal plane under the Frenet frame $\{T^*(s), N^*(s), {B_1}^*(s), {B_2}^*(s)\}$ as follows:

$$\text{(i)}\quad ak_2 + bk_3 \neq 0, \qquad \text{(ii)}\quad 1 - ak_1 = h(ak_2 + bk_3),$$
$$\text{(iii)}\quad \mu k_2 = hk_1 + k_3, \qquad \text{(iv)}\quad h\mu k_1 - (\mu + \sigma)k_2 - k_3 = 0,$$

where $a, b, h \in \mathbb{R}$ and $\mu, \sigma$ are smooth functions.

**Proof.** Let $\zeta : I \subset \mathbb{R} \mapsto \mathbb{R}_1^4$ is a Bertrand curve with non-vanishing curvature functions in the $(1,3)$-normal plane, with the arc length parameter $s$. Then in $(1,3)$-normal plan $\zeta^*$ can be expressed as:

$$\zeta^*(s^*) = \zeta^*(f(s)) = \beta(s) + a(s)N(s) + b(s)B_2(s). \tag{3}$$

By taking derivative of equation (3) by $s$ using equation (2), we get

$$T^* f' = (1 - ak_1)T(s) + a^{'}(s)N(s) + b^{'}(s)B_2(s) + (ak_2 + bk_3)B_1(s). \tag{4}$$

By multiplying with $N$ and $B_2$ apiece, we obtain $a^{'} = 0$ and $b^{'} = 0$.
Hence equation (4) gets the form

$$f^{'}T^* = (1 - ak_1)T(s) + (a(s)k_2 + b(s)k_3)B_1(s). \tag{5}$$

Denoting

$$\delta = \frac{(1 - ak_1)}{f'} \quad \text{and} \quad \gamma = \frac{(ak_2 + bk_3)}{f'}. \tag{6}$$

We obtain

$$T^* = \delta T(s) + \gamma B_1(s). \tag{7}$$

Differentiating (7) with respect to $s$ by using Frenet formula (2)

$$\epsilon_2 f' k_1^* N^* = \delta' T(s) + (\delta k_1 + \gamma k_3)N(s) + \gamma' B_1(s) + \gamma k_2 B_2(s). \tag{8}$$

Multiplying by $T$ and $B_1$ apiece, we acquire

$$\delta' = 0, \ \gamma' = 0. \tag{9}$$

Using equation (6), we get

$$(1 - ak_1)\gamma = \delta(ak_2 + bk_3). \tag{10}$$

Assume $\gamma = 0$, so from (7), $T^* = \delta T$. Hence

$$T^* = \pm T. \tag{11}$$

Differentiating by $s$ using equation (2), we get

$$f' k_1^* N^* = \pm k_1 N. \tag{12}$$

Following equation (12), we see that $N$ is linearly dependent on $N^*$, which leads to a contradiction. Hence, $\gamma \neq 0$. From equation (6), we obtain the solution (i) considering $\gamma \neq 0$.

$$(ak_2 + bk_3) \neq 0. \tag{13}$$

From (10), we get the result (ii)

$$(1 - ak_1) = h(ak_2 + bk_3), \tag{14}$$

where $h = \frac{\delta}{\gamma}$ for $\gamma \neq 0$.
Multiplying (5) by itself, we get

$$(f^{'})^2 = (1 - ak_1)^2 + (ak_2 + bk_3)^2. \tag{15}$$

Using equation (14) in (15), we get

$$(f^{'})^2 = (ak_2 + bk_3)^2(h^2 + 1). \tag{16}$$

Using (9) in (8), we get

$$\epsilon_2 f' k_1^* N^* = (\delta k_1 + \gamma k_3) N(s) + \gamma k_2 B_2(s). \tag{17}$$

Multiplying (17) by itself, we obtain

$$\epsilon_2 (f')^2 (k_1^*)^2 = -2(\gamma k_2)(\delta k_1 + \gamma k_3). \tag{18}$$

Substituting (6) in (18), we obtain

$$(f')^2 (k_1^*)^2 = -2k_2 \frac{(ak_2 + bk_3)^2}{(f')^2} [hk_1 + k_3]. \tag{19}$$

Using (16) in (19), we get

$$(f')^2 (k_1^*)^2 = \frac{-2k_2 \epsilon_2}{h^2 + 1} [hk_1 + k_3]. \tag{20}$$

If we denote

$$\lambda_1 = \frac{\delta k_1 + \gamma k_3}{f' k_1^*} = \frac{(ak_2 + bk_3)}{(f')^2 k_1^*} [hk_1 + k_3], \tag{21}$$

$$\lambda_2 = \frac{\gamma k_2}{f' k_1^*} = \frac{(ak_2 + bk_3)}{(f')^2 k_1^*} [k_2], \tag{22}$$

we get

$$N^* = \lambda_1 N(s) + \lambda_2 B_2(s). \tag{23}$$

Differentiating (23) w.r.t $s$ using Frenet frame (2), we get

$$f' k_1^* T^* + f' k_2^* B_1^* = \lambda_1^{'} N(s) + \lambda_2^{'} B_2(s) - \lambda_1 k_1 T(s) + (\lambda_1 k_2 + \lambda_2 k_3) B_1(s). \tag{24}$$

Taking the dot product of $N$ and $B_2$ with the equation (24), individually, we get

$$\lambda_1^{'} = 0, \ \lambda_2^{'} = 0. \tag{25}$$

Dividing equation (21) by (22), since $\lambda_2 \neq 0$, we have result (iii)

$$\mu k_2 = hk_1 + k_3, \tag{26}$$

where $\mu = \frac{\lambda_1}{\lambda_2}$.
Using equation (25) in (24), we get

$$f' k_2^* B_1^* = f' k_1^* T^* + (\lambda_2 k_1 - \lambda_1 k_1) T(s) + (\lambda_1 k_2 + \lambda_2 k_3) B_1(s). \tag{27}$$

Using equation (5) in (27), we have the relation

$$f' k_2^* B_1^* = (hk_1^*(ak_2 + bk_3) - \lambda_1 k_1) T(s) + ((ak_2 + bk_3) k_1^* + \lambda_1 k_2 + \lambda_2 k_3) B_1(s). \tag{28}$$

From this, we may suppose like this

$$\frac{(ak_2+bk_3)k_1^*+\lambda_1k_2+\lambda_2k_3}{hk_1^*(ak_2+bk_3)-\lambda_1k_1}=-\frac{\delta}{\gamma}=-h. \tag{29}$$

$$(ak_2+bk_3)k_1^*[1+h^2]=h\lambda_1k_1-\lambda_1k_2-\lambda_2k_3. \tag{30}$$

Taking square on both side

$$(ak_2+bk_3)^2(k_1^*)^2[1+h^2]^2=[h\lambda_1k_1-\lambda_1k_2-\lambda_2k_3]^2. \tag{31}$$

Substituting value of $(k_1^*)^2$ from equation (19), we get

$$-2k_2\epsilon_2[hk_1+k_3]=[h\lambda_1k_1-\lambda_1k_2-\lambda_2k_3]^2. \tag{32}$$

Using equation (26), we get

$$-2(k_2)^2\epsilon_2\mu=\lambda_2^2[h\mu k_1-\mu k_2-k_3]^2. \tag{33}$$

Suppose $\lambda_3^2=-2\epsilon_2\mu$ and $\omega=h\mu k_1-\mu k_2-k_3$, using in above equation

$$\lambda_3^2(k_2)^2=\lambda_2^2\omega^2. \tag{34}$$

Taking square root on both side, we get

$$\omega=\frac{\lambda_3}{\lambda_2}k_2. \tag{35}$$

Finally, we get the result (iv)

$$h\mu k_1-(\mu+\sigma)k_2-k_3=0. \tag{36}$$

**Proposition 3.2.** Under the same assumptions as in theorem 3.1, suppose $\zeta^*$ be the Bertrand mate curve with non-vanishing curvatures. After that, we obtain the four possible cases listed below:
For the values of $k_1$, $k_2$ and $k_3$

$$ak_1+hak_2+hbk_3=0, \tag{37}$$

$$hk_1+k_2+k_3=0, \tag{38}$$

where $\mu=-1$. Multiplying equation (38) by $-ha$ and adding in equation (37), we get the relation

$$a(1-h^2)k_1+h(b-a)k_3=1. \tag{39}$$

**Case 1:** If $a = b$ and $h^2 - 1 \neq 0$, then

$$k_1 = \frac{1}{a(1-h^2)}. \tag{40}$$

Using equation (38), we get

$$k_3 = -\frac{h}{a(1-h^2)} - k_2. \tag{41}$$

Multiplying equation (38) by $-hb$ and adding in equation (37), we get the relation

$$(a - h^2 b)k_1 + h(a-b)k_2 = 1. \tag{42}$$

**Case 2:** For $a \neq b$ and $h^2 \neq 0$, then

$$k_2 = \frac{1}{h(a-b)} - \frac{a - h^2 b}{h(a-b)k_1}, \tag{43}$$

also

$$k_3 = \frac{(1-h^2)a}{h(a-b)}k_1 - \frac{1}{h(a-b)}. \tag{44}$$

**Case 3:** For $a \neq b$ and $1 - h^2 = 0$, then

$$k_3 = \frac{-h}{a-b}, \tag{45}$$

$$k_2 = h(\frac{1}{a-b} - k_1). \tag{46}$$

**Case 4:** If $k_1$, $k_2$, $k_3$ are constants and $det(A) \neq 0$, then we have the following solution

$$k_1 = \frac{\begin{vmatrix} 1 & h & hb \\ 0 & -\mu & 1 \\ 0 & -\mu-\sigma & -1 \end{vmatrix}}{\det(A)} = \frac{2\mu+\sigma}{\det(A)}, \tag{47}$$

$$k_2 = \frac{\begin{vmatrix} a & 1 & hb \\ h & 0 & 1 \\ h\mu & 0 & -1 \end{vmatrix}}{\det(A)} = \frac{h(1+\mu)}{\det(A)}, \tag{48}$$

$$k_3 = \frac{\begin{vmatrix} a & h & 1 \\ h & -\mu & 0 \\ h\mu & -\mu-\sigma & 0 \end{vmatrix}}{\det(A)} = \frac{h(\mu^2-\mu-\sigma)}{\det(A)}. \tag{49}$$

**Theorem 3.3.** Suppose $\zeta : I \subset \mathbb{R} \mapsto \mathbb{R}^4_1$ be a Bertrand curve under the Frenet frame $\{T, N, B_1, B_2\}$ with $\kappa_1,\ \kappa_2,$ and $\kappa_3 \neq 0$. We obtain a Bertrand mate curve $\zeta^*$ as $\zeta^* = \zeta + \lambda N$, where $\lambda$ in $\mathbb{R}^4_1$ is given as follows:

$$\lambda = \pm\sqrt{\frac{1}{2}(u \pm \sqrt{u^2 + 4v})}.$$

**Proof.** Since $\zeta$ is a Bertrand curve, this implies that $T(s) = d\zeta(s)/ds$. Using Frenet frame (2), we obtain

$$T' = k_1 N(s). \tag{50}$$

If we take the derivative by $s$, we obtain

$$T'' = k_1(-k_1 T(s) + k_2 B_1(s)) = -k_1^2 T(s) + k_1 k_2 B_1(s). \tag{51}$$

Using equation (2), we differentiate by $s$ and obtain

$$T''' = -k_1^2(k_1 N(s)) + k_1 k_2(k_3 N(s) + k_2 B_2(s)) = (-k_1^2 + k_2 k_3)T'(s) + k_1 k_2^2 B_2(s). \tag{52}$$

Again differentiating by $s$, we obtain

$$T'''' = (-k_1^2 + k_2 k_3)T''(s) + k_1 k_2^2 k_3 B_1(s). \tag{53}$$

Using equation (51), we get

$$T'''' = (-k_1^2 + k_2 k_3)T''(s) + k_2 k_3(T''(s) + k_1^2 T(s)). \tag{54}$$

This implies that

$$T'''' = (2k_2 k_3 - k_1^2)T''(s) + k_1^2 k_2 k_3 T(s). \tag{55}$$

$$T^4(s) - uT^2(s) - vT(s) = 0. \tag{56}$$

Hence the C.F will be

$$\lambda^4 - u\lambda^2 - v = 0. \tag{57}$$

$$\lambda^2 = \frac{1}{2}(u \pm \sqrt{u^2 + 4v}). \tag{58}$$

$$\lambda = \pm\sqrt{\frac{1}{2}(u \pm \sqrt{u^2 + 4v})}. \tag{59}$$

Hence $\lambda$ is obtained for Bertrand curve in $\mathbb{R}^4_1$.

**Example 1.** Suppose that a Bertrand curve $\Gamma(s) : I \subset \mathbb{R} \mapsto \mathbb{R}^4_1$ with the equation:

$$\Gamma(s) = (\alpha \sin(2s), \beta \cos(2s), \gamma \sin(2s), 2\eta s),$$

where $\alpha,\ \beta,\ \gamma$ and $\eta$ are constants. The non-vanishing curvatures for the Bertrand curve by using Frenet frame-(2) are $\kappa_1 = -4,\ \kappa_2 = -3,\ \kappa_3 = 2$ and the orthonormal vectors are follows:

$$\begin{cases} T(s) & = 2(\alpha \cos(2s), -\beta \sin(2s), \gamma \cos(2s), \eta), \\ N(s) & = (\alpha \sin(2s), \beta \cos(2s), \gamma \sin(2s), 0), \\ B_1(s) & = 2(\alpha \cos(2s), -\beta \sin(2s), \gamma \cos(2s), (\frac{4}{3})\eta), \\ B_2(s) & = 2(\alpha \sin(2s), \beta \cos(2s), \gamma \sin(2s), 0). \end{cases}$$

If we choose $\epsilon_1 = -1,\ \epsilon_2 = \epsilon_3 = \ \epsilon_4 = 1$, we obtain its mate curve $\Gamma^* : I^* \subset \mathbb{R} \mapsto \mathbb{R}^4_1$ as follows:

$$\Gamma^* = 2(\alpha \sin(2s), \beta \cos(2s), \gamma \sin(2s), (\eta + \frac{1}{3})s).$$

By using Frenet frame-(1) and doing straight calculation, we obtain

$$\begin{cases} T^*(s) & = 2(2\alpha \cos(2s), -2\beta \sin(2s), 2\gamma \cos(2s), \eta), \\ N^*(s) & = (\alpha \sin(2s), \beta \cos(2s), \gamma \sin(2s), 0), \\ B_1{}^*(s) & = -2(\alpha \cos(2s), -\beta \sin(2s), \gamma \cos(2s), (\frac{8}{15})\eta), \\ B_2{}^*(s) & = (\alpha \sin(2s), \beta \cos(2s), \gamma \sin(2s), 0). \end{cases}$$

Further, the non-vanishing curvature functions are listed below for $\Gamma^*$ .

$$\kappa_1{}^* = -8,\ \kappa_2{}^* = 15,\ \text{and}\ \kappa_3{}^* = 19.$$

As $N(s) = N^*(s)$, hence we consider $\Gamma^*$ and $\Gamma$ as Bertrand curves.
**Example 2.** Consider another Bertrand curve with the equation $\Phi(s) = \frac{1}{2}(\alpha \cosh(2s), \beta \sinh(2s), 2\gamma s, 2\eta s)$. Whose Bertrand mate curve for $\epsilon_1 = -1,\ \epsilon_2 = \epsilon_3 = \epsilon_4 = 1$ is denoted by $\Phi^*$ be given as:

$$\Phi^*(s) = ((\frac{3}{2})\alpha \cosh(2s), (\frac{3}{2})\beta \sinh(2s), (s - \frac{1}{2})\gamma, (s - \frac{1}{2})\eta),$$

where $\alpha,\ \beta,\ \gamma$ and $\eta$ are arbitrary constants. Using Frenet frame-(1), we obtain orthonormal vectors as follows:

$$\begin{cases} T^*(s) & = (3\alpha \sinh(2s), 3\beta \cosh(2s), \gamma, \eta), \\ N^*(s) & = (\alpha \cosh(2s), \beta \sinh(2s), 0, 0), \\ B_1{}^*(s) & = (10\alpha \sinh(2s), 10\beta \cosh(2s), 3\gamma, 3\eta), \\ B_2{}^*(s) & = (2\alpha \cosh(2s), 2\beta \sinh(2s), 0, 0). \end{cases}$$

And the non-vanishing curvatures are obtain as follows:$\kappa_1^* = 6,\ \kappa_2^* = 2,$ and $\kappa_3^* = 11$. Further, the curvature functions for $\Phi(s)$ obtained by using Frenet frame-(2) as follows: $\kappa_1 = 2,\ \kappa_2 = 4,$ and $\kappa_3 = -2$.

$$\begin{cases} T(s) &= (\alpha \sinh(2s), \beta \cosh(2s), \gamma, \eta), \\ N(s) &= (\alpha \cosh(2s), \beta \sinh(2s), 0, 0), \\ B_1(s) &= (\alpha \sinh(2s), \beta \cosh(2s), (\frac{1}{2})\gamma, (\frac{1}{2})\eta), \\ B_2(s) &= 2(\alpha \cosh(2s), \beta \sinh(2s), 0, 0). \end{cases}$$

Since $N^* = N$ implies that $\Phi^*(s)$ and $\Phi(s)$ are Bertrand and Bertrand mate curve.

## 4. Conclusion

This study has introduced a novel Frenet frame for the analysis of Bertrand curves in 4-dimensional Minkowski space ($\mathbb{R}_1^4$), providing a significant advancement in the geometric understanding of these curves. By applying this new frame to the $(1, 3)$-normal plane, we derived explicit relationships between the curvature functions $\kappa_1$, $\kappa_2$, and $\kappa_3$, and established conditions under which Bertrand curves and their mate curves exist. Our results include new formulas for the curvatures and the parameter $\lambda$, which defines the relationship between a Bertrand curve and its mate.

We investigated four distinct cases of Bertrand curves, each characterized by specific conditions on the curvature functions, and provided explicit solutions for these cases. The examples presented in this work demonstrate the practical application of our theoretical findings, showcasing the construction of Frenet frames and the computation of curvature functions for specific curve pairs in $\mathbb{R}_1^4$. These results not only enhance the theoretical foundation of Bertrand curves in Minkowski space but also provide a framework for further exploration in differential geometry and related fields.

The findings of this study open several avenues for future research. One direction is the extension of this framework to higher-dimensional Minkowski spaces or other pseudo-Riemannian manifolds. Additionally, the application of these results to physical models, such as spacetime structures in general relativity or kinematic modeling in engineering, could yield valuable insights. Further investigation into the geometric properties of Bertrand curves, such as their behavior under deformations or their relationship with other special curves (e.g., helices, slant helices, or rectifying curves), could also be pursued. Finally, the development of computational tools to automate the construction and analysis of Bertrand curves and their mate curves would facilitate their application in practical scenarios.

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