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A note on the generalized ML function as the kernel of the extended wright function

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This paper introduces a unified generalization of the Gamma and Beta functions by employing the generalized Mittag-Leffler function as the kernel. The resulting extended Wright function incorporates a newly proposed generalized Beta function that enhances analytical flexibility in modeling fractional systems. Several integral transforms are established along with Riemann–Liouville and Hilfer fractional derivative properties. These results generalize many classical identities and offer potential applications in fractional differential equations, signal processing, and viscoelastic models.

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1. Introduction and Preliminaries

The Wright, Mittag-Leffler, and Beta functions are essential in fractional calculus and the modeling of complex systems in physics and engineering, (see (2; 16; 19; 21; 26; 30)). The majority of current generalizations either employ restrictive kernels or treat these functions independently. This paper proposes a unified framework that embeds the generalized Mittag-Leffler function $E_{\alpha,\gamma,l}(z)$ as a kernel within the extended Beta and Wright functions, thereby enhancing their analytical flexibility. The results provide a comprehensive foundation for applications in viscoelasticity, diffusion, and signal analysis, where nonlocal effects are significant. The increasing significance of such generalized special functions in applied mathematics is demonstrated by a number of recent research (see (14; 15; 20; 24; 25; 31)). In many applications of fractional calculus, the choice of kernel plays a crucial role in determining the analytical behavior of the associated operators. Classical kernels based on the exponential function or the two parameter Mittag-Leffler function often lack sufficient flexibility to capture the complex memory structures observed in viscoelasticity, anomalous diffusion, and signal processing. Consequently, the development of generalized kernels that offer greater flexibility while preserving analytical tractability is gaining increasing popularity.

Classical kernels based on the exponential function or the Mittag-Leffler function play an important role in modeling memory and hereditary effects in many physical systems. In particular, the one and two parameter Mittag-Leffler functions, $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$, have been widely used in fractional calculus. However, such classical kernels often lack sufficient flexibility to capture complex memory structures arising in various applications, including physics, engineering, and signal processing. Therefore, there is a growing interest in developing generalized kernels that provide additional degrees of freedom while preserving analytical tractability. Such reliance on classical kernels restricts the range of fractional operators that can be represented. Specifically, these kernels offer limited flexibility for controlling convergence regions of integral transforms and modeling multiscale memory effects. In order to overcome these constraints, we extend the admissible parameter domain and enhance the analytical structure by adding further parameters (γ, l, λ) to the generalized function $E_{\alpha,\gamma,l}(z)$.

The classical Euler Beta function is defined by

$$B(\phi_1,\phi_2) = \int_0^1 x^{\phi_1-1}(1-x)^{\phi_2-1}\,dx, \quad \Re(\phi_1), \Re(\phi_2) > 0. \tag{1}$$

Choudhary *et al.* (4) proposed an exponential extension as

$$B_\lambda(\phi_1,\phi_2) = \int_0^1 x^{\phi_1-1}(1-x)^{\phi_2-1} e^{\left(-\frac{\lambda}{x(1-x)}\right)} dx, \quad \Re(\lambda) \geq 0. \tag{2}$$

Shadab *et al.* (29) offered a further generalization of this by employing the Mittag-Leffler kernel as

$$B_{\lambda}^{(\alpha)}(\phi_1,\phi_2)=\int_0^1 x^{\phi_1-1}(1-x)^{\phi_2-1}E_{\alpha}\left(-\frac{\lambda}{x(1-x)}\right)dx,\quad \Re(\lambda)\geq 0,\quad \alpha\in\Re_0^+, \tag{3}$$

where the classical Mittag-Leffler function (7; 23) is

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\quad \alpha\in\Re_0^+, z\in\mathbb{C}. \tag{4}$$

The Wright function (6) is given by

$$W_{\xi,\eta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\,\Gamma(\xi n+\eta)},\quad \xi,\eta\in\mathbb{C},\ \Re(\xi)>-1. \tag{5}$$

The general form, as proposed by El-Shahed and Salem (28), is

$$W_{\xi,\eta}^{(\varsigma,\varrho)}(z)=\sum_{n=0}^{\infty}\frac{(\varsigma)_n}{(\varrho)_n}\frac{z^n}{n!\,\Gamma(\xi n+\eta)},\quad \Re(\xi)>-1,\ \Re(\varrho)>\Re(\varsigma)>0, \tag{6}$$

where $(\varsigma)_n=\varsigma(\varsigma+1)\cdots(\varsigma+n-1)$ is the Pochhammer symbol.

The generalized Wright function ${}_r\Psi_s(z)$, (1; 22) is defined by the series

$${}_r\Psi_s(z)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\Gamma(a_i+A_in)}{\prod_{j=1}^{s}\Gamma(b_j+B_jn)}\frac{z^n}{n!},$$

where $a_i,b_j\in\mathbb{C}$ and $A_i,B_j\in\Re$ with suitable convergence conditions $1+\sum_{j=1}^{s}B_j-\sum_{i=1}^{r}A_i\geq 0$.

In this study, we introduce a new generalized Beta function based on the generalized Mittag-Leffler kernel and derive several integral representations and summation identities in Sec. 2. In subsequent sections, corresponding extensions of the Wright function are obtained, along with their fundamental properties. This study differs from earlier investigations of Khan *et al.* (15) and Shadab *et al.* (29), where the classical Mittag-Leffler kernel was mainly utilized.

A number of recent studies have emphasized the growing importance of generalized special functions in applied mathematics (see (3; 5; 8; 9; 11; 12; 13)). For example, expansions of hypergeometric and Mittag-Leffler type functions and their applications in integral operators and fractional calculus have been the subject of contemporary research. Nevertheless, the majority of these research rely on classical or limited parameter kernels, which limit their ability to describe intricate fractional systems. In contrast, this work employs the generalized Mittag-Leffler function $E_{\alpha,\gamma,l}(z)$, introducing additional parameters that enlarge the admissible parameter domain. Consequently, an extended Wright function and a new generalized Beta function are derived, together with a number of integral transforms and fractional derivative properties.

Parameters $\alpha, \gamma, l, \lambda$ provide additional flexibility in the analytical structure of the kernel, α represents the fractional order, γ influences the spacing in the series coefficients, l introduces a structural shift, and λ acts as a scaling parameter. These parameters allow the proposed functions to model a wider class of fractional processes and extend previous results as special cases.

1.1. *The Generalized Mittag-Leffler Function $E_{\alpha,\gamma,l}(z)$*

The generalized Mittag-Leffler function $E_{\alpha,\gamma,l}(z)$ was introduced by Kilbas and Saigo (17) was represented by the following power series:

$$E_{\alpha,\gamma,l}(z) = \sum_{n=0}^{\infty} \left(\prod_{i=0}^{n-1} \frac{\Gamma[\alpha(i\gamma + l) + 1]}{\Gamma[\alpha(i\gamma + l + 1) + 1]} \right) z^n, \tag{7}$$

where the parameters satisfy $(0 < \alpha < 1, \gamma \in I^+(I^+ = 1, 2, 3, \ldots), l \in S\,(S = -1, 0, 1, \ldots))$.

Then $E_{\alpha,\gamma,l}(z)$ can also be expressed as a power series

$$E_{\alpha,\gamma,l}(z) = \sum_{n=0}^{\infty} c_n z^n, \quad c_n = \prod_{i=0}^{n-1} \frac{\Gamma[\alpha(i\gamma + l) + 1]}{\Gamma[\alpha(i\gamma + l + 1) + 1]}, \tag{8}$$

where by applying the ratio test $\frac{c_{n+1}}{c_n} \to 0$ as $n \to \infty$, the series converges for all z. Hence, $E_{\alpha,\gamma,l}(z)$ is an entire function.

If we take $\gamma = 1$, the above function reduced in the form of Mittag-leffler function as

$$E_{\alpha,1,l}(z) = \Gamma(\alpha l + 1)\, E_{\alpha,\alpha l+1}(z), \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{9}$$

where $E_\alpha(z) = E_{\alpha,1}(z)$. Classical kernels based on the exponential or the standard Mittag-Leffler function often provide limited flexibility for modeling complex multiscale phenomena. The generalized Mittag-Leffler function $E_{\alpha,\gamma,l}(z)$, incorporates additional parameters (γ, l) that improve control over convergence and fractional dynamics. The inclusion of this generalized kernel in Beta and Wright-type functions leads to a broader class of special functions and provides a unified analytical framework for representing various fractional operators and integral transforms. Consequently, it expands the applicability of these functions in fractional differential equations and related mathematical models. These features motivate the introduction of a new generalized Beta function in the following section.

2. Newly Generalized Beta Function and its Properties

The newly introduced generalized Gamma and Beta functions are as follows:

$$\Gamma_{l;\lambda}^{(\alpha,\gamma)}(s) = \int_0^\infty x^{s-1} E_{\alpha,\gamma,l}\left(-x - \frac{\lambda}{x}\right) dx, \tag{10}$$

$$B_{l;\lambda}^{\alpha,\gamma}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} E_{\alpha,\gamma,l}\left(\frac{-\lambda}{x(1-x)}\right) dx, \tag{11}$$

where $(\Re(p) > 0, \Re(q) > 0, \Re(\lambda) > 0, \gamma \in I^+, l \in S, 0 < \alpha < 1)$.

We use the same condition for the prescribed theorems, where $E_{\alpha,\gamma,l}(z)$ is the power series defined by Eq. (8). Parameters α, γ, l, and λ play an important role in controlling the analytical behavior of the generalized kernel. The parameter α determines the fractional order and governs the growth rate of the function. The parameter γ influences the spacing in the gamma function arguments appearing in the coefficients of the power series and thus affects convergence properties. The integer parameter l introduces an additional shift in the structure of the function, allowing a broader class of Mittag-Leffler type functions to be generated. Parameter λ acts as a scaling parameter in the generalized Beta and Wright kernels and controls the strength of the nonlocal interaction in the integral representations.

Theorem 2.1. *The following integral representation holds for* $B_{l;\lambda}^{\alpha,\gamma}(p,q)$:

$$B_{l;\lambda}^{\alpha,\gamma}(p,q) = 2\int_0^{\pi/2} \cos^{2p-1}\theta \sin^{2q-1}\theta\, E_{\alpha,\gamma,l}\left(\frac{-\lambda}{\cos^2\theta \sin^2\theta)}\right) d\theta. \tag{12}$$

Proof. Taking $x = \cos^2\theta$ in Eq. (11), we obtain the required result. □

Theorem 2.2. *The following relation holds*:

$$B_{l;\lambda}^{\alpha,\gamma}(p,q) = \int_0^\infty \frac{u^{(p-1)}}{(1+u)^{(p+q)}} E_{\alpha,\gamma,l}\left(\frac{-\lambda(1+u)^2}{u}\right) du. \tag{13}$$

Proof. Substituting $x = \frac{u}{1+u}$ in Eq. (11), the expression becomes the same as the required result. □

Theorem 2.3. *The following formula holds*:

$$B_{l;\lambda}^{\alpha,\gamma}(p,q) = 2^{(1-p-q)} \int_{-1}^1 (1+u)^{p-1}(1-u)^{q-1} E_{\alpha,\gamma,l}\left(\frac{-2^2\lambda}{(1+u)(1-u)}\right) du. \tag{14}$$

Proof. Putting $t = \frac{u+1}{2}$ in Eq. (11), we get the result. □

Theorem 2.4. *The following summation identity holds for* $B_{l;\lambda}^{\alpha,\gamma}(p,q)$:

$$B_{l;\lambda}^{\alpha,\gamma}(p,q) = \sum_{n=0}^m {}^mC_n\, B_{l;\lambda}^{\alpha,\gamma}(p+n, q+m-n), \quad m \in N. \tag{15}$$

Proof. In Eq. (11), using $x^{p-1}(1-x)^{q-1} = [x^p(1-x)^{q-1} + x^{p-1}(1-x)^q]$, we get

$$B_{l;\lambda}^{\alpha,\gamma}(p,q) = B_{l;\lambda}^{\alpha,\gamma}(p,q+1) + B_{l;\lambda}^{\alpha,\gamma}(p+1,q),$$

similarly if $x^{p-1}(1-x)^{q-1} = [x^{p-1}(1-x)^{q+1} + 2x^p(1-x)^q + x^{p+1}(1-x)^{q-1}]$, we have

$$B_{l;\lambda}^{\alpha,\gamma}(p,q) = B_{l;\lambda}^{\alpha,\gamma}(p,q+2) + 2B_{l;\lambda}^{\alpha,\gamma}(p+1,q+1) + B_{l;\lambda}^{\alpha,\gamma}(p+2,q).$$

The same process is repeatedly applied to the right side of Eq. (15), up to n times, and we yield the result. □

Theorem 2.5. *The following summation formula for* $B_{l;\lambda}^{\alpha,\gamma}(p,q)$ *holds*:

$$B_{l;\lambda}^{\alpha,\gamma}(p,1-q) = \sum_{k=0}^{\infty} \frac{(q)_k}{k!} B_{l;\lambda}^{\alpha,\gamma}(p+k,1). \tag{16}$$

Proof. Replacing q by $1-q$ in Eq. (11), and using the binomial series after changing the order, we get the required result. □

Theorem 2.6. *The following property establishes for* $B_{l;\lambda}^{\alpha,\gamma}(p,q)$:

$$B_{l;\lambda}^{\alpha,\gamma}(p,q) = \sum_{k=0}^{\infty} B_{l;\lambda}^{\alpha,\gamma}(p+k,q+1). \tag{17}$$

Proof. By using the binomial theorem, $(1-x)^{q-1} = (1-x)^q \sum_{k=0}^{\infty} x^k$, changing the order of summation and integration, and using (11), we get the required result. □

3. Main Result

3.1. *Extended wright function*

To construct the extended Wright function, we first recall that the ratio of the Pochhammer symbols can be expressed in terms of the Beta function as

$$\frac{(\varsigma)_n}{(\varrho)_n} = \frac{B(\varsigma+n, \varrho-\varsigma)}{B(\varsigma, \varrho-\varsigma)},$$

which follows from the relation between the Pochhammer symbol and the Beta function. Replacing the classical Beta function by the extended Beta function defined in (11), this relation can be written as

$$\frac{(\varsigma)_n}{(\varrho)_n} = \frac{B_{l;\lambda}^{\alpha,\gamma}(\varsigma+n, \varrho-\varsigma)}{B(\varsigma, \varrho-\varsigma)}.$$

This leads to the following construction of the extended Wright function:

$$W_{\zeta,\eta}^{\alpha,\gamma,l;\varsigma,\varrho}(y;\lambda) = \sum_{n=0}^{\infty} \frac{B_{l;\lambda}^{\alpha,\gamma}(\varsigma+n,\varrho-\varsigma)}{B(\varsigma,\varrho-\varsigma)} \frac{1}{\Gamma(\zeta n+\eta)} \frac{y^n}{n!}, \tag{18}$$

where $\eta, \varrho, \varsigma \in \mathbb{C}; \zeta > -1, \gamma \in I^+, l \in S, 0 < \alpha < 1, \Re(\lambda) > 0, \Re(\varrho) > \Re(\varsigma) > 0$. In the subsequent theorems, the parameters are assumed to satisfy the conditions stated in Eq. (18). These conditions will be omitted for brevity.

3.2. *Derivative properties of* $W_{\zeta,\eta}^{\alpha,\gamma,l;\varsigma,\varrho}(y;\lambda)$

Here, we consider some derivative properties of the extended Wright function, which is defined in Eq. (18).

Theorem 3.1. *Differentiation property holds for the extended Wright function*:

$$\left(\frac{d}{dy}\right)^m [y^{\eta-1} W_{\zeta,\eta}^{\alpha,\gamma,l;\varsigma,\varrho}(\mu y^{\zeta};\lambda)] = y^{\eta-m-1} W_{\zeta,\eta}^{\alpha,\gamma,l;\varsigma,\varrho} \times(\mu y^{\zeta-m};\lambda), \quad \mathcal{R}(\zeta-m) > 0,\ m \in N. \tag{19}$$

Proof. Using (18) and applying term-wise m times differentiation of left-hand side, we have

$$= \sum_{n=0}^{\infty} \frac{B_{l;\lambda}^{\alpha,\gamma}(\varsigma+n,\varrho-\varsigma)}{B(\varsigma,\varrho-\varsigma)} \frac{1}{\Gamma(\zeta n+\eta)} \frac{\mu^n}{n!} \left(\frac{d}{dy}\right)^m y^{(\zeta n+\eta-1)}$$

$$= y^{\eta-m-1} W_{\zeta,\eta-m}^{\alpha,\gamma,l;\varsigma,\varrho}(\mu y^{\zeta};\lambda).$$ □

Theorem 3.2. *The following property holds for* $W_{\zeta,\eta}^{\alpha,\gamma,l;\varsigma,\varrho}(y;\lambda)$:

$$\frac{d^m}{dy^m}\left[W_{\zeta,\eta}^{\alpha,\gamma,l;\varsigma,\varrho}(y;\lambda)\right] = \frac{(\varsigma)_m}{(\varrho)_m} W_{\zeta,\eta+\zeta m}^{(\alpha,\gamma,l;\varsigma+m,\varrho+m)}(y;\lambda), \quad m \in N. \tag{20}$$

Proof. Using Eq. (18) and applying term-wise differentiation of the extended Wright function as

$$\frac{d}{dy}[W_{\zeta,\eta}^{\alpha,\gamma,l;\varsigma,\varrho}(y;\lambda)] = \sum_{n=1}^{\infty} \frac{B_{l;\lambda}^{\alpha,\gamma}(\varsigma+n,\varrho-\varsigma)}{B(\varsigma,\varrho-\varsigma)} \frac{1}{\Gamma(\zeta n+\eta)} \frac{y^{(n-1)}}{(n-1)!}, \tag{21}$$

shifting $n \to n+1$ and using a known formula (see 26, p. 46), $B(q,r-q) = \frac{r}{q}B(q+1,r-q)$, we get

$$= \frac{\varsigma}{\varrho} W_{\zeta,\eta+\zeta}^{\alpha,\gamma,l;\varsigma+1,\varrho+1}(y;\lambda). \tag{22}$$

Repeated application of this transformation yields the required general form on the right-hand side of Eq. (20). □

Theorem 3.3. *The differentiation of the extended Wright function yields the following result*:

$$W_{\zeta,\eta}^{\alpha,\gamma,l;\varsigma,\varrho}(y;\lambda) = \eta\, W_{\zeta,\eta+1}^{\alpha,\gamma,l;\varsigma,\varrho}(y;\lambda) + \zeta\, y \frac{d}{dy} W_{\zeta,\eta+1}^{\alpha,\gamma,l;\varsigma,\varrho}(y;\lambda). \tag{23}$$

In particular, we have

$$W_{\zeta,\eta}^{\varsigma,\varrho}(y) = \eta\, W_{\zeta,\eta+1}^{\varsigma,\varrho}(y) + \zeta\, y \frac{d}{dy} W_{\zeta,\eta+1}^{\varsigma,\varrho}(y).$$

Proof. Taking right-hand side of Eq. (23) and using Eq. (18), we have

$$= \eta\, W_{\zeta,\eta+1}^{\alpha,\gamma,l;\varsigma,\varrho}(y;\lambda) + \sum_{n=0}^{\infty} \frac{B_{l;\lambda}^{\alpha,\gamma}(\varsigma+n,\varrho-\varsigma;\lambda)}{B(\varsigma,\varrho-\varsigma)} \frac{1}{\Gamma(\zeta n+1+\eta)} \zeta n \frac{y^n}{n!},$$

$$= \sum_{n=0}^{\infty} \frac{B_{l;\lambda}^{\alpha,\gamma}(\varsigma+n,\varrho-\varsigma;\lambda)}{B(\varsigma,\varrho-\varsigma;\lambda)} \frac{(\zeta n+\eta)}{\Gamma(\zeta n+1+\eta)} \frac{y^n}{n!}, \tag{24}$$

using the Gamma property, and $\lambda = 1, \gamma = 1, \Gamma(\alpha l+1) = 1$, we yield the result. □

4. Integral Transform of $W_{\zeta,\eta}^{\alpha,\gamma,l;\varsigma,\varrho}(y;\lambda)$

This section presents the derivation of several important integral transforms for the extended Wright function under the same parameter assumptions as in the definition of the extended Wright function.

4.1. *Euler beta transform*

For the function $f(y)$, the Euler Beta transform (20) is defined as

$$B\{f(y);p,q\} = \int_0^1 (1-y)^{q-1} y^{p-1} f(y)dy. \tag{25}$$

Theorem 4.1. *The Euler Beta transform for the extended Wright function holds for* $\Re(u) > 0, \Re(v) > 0$, *and is expressed as*

$$B\{W_{\zeta,\eta}^{\alpha,\gamma,l;\varsigma,\varrho}(ty^{\delta};\lambda);u,v\} = \Gamma(v)\, {}_r\Psi_s{}^{(\alpha,\gamma,l;\varsigma,\varrho)}\left[\begin{matrix}(u,\delta); \\ (\eta,\zeta),(u+v;\delta)\end{matrix} ;t\lambda\right]. \tag{26}$$

Proof. By taking the left-hand side and using the definition of the Euler Beta transform and Eq. (18), we have

$$= \sum_{n=0}^{\infty} \frac{B_{l;\lambda}^{\alpha,\gamma}(\varsigma+n,\varrho-\varsigma)}{B(\varsigma,\varrho-\varsigma)} \frac{1}{\Gamma(\zeta n+\eta)} \frac{t^n}{n!} \frac{\Gamma(u+\delta n)}{\Gamma(u+\delta n+v)} \Gamma(v),$$

which gives the right-hand side of Eq. (26). □

4.2. *Laplace transform*

The Laplace transform (27) of the function $f(z)$, is defined, as usual, by

$$L(f(z);s) = \int_0^{\infty} e^{-sz} f(z)dz, \quad \Re(s) > 0. \tag{27}$$

Theorem 4.2. *For $\Re(s) > 0$, the extended Wright function satisfies the following Laplace transform*:

$$\int_0^{\infty} y^{u-1} e^{-sy} W_{\zeta,\eta}^{\alpha,\gamma,l;\varsigma,\varrho}(ty^{\delta};\lambda)dy = \frac{1}{s^u} {}_1\Psi_1{}^{(\alpha,\gamma,l;\varsigma,\varrho)}\left[\begin{matrix}(u,\delta)\\(\eta,\zeta)\end{matrix};\frac{t}{s^{\delta}};\lambda\right]. \tag{28}$$

Proof. Considering Eq. (28) and applying Eqs. (18) and (27), with some simplification, we get the required result. □

4.3. *Mellin transform*

The Mellin transform (30) of an appropriately integrable function $f(\alpha)$ with index s is defined as usual, by

$$M\{f(\alpha);\alpha\to s\} := \int_0^{\infty} \alpha^{s-1} f(\alpha)d\alpha, \quad \Re(s) > 0, \tag{29}$$

whenever the improper integral in (28) exists.

Theorem 4.3. *The extended Wright function admits the following Mellin transform*:

$$M\{W_{\zeta,\eta}^{\alpha,\gamma,l;\varsigma,\varrho}(y;\lambda) : \lambda \to s\}$$

$$= \frac{\Gamma(\varrho)\,\Gamma_{l,0}^{(\alpha,\gamma)}(s)\Gamma(\varrho-\varsigma+s)}{\Gamma(\varsigma)\Gamma(\varrho-\varsigma)} {}_1\Psi_2\left[\begin{matrix}(\varsigma+s,1);\\(\varrho+2s,1),(\eta,\zeta);\end{matrix}\, y\right], \tag{30}$$

$\Re(s) > 0$, *where* $\Gamma_{l,0}^{(\alpha,\gamma)}$ *is a particular case of* (*13*) *for* $\lambda = 0$.

Proof. Applying the definition of Mellin transform and extended Wright function new generalized beta function, we have

$$M\{W^{\alpha,\gamma,l;\varsigma,\varrho}_{\zeta,\eta}(y;\lambda):\lambda\to s\} = \frac{1}{B(\varsigma,\varrho-\varsigma)}\sum_{n=0}^{\infty}\frac{y^n}{\Gamma(\zeta n+\eta)\,n!}$$

$$\times\int_0^{\infty}\lambda^{s-1}B^{\alpha,\gamma}_{l;\lambda}(\varsigma+n,\varrho-\varsigma)\,d\lambda, \tag{31}$$

to compute the right-hand side of the above, using Eq. (11) and the change in integration order, we obtain

$$= \int_0^1 x^{\varsigma+n-1}(1-x)^{\varrho-\varsigma-1}\left[\int_0^{\infty}\lambda^{s-1}E_{\alpha,\gamma,l}\left(\frac{-\lambda}{x(1-x)}\right)d\lambda\right]dx$$

$$\text{put}\quad \frac{\lambda}{x(1-x)} = u,\quad d\lambda = x(1-x)du$$

$$= \int_0^1 x^{\varsigma+n+s-1}(1-x)^{\varrho-\varsigma+s-1}\left[\int_0^{\infty}u^{s-1}E_{\alpha,\gamma,l}(-u)du\right]dx, \tag{32}$$

using $\Gamma^{(\alpha,\gamma)}_{l,0}$ from Eq. (10) and the definition of the classical beta function in Eq. (32), we have

$$= \Gamma^{(\alpha,\gamma)}_{l,0}(s)\,B(\varsigma+n+s,\varrho-\varsigma+s), \tag{33}$$

with the help of Eq. (33) in (30), we yield the desired result. □

4.4. *Whittaker transform*

Theorem 4.4. *The Whittaker transform for the extended Wright function holds under the following form*:

$$\int_0^{\infty} e^{(\frac{-\beta x}{2})}x^{\xi-1}W_{\lambda,\mu}(\beta x)W^{\alpha,\gamma,l;\varsigma,\varrho}_{\zeta,\eta}(tx^{\delta};b)dx$$

$$= \frac{\beta^{-\xi}}{B(\varsigma,\varrho-\varsigma)}\,{}_2\Psi_2{}^{(\alpha,\gamma,l;\varsigma,\varrho)}\left[\begin{matrix}(1/2+\mu+\xi,\delta)(1/2-\mu+\xi,\delta); & \dfrac{t}{\beta^{\delta}}\\ (\eta,\zeta),(1/2-\lambda+\delta); & \end{matrix}\right]. \tag{34}$$

Proof. Substituting $\beta x = v$ in the left-hand side of (33), and interchanging the order of summation and integration (since the series is absolutely convergent), we obtain

$$= \beta^{-\xi}\sum_{n=0}^{\infty}\frac{B^{\alpha,\gamma}_{l;\lambda}(\varsigma+n,\varrho-\varsigma)\,t^n}{B(\varsigma,\varrho-\varsigma)\,\Gamma(\zeta n+\eta)\,n!}\,\frac{1}{\beta^{\delta n}}\int_0^{\infty}e^{(\frac{-\nu}{2})}\nu^{\xi-1+\delta n}W_{\lambda,\mu}(\nu)\,d\nu,$$

using the formula

$$\int_0^\infty e^{(\frac{-x}{2})} x^{\nu-1} W_{\lambda,\mu}(\nu)\, d\nu = \frac{\Gamma(1/2-\mu+\nu)\Gamma((1/2+\mu+\nu)}{\Gamma(1-\lambda+\nu)}, \quad (\mathcal{R}(\nu \pm \mu) > -1/2),$$

in above expression, we get the desired solution. □

5. Fractional Properties of $W_{\zeta,\eta}^{\alpha,\gamma,l;\varsigma,\varrho}(y;b)$

Here, we develop some results involving the right-sided Riemann–Liouville fractional integral operator I_{a+}^{μ} and the derivative operator D_{a+}^{μ}, which are defined, respectively, as (see (18; 27))

$$I_{a+}^{\mu}(\varphi)(t) = \frac{1}{\Gamma(\mu)} \int_a^t \frac{\varphi(x)}{(t-x)^{1-\mu}}\, dx, \quad (\mu \in C, \Re(\mu) > 0) \tag{35}$$

and

$$(D_{a+}^{\mu}\varphi)(t) = \left(\frac{d}{dt}\right)^n (I_{a+}^{n-\mu}\varphi)(t), \quad (n = [\Re(\mu)] + 1; \mu \in C, \Re(\mu) > 0). \tag{36}$$

According to Hilfer (10), the generalized right-sided Riemann–Liouville fractional derivative operator $(D_{a+}^{\mu,\nu}\varphi)$ of order $0 \le \nu \le 1$ and $0 < \mu < 1$ with respect to t is defined as follows:

$$(D_{a+}^{\mu,\nu}\varphi)(t) = (I_{a+}^{\nu(1-\mu)} \frac{d}{dt}(I_{a+}^{(1-\nu)(1-\mu)}\varphi))(t) \quad (\mu \in C, \Re(\mu) > 0). \tag{37}$$

The following result concerning the right-sided fractional operator I_{a+}^{μ} is due to Mathai and Haubold (22) as

$$(I_{a+}^{\mu}[(x-a)^{\zeta-1}])(t) = \frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}(t-a)^{\mu+\zeta-1}, \quad (\mu, \zeta \in C, \Re(\zeta) > 0, \Re(\mu) > 0). \tag{38}$$

Also, Srivastava and Tomovski(31) established the following result:

$$(D_{a+}^{\varsigma,\lambda}[(v-a)^{\mu-1}])(x) = \frac{\Gamma(\mu)}{\Gamma(\mu-\varsigma)}(x-a)^{\mu-\varsigma-1}, \tag{39}$$

where $\varsigma, \mu \in C, \Re(\varsigma) > 0, \Re(\mu) > 0$ and $0 < \varsigma < 1, 0 \le \lambda \le 1$.

Theorem 5.1. *Let $p \in R_+, \Re(\eta) > 0, \Re(\mu) > 0$ under the parameter assumptions same as Sec. 3, and for $t > p$, the following relation holds true*:

$$(I_{a+}^{\mu}[(x-a)^{\eta-1} W_{\zeta,\eta}^{\alpha,\gamma,l;\varsigma;\varrho}(q(x-a)^{\zeta}; b)])(t) = (t-a)^{\eta+\mu-1}\, W_{\zeta,\eta+\mu}^{\alpha,\gamma,l;\varsigma;\varrho}(q(t-a)^{\zeta}; b), \tag{40}$$

$$(D^{\mu}_{a+}[(x-a)^{\eta-1}W^{\alpha,\gamma,l;\varsigma;\varrho}_{\zeta,\eta}(q(x-a)^{\zeta};b)])(t)$$
$$=(t-a)^{\eta-\mu-1}\,W^{\alpha,\gamma,l;\varsigma;\varrho}_{\zeta,\eta-\mu}(q(t-a)^{\zeta};b) \tag{41}$$

and

$$(D^{\mu,\nu}_{a+}[(x-a)^{\eta-1}W^{\alpha,\gamma,l;\varsigma;\varrho}_{\zeta,\eta}(q(x-a)^{\zeta};b)])(t)$$
$$=(t-a)^{\eta-\mu-1}\,W^{\alpha,\gamma,l;\varsigma;\varrho}_{\zeta,\eta-\mu}(q(t-a)^{\zeta};b). \tag{42}$$

Proof. Considering left-hand side and using Eq. (18), we have

$$I^{\mu}_{a+}[(x-a)^{\eta-1}\,W^{\alpha,\gamma,l;\varsigma;\varrho}_{\zeta,\eta}(q(x-a)^{\zeta};b)])(t)=\frac{1}{B(\varsigma,\varrho-\varsigma)}\sum_{k=0}^{\infty}$$
$$\times\frac{B^{\alpha,\gamma}_{l;\lambda}(\varsigma+n,\varrho-\varsigma;\lambda)\,q^{n}}{\Gamma(\zeta n+\eta)\,n!}$$
$$\times(I^{\mu}_{a+}[(x-a)^{\eta+\zeta n-1})(t).$$

Using the result of Eq. (37) in the right-hand side of the above, we get the desired result. Now, to prove Eq. (40), we have

$$(D^{\mu}_{a+}[(x-a)^{\eta-1}W^{\alpha,\gamma,l;\varsigma;\varrho}_{\zeta,\eta}(q(x-a)^{\zeta};b)])(t)=\left(\frac{d}{dt}\right)^{n}\{I^{n-\mu}_{0+}[(x-a)^{\eta-1}$$
$$\times W^{\alpha,\gamma,l;\varsigma;\varrho}_{\zeta,\eta}(q(x-a)^{\zeta};b)]\}(t)$$
$$=\left(\frac{d}{dt}\right)^{n}\{(t-a)^{\eta+n-\mu-1}$$
$$\times W^{\alpha,\gamma,l;\varsigma;\varrho}_{\zeta,\eta+n+\mu}(q(t-a)^{\zeta};b)\}$$

by using equation $\frac{d^n}{dt^n}(t^m)=\frac{m!}{(m-n)!}(t)^{m-n}$, we reach at the desired result.

Similarly, we can prove (41) with the help of Eqs. (18) and (38). □

6. Illustrative Application

Consider the fractional relaxation equation

$$D^{\alpha}_{t}y(t)=-\lambda y(t),\quad 0<\alpha<1.$$

The solution of this equation can be expressed in terms of the Mittag-Leffler function. Using the generalized Mittag-Leffler kernel introduced in this work, the solution can be written as

$$y(t)=E_{\alpha,\gamma,l}(-\lambda t^{\alpha}).$$

This representation demonstrates that the generalized Mittag-Leffler function provides a flexible framework for modeling fractional relaxation processes appearing in viscoelastic materials and anomalous diffusion systems.

7. Comparison and Discussion

This paper's approach builds upon previous research by Khan *et al.* (15) and Shadab *et al.* (29). Specifically, Khan *et al.* used kernels based on the classical Mittag-Leffler function to create an extended Beta function along with a related probability density function. In a similar vein, Shadab *et al.* (29) suggested extending the Beta function using the conventional Mittag-Leffler kernel.

In contrast, this work employs the generalized Mittag-Leffler function $E_{\alpha,\gamma,l}(z)$ as the kernel. The introduction of the additional parameters γ and l significantly enlarges the admissible parameter domain and provides greater flexibility in the analytical structure of the resulting functions. An extended Wright function and a new generalized Beta function are created using this generalized kernel. Moreover, several new integral transforms, summation identities, and integral representations are derived, along with properties of fractional derivatives involving the Hilfer and Riemann–Liouville operators.

When the parameters are specialized to $\gamma = 1$ and $l = 1$, the generalized Mittag-Leffler function reduces to the classical Mittag-Leffler function. Consequently, the present formulation provides a unified extension of the earlier results.

The graphical behavior of the extended Wright function and the generalized Mittag-Leffler function for various parameter values is shown in Figs. 1 and 2. The figures demonstrate how the shape and growth behavior of the functions are greatly affected by changes in the parameters α, γ, l, and λ. Specifically, γ and l alter the

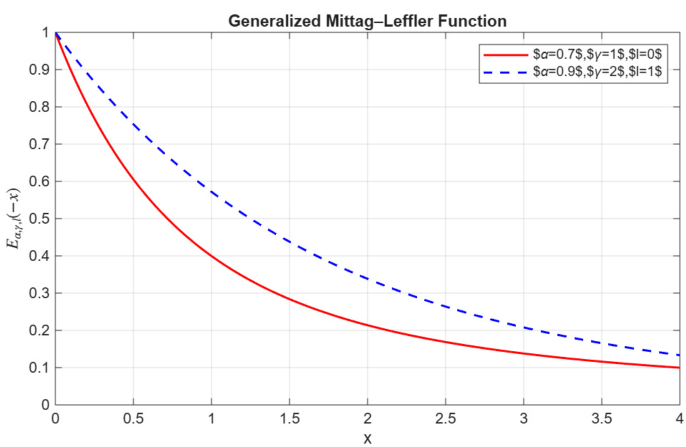

Fig. 1. Graph of the generalized Mittag-Leffler function $E_{\alpha,\gamma,l}(z)$ with a range of α, γ, l chosen at random.

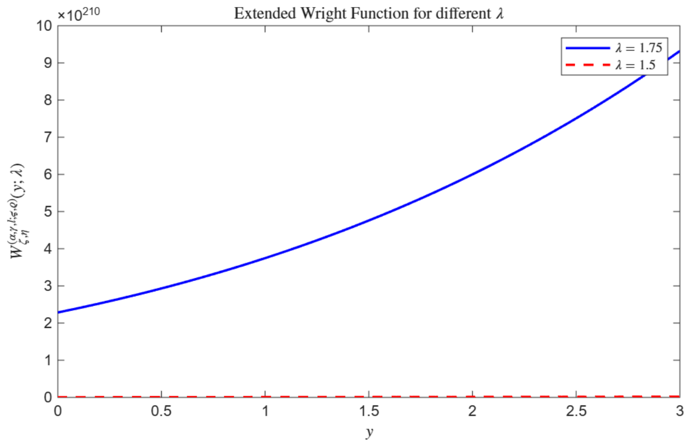

Fig. 2. Graph of the extended Wright function for different values of λ.

structure of the generalized Mittag-Leffler kernel, while α influences the rate of decay. The flexibility of the suggested functions in simulating various fractional phenomena is demonstrated by these graphical findings.

8. Conclusions

In this paper, newly generalized Beta and extended Wright functions were introduced using the generalized Mittag-Leffler kernel $E_{\alpha,\gamma,l}(z)$. Several analytical properties, including integral representations, summation identities, and integral transforms such as Euler Beta, Laplace, Mellin, and Whittaker transforms, were derived. Fractional derivative relations involving Riemann–Liouville and Hilfer operators were also established. The additional parameters $\alpha, \gamma, l, \lambda$ provide greater analytical flexibility and extend several previously known results. Potential applications of the proposed functions may arise in fractional differential equations, viscoelastic models, and anomalous diffusion processes.

Remark. If $\gamma = 1$ and $l = 1$, the generalized Mittag-Leffler function reduces to the classical Mittag-Leffler function. Consequently, the proposed generalized Beta and Wright functions reduce to the forms studied in earlier works by Khan *et al.* and Shadab *et al.* Therefore, the present results can be viewed as a natural generalization of these earlier formulations.

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