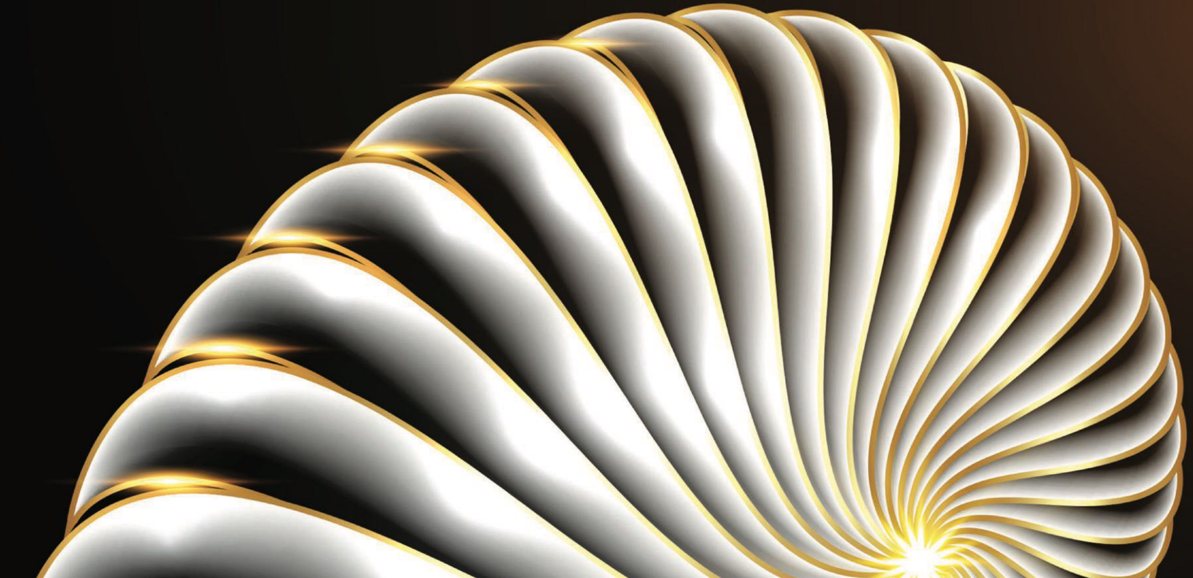
# Extended Hypergeometric Functions and Orthogonal Polynomials

*Edited by*

Praveen Agarwal
Clemente Cesarano



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# Introduction

Extended Hypergeometric Functions and Orthogonal Polynomials presents a comprehensive and accessible resource for researchers and graduate students interested in exploring the rich connections between extended hypergeometric functions, orthogonal polynomials, and multivariable polynomials. Integrating all three fields and their applications in Maple, Mathematica, and MATLAB, this book fosters interdisciplinary understanding and inspires new avenues of research in mathematics, engineering, physics, and computer science. It also provides a glimpse into future research directions in these areas, including potential applications in emerging fields of applied mathematics and interdisciplinary collaborations. Each chapter begins with an introduction, includes sections on theory, followed by sections on applications, and ends with exercises, problems, references, and suggested readings.

**Key Features**

- Provides an integrated and up-to-date resource on extended hypergeometric functions, orthogonal polynomials, and multivariable polynomials for researchers, students, and practicing engineers.
- Presents emerging trends, new developments, techniques, and applications in the fields of special functions and orthogonal polynomials.
- Gives special attention to practical techniques for working with these functions and polynomials, including computational methods and applications to solve real-world problems.

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CHAPTER

# 1

# The generalized $k$-hypergeometric function as the Meijer G-function

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## 1.1 Introduction and preliminaries

In the field of mathematics, special functions play a significant role. Although it is a vast discipline our focus is mainly on the generalizations of the special functions. In 2007, Diaz and Pariguan established new concepts of the $k$-gamma functions, $k$-beta functions, and the Pochhammer $k$-symbol [11]. These concepts are well known in the literature as generalizations of special functions [1–6].

The $k$-gamma function is given in the following manner:

$$\Gamma_k(v_0) = \int_0^\infty w^{v_0-1} e^{-w^k k^{-1}} dw \tag{1.1.1}$$

and

$$\Gamma_k(v_0 + k) = v_0 \Gamma_k(v_0) \tag{1.1.2}$$

$$\Gamma_k(v_0) = k^{\frac{v_0}{k}-1} \Gamma\left(\frac{v_0}{k}\right) \tag{1.1.3}$$

$$(v_0)_{(n,k)} = k^n \left(\frac{v_0}{k}\right)_k, \tag{1.1.4}$$

where $k > 0$ and $\Re(v_0) > 0$.

The Pochhammer $k$-symbol is defined in the following way:

$$(v_0)_{n,k} := \frac{\Gamma_k\,(v_0 + nk)}{\Gamma_k\,(v_0)} = \begin{cases} 1 & n = 0;\ v_0 \in \mathbb{C} \setminus \{0\}, \\ v_0\,(v_0 + k) \cdots (v_0 + (n-1)k) & n \in \mathbb{N};\ v_0 \in \mathbb{C}. \end{cases} \tag{1.1.5}$$

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The $k$-beta function is given in the following manner:

$$B_k(v_0, v_1) = \frac{1}{k}\int_0^1 w^{\frac{v_0}{k}-1}(1-w)^{\frac{v_1}{k}-1}dw, \tag{1.1.6}$$

where $\Re(v_0)$ and $\Re(v_1) > 0$.

The relationship between the $k$-gamma function and the $k$-beta function is:

$$B_k(v_0, v_1) = \frac{\Gamma_k(v_0)\Gamma_k(v_1)}{\Gamma_k(v_0+v_1)}, \tag{1.1.7}$$

where $\Re(v_0)$ and $\Re(v_1) > 0$.

For $k > 0$, the generalized $k$-hypergeometric function is defined as follows [18]:

$${}_pF_{q,k}[(u_1,k),(u_2,k)...(u_p,k);(v_1,k),(v_2,k)...(v_q,k);w] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(u_i)_{n,k}}{\prod_{i=1}^{q}(v_i)_{n,k}}\frac{w^n}{n!}, \tag{1.1.8}$$

where $w, u_i, v_i \in C(i = 1, 2, 3...p, j = 1, 2, 3, ...q)$ and $p, q \in N_0 = 0, 1, 2....$

The generalized $k$-hypergeometric function includes many elementary and special functions.

In particular, when $p = q = 0$ and $k = 1$ in (1.1.8) the exponential function is:

$${}_0F_{0,1}[-;-;w] = e^w = \sum_{n=0}^{\infty}\frac{w^n}{n!}. \tag{1.1.9}$$

If $p = 2$ and $q = 1$ then (1.1.8) gives the Gauss $k$-hypergeometric function:

$${}_2F_{1,k}[(v_0,k),(v_1,k);(v_2,k);w] = \sum_{n=0}^{\infty}\frac{(v_0)_{n,k}(v_1)_{n,k}}{(v_2)_{n,k}}\frac{w^n}{n!}, \tag{1.1.10}$$

where $\Re(v_0) > 0$, $\Re(v_1) > 0$, $\Re(v_2) > 0$, and $|w| < \frac{1}{k}$ and $(v_0)_{n,k}$ is the $k$-Pochhammer symbol.

While for $p = 3$ and $q = 2$ in (1.1.8) the $k$-hypergeometric function is defined as [17]:

$${}_3F_{2,k}[(v_0,k),(v_1,k),(v_2,k);(v_3,k),(v_4,k):w] = \sum_{n=0}^{\infty}\frac{(v_0)_{n,k}(v_1)_{n,k}(v_2)_{n,k}}{(v_3)_{n,k}(v_4)_{n,k}}\frac{w^n}{n!}, \tag{1.1.11}$$

where $\Re(v_0) > 0$, $\Re(v_1) > 0$, $\Re(v_2) > 0$, $\Re(v_3) > 0$, and $\Re(v_4) > 0$ and $(v_0)_{n,k}$ is the $k$-Pochhammer symbol.

It is known that the generalized $k$-hypergeometric series in (1.1.8) is absolutely convergent for all finite $z \in \mathbb{C}$ if $p < q + 1$ and for $|z| < 1$ if $p = q + 1$. In particular, the series in (1.1.10) and (1.1.11) are absolutely convergent for $|z| < 1$.

## 1.2 The generalized $k$-hypergeometric function as the Mellin–Barnes integral

Let $k > 0, a_i, b_j \in C$ $(i = 1, 2, 3...p,\ j = 1, 2, 3, ..., q)$, $b_j \neq -l(j = 1, 2, 3...q; l \in N_0)$, then the generalized $k$-hypergeometric function can be represented in terms of the Mellin–Barnes integral that is of the form [20]:

$$
{}_pF_{q,k}[(u_1,k),(u_2,k)...(u_p,k);(v_1,k),(v_2,k)...(v_q,k);w] = \frac{\prod_{j=1}^{q}\Gamma_k(v_j)}{\prod_{i=1}^{p}\Gamma_k(u_i)}\frac{1}{2\pi i}
\int_L \frac{\Gamma(s)\prod_{i=1}^{p}\Gamma_k(u_i - sk)}{\prod_{j=1}^{q}\Gamma_k(v_j - sk)}(-w)^{-s}ds, \tag{1.2.1}
$$

with the special chosen contour $L$.

In particular, the $k$-hypergeometric series in (1.1.10) and (1.1.11) have the representations [20]:

$$
{}_2F_{1,k}[(u_1,k),(u_2,k);(v_1,k);w] = \frac{\Gamma_k(v_1)}{\Gamma_k(u_1)\Gamma_k(u_2)}\frac{1}{2\pi i}
\int_L \frac{\Gamma(s)\Gamma_k(u_1 - sk)\Gamma_k(u_2 - sk)}{\Gamma_k(v_1 - sk)}(-w)^{-s}ds, \tag{1.2.2}
$$

with $v_1 \neq -l, l \in N_0$ and

$$
{}_3F_{2,k}[(u_1,k),(u_2,k),(u_3,k);(v_1,k),(v_2,k);w] = \frac{\Gamma_k(v_1)\Gamma_k(v_2)}{\Gamma_k(u_1)\Gamma_k(u_2)\Gamma_k(u_3)}\frac{1}{2\pi i}
\int_L \frac{\Gamma(s)\Gamma_k(u_1 - sk)\Gamma_k(u_2 - sk)\Gamma_k(u_3 - sk)}{\Gamma_k(v_1 - sk)\Gamma_k(v_2 - sk)}(-w)^{-s}ds, \tag{1.2.3}
$$

with $v_j \neq -l,\ j = 1, 2, l \in N_0$, respectively.

In this chapter, we use the representation (1.2.1) as the definition of the generalized $k$-hypergeometric function. Choosing the contour of integration $L$, we give conditions for the existence of such a function for all finite $z \neq 0$ and for all ranges of parameters in respective cases $p < q + 1$, $p = q + 1$, and $p > q + 1$. We deduce the representations for ${}_pF_{q,k}(z)$ as the Meijer G-function.

## 1.3 Existence conditions for the generalized $k$-hypergeometric function ${}_pF_{q,k}(w)$

In this section, we give the conditions for the existence of the extended generalized $k$-hypergeometric function ${}_pF_{q,k}(w)$ defined by the Mellin–Barnes integral of the form (1.2.1).

The conditions will be different for the infinite contour L, which has one of the following forms:
1. $L = L_{-\infty}$ is a left loop situated in a horizontal strip starting at the point $-\infty + i\phi_1$ and terminating at the point $-\infty + i\phi_2$ with $-\infty < \phi_1 < \phi_2 < \infty$;
2. $L = L_{+\infty}$ is a right loop situated in a horizontal strip starting at the point $+\infty + i\phi_1$ and terminating at the point $+\infty + i\phi_2$ with $-\infty < \phi_1 < \phi_2 < \infty$;
3. $L = L_{i\gamma\infty}$ is a contour starting at the point $\gamma - i\infty$ and terminating at the point $\gamma + i\infty$, where $\gamma \in R$.

We suppose that $u_i, v_j \in C$ $(i = 1, 2, 3...p,\ j = 1, 2, 3, ..., q)$ are such that $v_j \neq -l(j = 1, 2, 3...q; l \in N_0)$ and the poles

$$c_l = -l, l \in N_0 \tag{1.3.1}$$

of the gamma function $\Gamma(s)$ and the poles

$$u_{in_1} = \frac{u_i + n_1}{k}, (i = 1, 2, 3...p; n_1 \in N_0) \tag{1.3.2}$$

of the $k$-gamma function $\Gamma_k(u_i - sk)$, $(i = 1, 2, 3...p)$ do not coincide:

$$u_{in_1} = \frac{u_i + n_1}{k} \neq -l, (i = 1, 2, 3...p; l \in N_0). \tag{1.3.3}$$

We also suppose that the poles $u_i n_1$ in (1.3.2) are simple:

$$\frac{u_i + n_1}{k} \neq \frac{u_j + n_2}{k}, (i \neq j, i = 1, 2, 3...p, j = 1, 2, 3...q; n_1, n_2 \in N_0). \tag{1.3.4}$$

We shall use the notation:

$$\mu_k = \sum_{j=1}^{q} \frac{v_j}{k} - \sum_{i=1}^{p} \frac{u_i}{k} + \frac{p-q}{2}, \tag{1.3.5}$$

when $L = L_{-\infty}$, the existence of the extended generalized $k$-hypergeometric function ${}_pF_{q,k}(w)$ defined by the Mellin–Barnes integral (1.2.1) is given by the following result.

**Theorem 1.3.1.** *Let $k > 0, u_i, v_j \in C(i = 1, 2, 3...p, j = 1, 2, 3, ..., q)$ be such that $v_j \neq -l(j = 1, 2, 3...q; l \in N_0)$ and the conditions in (1.3.3) and (1.3.4) are satisfied. Let one of the following conditions hold:*

$$p < q + 1, w \neq 0 \tag{1.3.6}$$

$$q + 1 = p, 0 < |w| < 1 \tag{1.3.7}$$

$$q + 1 = p, |w| = 1, Re\left(\sum_{j=1}^{q} \frac{v_j}{k} - \sum_{i=1}^{p} \frac{u_i}{k}\right) > 0. \tag{1.3.8}$$

*Then, there exists ${}_pF_{q,k}(w)$ defined by the Mellin–Barnes integral (1.2.1), where the path of integration $L = L_{-\infty}$ separates all poles $c_l$ in (1.3.1) to the left and poles $u_{in_1}$ in (1.3.2) to the right.*

**Corollary 1.3.2.** *Let $k > 0, u_1, u_2, v_1 \in C$ be such that $v_1 \neq -l (l \in N_0)$ and the conditions (1.3.3) and (1.3.4) with $p = 2$ are satisfied. Let either of the conditions hold:*

$$0 < |w| < 1 \tag{1.3.9}$$

$$|w| = 1, Re\left(\frac{v_1}{k} - \frac{u_1}{k} - \frac{u_2}{k}\right) > 0. \tag{1.3.10}$$

*Then, there exists ${}_2F_{1,k}(w)$ defined by the Mellin–Barnes integral (1.2.2), where the path of integration $L = L_{-\infty}$ separates all poles $c_l$ in (1.3.1) to the left and all poles $u_{in_1}$ in (1.3.2) to the right.*

**Corollary 1.3.3.** *Let $k > 0, u_1, u_2, u_3, v_1, v_2 \in C$ be such that $v_1, v_2 \neq -l (l \in N_0)$ and the conditions (1.3.3) and (1.3.4) with $p = 3$ are satisfied. Let either of the conditions hold:*

$$0 < |w| < 1 \tag{1.3.11}$$

$$|w| = 1, Re\left(\frac{v_1}{k} + \frac{v_2}{k} - \frac{u_1}{k} - \frac{u_2}{k} - \frac{u_3}{k}\right) > 0. \tag{1.3.12}$$

*Then, there exists ${}_3F_{2,k}(w)$ defined by the Mellin–Barnes integral (1.2.3), where the path of integration $L = L_{-\infty}$ separates all poles $c_l$ in (1.3.1) to the left and all poles $u_{in_1} (i = 1, 2, 3)$ in (1.3.2) to the right.*

The next result yields the existence of the extended generalized $k$-hypergeometric function ${}_pF_{q,k}(w)$ defined by the Mellin–Barnes integral (1.2.1) for $L = L_{+\infty}$.

**Theorem 1.3.4.** *Let $k > 0, u_i, v_j \in C (i = 1, 2, 3...p, j = 1, 2, 3, ..., q)$ be such that $v_j \neq -l (j = 1, 2, 3...q; l \in N_0)$ and the conditions in (1.3.3) and (1.3.4) are satisfied. Let one of the following conditions hold:*

$$q + 1 < p, w \neq 0 \tag{1.3.13}$$

$$q + 1 = p, |w| > 1 \tag{1.3.14}$$

$$q + 1 = p, |w| = 1, Re\left(\sum_{j=1}^{q} \frac{v_j}{k} - \sum_{i=1}^{p} \frac{u_i}{k}\right) > 0. \tag{1.3.15}$$

*Then, there exists ${}_pF_{q,k}(w)$ defined by the Mellin–Barnes integral (1.2.1), where the path of integration $L = L_{+\infty}$ separates all poles $c_l$ in (1.3.1) to the left and poles $u_{in_1}$ in (1.3.2) to the right.*

**Corollary 1.3.5.** *Let $k > 0, u_1, u_2, v_1 \in C$ be such that $v_1 \neq -l (l \in N_0)$ and the conditions (1.3.3) and (1.3.4) with $p = 2$ are satisfied. Let either of the conditions hold:*

$$|w| = 1, Re\left(\frac{v_1}{k} - \frac{u_1}{k} - \frac{u_2}{k}\right) > 0 \tag{1.3.16}$$

$$|w| > 1. \tag{1.3.17}$$

*Then, there exists* ${}_2F_{1,k}(w)$ *defined by the Mellin–Barnes integral* (1.2.2)*, where the path of integration* $L = L_{+\infty}$ *separates all poles* $c_l$ *in* (1.3.1) *to the left and all poles* $u_{in_1}$ *in* (1.3.2) *to the right.*

**Corollary 1.3.6.** *Let* $k > 0, u_1, u_2, u_3, v_1, v_2 \in C$ *be such that* $v_1, v_2 \neq -l (l \in N_0)$ *and the conditions in* (1.3.3) *and* (1.3.4) *with* $p = 3$ *are satisfied. Let either of the conditions hold:*

$$|w| > 1 \tag{1.3.18}$$

$$|w| = 1, Re\left(\frac{v_1}{k} + \frac{v_2}{k} - \frac{u_1}{k} - \frac{u_2}{k} - \frac{u_3}{k}\right) > 0. \tag{1.3.19}$$

*Then, there exists* ${}_3F_{2,k}(w)$ *defined by the Mellin–Barnes integral* (1.2.3)*, where the path of integration* $L = L_{-\infty}$ *separates all poles* $c_l$ *in* (1.3.1) *to the left and all poles* $u_{in_1}(i = 1, 2, 3)$ *in* (1.3.2) *to the right.*

Finally, we deduce the existence of the extended generalized $k$-hypergeometric function ${}_pF_{q,k}(w)$ defined by the Mellin–Barnes integral (1.2.1) for $L = L_{i\gamma\infty}$.

**Theorem 1.3.7.** *Let* $k > 0, u_i, v_j \in C(i = 1, 2, 3...p, j = 1, 2, 3, ..., q)$ *be such that* $v_j \neq -l(j = 1, 2, 3...q; l \in N_0)$ *and the conditions in* (1.3.3) *and* (1.3.4) *are satisfied. Let one of the following conditions hold:*

$$p + 1 > p, |arg(-w)| < \frac{(p - q + 1)\pi}{2}, w \neq 0 \tag{1.3.20}$$

$$p + 1 = q, Re\left(\sum_{j=1}^{q} \frac{v_j}{k} - \sum_{i=1}^{p} \frac{u_i}{k}\right) > 2\gamma. \tag{1.3.21}$$

*Then, there exists* ${}_pF_{q,k}(w)$ *defined by the Mellin–Barnes integral* (1.2.1)*, where the path of integration* $L = L_{i\gamma\infty}$ *separates all poles* $c_l$ *in* (1.3.1) *to the left and all poles* $u_{in_1}$ *in* (1.3.2) *to the right.*

**Corollary 1.3.8.** *Let* $k > 0, u_1, u_2, v_1 \in C$ *be such that* $v_1 \neq -l(l \in N_0)$ *and the conditions* (1.3.3) *and* (1.3.4) *with* $p = 2$ *are satisfied and let* $w \in C$ *be such that* $|arg(-w)| < \pi$*. Then, there exists* ${}_2F_{1,k}(w)$ *defined by the Mellin–Barnes integral* (1.2.2)*, where the path of integration* $L = L_{i\gamma\infty}$ *separates all poles* $c_l$ *in* (1.3.1) *to the left and all poles* $u_{in_1}(i = 1, 2)$ *in* (1.3.2) *to the right.*

**Corollary 1.3.9.** *Let* $k > 0, u_1, u_2, u_3, v_1, v_2 \in C$ *be such that* $v_1, v_2 \neq -l(l \in N_0)$ *and the conditions in* (1.3.3) *and* (1.3.4) *with* $p = 3$ *are satisfied and let* $w \in C$ *be such that* $|arg(-w)| < \pi$*. Then, there exists* ${}_3F_{2,k}(w)$ *defined by the Mellin–Barnes integral* (1.2.3)*, where the path of integration* $L = L_{i\gamma\infty}$ *separates all poles* $c_l$ *in* (1.3.1) *to the left and all poles* $u_{in_1}(i = 1, 2, 3)$ *in* (1.3.2) *to the right.*

**Remark.** Theorem 1.3.1 allows us to define the extended generalized $k$-hypergeometric function ${}_pF_{q,k}(w)$ defined by (1.2.1) with $L = L_{+\infty}$ for the range of parameters $p, q \in N_0$ such that $p > q+1$ for all $w \in C$ and $p = q+1$ for $|w| > 1$. This representation can be considered as an extension of the classical generalized $k$-hypergeometric function defined by the series in (1.1.8) from the usual range of parameters and variable w; $p < q+1$ for all $w \in C$ and $p = q+1$ for $|w| = 1$. Similarly, in the case $p = q+1$ the relation (1.2.1) can be considered as the extension of (1.1.8) from $|w| < 1$ to $|w| > 1$.

**Remark.** The above approach, giving meaning to the extended generalized $k$-hypergeometric function ${}_pF_{q,k}(w)$ when $p > q+1$, is based on the Mellin–Barnes integral representation (1.2.1).

**Remark.** The representation (1.2.2) for the Gauss $k$-hypergeometric function ${}_2F_{1,k}(w)$, is defined by the $k$-hypergeometric series (1.1.10).

## 1.4 Extended generalized $k$-hypergeometric function ${}_pF_{q,k}(z)$ as the Meijer $G$-function

Now, we apply the results in Theorems 1.3.1–1.3.3 of Section 1.3 to represent the extended generalized $k$-hypergeometric function ${}_pF_{q,k}(w)$ defined by (1.2.1) with $u_i, v_j \in C(i = 1,2,3...p,\ j = 1,2,3,...,q)$ as a special case of the Meijer $G_k$-function $G^{m,n}_{p,q,k}(w)$. This function, for $k > 0, m, n, p, q \in N_0$ such that $0 \le n \le p$, $1 \le m \le q$, and $u_i, v_j \in C$, is defined by means of a Mellin–Barnes-type integral in the following manner:

$$G^{m,n}_{p,q,k}[(u_1,k),(u_2,k)...(u_p,k);(v_1,k),(v_2,k)...(v_q,k);w] = \frac{1}{2\pi i}\int_L f(s)(w)^{-s}ds \tag{1.4.1}$$

$$f(s) = \frac{\prod_{j=1}^{m}\Gamma_k(v_j+sk)\prod_{i=1}^{n}\Gamma_k(1-u_i-sk)}{\prod_{i=n+1}^{p}\Gamma_k(u_i+sk)\prod_{j=m+1}^{q}\Gamma_k(1-v_j-sk)}. \tag{1.4.2}$$

Here,

$$w^{-s} = exp[-s(log(|w|)) + iarg(w)], w \ne 0, i = \sqrt{-1}, \tag{1.4.3}$$

where $log|w|$ represents the natural logarithm of $|w|$ and $arg(w)$ is not necessarily the principal value.

Let the poles

$$v_{jr_1} = -\frac{v_j + r_1}{k}(j = 1,2,3...m, r_1 \in N_0) \tag{1.4.4}$$

of the gamma function $\Gamma_k(b_j + sk)$ and the poles

$$u_{in_1} = \frac{1 - u_i + n_1}{k}(i = 1,2,3...n, n_1 \in N_0) \tag{1.4.5}$$

of the gamma function $\Gamma_k(1-u_i-sk)$ not coincide:

$$\frac{v_j+r_1}{k} \neq \frac{u_i-n_1-1}{k} (i=1,2,3...n,\ j=1,2,3...m, r_1, n_1 \in N_0). \tag{1.4.6}$$

Let (1.4.1) be one of the above contours $L=L_{-\infty}$, $L=L_{+\infty}$ or $L=L_{i\gamma\infty}$ that separates all poles $v_{jr_1}$ in (1.4.4) to the left and all poles $u_{in_1}$ in (1.4.5) to the right of $L$.

According to (1.2.1), (1.4.1), and (1.4.2), we obtain the representation of the extended generalized $k$-hypergeometric function ${}_pF_{q,k}(z)$ as a $G_k$ function of the form:

$$\begin{aligned} {}_pF_{q,k}[(u_1,k),(u_2,k)...(u_p,k);(v_1,k),(v_2,k)...(v_q,k);w] &= \frac{\prod_{j=1}^{q}\Gamma_k(v_j)}{\prod_{i=1}^{p}\Gamma_k(u_i)} \\ G^{1,p}_{p,q+1,k}[(1-u_1,k),(1-u_2,k)...(1-u_p,k);(0,k),(1-v_1,k),(1-v_2,k)... \\ (1-v_q,k);-z]. \end{aligned} \tag{1.4.7}$$

From Theorems (1.2.1)–(1.2.3) we deduce the conditions for these representations.

**Theorem 1.4.1.** *Let* $k>0$, $p,q \in N_0, u_i, v_j \in C (i=1,2,3...p,\ j=1,2,3,...,q)$ *be such that* $v_j \neq -l (j=1,2,3...q; l \in N_0)$ *and the conditions in* (1.3.3) *and* (1.3.4) *are satisfied, and let* $\gamma \in R$. *Let L be the contour that separates all poles* $c_l$ *in ((1.3.1)) to the left and all poles* $u_i n_1$ *in ((1.3.2)) to the right.*

*Let one of the following conditions be valid:*
*(a)* $L=L_{-\infty}$ *and either of the conditions in* (1.3.6), (1.3.7), *and* (1.3.8) *holds.*
*(b)* $L=L_{+\infty}$ *and either of the conditions in* (1.3.13), (1.3.14), *and* (1.3.15) *holds.*
*(c)* $L=L_{i\gamma\infty}$ *and either of the conditions in* (1.3.20) *and* (1.3.21) *holds.*

*Then, the extended generalized k-hypergeometric function* ${}_pF_{q,k}(w)$ *defined by* (1.2.1) *is represented as a* $G_k$*-function by* (1.4.7).

**Corollary 1.4.2.** *Let* $k>0, u_1, u_2, v_1 \in C$ *be such that* $v_1 \neq -l; l \in N_0)$ *and the conditions in* (1.3.3) *and* (1.3.4) *with* $p=2$ *are satisfied, and let* $\gamma \in R$. *Let L be the contour that separates all poles* $c_l$ *in* (1.3.1) *to the left and all poles* $a_i n_1 (i=1,2)$ *in* (1.3.2) *to the right.*

*Let one of the following conditions be valid:*
*(a)* $L=L_{-\infty}$ *and either of the conditions in* (1.3.9) *and* (1.3.10) *holds.*
*(b)* $L=L_{+\infty}$ *and either of the conditions in* (1.3.16) *and* (1.3.17) *holds.*
*(c)* $L=L_{i\gamma\infty}$ *and* $z \in C$ *is such that* $|arg(-z)|<\pi$.

*Then,* ${}_2F_{1,k}(w)$ *defined by* (1.2.2) *is represented as a* $G_k$*-function by:*

$$\begin{aligned} {}_2F_{1,k}[(u_1,k),(u_2,k);(v_1,k);w] &= \frac{\Gamma_k(v_1)}{\Gamma_k(u_1)\Gamma_k(u_2)} \\ G^{1,2}_{2,2,k}[(1-u_1,k),(1-u_2,k);(0,k),(1-v_1,k);-w]. \end{aligned} \tag{1.4.8}$$

**Corollary 1.4.3.** *Let $k>0, u_1, u_2, u_3, v_1, v_2 \in C$ be such that $b_j \neq -l$; $(j=1,2)l \in N_0)$ and the conditions in (1.3.3) and (1.3.4) with $p=3$ are satisfied, and let $\gamma \in R$. Let $L$ be the contour that separates all poles $c_l$ in ((1.3.1)) to the left and all poles $a_i n_1 (i=1,2,3)$ in ((1.3.2)) to the right.*

*Let one of the following conditions be valid:*
*(a) $L=L_{-\infty}$ and either of the conditions in (1.3.11) and (1.3.12) holds.*
*(b) $L=L_{+\infty}$ and either of the conditions in (1.3.18) and (1.3.19) holds.*
*(c) $L=L_{i\gamma\infty}$ and $z \in C$ is such that $|arg(-w)|<\pi$.*

*Then, ${}_3F_{2,k}(w)$ defined by (1.2.3) is represented as a $G_k$-function by:*

$$ {}_3F_{2,k}[(u_1,k),(u_2,k),(u_3,k);(v_1,k),(v_2,k);w] = \frac{\Gamma_k(v_1)\Gamma_k(v_2)}{\Gamma_k(u_1)\Gamma_k(u_2)\Gamma_k(u_3)} $$
$$ G^{1,3}_{3,3,k}[(1-u_1,k),(1-u_2,k),(1-u_3,k);(0,k),(1-v_1,k),(1-v_2,k);-w]. \tag{1.4.9} $$

**Remark.** The conditions (1.3.6) and (1.3.7) for the representation of ${}_pF_{q,k}(w)$ given in Theorem 1.4.1(a) coincide, except for the point $w=0$, with the well-known conditions of convergence of the generalized $k$-hypergeometric series (1.1.8).

**Remark.** Theorems 1.3.4 and 1.4.1(b) allow us to define the extended generalized $k$-hypergeometric function ${}_pF_{q,k}(w)$ defined by (1.2.1) and (1.4.7) for the parameters $p, q$ and variable $w \in C$: $p>q+1$ for all complex $w \neq 0$ and $p=q+1$ for $|w|>1$. This representation can be considered as an extension of the generalized $k$-hypergeometric function defined by the series in (1.1.8) from the usual range of parameters $p, q$ and variable $w$; $p<q+1$ for all $w \in C$ and $p=q+1$ for $|w|<1$. Similarly, in the case $p=q+1$ the relation (1.2.1) and (1.4.7) can be considered as an extension of the generalized $k$-hypergeometric function (1.1.8) from $|w|<1$ to $|w|>1$.

## 1.5 Series representations of the generalized $k$-hypergeometric function ${}_pF_{q,k}(w)$

In this section we prove the series representation for the extended generalized $k$-hypergeometric function ${}_pF_{q,k}(w)$ defined by (1.2.1) and (1.4.7). These representations are different for $L_{-\infty}$ and $L_{+\infty}$. From Theorem 1.3.1 we deduce the result (1.1.8) that yields the series representation of the extended generalized $k$-hypergeometric function at zero.

**Theorem 1.5.1.** *Let $k>0$, $p, q \in N_0, u_i, v_j \in C(i=1,2,3...p, j=1,2,3,...,q)$ be such that $v_j \neq -l(j=1,2,3...q; l \in N_0)$ and let the conditions in (1.3.3) and (1.3.4) be satisfied. Let either one of the conditions in (1.3.6), (1.3.7), or (1.3.8) hold and let $L=L_{-\infty}$ be the contour that separates all poles $c_l$ in (1.3.1) to the left and all* *poles $u_{in_1}$ in (1.3.2) to the right. Then, the generalized $k$-hypergeometric function ${}_pF_{q,k}(w)$ defined by (1.2.1) with $L = L_{-\infty}$ has the power-series Eq. (1.1.8), that is,*

$$ {}_pF_{q,k}[(u_1,k),(u_2,k)\dots(u_p,k);(v_1,k),(v_2,k)\dots(v_q,k);w] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(u_i)_{n,k}}{\prod_{i=1}^{q}(v_i)_{n,k}} \frac{w^n}{n!}. \tag{1.5.1} $$

*Proof.* We write ${}_pF_{q,k}(w)$ defined by (1.2.1) in the form:

$$ {}_pF_{q,k}[(u_1,k),(u_2,k)\dots(u_p,k);(v_1,k),(v_2,k)\dots(v_q,k);w] = \frac{\prod_{j=1}^{q}\Gamma_k(v_j)}{\prod_{i=1}^{p}\Gamma_k(u_i)} \frac{1}{2\pi i} \int_L \frac{\Gamma(s)\prod_{i=1}^{p}\Gamma_k(u_i - sk)}{\prod_{j=1}^{q}\Gamma_k(v_j - sk)} (-z)^{-s} ds. \tag{1.5.2} $$

Applying the usual procedure, we evaluate the above integral as a sum of the residues of the integrand in (1.5.2) at the simple poles $c_l$ in (1.3.1). Taking the residue at $z = -v$ of $\Gamma(z)$:

$$ \lim_{z \to -v} (z+v)\Gamma(z) = \frac{(-1)^v}{v!}, v = 0, 1, 2\dots \tag{1.5.3} $$

we have

$$ \begin{aligned} &{}_pF_{q,k}[(u_1,k),(u_2,k)\dots(u_p,k);(v_1,k),(v_2,k)\dots(v_q,k);w] \\ &= \frac{\prod_{j=1}^{q}\Gamma_k(v_j)}{\prod_{i=1}^{p}\Gamma_k(u_i)} \sum_{n=0}^{\infty} Res_{s=-n}\left( \frac{\Gamma(s)\prod_{i=1}^{p}\Gamma_k(u_i - sk)}{\prod_{j=1}^{q}\Gamma_k(v_j - sk)} (-z)^{-s} \right). \qquad (1.5.4) \\ &= \frac{\prod_{j=1}^{q}\Gamma_k(v_j)}{\prod_{i=1}^{p}\Gamma_k(u_i)} \sum_{n=0}^{\infty} \left( \frac{\prod_{i=1}^{p}\Gamma_k(u_i + nk)}{\prod_{j=1}^{q}\Gamma_k(v_j + nk)} \frac{z^n}{n!} \right). \qquad (1.5.5) \end{aligned} $$

From here, in accordance with the formula $\Gamma_k(a+nk) = \Gamma_k(a)(a)_{n,k}$:

$$ {}_pF_{q,k}[(u_1,k),(u_2,k)\dots(u_p,k);(v_1,k),(v_2,k)\dots(v_q,k);w] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(u_i)_{n,k}}{\prod_{i=1}^{q}(v_i)_{n,k}} \frac{w^n}{n!}. \tag{1.5.6} $$

□

## 1.6 Conclusion

We conclude this chapter by remarking that we represented the generalized $k$-hypergeometric function ${}_pF_{q,k}$ in terms of the known Meijer G-function to extend to the range of parameters of such a special function is so that it is convergent. Furthermore, we established the corresponding series representation for such an extension. Our results are an important contribution to the field of special functions and their extensions [7–10,12–16,19].

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CHAPTER

# Matrix analog of a new class of hypergeometric functions

# 2

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## 2.1 Introduction

In 1998, Jódar and Cortés [7,8] introduced Beta and Gamma matrix functions and studied many properties and applications of these functions. For further study of the special functions we recommend the following papers [1–6].

In this chapter, we used the following notations: $I$ and $O$ represent the Identity and Zero matrices in $\mathbb{C}^{r\times r}$, respectively, and where $\mathbb{C}^{r\times r}$ is the vector space of $r$-square matrices with complex entries.

Consider matrix $A \in \mathbb{C}^{r\times r}$ the spectrum, represented by $\sigma(A)$, and it is the collection of all eigenvalues of the matrix A.

A matrix $A \in \mathbb{C}^{r\times r}$ is a positive stable matrix if $\Re(\mu) > 0 \; \forall \; \mu \in \sigma(A)$.

The Gamma and Beta matrix functions are given as follows [7]:

$$\Gamma(M) = \int_0^\infty e^{-t} t^{M-I} dt, \tag{2.1.1}$$

$$B(M, N) = \int_0^1 t^{M-I}(1-t)^{N-I} dt, \tag{2.1.2}$$

where $M$ and $n$ are positive stable matrices in $\mathbb{C}^{r\times r}$.

Also, notice that if $M$, $N$ and $M+N$ are positive stable matrices in $\mathbb{C}^{r\times r}$ and $M$ and $n$ are commutative with each other, i.e., $MN = NM$, then we have that the following relation holds true:

$$B(M, N) = \Gamma(M)\Gamma(N)\Gamma^{-1}(M+N). \tag{2.1.3}$$

Also, if $M + mI$ is an invertible matrix $\forall \; m \geq 0$, then the reciprocal Gamma matrix function is given as follows [7]:

$$\Gamma^{-1}(M) = M(M+I)....(M+(k-1)I)\Gamma^{-1}(M+kI), \quad k \geq 1. \tag{2.1.4}$$

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The Pochhammer matrix symbol is given as [8]:

$$(M)_k = \begin{cases} I, & if\ k=0, \\ M(M+I)...(M+(k-1)I), & if\ \ k \geq 1. \end{cases} \tag{2.1.5}$$

By using the above Pochhammer matrix symbol definition and Eq. (2.1.4), we can notice that:

$$(M)_k = \Gamma^{-1}(M)\Gamma(M+kI), \ \ k \geq 1. \tag{2.1.6}$$

In the sequence, Jódar and Cortés [8] also studied the Gauss hypergeometric and Confluent hypergeometric matrix functions as follows:

$${}_2F_1(M,N,P;x) = F(M,N,P;x) = \sum_{k=0}^{\infty} (M)_k (N)_k (P)_k^{-1} \frac{x^k}{k!}, \tag{2.1.7}$$

$${}_1F_1(N,P;x) = \Phi(N,P;x) = \sum_{k=0}^{\infty} (N)_k (P)_k^{-1} \frac{x^k}{k!}, \tag{2.1.8}$$

where matrices $M$, $N$, and $P$ in $\mathbb{C}^{r\times r}$ are such that $M+mI$ is invertible for all integers $m \geq 0$.

The above-defined series (2.1.7), converges absolutely for $|\ x\ |< 1$ and for $x=1$, if $\beta(M)+\beta(N)<\alpha(P)$, where $\beta(M)=\max\{\Re(x)|x\in\sigma(M)\}$, $\beta(M)=\min\{\Re(x)|x\in\sigma(M)\}$, and $\alpha(M)=-\beta(-M)$.

Dwivedi and Sahai [10] studied the generalized hypergeometric matrix function as follows:

$${}_pF_q(Q_1,....Q_p,R_1,.....R_q;z) = \sum_{k=0}^{\infty} (Q_1)_k,....(Q_p)_k(R_1)_k^{-1},.....(R_q)_k^{-1}\frac{z^k}{k!}, \tag{2.1.9}$$

where $Q_i, R_j \in \mathbb{C}^{r\times r}$, $1\leq i\leq p$, $1\leq j\leq q$, such that $R_j+mI$ are invertible for all integers $m\geq 0$

Very recently, Agarwal and Goyal [9] introduced a new class of hypergeometric function ${}_pG_q^{\eta,\zeta,m,\xi}(a,b;z)$ as follows:

$$\begin{aligned} {}_pG_q^{\eta,\zeta,m,\xi}(w) &= {}_pG_q^{\eta,\zeta,m,\xi}(u_1,...,u_p,v_1,...,v_q;w) \\ &= {}_pG_q^{\eta,\zeta,m,\xi}\left[w\left|\begin{matrix}u_1,...,u_p\\ v_1,...,v_q\end{matrix}\right.\right] = \sum_{k=0}^{\infty}\frac{(u_1)_k....(u_p)_k(\xi)_{mk}}{(v_1)_k...(v_q)_k\Gamma(\eta k+\zeta)}\frac{w^k}{k!}, \end{aligned} \tag{2.1.10}$$

where $\eta,\zeta,\xi\in\mathbb{C}$ such that $w\in\mathbb{C}$, $\Re(\eta),\Re(\zeta),\Re(\xi)>0$, $m\in(0,1)\cup\mathbb{N}$ and $(u_i)_k$ and $(v_j)_k$, $i=1,..p$, $j=1,...q$ are the Pochhammer symbols.

## 2.2 Main results

In this section, we introduce a matrix analog of a new class of the hypergeometric function ${}_pG_q^{\eta,\zeta,m,\xi}(w)$.

**Definition 2.2.1.** Let $R$, $S$ $A_i$, $B_j$ $1 \le i \le p$, $1 \le j \le q$, be positive stable matrices in $\mathbb{C}^{r\times r}$ such that $B_j + kI$, $1 \le j \le q$, and $R + kI$ are invertible for all integers $k \ge 0$. Then, the matrix analog of a new class of hypergeometric function ${}_pG_q^{\eta,\zeta,m,\xi}(z)$ with $z \in \mathbb{C}$ is defined by:

$$
\begin{aligned}
{}_pG_q^{\alpha,R,S,\beta}(z) &= {}_pG_q^{\alpha,R,S,\beta} A_1, ..., A_p, B_1, ..., B_q; z) \\
= {}_pG_q^{\alpha,R,S,\beta}\left[z \left| \begin{matrix} A_1, ..., A_p \\ B_1, ..., B_q \end{matrix} \right.\right] &= \sum_{m=0}^{\infty} \frac{(A_1)_m, ...., (A_p)_m (S)_{\beta m}}{(B_1)_m ... (B_q)_m \Gamma(\alpha m I + R)} \frac{z^m}{m!},
\end{aligned} \tag{2.2.1}
$$

where $\beta, \alpha, \in \mathbb{C}$ such that $\Re(\beta), \Re(\alpha) > 0$, $\beta \in (0,1) \cup \mathbb{N}$ and $(A_i)_m$, and $(B_j)_m$, $i = 1, ..p$, $j = 1, ...q$ are the Pochhammer matrix symbols.

**Remark.** The matrix analog of the new class of the hypergeometric function ${}_pG_q^{\eta,\zeta,m,\xi}(z)$ can also be written as:

$$
{}_pG_q^{\alpha,R,S,\beta}\left[z \left| \begin{matrix} A_1, ..., A_p \\ B_1, ..., B_q \end{matrix} \right.\right] = \sum_{m=0}^{\infty} \frac{(A_1)_m, ...., (A_p)_m \Gamma(\beta m I + S)}{(B_1)_m ... (B_q)_m \Gamma(\alpha m I + R)\Gamma(S)} \frac{z^m}{m!}. \tag{2.2.2}
$$

**Remark.** (i) If we substitute $R = S = I$ and $\beta = 1$, then series (2.1.10) represents the matrix analog of the M-series given in [11]:

$$
{}_pG_q^{\alpha,I,I,1}\left[z \left| \begin{matrix} A_1, ..., A_p \\ B_1, ..., B_q \end{matrix} \right.\right] = {}_pM_q^{\alpha}\left[z \left| \begin{matrix} A_1, ..., A_p \\ B_1, ..., B_q \end{matrix} \right.\right]. \tag{2.2.3}
$$

(ii) If we put $S = I$ and $\beta = 1$, then series (2.1.10) represents the matrix analog of the modified M-series given in [12]:

$$
{}_pG_q^{\alpha,R,I,1}\left[z \left| \begin{matrix} A_1, ..., A_p \\ B_1, ..., B_q \end{matrix} \right.\right] = {}_pM_q^{\alpha,R}\left[z \left| \begin{matrix} A_1, ..., A_p \\ B_1, ..., B_q \end{matrix} \right.\right]. \tag{2.2.4}
$$

(iii) If we consider $\beta = 1$, then series (2.1.10) represents the matrix analog of the K-series defined in [13]:

$$
{}_pG_q^{\alpha,R,S,1}\left[z \left| \begin{matrix} A_1, ..., A_p \\ B_1, ..., B_q \end{matrix} \right.\right] = {}_pM_q^{\alpha,R,S}\left[z \left| \begin{matrix} A_1, ..., A_p \\ B_1, ..., B_q \end{matrix} \right.\right]. \tag{2.2.5}
$$

## 2.3 Special cases

### 2.3.1 Relation to generalized hypergeometric function

Assume $\alpha = \beta = 1$ and $R = S = I$, then series (2.1.10) is reduced to the matrix analog generalized hypergeometric function defined in (2.1.9):

$$ {}_pG_q^{1,I,I,1}\left[z\left|\begin{matrix}A_1,\dots,A_p\\ B_1,\dots,B_q\end{matrix}\right.\right] = {}_pF_q\left[z\left|\begin{matrix}A_1,\dots,A_p\\ B_1,\dots,B_q\end{matrix}\right.\right] = \sum_{m=0}^{\infty}\frac{(A_1)_m,\dots.,(A_p)_m}{(B_1)_m\dots(B_q)_m}\frac{z^m}{m!}. \tag{2.3.1}$$

Consider $|z| < 1$, $p = 2$, and $q = 1$, then (2.3.1) reduces to the matrix analog of the Gauss hypergeometric function ${}_2F_1(z)$ given in (2.1.1):

$$ {}_2G_1^{1,I,I,1}\left[z\left|\begin{matrix}A_1,A_2\\ B_1\end{matrix}\right.\right] = {}_2F_1\left[z\left|\begin{matrix}A_1,A_2\\ B_1\end{matrix}\right.\right] = \sum_{m=0}^{\infty}\frac{(A_1)_m,(A_2)_m}{(B_1)_m}\frac{z^m}{m!}. \tag{2.3.2}$$

Let $p = 1$ and $q = 1$, then (2.3.1) reduces to the matrix analog confluent hypergeometric function ${}_1\Phi_1(w)$ defined in (2.1.8):

$$ {}_1G_1^{1,I,I,1}\left[z\left|\begin{matrix}A_1\\ B_1\end{matrix}\right.\right] = {}_1F_1\left[z\left|\begin{matrix}A_1\\ B_1\end{matrix}\right.\right] = \sum_{m=0}^{\infty}\frac{(A_1)_m}{(B_1)_m}\frac{z^m}{m!}. \tag{2.3.3}$$

Consider $p = 0 = q$, then (2.1.10) reduces to the matrix analog of the four-parameter Mittag-Leffler function $E_{\alpha,\mu}^{\gamma,\beta}(z)$ defined by Shukla and Prajapati in [14]:

$$ {}_0G_0^{\alpha,R,S,\beta}\left[z\left|\begin{matrix}-\\ -\end{matrix}\right.\right] = E_{\alpha,R}^{S,\beta}(z) = \sum_{m=0}^{\infty}\frac{(S)_{\beta m}}{\Gamma(\alpha m I + R)}\frac{z^m}{m!}. \tag{2.3.4}$$

In addition to the above conditions, if we put $\beta = 1$, then (2.1.10) becomes the matrix analog of the three-parameter Mittag-Leffler function $E_{\eta,\zeta}^{\xi}(z)$ studied by Prabhakar [15]:

$$ {}_0G_0^{\alpha,R,S,1}\left[z\left|\begin{matrix}-\\ -\end{matrix}\right.\right] = E_{\alpha,R}^{S}(z) = \sum_{m=0}^{\infty}\frac{(S)_m}{\Gamma(\alpha m I + R)}\frac{z^m}{m!}. \tag{2.3.5}$$

If we further substitute $S = I$, we obtain the matrix analog of Wiman's function $E_{\eta,\zeta}(z)$ defined in [16] as follows:

$$ {}_0G_0^{\alpha,R,I,1}\left[z\left|\begin{matrix}-\\ -\end{matrix}\right.\right] = E_{\alpha,R}(z) = \sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha m I + R)m!}. \tag{2.3.6}$$

From the definition of the matrix analog of the ${}_pG_q^{\eta,\zeta,m,\xi}(z)$ function, we can observe that it is closely related to the matrix analog of the Wright hypergeometric function given [7] as follows:

$$ {}_pG_q^{\alpha,R,S,\beta}\left[z\left|\begin{matrix}A_1,\ldots,A_p\\B_1,\ldots,B_q\end{matrix}\right.\right]=\Omega.{}_{p+1}\Psi_{q+1}\left[z\left|\begin{matrix}(A_1,I),\cdots,\left(A_p,I\right),(S,\beta I)\\(B_1,1),\cdots,\left(B_q,1\right),(R,\alpha I)\end{matrix}\right.\right], \tag{2.3.7}$$

where $\Omega=\frac{\Gamma(B_1),\cdots,\Gamma(B_q)}{\Gamma(S)\Gamma(A_1),\cdots,\Gamma(A_p)}$.

### 2.3.2 Relation to the Fox H-function

Consider the matrix analog of the ${}_pG_q^{\eta,\zeta,m,\xi}(z)$ function: we found that it is related to the matrix version of the Fox $H$-function defined in [11] as follows:

$$ {}_pG_q^{\alpha,R,S,\beta}\left[z\left|\begin{matrix}A_1,\ldots,A_p\\B_1,\ldots,B_q\end{matrix}\right.\right]$$
$$=\Omega.H_{p+1,q+2}^{1,p+1}\left[-w\left|\begin{matrix}(1-A_1,1),\ldots,\left(1-A_p,1\right),(1-S,\beta I)\\(0,1),(1-B_1,1),\ldots,\left(1-B_q,1\right),(1-R,\alpha I)\end{matrix}\right.\right], \tag{2.3.8}$$

where $\Omega=\frac{\Gamma(B_1),\cdots,\Gamma(B_q)}{\Gamma(S)\Gamma(A_1),\cdots,\Gamma(A_p)}$.

### 2.3.3 Some properties of the matrix version of the ${}_pG_q^{\eta,\zeta,m,\xi}(z)$ function

**Theorem 2.3.1.** *The following integral representation for the matrix analog of the ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function holds true:*

$$ {}_pG_q^{\alpha,R,S,\beta}\left[z\left|\begin{matrix}A_1,\ldots,A_p\\B_1,\ldots,B_q\end{matrix}\right.\right]$$
$$=\frac{\Gamma(B_1)}{\Gamma(A_1)\Gamma(B_1-A_1)}\int_0^1 t^{A_1-1}(1-t)^{B_1-A_1-1}\,{}_{p-1}G_{q-1}^{\alpha,R,S,\beta}\left[zt\left|\begin{matrix}A_2,\ldots,A_p\\B_2,\ldots,B_q\end{matrix}\right.\right], \tag{2.3.9}$$

*where $R$, $S$ $A_i$, $B_j$ $1\le i\le p$, $1\le j\le q$, are positive stable matrices in $\mathbb{C}^{r\times r}$ such that $B_j+kI$, $1\le j\le q$, and $R+kI$ are invertible for all integers $k\ge 0$ and $\beta,\alpha,\in\mathbb{C}$ such that $\Re(\beta),\Re(\alpha)>0$, $\beta\in(0,1)\cup\mathbb{N}$, and $(A_i)_m$ and $(B_j)_m$, $i=1,..p$, $j=1,...q$ are the Pochhammer matrix symbols.*

*Proof.* From the properties of the Pochhammer matrix symbol given in [8], we obtain:

$$\frac{(A_1)_m}{(B_1)_m}=\frac{\Gamma(A_1+mI)\Gamma(B_1)}{\Gamma(B_1+mI)\Gamma(A_1)}=\frac{\Gamma(A_1+mI)\Gamma(B_1)\Gamma(B_1-A_1)}{\Gamma(B_1+mI)\Gamma(A_1)\Gamma(B_1-A_1)}. \tag{2.3.10}$$

By using the relation between the Beta and Gamma matrix functions, we have:

$$\frac{(A_1)_m}{(B_1)_m} = \frac{\Gamma(B_1)}{\Gamma(A_1)\Gamma(B_1 - A_1)} B(A_1 + mI, B_1 - A_1). \tag{2.3.11}$$

Then, using the integral representation of the Beta matrix function [7], we have:

$$\frac{(A_1)_m}{(B_1)_m} = \frac{\Gamma(B_1)}{\Gamma(A_1)\Gamma(B_1 - A_1)} \int_0^1 t^{A_1+mI-I}(1-t)^{B_1-A_1-I} dt. \tag{2.3.12}$$

Substituting the value of $\frac{(A_1)_m}{(B_1)_m}$ in the definition of the matrix analog of the ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function 2.2.1, we obtain:

$$\begin{aligned}
&{}_pG_q^{\alpha,R,S,\beta}\left[z\left| \begin{matrix} A_1, \dots, A_p \\ B_1, \dots, B_q \end{matrix}\right.\right] \\
&= \sum_{m=0}^{\infty} \frac{(A_2)_m \dots (A_p)_m \Gamma(\beta m I + S)}{(B_2)_m \dots (B_q)_m \Gamma(\alpha m I + R)\Gamma(S)} \\
&\left\{ \frac{\Gamma(B_1)}{\Gamma(A_1)\Gamma(B_1 - A_1)} \int_0^1 t^{A_1+mI-I}(1-t)^{B_1-A_1-I} dt \right\} \frac{z^m}{m!}.
\end{aligned} \tag{2.3.13}$$

On interchanging the order of integration and summation with some calculation, we have:

$$\begin{aligned}
&{}_pG_q^{\alpha,R,S,\beta}\left[z\left| \begin{matrix} A_1, \dots, A_p \\ B_1, \dots, B_q \end{matrix}\right.\right] \\
&= \frac{\Gamma(B_1)}{\Gamma(A_1)\Gamma(B_1 - A_1)} \int_0^1 t^{A_1-I}(1-t)^{B_1-A_1-I} \\
&\left( \sum_{m=0}^{\infty} \frac{(A_2)_m \dots (A_p)_m \Gamma(\beta m I + S)}{(B_2)_m \dots (B_q)_m \Gamma(\alpha m + R)\Gamma(S)} \frac{(zt)^m}{m!} \right) dt.
\end{aligned} \tag{2.3.14}$$

By the definition of the matrix analog of the ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function 2.2.1, we obtain our desired result:

$$\begin{aligned}
&{}_pG_q^{\alpha,R,S,\beta}\left[z\left| \begin{matrix} A_1, \dots, A_p \\ B_1, \dots, B_q \end{matrix}\right.\right] \\
&= \frac{\Gamma(B_1)}{\Gamma(A_1)\Gamma(B_1 - A_1)} \int_0^1 t^{A_1-1}(1-t)^{B_1-A_1-1} {}_{p-1}G_{q-1}^{\alpha,R,S,\beta}\left[zt\left| \begin{matrix} A_2, \dots, A_p \\ B_2, \dots, B_q \end{matrix}\right.\right].
\end{aligned} \tag{2.3.15}$$

□

**Theorem 2.3.2.** *Assume $R$ and $S$ are two commutative matrices in $\mathbb{C}^{r\times r}$ such that $R$, $S$, and $R - S$ are positive stable. Then, the following integral representation for* *the matrix analog of the* ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ *function holds true:*

$$
{}_pG_q^{\alpha,R,S,\alpha}\left[z\left|\begin{matrix}A_1,\ldots,A_p\\B_1,\ldots,B_q\end{matrix}\right.\right]=\frac{1}{\Gamma(S)\Gamma(R-S)}\left\{\int_0^1 t^{S-I}(1-t)^{R-S-I}\,{}_pF_q[zt^\alpha]dt\right\},
\tag{2.3.16}
$$

*where* $\alpha\in\mathbb{C}$ *such that* $\Re(\alpha)>0$ *and* $(A_i)_m$ *and* $(B_j)_m$, $i=1,..p,\ j=1,...q$ *are the Pochhammer matrix symbols.*

*Proof.* From the definition of the matrix analog of the ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function 2.2.1, we have:

$$
{}_pG_q^{\alpha,R,S,\alpha}\left[z\left|\begin{matrix}A_1,\ldots,A_p\\B_1,\ldots,B_q\end{matrix}\right.\right]=\sum_{m=0}^{\infty}\frac{(A_1)_m,\ldots.,(A_p)_m\Gamma(\alpha mI+S)}{(B_1)_m\ldots(B_q)_m\Gamma(\alpha mI+R)\Gamma(S)}\frac{z^m}{m!}.
\tag{2.3.17}
$$

Then, from the relation between the Gamma and Beta matrix functions, we have:

$$
B(\alpha mI+S,R-S)=\frac{\Gamma(\alpha mI+S)\Gamma(R-S)}{\Gamma(\alpha mI+R)}.
\tag{2.3.18}
$$

On using the above result in Eq. (2.3.17), we obtain:

$$
{}_pG_q^{\alpha,R,S,\alpha}\left[z\left|\begin{matrix}A_1,\ldots,A_p\\B_1,\ldots,B_q\end{matrix}\right.\right]=\sum_{m=0}^{\infty}\frac{(A_1)_m,\ldots.,(A_p)_m B(\alpha mI+S,R-S)}{(B_1)_m\ldots(B_q)_m\Gamma(R-S)\Gamma(S)}\frac{z^m}{m!}.
\tag{2.3.19}
$$

By using the integral representation of the Beta matrix function (2.1.2), we obtain:

$$
\begin{aligned}
&{}_pG_q^{\alpha,R,S,\alpha}\left[z\left|\begin{matrix}A_1,\ldots,A_p\\B_1,\ldots,B_q\end{matrix}\right.\right]\\
&=\sum_{m=0}^{\infty}\frac{(A_1)_m,\ldots.,(A_p)_m}{(B_1)_m\ldots(B_q)_m\Gamma(R-S)\Gamma(S)}\left\{\int_0^1 t^{S+(\alpha m-1)I}(1-t)^{(R-S)-I}dt\right\}\frac{z^m}{m!}.
\end{aligned}
\tag{2.3.20}
$$

On interchanging summation and integration, we have:

$$
\begin{aligned}
&{}_pG_q^{\alpha,R,S,\alpha}\left[z\left|\begin{matrix}A_1,\ldots,A_p\\B_1,\ldots,B_q\end{matrix}\right.\right]\\
&=\frac{1}{\Gamma(R-S)\Gamma(S)}\int_0^1 t^{S-I}(1-t)^{(R-S)-I}\sum_{m=0}^{\infty}\frac{(A_1)_m,\ldots.,(A_p)_m}{(B_1)_m\ldots(B_q)_m}\frac{(zt^\alpha)^m}{m!}dt.
\end{aligned}
\tag{2.3.21}
$$

Then, using Eq. (2.1.7), we obtain our desired result:

$$ {}_pG_q^{\alpha,R,S,\alpha}\left[z\left|\begin{matrix}A_1,\ldots,A_p\\B_1,\ldots,B_q\end{matrix}\right.\right]=\frac{1}{\Gamma(S)\Gamma(R-S)}\left\{\int_0^1 t^{S-I}(1-t)^{R-S-I}\,{}_pF_q[zt^{\alpha}]dt\right\}. \tag{2.3.22}$$

□

**Corollary 2.3.3.** *The following result holds true:*

$$E^{\alpha,R,S,\alpha}(z)=E_{\alpha,R}^{S,\alpha}(z)=\frac{1}{\Gamma(S)\Gamma(R-S)}\left\{\int_0^1 t^{S-I}(1-t)^{R-S-I}e^{zt^{\alpha}}dt\right\}, \tag{2.3.23}$$

*where* $\alpha\in\mathbb{C}$ *such that* $\Re(\alpha)>0$.

*Proof.* Consider $p=0=q$, and with the help of Eq. (2.3.4) we obtain our desired result. □

**Theorem 2.3.4.** *Assume* $R$, $S$, *and* $T$ *are commutative matrices in* $\mathbb{C}^{r\times r}$ *such that* $R$, $S$, $T$, *and* $R+T$ *are positive stable. Then, the following integral representation for the matrix analog of the* ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ *function holds true:*

$$\begin{aligned}&\int_w^z (z-v)^{T-I}(v-w)^{R-I}\,{}_pG_q^{\alpha,R,S,\beta}\left[x(v-w)^{\alpha}\left|\begin{matrix}A_1,\ldots,A_p\\B_1,\ldots,B_q\end{matrix}\right.\right]dv=\\&\qquad\Gamma(T)(z-w)^{R+T-I}\,{}_pG_q^{\alpha,R+T,S,\beta}\left[x(z-w)^{\alpha}\left|\begin{matrix}A_1,\ldots,A_p\\B_1,\ldots,B_q\end{matrix}\right.\right],\end{aligned} \tag{2.3.24}$$

*where* $\beta,\alpha,\in\mathbb{C}$ *such that* $\Re(\beta),\Re(\alpha)>0$, $\beta\in(0,1)\cup\mathbb{N}$.

*Proof.* Consider the left-hand side of the above equation and using the definition of the matrix analog of the ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function 2.2.1, we obtain:

$$\begin{aligned}&\int_w^z (z-v)^{T-I}(v-w)^{R-I}\,{}_pG_q^{\alpha,R,S,\beta}\left[x(v-w)^{\alpha}\left|\begin{matrix}A_1,\ldots,A_p\\B_1,\ldots,B_q\end{matrix}\right.\right]dv=\\&\int_w^z (z-v)^{T-I}(v-w)^{R-I}\left\{\sum_{m=0}^{\infty}\frac{(A_1)_m,\ldots.,(A_p)_m\Gamma(\beta mI+S)}{(B_1)_m\ldots(B_q)_m\Gamma(\alpha mI+R)\Gamma(S)}\frac{x^m(v-w)^{\alpha m}}{m!}\right\}dv.\end{aligned} \tag{2.3.25}$$

On interchanging summation and integration, we have

$$\begin{aligned}&\int_w^z (z-v)^{T-I}(v-w)^{R-I}\,{}_pG_q^{\alpha,R,S,\beta}\left[x(v-w)^{\alpha}\left|\begin{matrix}A_1,\ldots,A_p\\B_1,\ldots,B_q\end{matrix}\right.\right]dv=\\&\frac{1}{\Gamma(S)}\sum_{m=0}^{\infty}\frac{(A_1)_m,\ldots.,(A_p)_m\Gamma(\beta mI+S)}{(B_1)_m\ldots(B_q)_m\Gamma(\alpha mI+R)}\frac{x^m}{m!}\int_w^z (z-v)^{T-I}(v-w)^{R+(\alpha m-1)I}dv.\end{aligned} \tag{2.3.26}$$

By changing the variable $v$ to $s = \frac{v-w}{z-w}$ and after some calculations, we have:

$$\int_w^z (z-v)^{T-I}(v-w)^{R-I}\,{}_pG_q^{\alpha,R,S,\beta}\left[x(v-w)^\alpha \middle| \begin{matrix} A_1,\dots,A_p \\ B_1,\dots,B_q \end{matrix}\right] dv$$
$$= \frac{1}{\Gamma(S)}(z-w)^{R+T-I} \sum_{m=0}^{\infty} \frac{(A_1)_m,\dots.,(A_p)_m \Gamma(\beta m I + S)}{(B_1)_m \dots (B_q)_m \Gamma(\alpha m I + R)} \frac{(x(z-w)^\alpha)^m}{m!}$$
$$\int_0^1 s^{R+(\alpha m-1)I}(1-s)^{T-I} ds$$
$$= \frac{1}{\Gamma(S)}(z-w)^{R+T-I} \sum_{m=0}^{\infty} \frac{(A_1)_m,\dots.,(A_p)_m \Gamma(\beta m I + S)}{(B_1)_m \dots (B_q)_m \Gamma(\alpha m I + R)} \frac{(x(z-w)^\alpha)^m}{m!}$$
$$B(T, R+\alpha m I). \tag{2.3.27}$$

On using the Beta and Gamma matrix functions properties, we have:

$$\int_w^z (z-v)^{T-I}(v-w)^{R-I}\,{}_pG_q^{\alpha,R,S,\beta}\left[x(v-w)^\alpha \middle| \begin{matrix} A_1,\dots,A_p \\ B_1,\dots,B_q \end{matrix}\right] dv$$
$$= \frac{\Gamma(T)}{\Gamma(S)}(z-w)^{R+T-I} \sum_{m=0}^{\infty} \frac{(A_1)_m,\dots.,(A_p)_m \Gamma(\beta m I + S)}{(B_1)_m \dots (B_q)_m \Gamma(\alpha m I + R + T)} \frac{(x(z-w)^\alpha)^m}{m!}. \tag{2.3.28}$$

Then, by using Definition 2.2.1, we obtain our desired result:

$$\int_w^z (z-v)^{T-I}(v-w)^{R-I}\,{}_pG_q^{\alpha,R,S,\beta}\left[x(v-w)^\alpha \middle| \begin{matrix} A_1,\dots,A_p \\ B_1,\dots,B_q \end{matrix}\right] dv = \Gamma(T)(z-w)^{R+T-I}\,{}_pG_q^{\alpha,R+T,S,\beta}\left[x(z-w)^\alpha \middle| \begin{matrix} A_1,\dots,A_p \\ B_1,\dots,B_q \end{matrix}\right]. \tag{2.3.29}$$

□

**Corollary 2.3.5.** *The following result holds true:*

$$\int_w^z (z-v)^{T-I}(v-w)^{R-I} E^{\alpha,R,S,\alpha}(x(v-w)^\alpha) dv = \Gamma(T)(z-w)^{R+T-I} E^{\alpha,R+T,S,\alpha}(x(z-w)^\alpha), \tag{2.3.30}$$

*where* $\beta, \alpha, \in \mathbb{C}$ *such that* $\Re(\beta), \Re(\alpha) > 0$, $\beta \in (0,1) \cup \mathbb{N}$.

*Proof.* Consider $p = 0 = q$, and with the help of Eq. (2.3.4) we obtain our desired result. □

**Theorem 2.3.6.** *For the matrix analog of the* ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ *function the following derivative formula holds true:*

$$\left(\frac{d}{dz}\right)^m {}_pG_q^{\alpha,R,S,\beta}\left[z\left|\begin{matrix}A_1,\ldots,A_p\\B_1,\ldots,B_q\end{matrix}\right.\right]=$$
$$\frac{\Gamma(S+\beta mI)}{\Gamma(S)}\frac{(A_1)_m\ldots.(A_p)_m}{(B_1)_m\ldots.(B_q)_m}\,{}_pG_q^{\alpha,R+\alpha mI,S+\beta mI,\beta}\left[z\left|\begin{matrix}A_1+m,\ldots,A_p+m\\B_1+m,\ldots,B_q+m\end{matrix}\right.\right], \tag{2.3.31}$$

*where* $\beta,\alpha,\in\mathbb{C}$ *such that* $\Re(\beta),\Re(\alpha)>0$, $\beta\in(0,1)\cup\mathbb{N}$, *and* $(A_i)_m$ *and* $(B_j)_m$, $i=1,..p$, $j=1,\ldots q$ *are the Pochhammer matrix symbols.*

*Proof.* From the definition of the matrix analog of the ${}_pG_q^{\eta,\zeta,k,\xi}(w)$ function 2.2.1, we have:

$$\begin{aligned}(\frac{d}{dz})^m {}_pG_q^{\alpha,R,S,\beta}\left[z\left|\begin{matrix}A_1,\ldots,A_p\\B_1,\ldots,B_q\end{matrix}\right.\right]&=(\frac{d}{dz})^m\sum_{n=0}^{\infty}\frac{(A_1)_n,\ldots.,(A_p)_n\Gamma(\beta nI+S)}{(B_1)_n\ldots(B_q)_n\Gamma(\alpha nI+R)\Gamma(S)}\frac{z^n}{n!}\\&=\sum_{n=0}^{m}\frac{(A_1)_n,\ldots.,(A_p)_n\Gamma(\beta nI+S)}{(B_1)_n\ldots(B_q)_n\Gamma(\alpha nI+R)\Gamma(S)}\frac{z^{n-m}}{(n-m)!}.\end{aligned} \tag{2.3.32}$$

Then, replacing $n-m$ by $n$, we obtain:

$$\begin{aligned}&\left(\frac{d}{dz}\right)^m {}_pG_q^{\alpha,R,S,\beta}\left[z\left|\begin{matrix}A_1,\ldots,A_p\\B_1,\ldots,B_q\end{matrix}\right.\right]\\&=\sum_{n=0}^{\infty}\frac{(A_1)_{n+m},\ldots.,(A_p)_{n+m}\Gamma(\beta(n+m)I+S)}{(B_1)_{n+m}\ldots(B_q)_{n+m}\Gamma(\alpha(n+m)I+R)\Gamma(S)}\frac{z^n}{(n)!}.\end{aligned} \tag{2.3.33}$$

By using the property of the Pochhammer matrix symbol $(A)_{n+m}=(A)_m(A+mI)_n$, and using the Definition 2.2.1, we obtain our desired result:

$$(\frac{d}{dz})^m {}_pG_q^{\alpha,R,S,\beta}\left[z\left|\begin{matrix}A_1,\ldots,A_p\\B_1,\ldots,B_q\end{matrix}\right.\right]=$$
$$\frac{\Gamma(S+\beta mI)}{\Gamma(S)}\frac{(A_1)_m\ldots.(A_p)_m}{(B_1)_m\ldots.(B_q)_m}\,{}_pG_q^{\alpha,R+\alpha mI,S+\beta mI,\beta}\left[z\left|\begin{matrix}A_1+mI,\ldots,A_p+mI\\B_1+mI,\ldots,B_q+mI\end{matrix}\right.\right]. \tag{2.3.34}$$

□

**Corollary 2.3.7.** *For the matrix analog of the* ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ *function the following result holds true:*

$$\frac{d}{dz}{}_pG_q^{\alpha,R,S,\beta}\left[z\left|\begin{matrix}A_1,\dots,A_p\\B_1,\dots,B_q\end{matrix}\right.\right]= \frac{\Gamma(S+\beta I)}{\Gamma(S)}\frac{A_1\dots.A_p}{B_1\dots.B_q}{}_pG_q^{\alpha,R+\alpha I,S+\beta I,\beta}\left[z\left|\begin{matrix}A_1+I,\dots,A_p+I\\B_1+I,\dots,B_q+I\end{matrix}\right.\right]. \tag{2.3.35}$$

**Theorem 2.3.8.** *For the matrix analog of the* ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ *function the following derivative formula holds true:*

$$\left(\frac{d}{dz}\right)^n\left\{z^{R-I}{}_pG_q^{\alpha,R,S,\beta}\left[wz^\alpha\left|\begin{matrix}A_1,\dots,A_p\\B_1,\dots,B_q\end{matrix}\right.\right]\right\} = z^{R-(n+1)I}{}_pG_q^{\alpha,R-nI,S,\beta}\left[wz^\alpha\left|\begin{matrix}A_1,\dots,A_p\\B_1,\dots,B_q\end{matrix}\right.\right], \tag{2.3.36}$$

*where* $\beta,\alpha,\in\mathbb{C}$ *such that* $\Re(\beta),\Re(\alpha)>0$, $\beta\in(0,1)\cup\mathbb{N}$, *and* $(A_i)_m$ *and* $(B_j)_m$, $i=1,..p$, $j=1,\dots q$ *are the Pochhammer matrix symbols.*

*Proof.* From the definition of the matrix analog of the ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function 2.2.1, we have:

$$\begin{aligned}&\left(\frac{d}{dz}\right)^n\left\{z^{R-I}{}_pG_q^{\alpha,R,S,\beta}\left[wz^\alpha\left|\begin{matrix}A_1,\dots,A_p\\B_1,\dots,B_q\end{matrix}\right.\right]\right\}\\ &=\left(\frac{d}{dz}\right)^n\left\{z^{R-I}\sum_{m=0}^{\infty}\frac{(A_1)_m,\dots.,(A_p)_m\Gamma(\beta mI+S)}{(B_1)_m\dots(B_q)_m\Gamma(\alpha mI+R)\Gamma(S)}\frac{w^m(z^\alpha)^m}{m!}\right\}\\ &=\sum_{m=0}^{\infty}\frac{(A_1)_m,\dots.,(A_p)_m\Gamma(\beta mI+S)}{(B_1)_m\dots(B_q)_m\Gamma(\alpha mI+R)\Gamma(S)}\frac{w^m}{m!}\left(\frac{d}{dz}\right)^n[z^{R+(\alpha m-1)I}].\end{aligned} \tag{2.3.37}$$

On differentiating each term under the sign of summation and then by using the Definition 2.2.1, we obtain our desired result:

$$\left(\frac{d}{dz}\right)^n\left\{z^{R-I}{}_pG_q^{\alpha,R,S,\beta}\left[wz^\alpha\left|\begin{matrix}A_1,\dots,A_p\\B_1,\dots,B_q\end{matrix}\right.\right]\right\} = z^{R-(n+1)I}{}_pG_q^{\alpha,R-nI,S,\beta}\left[wz^\alpha\left|\begin{matrix}A_1,\dots,A_p\\B_1,\dots,B_q\end{matrix}\right.\right]. \tag{2.3.38}$$

□

**Theorem 2.3.9.** *The following differential property holds true:*

$${}_pG_q^{\alpha,R,S,\beta}\left[z\left|\begin{matrix}A_1,\dots,A_p\\B_1,\dots,B_q\end{matrix}\right.\right] = R\,{}_pG_q^{\alpha,R+I,S,\beta}\left[z\left|\begin{matrix}A_1,\dots,A_p\\B_1,\dots,B_q\end{matrix}\right.\right]+\alpha z\frac{d}{dz}{}_pG_q^{\alpha,R+I,S,\beta}\left[z\left|\begin{matrix}A_1,\dots,A_p\\B_1,\dots,B_q\end{matrix}\right.\right], \tag{2.3.39}$$

*where $\beta, \alpha, \in \mathbb{C}$ such that $\Re(\beta), \Re(\alpha) > 0$, $\beta \in (0,1) \cup \mathbb{N}$, and $(A_i)_m$ and $(B_j)_m$, $i = 1, ..p$, $j = 1, ...q$ are the Pochhammer matrix symbols.*

*Proof.* From the above Corollary (2.3.35), we have:

$$\begin{aligned} &\alpha z \frac{d}{dz} {}_pG_q^{\alpha, R+I, S, \beta}\left[z \middle| \begin{matrix} A_1, ..., A_p \\ B_1, ..., B_q \end{matrix}\right] \\ &= \sum_{m=0}^{\infty} \frac{\alpha m (A_1)_m, ...., (A_p)_m \Gamma(\beta m I + S)}{(B_1)_m ... (B_q)_m \Gamma(\alpha m I + R + I)\Gamma(S)} \frac{z^m}{m!} \\ &= \sum_{m=0}^{\infty} \frac{(\alpha m I + R - R)(A_1)_m, ...., (A_p)_m \Gamma(\beta m I + S)}{(B_1)_m ... (B_q)_m \Gamma(\alpha m I + R + I)\Gamma(S)} \frac{z^m}{m!}. \end{aligned} \tag{2.3.40}$$

After some re-arrangement, we obtain:

$$\begin{aligned} &\alpha z \frac{d}{dz} {}_pG_q^{\alpha, R+I, S, \beta}\left[z \middle| \begin{matrix} A_1, ..., A_p \\ B_1, ..., B_q \end{matrix}\right] \\ &= \sum_{m=0}^{\infty} \frac{(\alpha m I)(A_1)_m, ...., (A_p)_m \Gamma(\beta m I + S)}{(B_1)_m ... (B_q)_m \Gamma(\alpha m I + R + I)\Gamma(S)} \frac{z^m}{m!} \\ &- \sum_{m=0}^{\infty} \frac{R(A_1)_m, ...., (A_p)_m \Gamma(\beta m I + S)}{(B_1)_m ... (B_q)_m \Gamma(\alpha m I + R + I)\Gamma(S)} \frac{z^m}{m!} \\ &= \sum_{m=0}^{\infty} \frac{(\alpha m I + R)(A_1)_m, ...., (A_p)_m \Gamma(\beta m I + S)}{(B_1)_m ... (B_q)_m \Gamma(\alpha m I + R + I)\Gamma(S)} \frac{z^m}{m!} \\ &- R_p G_q^{\alpha, R+I, S, \beta}\left[z \middle| \begin{matrix} A_1, ..., A_p \\ B_1, ..., B_q \end{matrix}\right]. \end{aligned} \tag{2.3.41}$$

Then, we have:

$$\begin{aligned} &\alpha z \frac{d}{dz} {}_pG_q^{\alpha, R+I, S, \beta}\left[z \middle| \begin{matrix} A_1, ..., A_p \\ B_1, ..., B_q \end{matrix}\right] + R_p G_q^{\alpha, R+I, S, \beta}\left[z \middle| \begin{matrix} A_1, ..., A_p \\ B_1, ..., B_q \end{matrix}\right] \\ &= \sum_{m=0}^{\infty} \frac{(\alpha m I + R)(A_1)_m, ...., (A_p)_m \Gamma(\beta m I + S)}{(B_1)_m ... (B_q)_m \Gamma(\alpha m I + R + I)\Gamma(S)} \frac{z^m}{m!}. \end{aligned} \tag{2.3.42}$$

On some simplification, we obtain our desired result:

$$\begin{aligned} &{}_pG_q^{\alpha, R, S, \beta}\left[z \middle| \begin{matrix} A_1, ..., A_p \\ B_1, ..., B_q \end{matrix}\right] \\ &= R_p G_q^{\alpha, R+I, S, \beta}\left[z \middle| \begin{matrix} A_1, ..., A_p \\ B_1, ..., B_q \end{matrix}\right] + \alpha z \frac{d}{dz} {}_pG_q^{\alpha, R+I, S, \beta}\left[z \middle| \begin{matrix} A_1, ..., A_p \\ B_1, ..., B_q \end{matrix}\right]. \end{aligned} \tag{2.3.43}$$

□

**Lemma 2.3.10.** *[17] Suppose $U(p,q)$ and $V(p,q)$ are matrices in $\mathbb{C}^{r\times r}$. Then, the following series relations are satisfied:*

$$\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}U(p,q)=\sum_{q=0}^{\infty}\sum_{p=0}^{q}U(p,q-p)$$

*and*

$$\sum_{q=0}^{\infty}\sum_{p=0}^{q}V(p,q)=\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}V(p,q+p).$$

**Theorem 2.3.11.** *The following finite summation formula for the matrix analog of the ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function holds true:*

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}\Gamma(S-\beta kI)\,{}_pG_q^{\alpha,R-\alpha kI,S-\beta kI,\alpha}\left[z\left|\begin{matrix}A_1,\dots,A_p\\ B_1,\dots,B_q\end{matrix}\right.\right]\mu^k\\ &=\Gamma(S)\,{}_pF_q(A_1,\dots.A_p,B_1,\dots..B_q;\mu z)\,{}_pG_q^{\alpha,R,S,\alpha}\left[z\left|\begin{matrix}A_1+kI,\dots,A_p+kI\\ B_1+kI,\dots,B_q+kI\end{matrix}\right.\right],\end{aligned}\tag{2.3.44}$$

*where $R$ and $S$ are two positive stable matrices in $\mathbb{C}^{r\times r}$ such that $R+kI$ is invertible for all $k\geq 0$ and $\alpha\in\mathbb{C}$ such that $\Re(\alpha)>0$ and $(A_i)_m$ and $(B_j)_m$, $i=1,..p$, $j=1,...q$ are the Pochhammer matrix symbols.*

*Proof.* To prove our result, consider the left-hand side of (2.3.44) and using the definition of the matrix analog of the ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function 2.2.1, we obtain:

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}\Gamma(S-\beta kI)\,{}_pG_q^{\alpha,R-\alpha kI,S-\beta kI,\alpha}\left[z\left|\begin{matrix}A_1,\dots,A_p\\ B_1,\dots,B_q\end{matrix}\right.\right]\mu^k\\ &=\sum_{k=0}^{m}\binom{m}{k}\Gamma(S-\beta kI)\left\{\sum_{m=0}^{\infty}\frac{(A_1)_m,\dots.,(A_p)_m\Gamma(\beta mI+S-\beta kI)}{(B_1)_m\dots(B_q)_m\Gamma(\alpha mI+R-\alpha kI)\Gamma(S-\beta kI)}\frac{z^m}{m!}\right\}\mu^k.\end{aligned}\tag{2.3.45}$$

On re-arranging the terms, we obtain:

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}\Gamma(S-\beta kI)\,{}_pG_q^{\alpha,R-\alpha kI,S-\beta kI,\alpha}\left[z\left|\begin{matrix}A_1,\dots,A_p\\ B_1,\dots,B_q\end{matrix}\right.\right]\mu^k\\ &=\sum_{k=0}^{m}\sum_{m=0}^{\infty}\frac{(A_1)_m,\dots.,(A_p)_m\Gamma(S+\beta(m-k)I)}{(B_1)_m\dots(B_q)_m\Gamma(R+\alpha(m-k)I)}\frac{z^m\mu^k}{k!(m-k)!}.\end{aligned}\tag{2.3.46}$$

Now, replacing $m$ by $m+k$ and using Lemma 2.3.10, we obtain:

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}\Gamma(S-\beta kI)_pG_q^{\alpha,R-\alpha kI,S-\beta kI,\alpha}\left[z\left|\begin{matrix}A_1,\dots,A_p\\B_1,\dots,B_q\end{matrix}\right.\right]\mu^k\\&=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(A_1)_{m+k},\dots,(A_p)_{m+k}\Gamma(S+\beta mI)}{(B_1)_{m+k}\dots(B_q)_{m+k}\Gamma(R+\alpha mI)}\frac{z^{m+k}\mu^k}{k!m!}.\end{aligned}\tag{2.3.47}$$

Now, using the properties of the Pochhammer matrix symbols and in view of definitions (2.1.7) and 2.2.1, we obtain our desired result:

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}\Gamma(S-\beta kI)_pG_q^{\alpha,R-\alpha kI,S-\beta kI,\alpha}\left[z\left|\begin{matrix}A_1,\dots,A_p\\B_1,\dots,B_q\end{matrix}\right.\right]\mu^k\\&=\Gamma(S)_pF_q(A_1,\dots A_p,B_1,\dots B_q;\mu z)_pG_q^{\alpha,R,S,\alpha}\left[z\left|\begin{matrix}A_1+kI,\dots,A_p+kI\\B_1+kI,\dots,B_q+kI\end{matrix}\right.\right].\end{aligned}\tag{2.3.48}$$

□

**Corollary 2.3.12.** *The following result holds true:*

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}\Gamma(S-\beta kI)E^{\alpha,R-\alpha kI,S-\beta kI,\alpha}(z)\mu^k\\&=\Gamma(S)e^{\mu z}E^{\alpha,R,S,\alpha}(z).\end{aligned}\tag{2.3.49}$$

*Proof.* Consider $p=0=q$, and with the help of Eq. (2.3.4) we obtain our desired result. □

**Theorem 2.3.13.** *The following finite summation formula for the matrix analog of the* ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ *function holds true:*

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}(T)_k\Gamma(S-\beta kI)_pG_q^{\alpha,R-\alpha kI,S-\beta kI,\alpha}\left[z\left|\begin{matrix}A_1,\dots,A_p\\B_1,\dots,B_q\end{matrix}\right.\right]\mu^k\\&=\Gamma(S)_{p+1}F_q(T,A_1,\dots A_p,B_1,\dots B_q;\mu z)_pG_q^{\alpha,R,S,\alpha}\left[z\left|\begin{matrix}A_1+kI,\dots,A_p+kI\\B_1+kI,\dots,B_q+kI\end{matrix}\right.\right],\end{aligned}\tag{2.3.50}$$

*where $R$, $S$, and $T$ are positive stable matrices in $\mathbb{C}^{r\times r}$ such that $R+kI$ is invertible for all $k\geq 0$ and $|\mu z|<1$ and $\alpha\in\mathbb{C}$ such that $\Re(\alpha)>0$ and $(A_i)_m$ and $(B_j)_m$, $i=1,..p$, $j=1,...q$ are the Pochhammer matrix symbols.*

*Proof.* To prove our result, consider the left-hand side of (2.3.50) and using the definition of the matrix analog of the ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function 2.2.1, we obtain:

$$\sum_{k=0}^{m}\binom{m}{k}(T)_k\Gamma(S-\beta kI)\,{}_pG_q^{\alpha,R-\alpha kI,S-\beta kI,\alpha}\left[z\left|\begin{matrix}A_1,\dots,A_p\\B_1,\dots,B_q\end{matrix}\right.\right]\mu^k$$
$$=\sum_{k=0}^{m}\binom{m}{k}(T)_k\Gamma(S-\beta kI) \tag{2.3.51}$$
$$\left\{\sum_{m=0}^{\infty}\frac{(A_1)_m,\dots,(A_p)_m\Gamma(\beta mI+S-\beta kI)}{(B_1)_m\dots(B_q)_m\Gamma(\alpha mI+R-\alpha kI)\Gamma(S-\beta kI)}\frac{z^m}{m!}\right\}\mu^k.$$

On re-arranging the terms, we obtain:

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}(T)_k\Gamma(S-\beta kI)\,{}_pG_q^{\alpha,R-\alpha kI,S-\beta kI,\alpha}\left[z\left|\begin{matrix}A_1,\dots,A_p\\B_1,\dots,B_q\end{matrix}\right.\right]\mu^k\\&=\sum_{k=0}^{m}\sum_{m=0}^{\infty}(T)_k\frac{(A_1)_m,\dots,(A_p)_m\Gamma(S+\beta(m-k)I)}{(B_1)_m\dots(B_q)_m\Gamma(R+\alpha(m-k)I)}\frac{z^m\mu^k}{k!(m-k)!}.\end{aligned} \tag{2.3.52}$$

Now, replacing $m$ by $m+k$ and using Lemma 2.3.10, we obtain:

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}(T)_k\Gamma(S-\beta kI)\,{}_pG_q^{\alpha,R-\alpha kI,S-\beta kI,\alpha}\left[z\left|\begin{matrix}A_1,\dots,A_p\\B_1,\dots,B_q\end{matrix}\right.\right]\mu^k\\&=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(T)_k\frac{(A_1)_{m+k},\dots,(A_p)_{m+k}\Gamma(S+\beta mI)}{(B_1)_{m+k}\dots(B_q)_{m+k}\Gamma(R+\alpha mI)}\frac{z^{m+k}\mu^k}{k!m!}.\end{aligned} \tag{2.3.53}$$

Now, using the properties of the Pochhammer matrix symbols and in view of definitions (2.1.7) and 2.2.1, we obtain our desired result:

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}(T)_k\Gamma(S-\beta kI)\,{}_pG_q^{\alpha,R-\alpha kI,S-\beta kI,\alpha}\left[z\left|\begin{matrix}A_1,\dots,A_p\\B_1,\dots,B_q\end{matrix}\right.\right]\mu^k\\&=\Gamma(S)\,{}_{p+1}F_q(T,A_1,\dots.A_p,B_1,\dots\dots B_q;\mu z)\,{}_pG_q^{\alpha,R,S,\alpha}\left[z\left|\begin{matrix}A_1+kI,\dots,A_p+kI\\B_1+kI,\dots,B_q+kI\end{matrix}\right.\right].\end{aligned} \tag{2.3.54}$$

□

**Corollary 2.3.14.** *The following result holds true:*

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}(T)_k\Gamma(S-\beta kI)E^{\alpha,R-\alpha kI,S-\beta kI,\alpha}(z)\mu^k\\&=\Gamma(S)(1-\mu z)^{-T}E^{\alpha,R,S,\alpha}(z).\end{aligned} \tag{2.3.55}$$

*Proof.* Consider $p = 0 = q$, and with the help of Eq. (2.3.4) we obtain our desired result. □

## 2.4 Conclusion

We conclude our research by remarking that all results are true and novel. First, we established the matrix analog of the ${}_pG_q^{\eta,\zeta,m,\xi}(w)$ function and then discussed many of its important properties.


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CHAPTER

# 3 Some extended-type hypergeometric functions of two and three variables

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## 3.1 Introduction and preliminaries

Euler's beta function, gamma function, and hypergeometric functions are the important members of the family of special functions and they play a significant role in the whole theory of special functions. These hypergeometric functions together with their extensions have many applications in research fields such as engineering, chemical, statistics, fractional calculus, and physical problems. In the last decades, many generalizations and extensions of the most popular special functions have been presented by many authors [1–4] and [19–24]. Similarly, multivariable hypergeometric functions such as Appell and Lauricella functions and their many extensions appear in many core areas of mathematics and their applications. In [5–8], many researchers have contributed their works on multivariable hypergeometric functions and their properties in detail. In the past few years, several authors and researchers [9–11] have introduced some fascinating generalizations of the Appell and Lauricella functions by using special functions in their kernels. Inspired by the above works, here we establish new extensions of the Appell hypergeometric function of two variables and the Lauricella hypergeometric function of three variables and obtained their relation with other well-known special functions and their extensions.

Let us define the hypergeometric functions of two and three variables, *i.e.,* Appell's functions $F_1(a_1; b_1, c_1; d_1; x_1, y_1)$ and $F_2(a_1; b_1, c_1; d_1, e_1; x_1, y_1)$, and Lauricella's function $F_D^3(a_1, b_1, c_1, d_1; e_1; x_1, y_1, z_1)$, respectively, which are defined as (see for more details, [12] and [13]):

$$F_1(a_1; b_1, c_1; d_1; x_1, y_1) = \sum_{m,n=0}^{\infty} \frac{B(a_1+m+n, d_1-a_1)(b_1)_m(c_1)_n}{B(a_1, d_1-a_1)} \frac{x_1^m}{m!} \frac{y_1^n}{n!}, \tag{3.1.1}$$

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where $\max\{| x_1 |, | y_1 |\} < 1$ and $B(a_1, a_2)$ denotes Euler's beta function defined as [12]:

$$B(a_1, a_2) = \int_0^1 s^{a_1-1}(1-s)^{a_2-1} ds, \qquad \Re(a_1), \Re(a_2) > 0 \tag{3.1.2}$$

and $(a_1)_l$ denotes the Pochhammer symbol defined as [14]:

$$(a_1)_l := \frac{\Gamma(a_1 + l)}{\Gamma(a_1)} = \begin{cases} 1 & (l = 0; a_1 \in \mathbb{C} \setminus \{0\}) \\ a_1 (a_1+1) \cdots (a_1 + l - 1) & (l \in \mathbb{N}; a_1 \in \mathbb{C}), \end{cases}$$

$$F_2(a_1; b_1, c_1; d_1, e_1; x_1, y_1) = \sum_{m,n=0}^{\infty} \frac{(a_1)_{m+n} B(b_1 + m, d_1 - b_1) B(c_1 + n, e_1 - c_1)}{B(b_1, d_1 - b_1) B(c_1, e_1 - c_1)} \frac{x_1^m}{m!} \frac{y_1^n}{n!}, \tag{3.1.3}$$

where $| x_1 | + | y_1 | < 1$,

$$F_D^3(a_1, b_1, c_1, d_1; e_1; x_1, y_1, z_1) = \sum_{m,n,t=0}^{\infty} \frac{B(a_1 + m + n + t, e_1 - a_1)(b_1)_m (c_1)_n (d_1)_t}{B(a_1, e_1 - a_1)} \frac{x_1^m}{m!} \frac{y_1^n}{n!} \frac{z_1^t}{t!}, \tag{3.1.4}$$

where $\sqrt{| x_1 |} + \sqrt{| y_1 |} + \sqrt{| z_1 |} < 1$.

In the recent past, Goyal et al. [15] studied the generalization of Euler's beta function by using the 2-parameter Mittag-Leffler function, thus examining many basic properties and relationships of that generalized beta function. The generalized beta function is defined as:

$$B_{(u_1,u_2)}^{(u)}(y_1, y_2) = \int_0^1 s^{y_1-1}(1-s)^{y_2-1} E_{u_1,u_2}\left(\frac{-u}{s(1-s)}\right) ds, \tag{3.1.5}$$

where $\Re(y_1) > 0$, $\Re(y_2) > 0$ $\Re(u_1) > 0$, $\Re(u_2) > 0$, $u \geq 0$, and $E_{u_1,u_2}(w)$ is the 2-parameter Mittag-Leffler function defined by:

$$E_{y_1,y_2}(w) = \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(k y_1 + y_2)}, \tag{3.1.6}$$

where $\Re(y_1) \geq 0$, $\Re(y_2) \geq 0$, and $w \in \mathbb{C}$.

The above generalized Euler's beta function has an important role in the establishment of our new extended functions due to the 2-parameter Mittag-Leffler function used in the kernel.

Very recently, inspired by the above extension, Jain et al. [16] generalized the Gauss hypergeometric function by using the generalized Euler's beta function given in (3.1.5). They also studied various properties like differentiation formulas, summation formulas, Mellin transforms, and the recurrence relations of generalized Gauss hypergeometric functions. The generalized Gauss hypergeometric function is defined as:

$$F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; x) = \sum_{k=0}^{\infty} \frac{B^{(r)}_{(r_1,r_2)}(p_1 + k, p_2 - p_1)}{B(p_1, p_2 - p_1)} (p_0)_n \frac{x^k}{k!}, \tag{3.1.7}$$

where $\Re(p_2) > \Re(p_1) > 0$, $\Re(r_1) > 0$, $\Re(r_2) > 0$, $r \geq 0$, $| x |< 1$, and $B^{(r)}_{(r_1,r_2)}(x_1, x_2)$ is the extended beta function (3.1.5).

Also, the Euler-type integral representation of the above generalized Gauss hypergeometric function is defined as:

$$\begin{aligned} &F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; x) \\ &= \frac{1}{B(p_1, p_2 - p_1)} \int_0^1 s^{p_1-1}(1-s)^{p_2-p_1-1}(1-xs)^{-p_0} E_{r_1,r_2}\left(\frac{-r}{s(1-s)}\right) ds, \end{aligned} \tag{3.1.8}$$

where $\Re(p_2) > \Re(p_1) > 0$, $\Re(r_1) > 0$, $\Re(r_2) > 0$, $r \geq 0$, and $| x |< 1$.

In 2010, Özarslan et al. [9] generalized Appell's hypergeometric functions of two variables, $F_1(a_1; b_1, c_1; d_1; x_1, y_1; p_1)$, $F_2(a_1; b_1, c_1; d_1, e_1; x_1, y_1; p_1)$ and generalized Lauricella's hypergeometric function of three variables, $F^3_{D,p}(a_1, b_1, c_1, d_1; e_1; x_1, y_1, z_1)$ by using the generalized Euler beta function $B_{p_1}(x_1, y_1)$ given in [2], and studied several properties of these generalized functions.

The generalized Appell's hypergeometric functions of two variables $F_1(a_1; b_1, c_1; d_1; x_1, y_1; p_1)$ and $F_2(a_1; b_1, c_1; d_1, e_1; x_1, y_1; p_1)$ are defined as [9]:

$$F_1(a_1; b_1, c_1; d_1; x_1, y_1; p_1) = \sum_{m,n=0}^{\infty} \frac{B_{p_1}(a_1 + m + n, d_1 - a_1)(b_1)_m (c_1)_n}{B(a_1, d_1 - a_1)} \frac{x_1^m}{m!} \frac{y_1^n}{n!}, \tag{3.1.9}$$

where $\max\{| x_1 |, | y_1 |\} < 1$,

$$\begin{aligned} &F_2(a_1; b_1, c_1; d_1, e_1; x_1, y_1; p_1) \\ &\quad = \sum_{m,n=0}^{\infty} \frac{(a_1)_{m+n} B_{p_1}(b_1 + m, d_1 - b_1) B_{p_1}(c_1 + n, e_1 - c_1)}{B(b_1, d_1 - b_1) B(c_1, e_1 - c_1)} \frac{x_1^m}{m!} \frac{y_1^n}{n!}, \end{aligned} \tag{3.1.10}$$

where $| x_1 | + | y_1 | < 1$.

The generalized Lauricella's hypergeometric function of three variables, $F^3_D(a_1, b_1, c_1, d_1; e_1; x_1, y_1, z_1; p_1)$ is defined as [9]:

$$\begin{aligned} &F^3_{D,p_1}(a_1, b_1, c_1, d_1; e_1; x_1, y_1, z_1) \\ &\quad = \sum_{m,n,t=0}^{\infty} \frac{B_{p_1}(a_1 + m + n + t, d_1 - a_1)(b_1)_m (c_1)_n (d_1)_t}{B(a_1, e_1 - a_1)} \frac{x_1^m}{m!} \frac{y_1^n}{n!} \frac{z_1^t}{t!}, \end{aligned} \tag{3.1.11}$$

where $\sqrt{| x_1 |} + \sqrt{| y_1 |} + \sqrt{| z_1 |} < 1$.

**Remark.** If we substitute $p_1 = 0$ in the above Eqs. (3.1.9), (3.1.10), and (3.1.11) then we obtain the original functions given by (3.1.1), (3.1.3), and (3.1.4), respectively.

In the last decades, many researchers have worked in fractional calculus in many applications in fields of engineering and science such as electrical networks, statistics probability, and fluid dynamics, particularly fluid flows. In 2001, Srivastava and Saxena [17] studied fractional calculus and its application systematically. Also, Srivastava and Manocha [8], described the benefits of fractional derivatives in the generating function theory of special functions. In this chapter, motivated by the above, we have also obtained some generating functions for the generalized hypergeometric functions $F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z)$ by using the generalized Riemann–Liouville fractional derivative operator defined by Jain et al. [18].

In 2021, Jain et al. [18] generalized the Riemann–Liouville-type fractional derivative operator by using the 2-parameter Mittag-Leffler function in its kernel and studied various basic properties of that generalized fractional derivative operator.

Now, here we recall the generalized Riemann–Liouville-type fractional derivative operator defined as [18]:

$$D^{u,(r)}_{x,(r_1,r_2)}[g(x)] = \begin{cases} \frac{1}{\Gamma(-u)} \int_0^x (x-s)^{-u-1} E_{r_1,r_2}\left(\frac{-rx^2}{s(x-s)}\right) g(s)ds, & (\Re(u) < 0) \\ \frac{d^k}{dx^k}\{D^{u-k,(r)}_{x,(r_1,r_2)} g(x)\}, & (k-1 \leq \Re(u) < k,\ k \in \mathbb{N}), \end{cases} \tag{3.1.12}$$

where $(\min\{\Re(r_1), \Re(r_2)\} > 0$, $\Re(r) > 0$, and $E_{r_1,r_2}(x)$ is the 2-parameter Mittag-Leffler function).

The following theorem is very important to obtain our main results.

**Theorem 3.1.1.** *([18]) The following result holds true:*

$$\sum_{m=0}^{\infty} \frac{(k)_m}{m!} F^{(r)}_{(r_1,r_2)}(k+m, n, l; x)s^m = (1-s)^{-k} F^{(r)}_{(r_1,r_2)}\left(k, n, l; \frac{x}{(1-s)}\right), \tag{3.1.13}$$

*where* $\mid \frac{x}{(1-s)} \mid < 1$, $\Re(k) > 0$, *and* $\Re(n) > \Re(l) > 0$.

## 3.2 Main results

In this section, we establish new extensions of Appell's hypergeometric functions $F^{(r)}_{1,(r_1,r_2)}(a_1; b_1, c_1; d_1; x_1, y_1)$ and $F^{(r)}_{2,(r_1,r_2)}(a_1; b_1, c_1; d_1, e_1; x_1, y_1)$, and Lauricella's hypergeometric function $F^{3,(r)}_{D,(r_1,r_2)}(a_1, b_1, c_1, d_1; e_1; x_1, y_1, z_1)$ by the use of the generalized Euler's beta function given in (3.1.5). We have also obtained integral representations of extended Appell's hypergeometric functions and Lauricella's hypergeometric function.

**Definition 3.2.1.** Let $\Re(r_1) > 0$, $\Re(r_2) > 0$, and $\Re(r) > 0$, then the new extensions of Appell's hypergeometric functions, $F_1(a_1; b_1, c_1; d_1; x_1, y_1)$ and $F_2(a_1; b_1, c_1; d_1, e_1; x_1, y_1)$ are defined as:

$$F_{1,(r_1,r_2)}^{(r)}(a_1; b_1, c_1; d_1; x_1, y_1) = \sum_{m,n=0}^{\infty} \frac{B_{(r_1,r_2)}^{(r)}(a_1+m+n, d_1-a_1)(b_1)_m(c_1)_n}{B(a_1, d_1-a_1)} \frac{x_1^m}{m!} \frac{y_1^n}{n!}, \tag{3.2.1}$$

where $\max\{|x_1|, |y_1|\} < 1$,

$$F_{2,(r_1,r_2)}^{(r)}(a_1; b_1, c_1; d_1, e_1; x_1, y_1) = \sum_{m,n=0}^{\infty} \frac{(a_1)_{m+n} B_{(r_1,r_2)}^{(r)}(b_1+m, d_1-b_1) B(c_1+n, e_1-c_1)}{B_{(r_1,r_2)}^{(r)}(b_1, d_1-b_1) B(c_1, e_1-c_1)} \frac{x_1^m}{m!} \frac{y_1^n}{n!}, \tag{3.2.2}$$

where $|x_1| + |y_1| < 1$.

**Definition 3.2.2.** Let $\Re(r_1) > 0$, $\Re(r_2) > 0$, and $\Re(r) > 0$. Then, the new extension of Lauricella's hypergeometric function $F_D^3(a_1, b_1, c_1, d_1; e_1; x_1, y_1, z_1)$ is defined as:

$$F_{D,(r_1,r_2)}^{3,(r)}(a_1, b_1, c_1, d_1; e_1; x_1, y_1, z_1) = \sum_{m,n,t=0}^{\infty} \frac{B_{(r_1,r_2)}^{(r)}(a_1+m+n+t, e_1-a_1)(b_1)_m(c_1)_n(d_1)_t}{B(a_1, e_1-a_1)} \frac{x_1^m}{m!} \frac{y_1^n}{n!} \frac{z_1^t}{t!}, \tag{3.2.3}$$

where $\sqrt{|x_1|} + \sqrt{|y_1|} + \sqrt{|z_1|} < 1$.

**Remark.** (i) If we consider $r_2 = r_1 = 1$, then the new extended Appell's hypergeometric functions defined in (3.2.1) and (3.2.2) and the new extended Lauricella's hypergeometric function defined in (3.2.3) reduce to the extended Appell's hypergeometric functions given by (3.1.9) and (3.1.10) and the extended Lauricella's hypergeometric function given by (3.1.11), respectively.

(ii) If we substitute $r_2 = r_1 = 1$ and $r = 0$ then the new extended Appell's hypergeometric functions defined in (3.2.1) and (3.2.2) and the new extended Lauricella's hypergeometric function defined in (3.2.3) reduce to the original Appell's hypergeometric functions given by (3.1.1) and (3.1.3) and Lauricella's hypergeometric function given by (3.1.4), respectively.

Then, we proceed to obtain the integral representations of the functions $F_{1,(r_1,r_2)}^{(r)}(a_1; b_1, c_1; d_1; x_1, y_1)$, $F_{2,(r_1,r_2)}^{(r)}(a_1; b_1, c_1; d_1, e_1; x_1, y_1)$, and $F_{D,(r_1,r_2)}^{3,(r)}(a_1, b_1, c_1, d_1; e_1; x_1, y_1, z_1)$.

**Theorem 3.2.3.** *For $F^{(r)}_{1,(r_1,r_2)}(a_1; b_1, c_1; d_1; x_1, y_1)$, the following integral representation holds true:*

$$F^{(r)}_{1,(r_1,r_2)}(a_1; b_1, c_1; d_1; x_1, y_1) = \frac{\Gamma(d_1)}{\Gamma(a_1)\Gamma(d_1-a_1)} \int_0^1 t^{a_1-1}(1-t)^{d_1-a_1-1}(1-x_1t)^{-b_1}(1-y_1t)^{-c_1} E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) dt, \tag{3.2.4}$$

*where* $\Re(r_1) > 0$, $\Re(r_2) > 0$, *and* $\Re(r) > 0$, $\Re(d_1) > \Re(a_1) > 0$ *with* $|\, x_1 \,| < 1$ *and* $|\, y_1 \,| < 1$.

*Proof.* From (3.2.1), we have

$$F^{(r)}_{1,(r_1,r_2)}(a_1; b_1, c_1; d_1; x_1, y_1) = \sum_{m,n=0}^{\infty} \frac{B^{(r)}_{(r_1,r_2)}(a_1+m+n, d_1-a_1)(b_1)_m(c_1)_n}{B(a_1, d_1-a_1)} \frac{x_1^m}{m!} \frac{y_1^n}{n!}. \tag{3.2.5}$$

Now, using the definition given in (3.1.5) in the above equation, we obtain:

$$F^{(r)}_{1,(r_1,r_2)}(a_1; b_1, c_1; d_1; x_1, y_1) = \frac{1}{B(a_1, d_1-a_1)} \sum_{m,n=0}^{\infty} \left\{ \int_0^1 t^{a_1+m+n-1}(1-t)^{d_1-a_1-1} E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) dt \right\} (b_1)_m (c_1)_n \frac{x_1^m}{m!} \frac{y_1^n}{n!}. \tag{3.2.6}$$

On interchanging summation and integration signs in the above equation and manipulating some terms, we have:

$$F^{(r)}_{1,(r_1,r_2)}(a_1; b_1, c_1; d_1; x_1, y_1) = \frac{1}{B(a_1, d_1-a_1)} \int_0^1 t^{a_1-1}(1-t)^{d_1-a_1-1} E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) \left\{ \sum_{m,n=0}^{\infty} (b_1)_m (c_1)_n \frac{(x_1t)^m}{m!} \frac{(y_1t)^n}{n!} \right\} dt. \tag{3.2.7}$$

Now, using the identities in the above equation:

$$(1-zt)^{-p} = \sum_{k=0}^{\infty} \frac{(p)_k}{k!}(zt)^k, \,|\, z \,| < 1 \tag{3.2.8}$$

and

$$B(x_1, y_1) = \frac{\Gamma(x_1)\Gamma(y_1)}{\Gamma(x_1 + y_1)}. \tag{3.2.9}$$

Then, we obtain our desired result:

$$\begin{aligned} &F^{(r)}_{1,(r_1,r_2)}(a_1, b_1, c_1; d_1; x_1, y_1) \\ &\quad = \frac{\Gamma(d_1)}{\Gamma(a_1)\Gamma(d_1 - a_1)} \\ &\qquad \int_0^1 t^{a_1-1}(1-t)^{d_1-a_1-1}(1-x_1t)^{-b_1}(1-y_1t)^{-c_1} E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) dt. \end{aligned} \tag{3.2.10}$$

□

**Theorem 3.2.4.** *For $F^{(r)}_{2,(r_1,r_2)}(a_1; b_1, c_1; d_1, e_1; x_1, y_1)$, the following integral representation holds true:*

$$\begin{aligned} F^{(r)}_{2,(r_1,r_2)}(a_1; b_1, c_1; d_1, e_1; x_1, y_1) &= \frac{\Gamma(d_1)\Gamma(e_1)}{\Gamma(b_1)\Gamma(c_1)\Gamma(d_1-b_1)\Gamma(e_1-c_1)} \times \\ &\int_0^1 \int_0^1 t^{b_1-1}(1-t)^{d_1-b_1-1} s^{c_1-1}(1-s)^{e_1-c_1-1}(1-xt-ys)^{-a_1} \\ &E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) E_{r_1,r_2}\left(\frac{-r}{s(1-s)}\right) dt ds, \end{aligned} \tag{3.2.11}$$

*where* $\Re(r_1) > 0$, $\Re(r_2) > 0$, *and* $\Re(r) > 0$, $\Re(d_1) > \Re(b_1) > 0$ *and* $\Re(e_1) > \Re(c_1) > 0$ *with* $|x| + |y| < 1$.

*Proof.* Similarly, on following the same steps as the proof of Theorem 3.2.3, and using (3.1.5) in (3.2.2) with the identity

$$\sum_{n=0}^{\infty} f(n) \frac{(a+b)^n}{n!} = \sum_{k,l=0}^{\infty} f(k+l) \frac{a^k}{k!} \frac{b^l}{l!}, \tag{3.2.12}$$

we obtain our desired result:

$$\begin{aligned} F^{(r)}_{2,(r_1,r_2)}(a_1, b_1, c_1; d_1, e_1; x_1, y_1) &= \frac{\Gamma(d_1)\Gamma(e_1)}{\Gamma(b_1)\Gamma(c_1)\Gamma(d_1-b_1)\Gamma(e_1-c_1)} \times \\ &\int_0^1 \int_0^1 t^{b_1-1}(1-t)^{d_1-b_1-1} s^{c_1-1}(1-s)^{e_1-c_1-1}(1-xt-ys)^{-a_1} \\ &E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) E_{r_1,r_2}\left(\frac{-r}{s(1-s)}\right) dt ds. \end{aligned} \tag{3.2.13}$$

□

**Theorem 3.2.5.** *For* $F^{3,(r)}_{D,(r_1,r_2)}(a_1,b_1,c_1,d_1;e_1;x_1,y_1,z_1)$*, the following integral representation holds true:*

$$F^{3,(r)}_{D,(r_1,r_2)}(a_1,b_1,c_1,d_1;e_1;x_1,y_1,z_1)$$
$$=\frac{\Gamma(e_1)}{\Gamma(a_1)\Gamma(e_1-a_1)}\int_0^1 t^{a_1-1}(1-t)^{e_1-a_1-1}(1-x_1t)^{-b_1}(1-y_1t)^{-c}(1-z_1t)^{-d_1}$$
$$E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right)dt, \tag{3.2.14}$$

*where* $\Re(r_1)>0$, $\Re(r_2)>0$, *and* $\Re(r)>0$, $\Re(e_1)>\Re(a_1)>0$ *with* $|x_1|<1$, $|y_1|<1$ *and* $|z_1|<1$.

*Proof.* By following the same line of proof as Theorem 3.2.3, we obtain our desired result. □

Now, here we obtain some results of the extended Riemann–Liouville-type fractional derivative operator and some generating functions of the extended hypergeometric function by the use of the extended Riemann–Liouville-type fractional derivative operator.

**Theorem 3.2.6.** *Consider* $\Re(u)<0$, $\Re(k)>0$, $\Re(l)>0$, $\Re(m)>0$ $|az|<1$, *and* $|bz|<1$, *then*

$$D^{k-u,(r)}_{z,(r_1,r_2)}[z^{k-1}(1-az)^{-l}(1-bz)^{-m}]=\frac{\Gamma(k)}{\Gamma(u)}z^{u-1}F^{(r)}_{1,(r_1,r_2)}(k;l,m;u;az,bz). \tag{3.2.15}$$

*Proof.* From (3.1.12), we obtain:

$$D^{k-u,(r)}_{z,(r_1,r_2)}[z^{k-1}(1-az)^{-l}(1-bz)^{-m}]=$$
$$\frac{1}{\Gamma(u-k)}\int_0^z (z-t)^{u-k-1}E_{r_1,r_2}\left(\frac{-rz^2}{t(z-t)}\right)t^{k-1}(1-at)^{-l}(1-bt)^{-m}dt. \tag{3.2.16}$$

On taking out $z$ from the integral, we have:

$$D^{k-u,(r)}_{z,(r_1,r_2)}[z^{k-1}(1-az)^{-l}(1-bz)^{-m}]=$$
$$\frac{z^{u-k-1}}{\Gamma(u-k)}\int_0^z\left(1-\frac{t}{z}\right)^{u-k-1}E_{r_1,r_2}\left(\frac{-rz^2}{t(z-t)}\right)t^{k-1}(1-at)^{-l}(1-bt)^{-m}dt. \tag{3.2.17}$$

Then, setting the value of $t = xz$ in the above equation and changing the limit from $t = 0, t = z$ to $x = 0, x = 1, dt = zdx$ with some re-arranging of the terms, we obtain:

$$D_{z,(r_1,r_2)}^{k-u,(r)}[z^{k-1}(1-az)^{-l}(1-bz)^{-m}] = \frac{z^{u-1}}{\Gamma(u-k)} \int_0^1 (1-x)^{u-k-1} E_{r_1,r_2}\left(\frac{-r}{x(1-x)}\right) x^{k-1}(1-axz)^{-l}(1-bxz)^{-m} dx. \tag{3.2.18}$$

From the integral formula of the extended Appell's hypergeometric function $F_{1,(r_1,r_2)}^{(r)}(a_1; b_1, c_1; d_1; x_1, y_1)$ (3.1.9), we obtain our desired result:

$$D_{z,(r_1,r_2)}^{k-u,(r)}[z^{k-1}(1-az)^{-l}(1-bz)^{-m}] = \frac{\Gamma(k)}{\Gamma(u)} z^{u-1} F_{1,(r_1,r_2)}^{(r)}(k; l, m; u; az, bz). \tag{3.2.19}$$

□

**Theorem 3.2.7.** *Consider* $\Re(u) < 0$, $\Re(k) > 0$, $\Re(l) > 0$, $\Re(m) > 0$, $\Re(n) > 0$, $| az | < 1$, $| bz | < 1$, *and* $| cz | < 1$, *then*

$$\begin{aligned} &D_{z,(r_1,r_2)}^{k-u,(r)}[z^{k-1}(1-az)^{-l}(1-bz)^{-m}(1-cz)^{-n}] \\ &= \frac{\Gamma(k)}{\Gamma(u)} z^{u-1} F_{D,(r_1,r_2)}^{3,(r)}(k, l, m, n; u; az, bz, cz). \end{aligned} \tag{3.2.20}$$

*Proof.* By similar steps as in the proof of Theorem 3.2.6, we can obtain our desired result. □

**Theorem 3.2.8.** *Consider* $\Re(u) < 0$, $\Re(k) > 0$, $\Re(l) > 0$, *and* $| \frac{x}{1-z} | < 1$, *then*

$$\begin{aligned} &D_{z,(r_1,r_2)}^{k-u,(r)}\left\{z^{k-1}(1-z)^{-l} F_{(r_1,r_2)}^{(r)}\left(l, m, p, \frac{x}{1-z}\right)\right\} \\ &= \frac{1}{B(m, p-m)\Gamma(u-k)} z^{u-1} F_{2,(r_1,r_2)}^{(r)}(l; m, k; p, u; x, z). \end{aligned} \tag{3.2.21}$$

*Proof.* By using (3.1.7), in the definition of the extended Riemann–Liouville fractional derivative operator (3.1.12), we have:

$$\begin{aligned} &D_{z,(r_1,r_2)}^{k-u,(r)}\left\{z^{k-1}(1-z)^{-l} F_{(r_1,r_2)}^{(r)}\left(l, m, p, \frac{x}{1-z}\right)\right\} = \\ &D_{z,(r_1,r_2)}^{k-u,(r)}\left\{z^{k-1}(1-z)^{-l}\left\{\frac{1}{B(m, p-m)} \sum_{n=0}^{\infty} \frac{(l)_n B_{(r_1,r_2)}^{(r)}(m+n, p-m)}{n!}\left(\frac{x}{1-z}\right)^n\right\}\right\} \\ &= \frac{1}{B(m, p-m)} D_{z,(r_1,r_2)}^{k-u,(r)}\left\{z^{k-1} \sum_{n=0}^{\infty} \frac{(l)_n B_{(r_1,r_2)}^{(r)}(m+n, p-m) x^n (1-z)^{-l-n}}{n!}\right\} \end{aligned}$$

$$= \frac{1}{B(m,p-m)} D_{z,(r_1,r_2)}^{k-u,(r)} \left\{ z^{m+k-1} \sum_{m,n=0}^{\infty} \frac{(l)_n (l+n)_m B_{(r_1,r_2)}^{(r)}(m+n,p-m) x^n}{n! m!} \right\}$$
$$= \frac{1}{B(m,p-m)} \sum_{m,n=0}^{\infty} B_{(r_1,r_2)}^{(r)}(m+n,p-m) \frac{x^n}{n!} \frac{(l)_n (l+n)_m}{m!} D_{z,(r_1,r_2)}^{k-u,(r)} \{z^{m+k-1}\}. \tag{3.2.22}$$

Now, using the result given in [18]

$$D_{z,(r_1,r_2)}^{u,(r)}[z^k] = \frac{B_{(r_1,r_2)}^{(r)}(k+1,-u)}{\Gamma(-u)} z^{k-u}. \tag{3.2.23}$$

Then, we have

$$D_{z,(r_1,r_2)}^{k-u,(r)} \left\{ z^{k-1}(1-z)^{-l} F_{(r_1,r_2)}^{(r)} \left( l,m,p,\frac{x}{1-z} \right) \right\} =$$
$$\frac{1}{B(m,p-m)} \sum_{m,n=0}^{\infty} B_{(r_1,r_2)}^{(r)}(m+n,p-m) \frac{x^n}{n!} \frac{(l)_{m+n}}{m!} \frac{B_{(r_1,r_2)}^{(r)}(k+m,u-k)}{\Gamma(u-k)} z^{u+m-1}. \tag{3.2.24}$$

Then, using the (3.2.2), we obtain our desired result:

$$D_{z,(r_1,r_2)}^{k-u,(r)} \left\{ z^{k-1}(1-z)^{-l} F_{(r_1,r_2)}^{(r)} \left( l,m,p,\frac{x}{1-z} \right) \right\}$$
$$= \frac{1}{B(m,p-m)\Gamma(u-k))} z^{u-1} F_{2,(r_1,r_2)}^{(r)}(l;m,k;p,u;x,z). \tag{3.2.25}$$

□

**Theorem 3.2.9.** *For the generalized hypergeometric function $F_{(r_1,r_2)}^{(r)}(a,b,c,z)$ the following generating relation holds true:*

$$\sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!} F_{(r_1,r_2)}^{(r)}(d-m,l,m;x)t^m = (1-t)^{-\lambda} F_{1,(r_1,r_2)}^{(r)} \left( l;d,\lambda;m;x,\frac{-xt}{(1-t)} \right), \tag{3.2.26}$$

*where* $\left| \frac{x}{(1-t)} \right| < 1$, $\Re(\lambda) > 0$ *and* $\Re(m) > \Re(l) > 0$.

*Proof.* Let the series identity $[1-(1-x)t]^{-\lambda} = (1-t)^{-\lambda} \left(1+\frac{xt}{(1-t)}\right)^{-\lambda}$.

Then, by using the binomial expansion of $(1-(1-x)t)^{-\lambda}$ on the left-hand side of the above equation, we obtain:

$$\sum_{m=0}^{\infty} \frac{(\lambda)_m (1-x)^m t^m}{m!} = (1-t)^{-\lambda} \left(1 - \frac{-xt}{(1-t)}\right)^{-\lambda}. \tag{3.2.27}$$

Now, multiplying both sides by $x^{l-1}(1-x)^{-d}$ and using the extended Riemann–Liouville fractional derivative operator $D_{x,(r_1,r_2)}^{l-m,(r)}$, we have:

$$D_{x,(r_1,r_2)}^{l-m,(r)} \left[ \sum_{m=0}^{\infty} \frac{(\lambda)_m (1-x)^{m-d} x^{l-1} t^m}{m!} \right] = (1-t)^{-\lambda} D_{x,(r_1,r_2)}^{l-m,(r)} \left[ x^{l-1}(1-x)^{-d} \left(1 - \frac{-xt}{(1-t)}\right)^{-\lambda} \right]. \tag{3.2.28}$$

After some calculations, we have:

$$\sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!} D_{x,(r_1,r_2)}^{l-m,(r)} \left[ x^{l-1}(1-x)^{-(-m+d)} \right] t^m = (1-t)^{-\lambda} D_{x,(r_1,r_2)}^{l-m,(r)} \left[ x^{l-1}(1-x)^{-d} \left(1 - \frac{-xt}{(1-t)}\right)^{-\lambda} \right]. \tag{3.2.29}$$

Then, using Theorem 3.1.1 proved in [18] and Theorem 3.2.3, we obtain our desired result:

$$\sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!} F_{(r_1,r_2)}^{(r)}(d-m, l, m; x) t^m = (1-t)^{-\lambda} F_{1,(r_1,r_2)}^{(r)} \left( l; d, \lambda; m; x, \frac{-xt}{(1-t)} \right). \tag{3.2.30}$$

□

**Theorem 3.2.10.** *For the generalized hypergeometric function $F_{(r_1,r_2)}^{(r)}(a,b,c;z)$ the following result holds true:*

$$\sum_{n=0}^{\infty} \frac{(k)_n}{n!} F_{(r_1,r_2)}^{(r)}(p, -n, q; y) F_{(r_1,r_2)}^{(r)}(k+n, l, m; x) t^n = (1-t)^{-k} \frac{B(p, q-p)}{B(l, m-l)} F_{2,(r_1,r_2)}^{(r)} \left( k; l, p; m, q; \frac{x}{(1-t)}, \frac{-yt}{(1-t)} \right), \tag{3.2.31}$$

*where* $\left| \frac{x}{(1-t)} \right| < 1$, $\left| \frac{y}{(1-t)} \right| < 1$, $\Re(k) > 0$, *and* $\Re(q) > \Re(p) > 0$.

*Proof.* By using the result, proved in [18], we have:

$$\sum_{n=0}^{\infty}\frac{(k)_n}{n!}F^{(r)}_{(r_1,r_2)}(k+n,l,m;x)t^n=(1-t)^{-k}F^{(r)}_{(r_1,r_2)}\left(k,l,m;\frac{x}{(1-t)}\right). \quad (3.2.32)$$

Now, changing the value $t$ by $(1-y)t$ in the above equation, and multiplying the resulting equation by $y^{p-1}$, we have:

$$\sum_{n=0}^{\infty}\frac{(k)_n}{n!}F^{(r)}_{(r_1,r_2)}(k+n,l,m;x)(1-y)^n y^{p-1}t^n$$
$$=(1-(1-y)t)^{-k}y^{p-1}F^{(r)}_{(r_1,r_2)}\left(k,l,m;\frac{x}{(1-(1-y)t)}\right). \quad (3.2.33)$$

Then, applying the extended fractional derivative operator $D^{p-q,(r)}_{y,(r_1,r_2)}$ to both sides, we obtain:

$$D^{p-q,(r)}_{y,(r_1,r_2)}\left\{\sum_{n=0}^{\infty}\frac{(k)_n}{n!}F^{(r)}_{(r_1,r_2)}(k+n,l,m;x)(1-y)^n y^{p-1}t^n\right\}$$
$$=D^{p-q,(r)}_{y,(r_1,r_2)}\left\{(1-(1-y)t)^{-k}y^{p-1}F^{(r)}_{(r_1,r_2)}\left(k,l,m;\frac{x}{(1-(1-y)t)}\right)\right\}. \quad (3.2.34)$$

After some calculations, we have:

$$\sum_{n=0}^{\infty}\frac{(k)_n}{n!}D^{p-q,(r)}_{y,(r_1,r_2)}\left\{(1-y)^n y^{p-1}\right\}F^{(r)}_{(r_1,r_2)}(k+n,l,m;x)t^n$$
$$=(1-t)^{-k}D^{p-q,(r)}_{y,(r_1,r_2)}\left\{y^{p-1}\left(1-\frac{-yt}{1-t}\right)^{-k}F^{(r)}_{(r_1,r_2)}\left(k,l,m;\frac{\frac{x}{1-t}}{1-\frac{-yt}{1-t}}\right)\right\}. \quad (3.2.35)$$

Now, using the Theorems 3.1.1 and 3.2.4, we obtain our desired result:

$$\sum_{n=0}^{\infty}\frac{(k)_n}{n!}F^{(r)}_{(r_1,r_2)}(p,-n,q;y)F^{(r)}_{(r_1,r_2)}(k+n,l,m;x)t^n$$
$$=(1-t)^{-k}\frac{B(p,q-p)}{B(l,m-l)}F^{(r)}_{2,(r_1,r_2)}\left(k;l,p;m,q;\frac{x}{(1-t)},\frac{-yt}{(1-t)}\right). \quad (3.2.36)$$

□

## 3.3 Conclusion

In this work, we have introduced new extensions of the hypergeometric functions of two and three variables, i.e., Appell and Lauricella functions, respectively. Then, we have derived integral representations of all these extensions. After that, by using an extended Riemann–Liouville-type fractional derivative operator, we derived some relations between the generalized Gauss hypergeometric function with extended Appell's hypergeometric functions and Lauricella's hypergeometric function, and subsequently using these relations, we obtained some generating functions for generalized hypergeometric functions. Finally, we conclude our research by mentioning that all the results obtained in the present article are new and important. Moreover, the general massive one-loop Feynman integral can be represented as a meromorphic function of space–time dimensions using the extended type Appell and Lauricella functions for self-energy, vertex, and box integrals, respectively.

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CHAPTER

# 4 An extension of the $k$-gamma and $k$-beta matrix functions by use of the two-parameter $k$-Mittag-Leffler function

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## 4.1 Introduction and preliminaries

Jódar and Cortés [10,11] presented the theory of special matrix functions, proving certain basic and significant characteristics of the Gamma and Beta matrix functions as well as a limit formula for the Gamma function of a matrix. They examined the hypergeometric function with matrix arguments after spending some time with these results. Many scholars [12–15,17–22] have been attracted to study the subject of special functions with matrix arguments by the analyses of the extended Gamma, Beta, and Gauss hypergeometric matrix functions presented in [10,11]. Their study continues to increase in importance, both theoretically and practically.

Several $k$-special functions have been presented and studied, including the $k$-gamma function, $k$-beta function, and $k$-hypergeometric functions (see, for example, [1–7]). Certain $k$-special functions of matrix arguments have been presented and researched, including the $k$-gamma function of a matrix argument, the $k$-beta function of matrix arguments, and the $k$-hypergeometric functions of matrix arguments [8,9].

In the present chapter, we develop the $k$-gamma and $k$-beta functions of matrix arguments as well as examine some of their characteristics. In the following, $Z$, $N$, $R$, $R+$, and $C$ represent the classes of integers, positive integers, real numbers, positive real numbers, and complex numbers, respectively. For $n \in \mathbb{N}$, let $C^{n\times n}$ be the set of all $n$ by $n$ matrices of which the entries are in $C$. Let $\rho(P_1)$ be the set of all eigenvalues of $P_1 \in C^{n\times n}$. Let $p_1$ be a matrix in $C^{n\times n}$ such that $\Re(x) > 0$, $x \in \rho(P_1)$.

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The gamma matrix function is described in the following manner [10]:

$$\Gamma(P_1) = \int_0^\infty w^{P_1 - I} e^{-w} dw. \tag{4.1.1}$$

The Pochhammer matrix symbol is described in the following manner [10]:

$$(P_1)_n := \frac{\Gamma_k(P_1 + nI)}{\Gamma(P_1)} = \begin{cases} I & n = 0, \\ P_1(P_1 + I)\cdots(P_1 + (n-1)I) & n \in \mathbb{N};\ P_1 \in C^{n\times n}. \end{cases} \tag{4.1.2}$$

The beta matrix function is described in the following manner [10]:

$$B(P_1, P_2) = \int_0^1 w^{P_1 - I}(1-w)^{P_2 - I} dw, \tag{4.1.3}$$

where $P_1$ and $P_2$ are positive stable matrices in $C^{n\times n}$.

The link between the gamma matrix function and the beta matrix function is described as follows:

$$B(P_1, P_2) = \frac{\Gamma(P_1)\Gamma(P_2)}{\Gamma(P_1 + P_2)}. \tag{4.1.4}$$

In 2015 Mubeen et al. [9] investigated a novel concept of the $k$-gamma, $k$-beta functions, Pochhammer $k$-symbol, and studied several identities and derived identities that were satisfied by the gamma and beta matrix functions and the Pochhammer matrix symbol.

The $k$-gamma matrix function is described in the following manner [9]:

$$\Gamma_k(P_1) = \int_0^\infty w^{P_1 - I} e^{-w^k k^{-1}} dw \tag{4.1.5}$$

and

$$\Gamma_k(P_1 + kI) = P_1 \Gamma_k(P_1), \tag{4.1.6}$$

where $k > 0$ and $P_1$ is a positive stable matrix in $C^{n\times n}$.

The Pochhammer matrix $k$-symbol is described in the following manner [9]:

$$(P_1)_{n,k} := \frac{\Gamma_k(P_1 + nkI)}{\Gamma_k(P_1)} = \begin{cases} I & n = 0;\ P_1 \in C^{n\times n}, \\ P_1(P_1 + kI)\cdots(P_1 + (n-1)kI) & n \in \mathbb{N};\ P_1 \in C^{n\times n}. \end{cases} \tag{4.1.7}$$

The $k$-beta matrix function is described in the following manner [9]:

$$B_k(P_1, P_2) = \frac{1}{k}\int_0^1 w^{\frac{P_1}{k} - I}(1-w)^{\frac{P_2}{k} - I} dw, \tag{4.1.8}$$

where $k > 0$ and $P_1$ and $P_2$ are positive stable matrices in $C^{n\times n}$.

The link between the $k$-gamma matrix function and the $k$-beta matrix function is:

$$B_k(P_1, P_2) = \frac{\Gamma_k(P_1)\Gamma_k(P_2)}{\Gamma_k(P_1 + P_2)}. \tag{4.1.9}$$

To obtain our main result we need to define $k$-Mittag-Leffler functions [16] as follows:

$$E_{k,u_1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(n_1 + 1)}, \qquad k > 0, \Re(u_1) \geq 0, z \in C, \tag{4.1.10}$$

$$E_{k,u_1,u_2}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(nu_1 + u_2)}, \qquad k > 0, \Re(u_1) \geq 0, \Re(u_2) \geq 0, z \in C. \tag{4.1.11}$$

**Note.** If we consider variable $u_1 = u_2 = 1$ in Eq. (4.1.11), then we obtain the $k$-exponential function $E_k(z)$.

## 4.2 Main results

In the present chapter, we apply the 2-parameter $k$-Mittag-Leffler function to modify the $k$-gamma and $k$-beta matrix functions, which have been inspired and motivated by the previous generalizations and extensions.

**Definition 4.2.1.** Let $P_1$ be a positive stable matrix in $C^{n\times n}$, then the extended $k$-gamma matrix function can be described in the following manner:

$$\Gamma_{k,u}^{(u_1,u_2)}(P_1) = \int_0^{\infty} t^{P_1 - I} E_{k,u_1,u_2}\left(-\frac{t^k}{k} - \frac{u^k t^{-k}}{k}\right) dt, \tag{4.2.1}$$

where $k > 0$, $u \geq 0$, $\Re(u_1) > 0$, $\Re(u_2) > 0$ and $E_{k,u_1,u_2}(z)$ is defined in (4.1.11).

**Definition 4.2.2.** Let $P_1$ and $P_2$ be positive stable matrices in $C^{n\times n}$, then the extended $k$-beta matrix function can be described in the following manner:

$$B_{k,u}^{(u_1,u_2)}(P_1, P_2) = \frac{1}{k} \int_0^1 t^{\frac{P_1}{k} - I} (1-t)^{\frac{P_2}{k} - I} E_{k,u_1,u_2}\left(\frac{-u^k}{kt(1-t)}\right) dt, \tag{4.2.2}$$

where $k > 0$, $u \geq 0$, $\Re(u_1) > 0$, $\Re(u_2) > 0$ and $E_{k,u_1,u_2}(z)$ is defined in (4.1.11).

**Note.** If we set variable $u_2 = 1$ in the above Definitions 4.2.1 and 4.2.2, we obtain another expansion of the $k$-gamma and $k$-beta matrix functions as follows.

**Definition 4.2.3.** Let $P_1$ be a positive stable matrix in $C^{n\times n}$, then the extended $k$-gamma matrix function can be described in the following manner:

$$\Gamma_{k,u}^{(u_1)}(P_1) = \int_0^{\infty} t^{P_1 - I} E_{k,u_1}\left(-\frac{t^k}{k} - \frac{u^k t^{-k}}{k}\right) dt, \tag{4.2.3}$$

where $k > 0$, $u \geq 0$, $\Re(u_1) > 0 > 0$ and $E_{k,u_1}(z)$ is given in (4.1.10).

**Definition 4.2.4.** Let $P_1, P_2$ be positive stable matrices in $C^{n\times n}$, then the extended $k$-beta matrix function can be described in the following manner:

$$B_{k,u}^{(u_1)}(P_1, P_2) = \frac{1}{k}\int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{\frac{P_2}{k}-I} E_{k,u_1}\left(\frac{-u^k}{kt(1-t)}\right)dt, \tag{4.2.4}$$

where $k > 0$, $u \geq 0$, $\Re(u_1) > 0$ and $E_{k,u_1}(z)$ is given in (4.1.10).

**Remark.** (i) If we set $u_1 = u_2 = 1$ and $u = 0$ in Eqs. (4.2.1) and (4.2.2), then we obtain Eqs. (4.1.5) and (4.1.8), respectively:

$$\Gamma_{k,0}^{(1,1)}(P_1) = \Gamma_k(P_1) \tag{4.2.5}$$

and

$$B_{k,0}^{(1,1)}(P_1, P_2) = B_k(P_1, P_2). \tag{4.2.6}$$

(ii) If we put $u_1 = u_2 = k = 1$ and $u = 0$ in Eqs. (4.2.1) and (4.2.2), then we obtain the classical gamma and beta functions defined in [10]:

$$\Gamma_{1,0}^{(1,1)}(P_1) = \Gamma(P_1) \tag{4.2.7}$$

and

$$B_{1,0}^{(1,1)}(P_1, P_2) = B(P_1, P_2). \tag{4.2.8}$$

**Theorem 4.2.5** (Symmetric relation). *An extended $k$-beta matrix function defined in (4.2.2) satisfies the following symmetric relation for the positive stable matrices $P_1$ and $P_2$ such that $P_1P_2 = P_2P_1$:*

$$B_{k,u}^{(u_1,u_2)}(P_1, P_2) = B_{k,u}^{(u_1,u_2)}(P_2, P_1), \tag{4.2.9}$$

*where $k > 0$, $\Re(u) > 0$, $P_1, P_2 \in C^{r\times r}$ such that $\Re(u_1) > 0$, $\Re(u_2) > 0$.*

*Proof.* By the use of the extended $k$-beta matrix function (4.2.2) to the left-hand side of Eq. (4.2.9), we obtain:

$$B_{k,u}^{(u_1,u_2)}(P_1, P_2) = \frac{1}{k}\int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{\frac{P_2}{k}-I} E_{k,u_1,u_2}\left(\frac{-u^k}{kt(1-t)}\right)dt.$$

Consider $t = 1-x$ in the above equation and re-arranging terms, we obtain:

$$B_{k,u}^{(u_1,u_2)}(P_1, P_2) = \frac{1}{k}\int_0^1 x^{\frac{P_2}{k}-I}(1-x)^{\frac{P_1}{k}-I} E_{k,u_1,u_2}\left(\frac{-u^k}{kx(1-x)}\right)dx.$$

Then, after applying Eq. (4.2.2), we obtain our desired result:

$$B_{k,u}^{(u_1,u_2)}(P_1, P_2) = B_{k,u}^{(u_1,u_2)}(P_2, P_1).$$

□

**Corollary 4.2.6.** *The following outcome is correct:*

$$B_{k,u}^{(u_1)}(P_1, P_2) = B_{k,u}^{(u_1)}(P_2, P_1). \tag{4.2.10}$$

*Proof.* By Putting $u_2 = 1$, we obtain our desired result. □

**Corollary 4.2.7.** *The following outcome is correct:*

$$B_{k,u}(P_1, P_2) = B_{k,u}(P_2, P_1). \tag{4.2.11}$$

*Proof.* By setting $u_1 = u_2 = 1$, we obtain our desired result. □

**Theorem 4.2.8** (Functional relation). *An extended k-beta matrix function defined in* (4.2.2) *satisfies the following Functional relation for the positive stable matrices $P_1$ and $P_2$:*

$$B_{k,u}^{(u_1,u_2)}(P_1, P_2) = B_{k,u}^{(u_1,u_2)}(P_1 + kI, P_2) + B_{k,u}^{(u_1,u_2)}(P_1, P_2 + kI), \tag{4.2.12}$$

*where $k > 0$, $\Re(u) > 0$, $P_1, P_2 \in C^{n\times n}$ such that $\Re(u_1) > 0$, $\Re(u_2) > 0$.*

*Proof.* Consider the right-hand side of Eq. (4.2.12), and use the extended $k$-beta function (4.2.2), we obtain:

$$\begin{aligned}
&B_{k,u}^{(u_1,u_2)}(P_1 + kI, P_2) + B_{k,u}^{(u_1,u_2)}(P_1, P_2 + kI) = \\
&\quad \frac{1}{k}\int_0^1 t^{\frac{P_1+kI}{k}-I}(1-t)^{\frac{P_2}{k}-I} E_{k,u_1,u_2}\left(\frac{-u^k}{kt(1-t)}\right)dt \\
&\quad + \frac{1}{k}\int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{\frac{P_2+kI}{k}-I} E_{k,u_1,u_2}\left(\frac{-u^k}{kt(1-t)}\right)dt.
\end{aligned}$$

On re-arranging the terms, we obtain:

$$\begin{aligned}
&B_{k,u}^{(u_1,u_2)}(P_1 + kI, P_2) + B_{k,u}^{(u_1,u_2)}(P_1, P_2 + kI) = \\
&\quad \frac{1}{k}\int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{\frac{P_2}{k}-I}[t^I + (1-t)^I] E_{k,u_1,u_2}\left(\frac{-u^k}{kt(1-t)}\right)dt.
\end{aligned}$$

Note that, for $0 < t < 1$,

$$t^I = exp(Ilnt) = I + \sum_{n=1}^{\infty} I^n \frac{In^n t}{n!} = \left(\sum_{n=0}^{\infty} \frac{In^n t}{n!}\right) I = exp(lnt)I = tI.$$

Similarly,

$$(1-t)^I = (1-t)I(0 < t < 1).$$

Then,

$$(1-t)^I + t^I = tI + (1-t)I = I$$

$$B_{k,u}^{(u_1,u_2)}(P_1+kI,P_2)+B_{k,u}^{(u_1,u_2)}(P_1,P_2+kI)=$$
$$=\frac{1}{k}\int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{\frac{P_2}{k}-I}E_{k,u_1,u_2}\left(\frac{-u^k}{kt(1-t)}\right)dt.$$

Then, using Eq. (4.2.2), we obtain our desired result: □

$$B_{k,u}^{(u_1,u_2)}(P_1+kI,P_2)+B_{k,u}^{(u_1,u_2)}(P_1,P_2+kI)=B_{k,u}^{(u_1,u_2)}(P_1,P_2).$$

**Corollary 4.2.9.** *The following outcome is correct:*

$$B_{k,u}^{(u_1)}(P_1,P_2)=B_{k,u}^{(u_1)}(P_1+kI,P_2)+B_{k,u}^{(u_1)}(P_1,P_2+kI). \tag{4.2.13}$$

*Proof.* By setting $u_2=1$, we obtain our desired result. □

**Corollary 4.2.10.** *The following outcome is correct:*

$$B_{k,u}(P_1,P_2)=B_{k,u}(P_1+kI,P_2)+B_{k,u}(P_1,P_2+kI). \tag{4.2.14}$$

*Proof.* By Putting $u_1=u_2=1$, we obtain our desired result. □

**Theorem 4.2.11.** *An extended $k$-beta matrix function defined in (4.2.2) satisfies the following finite summation formula for the positive stable matrices $P_1$ and $P_2$:*

$$B_{k,u}^{(u_1,u_2)}(P_1,P_2)=\sum_{n=0}^{r}\binom{r}{n}B_{k,u}^{(u_1,u_2)}(P_1+nkI,P_2+rkI-nkI), \tag{4.2.15}$$

*where $k>0$, $\Re(u)>0$, $P_1,P_2\in C^{n\times n}$ such that $\Re(u_1)>0$, $\Re(u_2)>0$.*

*Proof.* From Eq. (4.2.12) we have:

$$B_{k,u}^{(u_1,u_2)}(P_1,P_2)=B_{k,u}^{(u_1,u_2)}(P_1+kI,P_2)+B_{k,u}^{(u_1,u_2)}(P_1,P_2+kI).$$

Then, applying the same result of Theorem 4.2.8 on every term of the right-hand side of the above equation we obtain:

$$B_{k,u}^{(u_1,u_2)}(P_1,P_2)=B_{k,u}^{(u_1,u_2)}(P_1+2kI,P_2)+2B_{k,u}^{(u_1,u_2)}(P_1+kI,P_2+kI)$$
$$+B_{k,u}^{(u_1,u_2)}(P_1,P_2+2kI).$$

Then, by continuing the similar method on the right-hand side of the equation and applying mathematical induction $r$-times, we have:

$$B_{k,u}^{(u_1,u_2)}(P_1,P_2)=\sum_{n=0}^{r}\frac{r!}{n!(r-n)!}B_{k,u}^{(u_1,u_2)}(P_1+nkI,P_2+rkI-nkI).$$

On re-arranging the terms we obtain our desired result:

$$B_{k,u}^{(u_1,u_2)}(P_1,P_2)=\sum_{n=0}^{r}\binom{r}{n}B_{k,u}^{(u_1,u_2)}(P_1+nkI,P_2+rkI-nkI). \tag{4.2.16}$$

□

**Corollary 4.2.12.** *The following outcome is correct:*

$$B_{k,u}^{(u_1)}(P_1,P_2)=\sum_{n=0}^{r}\binom{r}{n}B_{k,u}^{(u_1)}(P_1+nkI,P_2+rkI-nkI). \tag{4.2.17}$$

*Proof.* By substituting $u_2=1$, we obtain our desired result. □

**Corollary 4.2.13.** *The following outcome is correct:*

$$B_{k,u}(P_1,P_2)=\sum_{n=0}^{r}\binom{r}{n}B_{k,u}(P_1+nkI,P_2+rkI-nkI). \tag{4.2.18}$$

*Proof.* By setting $u_1=u_2=1$, we obtain our desired result. □

**Theorem 4.2.14.** *An extended k-beta matrix function defined in (4.2.2) satisfies the following infinite summation formula for the positive stable matrices $P_1$ and $P_2$:*

$$B_{k,u}^{(u_1,u_2)}(P_1,P_2)=\sum_{n=0}^{\infty}B_{k,u}^{(u_1,u_2)}(P_1+nkI,P_2+kI), \tag{4.2.19}$$

*where $k>0,\ \Re(u)>0,\ P_1,P_2\in C^{n\times n}$ such that $\Re(u_1)>0,\ \Re(u_2)>0$.*

*Proof.* By the definition of the extended $k$-beta matrix function 4.2.2, we have:

$$\begin{aligned}B_{k,u}^{(u_1,u_2)}(P_1,P_2)&=\frac{1}{k}\int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{\frac{P_2}{k}-I}E_{k,u_1,u_2}\left(\frac{-u^k}{kt(1-t)}\right)dt\\&=\frac{1}{k}\int_0^1 t^{\frac{P_1}{k}-1}(1-t)^{\frac{P_2}{k}}(1-t)^{-I}E_{k,u_1,u_2}\left(\frac{-u^k}{kt(1-t)}\right)dt.\end{aligned}$$

Using the binomial expansion formula in the above equation, we have:

$$(1-t)^{-I}=\sum_{n=0}^{\infty}t^{nI},\ |t|<I \tag{4.2.20}$$

$$B_{k,u}^{(u_1,u_2)}(P_1,P_2)=\frac{1}{k}\int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{\frac{P_2}{k}}\left(\sum_{n=0}^{\infty}t^{nI}\right)E_{k,u_1,u_2}\left(\frac{-u^k}{kt(1-t)}\right)dt.$$

By interchanging the order of summation and integration in the above, as convergence is uniform, we obtain:

$$B_{k,u}^{(u_1,u_2)}(P_1, P_2) = \sum_{n=0}^{\infty} \frac{1}{k} \int_0^1 t^{\frac{P_1+nkI}{k}-I}(1-t)^{\frac{P_2}{k}} E_{k,u_1,u_2}\left(\frac{-u^k}{kt(1-t)}\right)dt.$$

Then, using Eq. (4.2.2), we obtain our desired result:

$$B_{k,u}^{(u_1,u_2)}(P_1, P_2) = \sum_{n=0}^{\infty} B_{k,u}^{(u_1,u_2)}(P_1 + nkI, P_2 + kI). \tag{4.2.21}$$

□

**Corollary 4.2.15.** *The following outcome is correct:*

$$B_{k,u}^{(u_1)}(P_1, P_2) = \sum_{n=0}^{\infty} B_{k,u}^{(u_1)}(P_1 + nkI, P_2 + kI). \tag{4.2.22}$$

*Proof.* By setting $u_2 = 1$, we obtain our desired result. □

**Corollary 4.2.16.** *The following outcome is correct:*

$$B_{k,u}(P_1, P_2) = \sum_{n=0}^{\infty} B_{k,u}(P_1 + nkI, P_2 + kI). \tag{4.2.23}$$

*Proof.* By setting $u_1 = u_2 = 1$, we obtain our desired result. □

**Theorem 4.2.17.** *An extended $k$-beta matrix function defined in (4.2.2) satisfies the following infinite summation formula for the positive stable matrices $P_1$ and $P_2$:*

$$B_{k,u}^{(u_1,u_2)}(P_1, kI - P_2) = \sum_{n=0}^{\infty} \frac{(P_2)_{n,k}}{k^n n!} B_{k,u}^{(u_1,u_2)}(P_1 + nkI, kI), \tag{4.2.24}$$

*where $k > 0$, $\Re(u) > 0$, $P_1, P_2 \in C^{n\times n}$ such that $\Re(u_1) > 0$, $\Re(u_2) > 0$.*

*Proof.* Applying the definition of the extended $k$-beta matrix function 4.2.2 on the left-hand side of Eq. (4.2.24), we have:

$$\begin{aligned} B_{k,u}^{(u_1,u_2)}(P_1, kI - P_2) &= \frac{1}{k} \int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{\frac{kI-P_2}{k}-I} E_{k,u_1,u_2}\left(\frac{-u^k}{kt(1-t)}\right)dt \\ &= \frac{1}{k} \int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{-\frac{P_2}{k}} E_{k,u_1,u_2}\left(\frac{-u^k}{kt(1-t)}\right)dt. \end{aligned}$$

Then, using the binomial expansion formula and the Pochhammer $k$-symbol relation, we have:

$$(1-t)^{-\frac{P_2}{k}} = \sum_{n=0}^{\infty} \left(\frac{P_2}{k}\right)_n \frac{t^{nI}}{n!}, \quad |t| < I$$

and

$$\left(\frac{P_2}{k}\right)_n = \frac{(P_2)_{n,k}}{k^n}$$

$$(1-t)^{-\frac{P_2}{k}} = \sum_{n=0}^{\infty} \frac{(P_2)_{n,k}}{k^n n!} t^{nI}, \quad |t| < I$$

$$B_{k,u}^{(u_1,u_2)}(P_1, kI - P_2) = \frac{1}{k}\int_0^1 t^{\frac{P_1}{k}-I}\left(\sum_{n=0}^{\infty} \frac{(P_2)_{n,k}}{k^n n!} t^{nI}\right) E_{k,u_1,u_2}\left(\frac{-u^k}{kt(1-t)}\right) dt.$$

By interchanging the order of summation and integration in the above, as convergence is uniform, we have:

$$B_{k,u}^{(u_1,u_2)}(P_1, kI - P_2) = \sum_{n=0}^{\infty} \frac{(P_2)_{n,k}}{k^n n!} \frac{1}{k}\int_0^1 t^{\frac{P_1+nkI}{k}-I} E_{k,u_1,u_2}\left(\frac{-u^k}{kt(1-t)}\right) dt.$$

Then, using Eq. (4.2.2), we obtain our desired result:

$$B_{k,u}^{(u_1,u_2)}(P_1, kI - P_2) = \sum_{n=0}^{\infty} \frac{(P_2)_{n,k}}{k^n n!} B_{k,u}^{(u_1,u_2)}(P_1 + nkI, kI). \tag{4.2.25}$$

□

**Corollary 4.2.18.** *The following outcome is correct:*

$$B_{k,u}^{(u_1)}(P_1, kI - P_2) = \sum_{n=0}^{\infty} \frac{(P_2)_{n,k}}{k^n n!} B_{k,u}^{(u_1)}(P_1 + nkI, kI). \tag{4.2.26}$$

*Proof.* By putting $u_2 = 1$, we obtain our desired result. □

**Corollary 4.2.19.** *The following outcome is correct:*

$$B_{k,u}(P_1, kI - P_2) = \sum_{n=0}^{\infty} \frac{(P_2)_{n,k}}{k^n n!} B_{k,u}(P_1 + nkI, kI). \tag{4.2.27}$$

*Proof.* By setting $u_1 = u_2 = 1$, we obtain our desired result. □

**Theorem 4.2.20.** *An extended k-beta matrix function presented in 4.2.2 has a relationship with the k-beta matrix function for the positive stable matrices $P_1$ and $P_2$:*

$$B_{k,u}^{(u_1,u_2)}(P_1, P_2) = \sum_{n=0}^{\infty} \frac{(-u^k)^n}{k^n \Gamma_k(u_1 n + u_2)} B_k(P_1 - nkI, P_2 - nkI), \tag{4.2.28}$$

*where* $k > 0$, $\Re(u) > 0$, $P_1, P_2 \in C^{n\times n}$ *such that* $\Re(u_1) > 0$, $\Re(u_2) > 0$.

*Proof.* By the use of the extended $k$-beta matrix function given in (4.2.2) and substituting the value of the two-parameter $k$-Mittag-Leffler function (4.1.11), we have:

$$B_{k,u}^{(u_1,u_2)}(P_1, P_2) = \frac{1}{k}\int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{\frac{P_2}{k}-I}\left(\sum_{n=0}^{\infty}\frac{1}{\Gamma_k(u_1 n+u_2)}\frac{(-u^k)^n}{k^n t^n(1-t)^n}\right)dt.$$

By interchanging the order of summation and integration in the above, as convergence is uniform, we have:

$$B_{k,u}^{(u_1,u_2)}(P_1, P_2) = \sum_{n=0}^{\infty}\frac{(-u^k)^n}{k^n\Gamma_k(u_1 n+u_2)}\left(\frac{1}{k}\int_0^1 t^{\frac{P_1-nkI}{k}-I}(1-t)^{\frac{P_2-nkI}{k}-I}dt\right).$$

Then, using Eq. (4.1.8), we obtain our desired result:

$$B_{k,u}^{(u_1,u_2)}(P_1, P_2) = \sum_{n=0}^{\infty}\frac{(-u^k)^n}{k^n\Gamma_k(u_1 n+u_2)}B_k(P_1-nkI, P_2-nkI). \tag{4.2.29}$$

□

**Corollary 4.2.21.** *The following outcome is correct:*

$$B_{k,u}^{(u_1)}(P_1, P_2) = \sum_{n=0}^{\infty}\frac{(-u^k)^n}{k^n\Gamma_k(u_1 n+u_2)}B_k(P_1-nkI, P_2-nkI). \tag{4.2.30}$$

*Proof.* By setting $u_2 = 1$, we obtain our desired result. □

**Corollary 4.2.22.** *The following outcome is correct:*

$$B_{k,u}(P_1, P_2) = \sum_{n=0}^{\infty}\frac{(-u^k)^n}{k^n\Gamma_k(u_1 n+u_2)}B_k(P_1-nkI, P_2-nkI). \tag{4.2.31}$$

*Proof.* By setting $u_1 = u_2 = 1$, we obtain our desired result. □

**Theorem 4.2.23.** *The following derivative formula holds true:*

$$\frac{\partial}{\partial u}B_{k,u}^{(u_1,u_2)}(P_1, P_2) = \frac{1}{u}\int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{\frac{P_2}{k}-I}\sum_{n=0}^{\infty}\frac{n(-u^k)^n}{k^n t^n(1-t)^n\Gamma_k(u_1 n+u_2)}dt, \tag{4.2.32}$$

*where* $k > 0$, $\Re(u) > 0$, $P_1, P_2 \in C^{r\times r}$ *such that* $\Re(u_1) > 0$, $\Re(u_2) > 0$.

*Proof.* On differentiating the extended $k$-beta matrix function defined in 4.2.2 with respect to the variable $u$, we have:

$$\frac{\partial}{\partial u}B_{k,u}^{(u_1,u_2)}(P_1,P_2)=\frac{\partial}{\partial u}\left(\frac{1}{k}\int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{\frac{P_2}{k}-I}E_{k,u_1,u_2}\left(\frac{-u^k}{kt(1-t)}\right)dt\right).$$

Then, using the two-parameter $k$-Mittag-Leffler function (4.1.11):

$$\frac{\partial}{\partial u}B_{k,u}^{(u_1,u_2)}(P_1,P_2)$$
$$=\frac{1}{k}\int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{\frac{P_2}{k}-I}\left(\frac{\partial}{\partial u}\sum_{n=0}^{\infty}\frac{(-u^k)^n}{k^n t^n(1-t)^n\Gamma_k(u_1 n+u_2)}\right)dt.$$

By differentiating the above series with respect to $u$, and using the derivative formulas, we have:

$$\frac{\partial}{\partial u}(-u^k)^n=(-1)^n\frac{\partial}{\partial u}(u^{kn})=(-1)^n kn(u^{kn-1})=\frac{kn}{u}(-u^k)^n$$

$$\frac{\partial}{\partial u}B_{k,u}^{(u_1,u_2)}(P_1,P_2)=\frac{1}{k}\int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{\frac{P_2}{k}-I}\sum_{n=0}^{\infty}\frac{kn(-u^k)^n}{uk^n t^n(1-t)^n\Gamma_k(u_1 n+u_2)}dt.$$

On re-arranging the terms we obtain our desired result:

$$\frac{\partial}{\partial u}B_{k,u}^{(u_1,u_2)}(P_1,P_2)=\frac{1}{u}\int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{\frac{P_2}{k}-I}\sum_{n=0}^{\infty}\frac{n(-u^k)^n}{k^n t^n(1-t)^n\Gamma_k(u_1 n+u_2)}dt.$$

□

**Corollary 4.2.24.** *The following outcome is correct:*

$$\frac{\partial}{\partial u}B_{k,u}^{(u_1)}(P_1,P_2)=\frac{1}{u}\int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{\frac{P_2}{k}-I}\sum_{n=0}^{\infty}\frac{n(-u^k)^n}{k^n t^n(1-t)^n\Gamma_k(u_1 n+1)}dt. \tag{4.2.33}$$

*Proof.* By setting $u_2=1$, we obtain our desired result. □

**Corollary 4.2.25.** *The following outcome is correct:*

$$\frac{\partial}{\partial u}B_{k,u}(P_1,P_2)=\frac{1}{u}\int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{\frac{P_2}{k}-I}\sum_{n=0}^{\infty}\frac{n(-u^k)^n}{k^n t^n(1-t)^n\Gamma_k(n+1)}dt. \tag{4.2.34}$$

*Proof.* By setting $u_1=u_2=1$, we obtain our desired result. □

**Theorem 4.2.26.** *The given integral representations for the extended k-beta matrix function defined in 4.2.2 hold true:*

$$B_{k,u}^{(u_1,u_2)}(P_1,P_2)=\frac{2}{k}\int_0^{\frac{\pi}{2}}(cos\theta)^{\frac{2P_1}{k}-I}(sin\theta)^{\frac{2P_2}{k}-I}$$
$$E_{k,u_1,u_2}\left(\frac{-u^k}{k}(sec\theta)^2(cosec\theta)^2\right)d\theta$$

$$B_{k,u}^{(u_1,u_2)}(P_1,P_2)=\frac{2}{k}\int_0^{\frac{\pi}{2}}(sin\theta)^{\frac{2P_1}{k}-I}(cos\theta)^{\frac{2P_2}{k}-I}$$
$$E_{k,u_1,u_2}\left(\frac{-u^k}{k}(sec\theta)^2(cosec\theta)^2\right)d\theta$$
$$B_{k,u}^{(u_1,u_2)}(P_1,P_2)=\frac{2}{k}\int_0^{\infty}(sinh\theta)^{\frac{2P_1}{k}-I}(cosh\theta)^{I-\frac{2(P_1+P_2)}{k}}$$
$$E_{k,u_1,u_2}\left(\frac{-u^k}{k}(cosh\theta)^4(cosech\theta)^2\right)d\theta$$
$$B_{k,u}^{(u_1,u_2)}(P_1,P_2)=\frac{z^{I-\frac{(P_1+P_2)}{k}}}{k}\int_0^{z}w^{\frac{P_1}{k}-I}(z-w)^{\frac{P_2}{k}-I}E_{k,u_1,u_2}\left(\frac{-u^kz^2}{kw(z-w)}\right)dw, \tag{4.2.35}$$

*where* $k>0,\ \Re(u)>0,\ P_1,P_2\in C^{n\times n}$ *such that* $\Re(u_1)>0,\ \Re(u_2)>0.$

*Proof.* On substituting $t=(cos\theta)^2$, $t=(sin\theta)^2$, $t=(tanh\theta)^2$, and $t=\frac{w}{z}$, respectively, in the integral form of the extended $k$-beta matrix function (4.2.2), we obtain our desired results. □

**Corollary 4.2.27.** *The following outcome is correct:*

$$B_{k,u}^{(u_1)}(P_1,P_2)=\frac{2}{k}\int_0^{\frac{\pi}{2}}(cos\theta)^{\frac{2P_1}{k}-I}(sin\theta)^{\frac{2P_2}{k}-I}E_{k,u_1}\left(\frac{-u^k}{k}(sec\theta)^2(cosec\theta)^2\right)d\theta$$
$$B_{k,u}^{(u_1)}(P_1,P_2)=\frac{2}{k}\int_0^{\frac{\pi}{2}}(sin\theta)^{\frac{2P_1}{k}-I}(cos\theta)^{\frac{2P_2}{k}-I}E_{k,u_1}\left(\frac{-u^k}{k}(sec\theta)^2(cosec\theta)^2\right)d\theta$$
$$B_{k,u}^{(u_1)}(P_1,P_2)=\frac{2}{k}\int_0^{\infty}(sinh\theta)^{\frac{2P_1}{k}-I}(cosh\theta)^{I-\frac{2(P_1+P_2)}{k}}$$
$$E_{k,u_1}\left(\frac{-u^k}{k}(cosh\theta)^4(cosech\theta)^2\right)d\theta$$
$$B_{k,u}^{(u_1)}(P_1,P_2)=\frac{z^{I-\frac{(P_1+P_2)}{k}}}{k}\int_0^{z}w^{\frac{P_1}{k}-I}(z-w)^{\frac{P_2}{k}-I}E_{k,u_1}\left(\frac{-u^kz^2}{kw(z-w)}\right)dw. \tag{4.2.36}$$

*Proof.* By setting $u_2=1$, we obtain our desired result. □

**Corollary 4.2.28.** *The following outcome is correct:*

$$B_{k,u}(P_1,P_2)=\frac{2}{k}\int_0^{\frac{\pi}{2}}(cos\theta)^{\frac{2P_1}{k}-I}(sin\theta)^{\frac{2P_2}{k}-I}E_{k}\left(\frac{-u^k}{k}(sec\theta)^2(cosec\theta)^2\right)d\theta$$
$$B_{k,u}(P_1,P_2)=\frac{2}{k}\int_0^{\frac{\pi}{2}}(sin\theta)^{\frac{2P_1}{k}-I}(cos\theta)^{\frac{2P_2}{k}-I}E_{k}\left(\frac{-u^k}{k}(sec\theta)^2(cosec\theta)^2\right)d\theta$$
$$B_{k,u}(P_1,P_2)=\frac{2}{k}\int_0^{\infty}(sinh\theta)^{\frac{2P_1}{k}-I}(cosh\theta)^{I-\frac{2(P_1+P_2)}{k}}$$

$$E_k\left(\frac{-u^k}{k}(cosh\theta)^4(cosech\theta)^2\right)d\theta$$

$$B_{k,u}(P_1, P_2) = \frac{z^{I-\frac{(P_1+P_2)}{k}}}{k}\int_0^z w^{\frac{P_1}{k}-I}(z-w)^{\frac{P_2}{k}-I}E_k\left(\frac{-u^k z^2}{kw(z-w)}\right)dw. \quad (4.2.37)$$

*Proof.* By setting $u_1 = u_2 = 1$, we obtain our desired result. □

**Theorem 4.2.29.** *The following integral representations for the extended k-beta matrix function defined in 4.2.2 hold true:*

$$B_{k,u}^{(u_1,u_2)}(P_1, P_2) = \frac{(c-d)^{I-\frac{P_1+P_2}{k}}}{k}\int_d^c (w-d)^{\frac{P_1}{k}-I}(c-d)^{\frac{P_2}{k}-I} E_{k,u_1,u_2}\left(\frac{-u^k(c-d)^2}{k(w-d)(c-w)}\right)dw$$

$$B_{k,u}^{(u_1,u_2)}(P_1, P_2) = \frac{2^{I-\frac{P_1+P_2}{k}}}{k}\int_{-1}^1 (1+w)^{\frac{P_1}{k}-I}(1-w)^{\frac{P_2}{k}-I}E_{k,u_1,u_2}\left(\frac{-4u^k}{k(1-w^2)}\right)dw$$

$$B_{k,u}^{(u_1,u_2)}(P_1, P_2) = \frac{2^{I-\frac{P_1+P_2}{k}}}{k}\int_{-\infty}^{\infty} \frac{e^{\theta(\frac{P_1+P_2}{k})}}{(cosh\theta)^{\frac{P_1+P_2}{k}}}E_{k,u_1,u_2}\left(\frac{-4u^k}{k}(cosh\theta)^2\right)d\theta, \quad (4.2.38)$$

*where* $k > 0$, $\Re(u) > 0$, $P_1, P_2 \in C^{n\times n}$ *such that* $\Re(u_1) > 0$, $\Re(u_2) > 0$.

*Proof.* On substituting $t = \frac{w-d}{c-d}$, $t = \frac{1+w}{2}$, and $w = (tanh\theta)^2$, respectively, in the integral form of the extended $k$-beta matrix function (4.2.2), we obtain our desired results. □

**Corollary 4.2.30.** *The following outcome is correct:*

$$B_{k,u}^{(u_1)}(P_1, P_2) = \frac{(c-d)^{I-\frac{P_1+P_2}{k}}}{k}\int_d^c (w-d)^{\frac{P_1}{k}-I}(c-d)^{\frac{P_2}{k}-I} E_{k,u_1}\left(\frac{-u^k(c-d)^2}{k(w-d)(c-w)}\right)dw$$

$$B_{k,u}^{(u_1)}(P_1, P_2) = \frac{2^{I-\frac{P_1+P_2}{k}}}{k}\int_{-1}^1 (1+w)^{\frac{P_1}{k}-I}(1-w)^{\frac{P_2}{k}-I}E_{k,u_1}\left(\frac{-4u^k}{k(1-w^2)}\right)dw$$

$$B_{k,u}^{(u_1)}(P_1, P_2) = \frac{2^{I-\frac{P_1+P_2}{k}}}{k}\int_{-\infty}^{\infty} \frac{e^{\theta(\frac{P_1+P_2}{k})}}{(cosh\theta)^{\frac{P_1+P_2}{k}}}E_{k,u_1}\left(\frac{-4u^k}{k}(cosh\theta)^2\right)d\theta. \quad (4.2.39)$$

*Proof.* By setting $u_2 = 1$, we obtain our desired result. □

**Corollary 4.2.31.** *The following outcome is correct:*

$$B_{k,u}(P_1,P_2)=\frac{(c-d)^{I-\frac{P_1+P_2}{k}}}{k}\int_d^c (w-d)^{\frac{P_1}{k}-I}(c-d)^{\frac{P_2}{k}-I}$$
$$E_k\left(\frac{-u^k(c-d)^2}{k(w-d)(c-w)}\right)dw$$
$$B_{k,u}(P_1,P_2)=\frac{2^{I-\frac{P_1+P_2}{k}}}{k}\int_{-1}^1 (1+w)^{\frac{P_1}{k}-I}(1-w)^{\frac{P_2}{k}-I}E_k\left(\frac{-4u^k}{k(1-w^2)}\right)dw$$
$$B_{k,u}(P_1,P_2)=\frac{2^{I-\frac{P_1+P_2}{k}}}{k}\int_{-\infty}^{\infty}\frac{e^{\theta(\frac{P_1+P_2}{k})}}{(cosh\theta)^{\frac{P_1+P_2}{k}}}E_k\left(\frac{-4u^k}{k}(cosh\theta)^2\right)d\theta. \quad (4.2.40)$$

*Proof.* By setting $u_1=u_2=1$, we obtain our desired result. □

**Theorem 4.2.32.** *The following integral representations for the extended k-beta matrix function defined in 4.2.2 hold true:*

$$B_{k,u}^{(u_1,u_2)}(P_1,P_2)=\frac{1}{k}\int_0^\infty \frac{w^{\frac{P_1}{k}-I}}{(1+w)^{\frac{P_1+P_2}{k}}}E_{k,u_1,u_2}\left(\frac{-u^k}{k}\left(2+w+\frac{1}{w})\right)dw\right.$$
$$B_{k,u}^{(u_1,u_2)}(P_1,P_2)=\frac{1}{k}\int_0^1 \frac{t^{\frac{P_1}{k}-I}+t^{\frac{P_2}{k}-I}}{(1+t)^{\frac{P_1+P_2}{k}}}E_{k,u_1,u_2}\left(\frac{-u^k}{k}\left(2+t+\frac{1}{t}\right)\right)dt, \quad (4.2.41)$$

*where $k>0$, $\Re(u)>0$, $P_1,P_2\in C^{n\times n}$ such that $\Re(u_1)>0$, $\Re(u_2)>0$.*

*Proof.* On substituting $t=\frac{w}{1+w}$, respectively, in the integral form of the extended $k$-beta matrix function (4.2.2), we obtain our desired results. □

**Corollary 4.2.33.** *The following outcome is correct:*

$$B_{k,u}^{(u_1)}(P_1,P_2)=\frac{1}{k}\int_0^\infty \frac{w^{\frac{P_1}{k}-I}}{(1+w)^{\frac{P_1+P_2}{k}}}E_{k,u_1}\left(\frac{-u^k}{k}\left(2+w+\frac{1}{w}\right)\right)dw$$
$$B_{k,u}^{(u_1)}(P_1,P_2)=\frac{1}{k}\int_0^1 \frac{t^{\frac{P_1}{k}-I}+t^{\frac{P_2}{k}-I}}{(1+t)^{\frac{P_1+P_2}{k}}}E_{k,u_1}\left(\frac{-u^k}{k}\left(2+t+\frac{1}{t}\right)\right)dt. \quad (4.2.42)$$

*Proof.* By setting $u_2=1$, we obtain our desired result. □

**Corollary 4.2.34.** *The following outcome is correct:*

$$B_{k,u}(P_1,P_2)=\frac{1}{k}\int_0^\infty \frac{w^{\frac{P_1}{k}-I}}{(1+w)^{\frac{P_1+P_2}{k}}}E_k\left(\frac{-u^k}{k}\left(2+w+\frac{1}{w}\right)\right)dw$$
$$B_{k,u}(P_1,P_2)=\frac{1}{k}\int_0^1 \frac{t^{\frac{P_1}{k}-I}+t^{\frac{P_2}{k}-I}}{(1+t)^{\frac{P_1+P_2}{k}}}E_k\left(\frac{-u^k}{k}\left(2+t+\frac{1}{t}\right)\right)dt. \quad (4.2.43)$$

*Proof.* By setting $u_1 = u_2 = 1$, we obtain our desired result. □

**Theorem 4.2.35.** *The Mellin transform of the extended k-beta function defined in 4.2.2 is given as follows:*

$$M\left(B_{k,u}^{(u_1,u_2)}(P_1, P_2) : u \to r\right) = B_k(P_1 + r, P_2 + r)\Gamma_{k,0}^{u_1,u_2}(r), \tag{4.2.44}$$

*where* $k > 0$, $\Re(u) > 0$, $P_1, P_2 \in C^{r\times r}$ *such that* $\Re(u_1) > 0$, $\Re(u_2) > 0$.

*Proof.* By using the Mellin transform:

$$M\Big(f(t) : u \to r\Big) = \int_0^\infty u^{r-1} f(t)du$$

to the extended $k$-beta matrix function defined in 4.2.2, we have:

$$M\left(B_{k,u}^{(u_1,u_2)}(P_1, P_2) : u \to r\right)$$
$$= \int_0^\infty u^{r-1}\left(\frac{1}{k}\int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{\frac{P_2}{k}-I} E_{k,u_1,u_2}\left(\frac{-u^k}{kt(1-t)}\right)dt\right)du.$$

On changing the order of integration, we have:

$$= \frac{1}{k}\int_0^1 t^{\frac{P_1}{k}-I}(1-t)^{\frac{P_2}{k}-I}\left(\int_0^\infty u^{r-1} E_{k,u_1,u_2}\left(\frac{-u^k}{kt(1-t)}\right)du\right)dt.$$

Substituting $y = \frac{u}{t^{\frac{1}{k}}(1-t)^{\frac{1}{k}}}$, then $du = t^{\frac{1}{k}}(1-t)^{\frac{1}{k}}dy$, and we have:

$$= \frac{1}{k}\int_0^1 t^{\frac{P_1+r}{k}-I}(1-t)^{\frac{P_2+r}{k}-I}\left(\int_0^\infty y^{r-1} E_{k,u_1,u_2}\left(\frac{-y^k}{k}\right)dy\right)dt.$$

Then, applying the definition of the extended $k$-gamma matrix function defined in 4.2.1 (when $u = 0$) and the $k$-beta matrix function (4.1.4), we obtain our desired result:

$$M\left(B_{k,u}^{(u_1,u_2)}(P_1, P_2) : u \to r\right) = B_k(P_1 + r, P_2 + r)\Gamma_{k,0}^{u_1,u_2}(r).$$
□

**Corollary 4.2.36.** *The following outcome is correct:*

$$M\left(B_{k,u}^{(u_1)}(P_1, P_2) : u \to r\right) = B_k(P_1 + r, P_2 + r)\Gamma_{k,0}^{u_1}(r). \tag{4.2.45}$$

*Proof.* By setting $u_2 = 1$, we obtain our desired result. □

**Corollary 4.2.37.** *The following outcome is correct:*

$$M\left(B_{k,u}(P_1,P_2):u\to r\right)=B_k(P_1+r,P_2+r)\Gamma_{k,0}(r). \tag{4.2.46}$$

*Proof.* By setting $u_1=u_2=1$, we obtain our desired result. □

**Theorem 4.2.38.** *The given integral representations for the extended k-gamma matrix function defined in 4.2.1 hold true:*

$$\begin{aligned}
\Gamma_{k,u}^{u_1,u_2}(P_1)&=\int_0^{\frac{\pi}{2}}\frac{(sin\theta)^{P_1-I}}{(cos\theta)^{P_2-I}}E_{k,u_1,u_2}\left(\frac{-(sin\theta)^{2k}-u^k(cos\theta)^{2k}}{k(sin\theta cos\theta)^k}\right)d\theta\\
\Gamma_{k,u}^{u_1,u_2}(P_1)&=-\int_{-1}^{0}\frac{v^{P_1-I}}{(1+v)^{P_1+I}}E_{k,u_1,u_2}\left(\frac{-v^{2k}-u^k(1+v)^{2k}}{k(v(1+v))^k}\right)dv\\
\Gamma_{k,u}^{u_1,u_2}(P_1)&=-\int_{0}^{1}\frac{v^{P_1-I}}{(v-1)^{P_1+I}}E_{k,u_1,u_2}\left(\frac{-v^{2k}-u^k(v-1)^{2k}}{k(v(v-1))^k}\right)dv\\
\Gamma_{k,u}^{u_1,u_2}(P_1)&=(d-c)\int_{d}^{c}\frac{(t-d)^{P_1-I}}{(t-c)^{P_1+I}}E_{k,u_1,u_2}\left(\frac{-(t-d)^{2k}-u^k(t-c)^{2k}}{k((t-c)(t-d))^k}\right)dt\\
\Gamma_{k,u}^{u_1,u_2}(P_1)&=-2\int_{-1}^{1}\frac{(t+1)^{P_1-I}}{(t-1)^{P_1+I}}E_{k,u_1,u_2}\left(\frac{-(t+1)^{2k}-u^k(t-1)^{2k}}{k((t^2-1))^k}\right)dt\\
\Gamma_{k,u}^{u_1,u_2}(P_1)&=-z\int_{0}^{z}\frac{(t)^{P_1-I}}{(t-z)^{P_1+I}}E_{k,u_1,u_2}\left(\frac{-t^{2k}-u^k(t-z)^{2k}}{k(t(t-z))^k}\right)dt,
\end{aligned} \tag{4.2.47}$$

*where* $k>0,\ \Re(u)>0,\ P_1,P_2\in C^{r\times r}$ *such that* $\Re(u_1)>0,\ \Re(u_2)>0.$

*Proof.* On substituting $t=tan\theta$, $t=\frac{v}{1+v}$, $t=\frac{v}{v-1}$, $v=\frac{t-d}{c-d}$, $v=\frac{1+t}{2}$, and $v=\frac{t}{z}$, respectively, in the integral form of the extended $k$-gamma function (4.2.3), we obtain our desired results. □

**Corollary 4.2.39.** *The following outcome is correct:*

$$\begin{aligned}
\Gamma_{k,u}^{u_1}(P_1)&=\int_0^{\frac{\pi}{2}}\frac{(sin\theta)^{P_1-I}}{(cos\theta)^{P_2-I}}E_{k,u_1}\left(\frac{-(sin\theta)^{2k}-u^k(cos\theta)^{2k}}{k(sin\theta cos\theta)^k}\right)d\theta\\
\Gamma_{k,u}^{u_1}(P_1)&=-\int_{-1}^{0}\frac{v^{P_1-I}}{(1+v)^{P_1+I}}E_{k,u_1}\left(\frac{-v^{2k}-u^k(1+v)^{2k}}{k(v(1+v))^k}\right)dv\\
\Gamma_{k,u}^{u_1}(P_1)&=-\int_{0}^{1}\frac{v^{P_1-I}}{(v-1)^{P_1+I}}E_{k,u_1}\left(\frac{-v^{2k}-u^k(v-1)^{2k}}{k(v(v-1))^k}\right)dv\\
\Gamma_{k,u}^{u_1}(P_1)&=(d-c)\int_{d}^{c}\frac{(t-d)^{P_1-I}}{(t-c)^{P_1+I}}E_{k,u_1}\left(\frac{-(t-d)^{2k}-u^k(t-c)^{2k}}{k((t-c)(t-d))^k}\right)dt\\
\Gamma_{k,u}^{u_1}(P_1)&=-2\int_{-1}^{1}\frac{(t+1)^{P_1-I}}{(t-1)^{P_1+I}}E_{k,u_1}\left(\frac{-(t+1)^{2k}-u^k(t-1)^{2k}}{k((t^2-1))^k}\right)dt
\end{aligned} \tag{4.2.48}$$

$$\Gamma^{u_1}_{k,u}(P_1) = -z\int_0^z \frac{(t)^{P_1-I}}{(t-z)^{P_1+I}} E_{k,u_1}\left(\frac{-t^{2k}-u^k(t-z)^{2k}}{k(t(t-z))^k}\right)dt.$$

*Proof.* By setting $u_2 = 1$, we obtain our desired result. □

**Corollary 4.2.40.** *The following outcome is correct:*

$$\begin{aligned}
\Gamma_{k,u}(P_1) &= \int_0^{\frac{\pi}{2}} \frac{(sin\theta)^{P_1-I}}{(cos\theta)^{P_2-I}} E_k\left(\frac{-(sin\theta)^{2k}-u^k(cos\theta)^{2k}}{k(sin\theta cos\theta)^k}\right)d\theta \\
\Gamma_{k,u}(P_1) &= -\int_{-1}^0 \frac{v^{P_1-I}}{(1+v)^{P_1+I}} E_k\left(\frac{-v^{2k}-u^k(1+v)^{2k}}{k(v(1+v))^k}\right)dv \\
\Gamma_{k,u}(P_1) &= -\int_0^1 \frac{v^{P_1-I}}{(v-1)^{P_1+I}} E_k\left(\frac{-v^{2k}-u^k(v-1)^{2k}}{k(v(v-1))^k}\right)dv \qquad (4.2.49) \\
\Gamma_{k,u}(P_1) &= (d-c)\int_d^c \frac{(t-d)^{P_1-I}}{(t-c)^{P_1+I}} E_k\left(\frac{-(t-d)^{2k}-u^k(t-c)^{2k}}{k((t-c)(t-d))^k}\right)dt \\
\Gamma_{k,u}(P_1) &= -2\int_{-1}^1 \frac{(t+1)^{P_1-I}}{(t-1)^{P_1+I}} E_k\left(\frac{-(t+1)^{2k}-u^k(t-1)^{2k}}{k((t^2-1))^k}\right)dt \\
\Gamma_{k,u}(P_1) &= -z\int_0^z \frac{(t)^{P_1-I}}{(t-z)^{P_1+I}} E_k\left(\frac{-t^{2k}-u^k(t-z)^{2k}}{k(t(t-z))^k}\right)dt.
\end{aligned}$$

*Proof.* By setting $u_1 = u_2 = 1$, we obtain our desired result. □

## 4.3 Conclusion

In this chapter, we have presented novel extensions of the $k$-gamma and $k$-beta matrix functions by using the 2-parameter $k$-Mittag-Leffler function as the kernel in the integral formulas of the $k$-gamma and $k$-beta matrix functions. Then, we looked at some of the basic characteristics of these extensions and their special cases.

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CHAPTER

# A $k$-type Caputo fractional derivative operator and its properties

5

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## 5.1 Introduction and preliminaries

Special functions are used in the application of mathematics to physical and engineering problems [6–11]. In recent years, many authors considered several extensions of the well-known special functions.

In [3], Mubeen defind the confluent $k$-hypergeometric function ($C.H.F.^{(k)}$) for $k > 0$ as follows:

$$ {}_1F_{1,k}[(c_1,k);(c_2,k);w] = \phi[(c_1,k);(c_2,k);w] = \sum_{j=0}^{\infty} \frac{(c_1)_{j,k}}{(c_2)_{j,k}} \frac{w^j}{j!}, \tag{5.1.1}$$

where, $\min\{Re(c_1), Re(c_2)\} > 0$.

In the sequence, in 2016, Mubeen et al. [2] extended the $k$-gamma and $k$-beta functions by using a confluent $k$-hypergeometric function as a kernel and defined as follows:

$$\Gamma_{k,c}^{(c_1,c_2)}(e_1) = \int_0^{\infty} t^{e_1-1} {}_1F_{1,k}\left[(c_1,k);(c_2,k);-\frac{t^k}{k}-\frac{c^k t^{-k}}{k}\right]dt, \tag{5.1.2}$$

where $k > 0$, $\Re(e_1) > 0$, $c \geq 0$, $\Re(c_1) > 0$, $\Re(c_2) > 0$ and ${}_1F_{1,k}(z)$ is defined in (5.1.1) and

$$\mathfrak{B}_{k,c}^{(c_1,c_2)}(e_1,e_2) = \frac{1}{k}\int_0^1 t^{\frac{e_1}{k}-1}(1-t)^{\frac{e_2}{k}-1} {}_1F_{1,k}\left[(c_1,k);(c_2,k);\frac{-c^k}{kt(1-t)}\right]dt, \tag{5.1.3}$$

where $k > 0$, $\min\{\Re(e_1), \Re(e_2)\} > 0$, $c \geq 0$, $\Re(c_1) > 0$, $\Re(c_2) > 0$ and ${}_1F_{1,k}(z)$ is defined in (5.1.1).

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Similarly, for $k>0$, $G.H.F.^{(k)}$ is defined as [3]:

$$ {}_2F_{1,k}[(g_0,k),(g_1,k);(g_2,k);w]=\sum_{j=0}^{\infty}\frac{(g_0)_{j,k}(g_1)_{j,k}}{(g_2)_{j,k}}\frac{w^j}{j!}, \tag{5.1.4}$$

where, $Re(g_0)>0$, $\min\{Re(g_1), Re(g_2)\}>0$, and $|w|<\frac{1}{k}$ and $(g_0)_{j,k}$ is the $P.S.^{(k)}$.

Let $k>0$, $x,y\in C$ , $c_0,c_1,c_2,c_3,c_4\in C$ , and $m,n\in N$.

The Appell $k$-functions are defined by [5]

$$F_{1,k}[(c_0,k),(c_1,k),(c_2,k);(c_3,k);x,y]=\sum_{m,n=0}^{\infty}\frac{(c_0)_{m+n,k}(c_1)_{m,k}(c_2)_{n,k}}{(c_3)_{m+n,k}}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{5.1.5}$$

$$\begin{aligned}&F_{2,k}[(c_0,k),(c_1,k),(c_2,k);(c_3,k),(c_4,k);x,y]\\&=\sum_{m,n=0}^{\infty}\frac{(c_0)_{m+n,k}(c_1)_{m,k}(c_2)_{n,k}}{(c_3)_{m,k}(c_4)_{n,k}}\frac{x^m}{m!}\frac{y^n}{n!},\end{aligned} \tag{5.1.6}$$

where $c_3,c_4\neq 0,-1,-2,\ldots$ and $|x|<\frac{1}{k}$, $|y|<\frac{1}{k}$.

## 5.2 Main results

In this section, we introduce the extensions of the $k$-hypergeometric function and the Appell $k$-hypergeometric functions $F_{1,k}$ and $F_{2,k}$.

**Definition 5.2.1.** The extended $k$-hypergeometric function is defined as follows:

$$\begin{aligned}&F_{k,c}^{(c_1,c_2)}[(q_1,k),(q_2,k);(q_3,k);w]\\&=\sum_{n=0}^{\infty}\frac{(q_1)_{n,k}(q_2)_{n,k}}{(q_2-rk)_{n,k}}\frac{\mathfrak{B}_{k,c}^{(c_1,c_2)}(q_2-rk+nk,q_3-q_2+rk)}{B_k(q_2-rk,q_3-q_2+rk)}\frac{w^n}{n!},\end{aligned} \tag{5.2.1}$$

where $k>0$, $r\in\mathbb{N}$, $\Re(q_3)>\Re(q_2)>r$, $\Re(c_1)>0$, $\Re(c_2)>0$, and for all $|w|<\frac{1}{k}$ and $\mathfrak{B}_{k,c}^{(c_1,c_2)}(.,.)$ is the extended $k$-beta function defind in (5.1.1).

**Definition 5.2.2.** The extended Appell $k$-hypergeometric function $F_{1,k}$ is defined as follows:

$$\begin{aligned}F_{1,k,c}^{(c_1,c_2)}[(q_1,k),(q_2,k),(q_3,k);(q_4,k);x,y]&=\sum_{n,u=0}^{\infty}\frac{(q_1)_{n+u,k}(q_2)_{n,k}(q_3)_{u,k}}{(q_1-rk)_{n+u,k}}\\&\quad\frac{\mathfrak{B}_{k,c}^{(c_1,c_2)}(q_1-rk+nk+uk,q_4-q_1+rk)}{B_k(q_1-rk,q_4-q_1+rk)}\frac{x^n}{n!}\frac{y^u}{u!},\end{aligned} \tag{5.2.2}$$

where $k > 0$, $r \in \mathbb{N}$, $\Re(q_4) > \Re(q_1) > r$, $\Re(q_3) > 0$, $\Re(c_1) > 0$, $\Re(c_2) > 0$, and for all $|x| < \frac{1}{k}$, $|y| < \frac{1}{k}$ and $\mathfrak{B}_{k,c}^{(c_1,c_2)}(q_1, q_2)$ is the extended $k$-beta function defined in (5.1.1).

**Definition 5.2.3.** The extended Appell $k$-hypergeometric function $F_{2,k}$ is defined as follows:

$$F_{2,k,c}^{(c_1,c_2)}[(q_1,k),(q_2,k),(q_3,k);(q_4,k),(q_5,k);x,y]$$
$$= \sum_{n,u=0}^{\infty} \frac{(q_1)_{n+u,k}(q_2)_{n,k}(q_3)_{u,k}}{(q_2-rk)_{n,k}(q_3-rk)_{u,k}}$$
$$\frac{\mathfrak{B}_{k,c}^{(c_1,c_2)}(q_2-rk+nk, q_4-q_2+rk)}{B_k(q_2-rk, q_4-q_2+rk)} \frac{\mathfrak{B}_{k,c}^{(c_1,c_2)}(q_3-rk+uk, q_5-q_3+rk)}{B_k(q_3-rk, q_5-q_3+rk)} \frac{x^n}{n!} \frac{y^u}{u!}, \tag{5.2.3}$$

where $k > 0$, $r \in \mathbb{N}$, $\Re(q_4) > \Re(q_2) > r$, $\Re(q_5) > \Re(q_3) > r$, $\Re(c_1) > 0$, $\Re(c_2) > 0$, and for all $|x| + |y| < \frac{1}{k}$ and $\mathfrak{B}_{k,c}^{(c_1,c_2)}(.,.)$ is the extended $k$-beta function in (5.1.1).

**Theorem 5.2.4.** *The following integral representations are valid:*

$$F_{k,c}^{(c_1,c_2)}[(q_1,k),(q_2,k);(q_3,k);w] = \frac{1}{kB_k(q_2-rk, q_3-q_2+rk)}$$
$$\int_0^1 t^{\frac{q_2-rk}{k}-1}(1-t)^{\frac{q_3-q_2+rk}{k}-1}$$
$${}_1F_{1,k}\left[(c_1,k);(c_2,k);\frac{-c^k}{kt(1-t)}\right] {}_2F_{1,k}[(q_1,k),(q_2,k);(q_2-rk,k);wt]dt \tag{5.2.4}$$

$$F_{1,k,c}^{(c_1,c_2)}[(q_1,k),(q_2,k),(q_3,k);(q_4,k);x,y] = \frac{1}{kB_k(q_1-rk, q_4-q_1+rk)}$$
$$\int_0^1 t^{\frac{q_1-rk}{k}-1}(1-t)^{\frac{q_4-q_1+rk}{k}-1}$$
$${}_1F_{1,k}\left[(c_1,k);(c_2,k);\frac{-c^k}{kt(1-t)}\right]$$
$$F_{1,k}[(q_1,k),(q_2,k),(q_3,k);(q_1-rk,k);xt,yt]dt \tag{5.2.5}$$

$$F_{2,k,c}^{(c_1,c_2)}[(q_1,k),(q_2,k),(q_3,k);(q_4,k),(q_5,k);x,y]$$
$$= \frac{1}{k^2 B_k(q_2-rk, q_4-q_2+rk)B_k(q_3-rk, q_5-q_3+rk)}$$
$$\int_0^1\int_0^1 t^{\frac{q_2-rk}{k}-1}u^{\frac{q_3-rk}{k}-1}(1-t)^{\frac{q_4-q_2+rk}{k}-1}(1-u)^{\frac{q_5-q_3+rk}{k}-1}$$
$${}_1F_{1,k}\left[(c_1,k);(c_2,k);\frac{-c^k}{kt(1-t)}\right]$$

$$
{}_1F_{1,k}\left[(c_1,k);(c_2,k);\frac{-c^k}{ku(1-u)}\right]
F_{2,k}[(q_1,k),(q_2,k),(q_3,k);(q_2-rk,k),(q_3-rk,k);x,y]dtdu. \tag{5.2.6}
$$

*where,* $k>0$*,* $Re(q)>0$*,* $Re(q_1)>0$*,* $Re(q_2)>0$ *and for all* $|w|<\frac{1}{k}$*,* $|x|<\frac{1}{k}$*,* $|y|<\frac{1}{k}$ *and* $|x|+|y|<\frac{1}{k}$*,* $F_{1,k}$ *,* $F_{2,k}$ *are the Appell k-functions defined in (5.2.1) and (5.2.2), respectively.*

*Proof.* To prove (5.2.4), we start from Definition 5.2.1, we have:

$$
F_{k,c}^{(c_1,c_2)}[(q_1,k),(q_2,k);(q_3,k);w]
= \sum_{n=0}^{\infty}\frac{(q_1)_{n,k}(q_2)_{n,k}}{(q_2-rk)_{n,k}}\frac{\mathfrak{B}_{k,c}^{(c_1,c_2)}(q_2-rk+nk,q_3-q_2+rk)}{B_k(q_2-rk,q_3-q_2+rk)}\frac{w^n}{n!}.
$$

By using the properties of the $k$-beta function and the definition of the extended $k$-beta function (5.1.3), we have:

$$
=\sum_{n=0}^{\infty}\frac{(q_1)_{n,k}(q_2)_{n,k}}{(q_2-rk)_{n,k}}\frac{1}{B_k(q_2-rk,q_3-q_2+rk)}
\left(\frac{1}{k}\int_0^1 t^{\frac{q_2-rk+nk}{k}-1}(1-t)^{\frac{q_3-q_2+rk}{k}-1}{}_1F_{1,k}\left[(c_1,k);(c_2,k);\frac{-c^k}{kt(1-t)}\right]dt\right)\frac{w^n}{n!}.
$$

By interchanging the order of summation and integration in the above, as convergence is uniform, we obtain:

$$
=\frac{1}{kB_k(q_2-rk,q_3-q_2+rk)}
\int_0^1 t^{\frac{q_2-rk}{k}-1}(1-t)^{\frac{q_3-q_2+rk}{k}-1}{}_1F_{1,k}\left[(c_1,k);(c_2,k);\frac{-c^k}{kt(1-t)}\right]
\sum_{n=0}^{\infty}\frac{(q_1)_{n,k}(q_2)_{n,k}}{(q_2-rk)_{n,k}}\frac{(wt)^n}{n!}dt.
$$

After applying (5.1.4) in the above equations and a little simplification, we get

$$
=\frac{1}{kB_k(q_2-rk,q_3-q_2+rk)}\int_0^1 t^{\frac{q_2-rk}{k}-1}(1-t)^{\frac{q_3-q_2+rk}{k}-1}
{}_1F_{1,k}\left[(c_1,k);(c_2,k);\frac{-c^k}{kt(1-t)}\right]{}_2F_{1,k}[(q_1,k),(q_2,k);(q_2-rk,k);wt]dt.
$$

This completes the proof of (5.2.4).

To prove (5.2.5), we start from Definition 5.2.2, we have:

$$F_{1,k,c}^{(c_1,c_2)}[(q_1,k),(q_2,k),(q_3,k);(p_4,k);x,y]=\sum_{n,u=0}^{\infty}\frac{(q_1)_{n+u,k}(q_2)_{n,k}(q_3)_{u,k}}{(q_1-rk)_{n+u,k}}$$
$$\frac{\mathfrak{B}_{k,c}^{(c_1,c_2)}(q_1-rk+nk+uk,q_4-q_1+rk)}{B_k(q_1-rk,q_4-q_1+rk)}\frac{x^n}{n!}\frac{y^u}{u!}.$$

Using the properties of the $k$-beta function and the definition of the extended $k$-beta function (5.1.3), we have:

$$=\sum_{n,u=0}^{\infty}\frac{(q_1)_{n+u,k}(q_2)_{n,k}(q_3)_{u,k}}{(q_1-rk)_{n+u,k}}\frac{1}{B_k(q_1-rk,q_4-q_1+rk)}$$
$$\left(\frac{1}{k}\int_0^1 t^{\frac{q_1-rk+nk+uk}{k}-1}(1-t)^{\frac{q_4-q_1+rk}{k}-1}{}_1F_{1,k}\left[(c_1,k);(c_2,k);\frac{-c^k}{kt(1-t)}\right]dt\right)\frac{x^n}{n!}\frac{y^u}{u!}.$$

By interchanging the order of summation and integration in the above, as convergence is uniform, we obtain:

$$=\frac{1}{kB_k(q_1-rk,q_4-q_1+rk)}\int_0^1 t^{\frac{q_1-rk}{k}-1}(1-t)^{\frac{q_4-q_1+rk}{k}-1}$$
$${}_1F_{1,k}\left[(c_1,k);(c_2,k);\frac{-c^k}{kt(1-t)}\right]\sum_{n,u=0}^{\infty}\frac{(q_1)_{n+u,k}(q_2)_{n,k}(q_3)_{u,k}}{(q_1-rk)_{n+u,k}}\frac{(xt)^n}{n!}\frac{(yt)^u}{u!}dt.$$

Applying (5.1.5), we have

$$=\frac{1}{kB_k(q_1-rk,q_4-q_1+rk)}\int_0^1 t^{\frac{q_1-rk}{k}-1}(1-t)^{\frac{q_4-q_1+rk}{k}-1}$$
$${}_1F_{1,k}\left[(c_1,k);(c_2,k);\frac{-c^k}{kt(1-t)}\right]$$
$$F_{1,k}[(q_1,k),(q_2,k),(q_3,k);(q_1-rk,k);xt,yt]dt.$$

This completes the proof of (5.2.5).

To prove (5.2.6), we start from the R.H.S. of (5.2.3), using (5.1.3) and by interchanging the order of summation and integration (as convergence is uniform), we have:

$$=\frac{1}{k^2B_k(q_2-rk,q_4-q_2+rk)B_k(q_3-rk,q_5-q_3+rk)}$$
$$\int_0^1 t^{\frac{q_2-rk+nk}{k}-1}u^{\frac{q_3-rk+uk}{k}-1}(1-t)^{\frac{q_4-q_2+rk}{k}-1}(1-u)^{\frac{q_5-q_3+rk}{k}-1}$$

$$ {}_1F_{1,k}\left[(c_1,k);(c_2,k);\frac{-c^k}{kt(1-t)}\right] $$
$$ {}_1F_{1,k}\left[(c_1,k);(c_2,k);\frac{-c^k}{ku(1-u)}\right]\sum_{n,u=0}^{\infty}\frac{(q_1)_{n+u,k}(q_2)_{n,k}(q_3)_{u,k}}{(q_2-rk)_{n,k}(q_3-rk)_{u,k}}\frac{(xt)^n}{n!}\frac{(yt)^u}{u!}dtdu. $$

Applying (5.2.3) and after a little simplification, we have

$$ =\frac{1}{k^2B_k(q_2-rk,q_4-q_2+rk)B_k(q_3-rk,q_5-q_3+rk)} $$
$$ \int_0^1\int_0^1 t^{\frac{q_2-rk}{k}-1}u^{\frac{q_3-rk}{k}-1}(1-t)^{\frac{q_4-q_2+rk}{k}-1}(1-u)^{\frac{q_5-q_3+rk}{k}-1} $$
$$ {}_1F_{1,k}\left[(c_1,k);(c_2,k);\frac{-c^k}{kt(1-t)}\right] $$
$$ {}_1F_{1,k}\left[(c_1,k);(c_2,k);\frac{-c^k}{ku(1-u)}\right] $$
$$ F_{2,k,c}^{(c_1,c_2)}[(q_1,k),(q_2,k),(q_3,k);(q_2-rk,k),(q_3-rk,k);xt,yt]dtdu. $$

This completes the proof of (5.2.6). □

## 5.3 Extension of the Caputo $k$-fractional derivative operator

In recent years, fractional calculus has become an indispensable tool for the analysis of nonlocal and memory-dependent phenomena. The Caputo fractional derivative is particularly appealing due to its well-posed initial value formulation. Parallel to this, the development of $k$-fractional calculus has led to a unified framework encompassing various fractional operators. The present extension of the Caputo $k$-fractional derivative operator introduces a more general and flexible structure that subsumes several existing operators as particular cases. This extension facilitates deeper analytical investigations and provides an effective framework for the formulation and solution of generalized fractional differential equations arising in applied mathematics and engineering sciences.

We start from the classical Caputo fractional derivative which is defined by:

$$ D^w g(x)=\frac{1}{\Gamma(r-w)}\int_0^x (x-t)^{r-w-1}\frac{d^r}{dt^r}g(t)dt, \tag{5.3.1} $$

where $r-1<\Re(w)<r$, $n\in\mathbb{N}$.

The extended Caputo fractional derivative is defined by:

$$ D_x^{w,c}g(x)=\frac{1}{\Gamma(r-w)}\int_0^x (x-t)^{r-w-1}e^{\left(\frac{-cx^2}{t(x-t)}\right)}\frac{d^r}{dt^r}g(t)dt, \tag{5.3.2} $$

where $\Re(c)>0$ and $r-1<\Re(w)<r$, $n\in\mathbb{N}$.

In the case $c = 0$, the extended Caputo fractional derivative reduces to the classical Caputo fractional derivative:

$D_x^{w,0} g(x) = D^w g(x)$.

The Caputo $k$-fractional derivative is defined by [1]:

$$D_k^w g(x) = \frac{1}{k\Gamma_k(r - \frac{w}{k})} \int_0^x (x-t)^{r-\frac{w}{k}-1} \frac{d^r}{dt^r} g(t)dt, \tag{5.3.3}$$

where $r - 1 < \Re(w) < r$, $n \in \mathbb{N}$.

In the sequel, in 2024, Laxmi et al. [4] introduced an extension of the Caputo $k$-fractional derivative by use of the 2-parameter $k$-Mittag-Leffler function as kernel which is defined as follows:

$$D_{k,c,x}^{w,c_1,c_2} g(x) = \frac{1}{k\Gamma_k(r - \frac{w}{k})} \int_0^x (x-t)^{r-\frac{w}{k}-1} E_{k,c}^{c_1,c_2}\left(\frac{-c^k x^2}{kt(x-t)}\right) \frac{d^r}{dt^r} g(t)dt, \tag{5.3.4}$$

where $\Re(s) > 0$ and $r - 1 < \Re(\nu) < r$, $n \in \mathbb{N}$.

Here, inspired and motivated by all the above generalizations and extensions of Caputo fractional operators [12–15], we introduce an extension of the Caputo $k$-fractional derivative by use of the confluent $k$-hypergeometric function given by (5.1.1).

**Definition 5.3.1.** The extended Caputo $k$-fractional derivative is defined by:

$$\begin{aligned} D_{k,c,x}^{w,c_1,c_2} g(x) = \frac{1}{k\Gamma_k(r - \frac{w}{k})} \int_0^x (x-t)^{r-\frac{w}{k}-1} \\ {}_1F_{1,k}\left[(c_1,k);(c_2,k);\left(\frac{-c^k x^2}{kt(x-t)}\right)\right] \frac{d^r}{dt^r} g(t)dt, \end{aligned} \tag{5.3.5}$$

where $k > 0$, $\Re(c) > 0$, $\Re(c_1) > 0$, $\Re(c_2) > 0$, and $r - 1 < \Re(w) < r$, $n \in \mathbb{N}$.

**Theorem 5.3.2.** *Let $\Re(c) > 0$ and $r - 1 < \Re(w) < r$, then*

$$D_{k,c,x}^{w,c_1,c_2}(t^{\frac{\alpha_1}{k}}) = \frac{x^{\frac{\alpha_1 - w}{k}}}{k^r} \frac{\Gamma_k(rk - w)\Gamma_k(\alpha_1 + k)}{\Gamma_k(r - \frac{w}{k})\Gamma_k(\alpha_1 - w + k)} \frac{B_{k,c}^{(c_1,c_2)}(\alpha_1 - rk + k, rk - w)}{B_k(\alpha_1 - rk + k, rk - w)}. \tag{5.3.6}$$

*Proof.* Applying Definition 5.3.1 to the left-hand side of (5.3.6), we have

$$\begin{aligned} D_{k,c,x}^{w,c_1,c_2}(t^{\frac{\alpha_1}{k}}) = \frac{1}{k\Gamma_k(r - \frac{w}{k})} \int_0^x (x-t)^{r-\frac{w}{k}-1} \\ {}_1F_{1,k}\left[(c_1,k);(c_2,k);\left(\frac{-c^k x^2}{kt(x-t)}\right)\right] \frac{d^r}{dt^r} t^{\frac{\alpha_1}{k}} dt \end{aligned}$$

$$D^{w,c_1,c_2}_{k,c,x}(t^{\frac{\alpha_1}{k}}) = \frac{1}{k\Gamma_k(r-\frac{w}{k})}\int_0^x (x-t)^{r-\frac{w}{k}-1}\,{}_1F_{1,k}\left[(c_1,k);(c_2,k);\left(\frac{-c^k x^2}{kt(x-t)}\right)\right]$$
$$\left(\frac{\alpha_1}{k}\right)\left(\frac{\alpha_1}{k}-1\right)\left(\frac{\alpha_1}{k}-2\right)\ldots\left(\frac{\alpha_1}{k}-r+1\right)t^{\frac{\alpha_1}{k}-r}dt$$
$$D^{w,c_1,c_2}_{k,c,x}(t^{\frac{\alpha_1}{k}}) = \frac{1}{k\Gamma_k(r-\frac{w}{k})}\frac{\Gamma(\frac{\alpha_1}{k}+1)}{\Gamma(\frac{\alpha_1}{k}-r+1)}\int_0^x (x-t)^{\frac{rk-w}{k}-1}(t)^{\frac{\alpha_1-rk}{k}}$$
$${}_1F_{1,k}\left[(c_1,k);(c_2,k);\left(\frac{-c^k x^2}{kt(x-t)}\right)\right]dt.$$

Putting $t = xu$ then $dt = xdu$ into the above equation, we have:

$$D^{w,c_1,c_2}_{k,c,x}(t^{\frac{\alpha_1}{k}}) = \frac{1}{k\Gamma_k(r-\frac{w}{k})}\frac{\Gamma(\frac{\alpha_1}{k}+1)}{\Gamma(\frac{\alpha_1}{k}-r+1)}\int_0^1 (x-xu)^{\frac{rk-w}{k}-1}(xu)^{\frac{\alpha_1-rk+k}{k}-1}$$
$${}_1F_{1,k}\left[(c_1,k);(c_2,k);\left(\frac{-c^k u^2}{ku(1-u)}\right)\right]xdu.$$

By using the result in the above equation, $\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k})$, we have:

$$D^{w,c_1,c_2}_{k,c,x}(t^{\frac{\alpha_1}{k}}) = \frac{1}{k\Gamma_k(r-\frac{w}{k})}\frac{\Gamma_k(\alpha_1+k)k^{\frac{\alpha_1-rk+k}{k}-1}}{\Gamma_k(\alpha_1-rk+k)k^{\frac{\alpha_1+k}{k}-1}}x^{\frac{\alpha_1-w}{k}}$$
$$\int_0^1 (1-u)^{\frac{rk-w}{k}-1}(u)^{\frac{\alpha_1-rk+k}{k}-1}\,{}_1F_{1,k}\left[(c_1,k);(c_2,k);\left(\frac{-c^k u^2}{ku(1-u)}\right)\right]du.$$

Applying the definition of the $k$-beta function from (5.1.3), we have:

$$D^{w,c_1,c_2}_{k,c,x}(t^{\frac{\alpha_1}{k}}) = k^{-r}\frac{\Gamma_k(rk-w)\Gamma_k(\alpha_1+k)}{\Gamma_k(r-\frac{w}{k})\Gamma_k(\alpha_1-w+k)}\frac{\Gamma_k(\alpha_1-w+k)}{\Gamma_k(\alpha_1-rk+k)\Gamma_k(rk-w)}$$
$$x^{\frac{\alpha_1-w}{k}}B^{(c_1,c_2)}_{k,c}(\alpha_1-rk+k, rk-w), \tag{5.3.7}$$

$$D^{w,c_1,c_2}_{k,c,x}(t^{\frac{\alpha_1}{k}}) = \frac{x^{\frac{\alpha_1-w}{k}}}{k^r}\frac{\Gamma_k(rk-w)\Gamma_k(\alpha_1+k)}{\Gamma_k(r-\frac{w}{k})\Gamma_k(\alpha_1-w+k)}\frac{B^{(c_1,c_2)}_{k,c}(\alpha_1-rk+k, rk-w)}{B_k(\alpha_1-rk+k, rk-w)}.$$

It completes the proof of Theorem 5.3.2. □

**Theorem 5.3.3.** *Let $\Re(w) > 0$ and suppose that the function $g(x)$ is analytic at the origin with its Maclaurian expansion given by $g(x) = \sum_{n=0}^{\infty} b_n x^n$, where $|x| < \rho$ for some $n \in \mathbb{R}^+$, then*

$$D^{w,c_1,c_2}_{k,c,x}g(x) = D^{w,c_1,c_2}_{k,c,x}\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} b_n D^{w,c_1,c_2}_{k,c,x}(x^n). \tag{5.3.8}$$

*Proof.* Applying Definition 5.3.1 to the left-hand side of (5.3.6), and using the power-series expansion of $g(x)$, we have

$$D_{k,c,x}^{w,c_1,c_2}(g(x)) = \frac{1}{k\Gamma_k(r-\frac{w}{k})}\int_0^x (x-t)^{r-\frac{w}{k}-1} \\ {}_1F_{1,k}\left[(c_1,k);(c_2,k);\left(\frac{-c^k x^2}{kt(x-t)}\right)\right]\sum_{n=0}^{\infty} b_n \frac{d^r}{dt^r}t^n dt.$$

Since the power series converges uniformly and the integral converges absolutely, then the order of the integration and the summation can be changed, and we have:

$$D_{k,c,x}^{w,c_1,c_2}(g(x)) = \sum_{n=0}^{\infty} b_n\left(\frac{1}{k\Gamma_k(r-\frac{w}{k})}\int_0^x (x-t)^{r-\frac{w}{k}-1} \\ {}_1F_{1,k}\left[(c_1,k);(c_2,k);\left(\frac{-c^k x^2}{kt(x-t)}\right)\right]\frac{d^r}{dt^r}t^n dt\right).$$

Applying Definition 5.3.1, we have:

$$D_{k,c,x}^{w,c_1,c_2}(g(x)) = \sum_{n=0}^{\infty} b_n D_{k,c,x}^{w,c_1,c_2}(x^n).$$

This proves the above theorem. □

**Theorem 5.3.4.** *Let $g(x)$ be the analytic function on the disk $|x| < \rho$ and have a power-series expansion given by $g(x) = \sum_{n=0}^{\infty} b_n x^n$, then*

$$D_{k,c,x}^{w,c_1,c_2}(x^{\frac{\alpha_1}{k}-1}g(x)) = \frac{\Gamma_k(rk-w)\Gamma_k(\alpha_1)}{\Gamma_k(r-\frac{w}{k})\Gamma_k(\alpha_1-w)}\frac{x^{\frac{\alpha_1-w-k}{k}}}{k^r}\sum_{n=0}^{\infty} b_n\frac{(\alpha_1)_{n,k}}{(\alpha_1-w)_{n,k}} \\ \frac{B_{k,c}^{(c_1,c_2)}(\alpha_1+nk-w-rk, rk-w)}{B_k(\alpha_1+nk-w-rk, rk-w)}x^n, \tag{5.3.9}$$

*where $k > 0$, $\Re(c) > 0$, $\Re(c_1) > 0$, $\Re(c_2) > 0$.*

*Proof.* Applying Theorem 5.3.3 and using the power-series expansion of $g(x)$, we have from the left-hand side of (5.3.8):

$$D_{k,c,x}^{w,c_1,c_2}(x^{\frac{\alpha_1}{k}-1}g(x)) = \sum_{n=0}^{\infty} b_n D_{k,c,x}^{w,c_1,c_2}(x^{\frac{\alpha_1}{k}+n-1}).$$

Applying Theorem 5.3.2, we have:

$$D_{k,c,x}^{w,c_1,c_2}(x^{\frac{\alpha_1}{k}-1}g(x)) = \sum_{n=0}^{\infty} b_n \frac{\Gamma_k(rk-w)\Gamma_k(\alpha_1+nk-k+k)}{\Gamma_k(r-\frac{w}{k})\Gamma_k(\alpha_1+nk-k-w+k)} \frac{x^{\frac{(\alpha_1+nk-k-w)}{k}}}{k^r} \frac{B_{k,c}^{(c_1,c_2)}(\alpha_1+nk-k-w-rk+k, rk-w)}{B_k(\alpha_1+nk-k-w-rk+k, rk-w)}$$

$$D_{k,c,x}^{w,c_1,c_2}(x^{\frac{\alpha_1}{k}-1}g(x)) = \sum_{n=0}^{\infty} b_n \frac{\Gamma_k(rk-w)\Gamma_k(\alpha_1+nk)}{\Gamma_k(r-\frac{w}{k})\Gamma_k(\alpha_1+nk-w)} \frac{x^{\frac{(\alpha_1+nk-k-w)}{k}}}{k^r} \frac{B_{k,c}^{(c_1,c_2)}(\alpha_1+nk-w-rk, rk-w)}{B_k(\alpha_1+nk-w-rk, rk-w)}.$$

Applying the result of the $k$-Pochhammer symbol, we have:

$$D_{k,c,x}^{w,c_1,c_2}(x^{\frac{\alpha_1}{k}-1}g(x)) = \frac{\Gamma_k(rk-w)\Gamma_k(\alpha_1)}{\Gamma_k(r-\frac{w}{k})\Gamma_k(\alpha_1-w)} \frac{x^{\frac{\alpha_1-w-k}{k}}}{k^r} \sum_{n=0}^{\infty} b_n \frac{(\alpha_1)_{n,k}}{(\alpha_1-w)_{n,k}} \frac{B_{k,c}^{(c_1,c_2)}(\alpha_1+nk-w-rk, rk-w)}{B_k(\alpha_1+nk-w-rk, rk-w)} x^n.$$

It completes the proof. □

**Theorem 5.3.5.** *Let $r-1 < \Re(\alpha_1-w) < \Re(\alpha_1)$, then*

$$D_{k,c,x}^{\alpha_1-w,c_1,c_2}(x^{\frac{\alpha_1}{k}-1}(1-kx)^{\frac{-\beta_1}{k}}) = \frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1)}{\Gamma_k(r-\frac{\alpha_1-w}{k})\Gamma_k(w)} \frac{x^{\frac{\alpha_1-w-k}{k}}}{k^r} \sum_{n=0}^{\infty} \frac{(\beta_1)_{n,k}}{n!} \frac{(\alpha_1)_{n,k}}{(\alpha_1-rk)_{n,k}} \frac{B_{k,c}^{(c_1,c_2)}(\alpha_1-rk+nk, rk-\alpha_1+w)}{B_k(\alpha_1-rk, rk-\alpha_1+w)} x^n \tag{5.3.10}$$

$$= \frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1)}{\Gamma_k(r-\frac{\alpha_1-w}{k})\Gamma_k(w)} \frac{x^{\frac{\alpha_1-w-k}{k}}}{k^r} F_{k,c}^{(c_1,c_2)}[(\beta_1,k),(\alpha_1,k);(w,k);x], \tag{5.3.11}$$

*where $|x| < \frac{1}{k}$.*

*Proof.* By applying Theorem 5.3.3 and using the power-series expansion of $(1-kx)^{\frac{-\beta_1}{k}}$, we have from left-hand side of (5.3.9):

$$D_{k,c,x}^{\alpha_1-w,c_1,c_2}\left(x^{\frac{\alpha_1}{k}-1}(1-kx)^{\frac{-\beta_1}{k}}\right) = \sum_{n=0}^{\infty} \frac{(\beta_1)_{n,k}}{n!} D_{k,x}^{\alpha_1-w,c}(x^{\frac{\alpha_1}{k}+n-1}).$$

Applying Theorem 5.3.2, we have:

$$= \sum_{n=0}^{\infty} \frac{(\beta_1)_{n,k}}{n!} \frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1+nk-k+k)}{\Gamma_k(r-\frac{\alpha_1-w}{k})\Gamma_k(\alpha_1+nk-k-\alpha_1+w+k)} \frac{x^{\frac{(\alpha_1+nk-k-w)}{k}}}{k^r}$$
$$\frac{B_{k,c}^{(c_1,c_2)}(\alpha_1+nk-k-rk+k, rk-\alpha_1+w)}{B_k(\alpha_1+nk-k-rk+k, rk-\alpha_1+w)}$$
$$= \sum_{n=0}^{\infty} \frac{(\beta_1)_{n,k}}{n!} \frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1+nk)}{\Gamma_k(r-\frac{\alpha_1-w}{k})\Gamma_k(nk+w)} \frac{x^{\frac{(\alpha_1+nk-k-w)}{k}}}{k^r}$$
$$\frac{B_{k,c}^{(c_1,c_2)}(\alpha_1+nk-rk, rk-\alpha_1+w)}{B_k(\alpha_1+nk-rk, rk-\alpha_1+w)}.$$

By applying the definition of the $k$-Pochhammer symbol, we have:

$$= \frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1)}{\Gamma_k(r-\frac{\alpha_1-w}{k})\Gamma_k(w)} \frac{x^{\frac{\alpha_1-w-k}{k}}}{k^r} \sum_{n=0}^{\infty} \frac{(\beta_1)_{n,k}(\alpha_1)_{n,k}}{(w)_{n,k}}$$
$$\frac{B_{k,c}^{(c_1,c_2)}(\alpha_1-rk+nk, rk-\alpha_1+w)}{B_k(\alpha_1-rk+nk, rk-\alpha_1+w)} \frac{x^n}{n!}$$
$$= \frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1)}{\Gamma_k(r-\frac{\alpha_1-w}{k})\Gamma_k(w)} \frac{x^{\frac{\alpha_1-w-k}{k}}}{k^r} \sum_{n=0}^{\infty} \frac{(\beta_1)_{n,k}(\alpha_1)_{n,k}}{(\alpha_1-rk)_{n,k}}$$
$$\frac{B_{k,c}^{(c_1,c_2)}(\alpha_1-rk+nk, rk-\alpha_1+w)}{B_k(\alpha_1-rk, rk-\alpha_1+w)} \frac{x^n}{n!}.$$

Finally by applying Definition 5.2.1, we have:

$$= \frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1)}{\Gamma_k(r-\frac{\alpha_1-w}{k})\Gamma_k(w)} \frac{x^{\frac{\alpha_1-w-k}{k}}}{k^r} F_{k,c}^{(c_1,c_2)}[(\beta_1,k),(\alpha_1,k);(w,k);x].$$

It completes the proof of Theorem 5.3.5. □

**Theorem 5.3.6.** *Let* $r-1 < \Re(\alpha_1-w) < \Re(\alpha_1)$, *then*

$$D_{k,c,x}^{\alpha_1-w,c_1,c_2}(x^{\frac{\alpha_1}{k}-1}(1-kax)^{\frac{-\beta_1}{k}}(1-kbx)^{\frac{-\gamma_1}{k}}) = \frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1)}{\Gamma_k(r-\frac{\alpha_1-w}{k})\Gamma_k(w)} \frac{x^{\frac{\alpha_1-w-k}{k}}}{k^r}$$
$$\sum_{n=0}^{\infty} \frac{(\beta_1)_{m,k}(\gamma_1)_{n,k}(\alpha_1)_{m+n,k}}{(\alpha_1-rk)_{m+n,k}}$$
$$\frac{B_{k,c}^{(c_1,c_2)}(\alpha_1+mk+nk-rk, rk-\alpha_1+w)}{B_k(\alpha_1-rk, rk-\alpha_1+w)} \frac{(ax)^m}{m!} \frac{(bx)^n}{n!} \tag{5.3.12}$$

$$= \frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1)}{\Gamma_k(r-\frac{\alpha_1-w}{k})\Gamma_k(w)} \frac{x^{\frac{\alpha_1-w-k}{k}}}{k^r} F_{1,k,c}^{(c_1,c_2)}[(\alpha_1,k),(\beta_1,k),(\gamma_1,k);(w,k);ax,bx]. \tag{5.3.13}$$

*Proof.* Applying Theorem 5.3.3 and using the power-series expansion of $(1-kx)^{\frac{-\beta_1}{k}}$, we have:

$$D_{k,c,x}^{\alpha_1-w,c_1,c_2}\left(x^{\frac{\alpha_1}{k}-1}(1-kax)^{\frac{-\beta_1}{k}}(1-kbx)^{\frac{-\gamma_1}{k}}\right)$$
$$= \sum_{m,n=0}^{\infty} \frac{(\beta_1)_{m,k}}{m!}\frac{(\gamma_1)_{n,k}}{n!} a^m b^n D_{k,c,x}^{\alpha_1-w,c_1,c_2}(x^{\frac{\alpha_1}{k}+m+n-1}).$$

Using Theorem 5.3.2 and after a little simplification, we get:

$$= \sum_{m,n=0}^{\infty} \frac{(\beta_1)_{m,k}}{m!}\frac{(\gamma_1)_{n,k}}{n!} a^m b^n \frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1+mk+nk-k+k)}{\Gamma_k(r-\frac{\alpha_1-w}{k})\Gamma_k(\alpha_1+mk+nk-k-\alpha_1+w+k)}$$
$$\frac{x^{\frac{(\alpha_1+mk+nk-k-w)}{k}}}{k^r}$$
$$\frac{B_{k,c}^{(c_1,c_2)}(\alpha_1+mk+nk-k-rk+k, rk-\alpha_1+w)}{B_k(\alpha_1+mk+nk-k-rk+k, rk-\alpha_1+w)}$$
$$= \sum_{m,n=0}^{\infty} \frac{(\beta_1)_{m,k}}{m!}\frac{(\gamma_1)_{n,k}}{n!} a^m b^n \frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1+mk+nk)}{\Gamma_k(r-\frac{\alpha_1-w}{k})\Gamma_k(mk+nk+w)} \frac{x^{\frac{(\alpha_1+mk+nk-k-w)}{k}}}{k^r}$$
$$\frac{B_{k,c}^{(c_1,c_2)}(\alpha_1+mk+nk-rk, rk-\alpha_1+w)}{B_k(\alpha_1+mk+nk-rk, rk-\alpha_1+w)}$$
$$= \sum_{m,n=0}^{\infty} \frac{(\beta_1)_{m,k}}{m!}\frac{(\gamma_1)_{n,k}}{n!} a^m b^n \frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1+mk+nk)}{\Gamma_k(r-\frac{\alpha_1-w}{k})} \frac{x^{\frac{(\alpha_1+mk+nk-k-w)}{k}}}{k^r}$$
$$\frac{B_{k,c}^{(c_1,c_2)}(\alpha_1+mk+nk-rk, rk-\alpha_1+w)}{\Gamma_k(\alpha_1+mk+nk-rk)\Gamma_k(rk-\alpha_1+w)}.$$

Applying the definition of the $k$-Pochhammer symbol, we have:

$$= \frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1)}{\Gamma_k(r-\frac{\alpha_1-w}{k})\Gamma_k(w)} \frac{x^{\frac{\alpha_1-w-k}{k}}}{k^r} \sum_{m,n=0}^{\infty} \frac{(\beta_1)_{m,k}(\gamma_1)_{n,k}(\alpha_1)_{m+n,k}}{(\alpha_1-rk)_{m+n,k}}$$
$$\frac{B_{k,c}^{(c_1,c_2)}(\alpha_1+mk+nk-rk, rk-\alpha_1+w)}{B_k(\alpha_1-rk, rk-\alpha_1+w)} \frac{(ax)^m}{m!}\frac{(bx)^n}{n!}.$$

In last by applying Definition 5.2.2, we have:

$$= \frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1)}{\Gamma_k(r-\frac{\alpha_1-w}{k})\Gamma_k(w)} \frac{x^{\frac{\alpha_1-w-k}{k}}}{k^r} F_{1,k,c}^{(c_1,c_2)}[(\alpha_1,k),(\beta_1,k),(\gamma_1,k);(w,k);ax,bx].$$

Hence, the proof is complete. □

**Theorem 5.3.7.** *Let $r-1<\Re(\alpha_1-w)<\Re(\alpha_1)$, then*

$$D_{k,c,x}^{\alpha_1-w,c_1,c_2}\left(x^{\frac{\alpha_1}{k}-1}(1-kx)^{\frac{-\beta_1}{k}} F_{k,c}^{(c_1,c_2)}\left[(\beta_1,k),(\gamma_1,k);(\delta_1,k);\frac{t}{1-kx}\right]\right)=$$
$$\frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1)}{\Gamma_k(r-\frac{\alpha_1-w}{k})\Gamma_k(w)} \frac{x^{\frac{\alpha_1-w-k}{k}}}{k^r} \sum_{n,u=0}^{\infty} \frac{(\alpha_1)_{u,k}(\beta_1)_{n+u,k}(\gamma_1)_{n,k}}{(\alpha_1-rk)_{u,k}(\gamma_1-rk)_{n,k}}$$
$$\frac{B_{k,c}^{(c_1,c_2)}(\alpha_1-rk+uk,rk-\alpha_1+w)}{B_k(\alpha_1-rk,rk-\alpha_1+w)} \frac{B_{k,c}^{(c_1,c_2)}(\gamma_1-rk+nk,\delta_1-\gamma_1+rk)}{B_k(\gamma_1-rk,\delta_1-\gamma_1+rk)} \frac{t^n}{n!}\frac{x^u}{u!} \tag{5.3.14}$$

$$= \frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1)}{\Gamma_k(r-\frac{\alpha_1-w}{k})\Gamma_k(w)} \frac{x^{\frac{\alpha_1-w-k}{k}}}{k^r}$$
$$F_{2,k,c}^{(c_1,c_2)}[(\beta_1,k),(\gamma_1,k),(\alpha_1,k);(\delta_1,k),(w,k);x,y]. \tag{5.3.15}$$

*Proof.* Applying Definition 5.2.1, we have from left-hand side from (5.3.13):

$$D_{k,c,x}^{\alpha_1-w,c_1,c_2}\left(x^{\frac{\alpha_1}{k}-1}(1-kx)^{\frac{-\beta_1}{k}} F_{k,c}^{(c_1,c_2)}\left[(\beta_1,k),(\gamma_1,k);(\delta_1,k);\frac{t}{1-kx}\right]\right)=$$
$$D_{k,c,x}^{\alpha_1-w,c_1,c_2}\left(x^{\frac{\alpha_1}{k}-1}(1-kx)^{\frac{-\beta_1}{k}} \sum_{n=0}^{\infty} \frac{(\beta_1)_{n,k}(\gamma_1)_{n,k}}{(\gamma_1-rk)_{n,k}}\right.$$
$$\left.\frac{B_{k,c}^{(c_1,c_2)}(\gamma_1-rk+nk,\delta_1-\gamma_1+rk)}{B_k(\gamma_1-rk,\delta_1-\gamma_1+rk)}\left(\frac{t}{1-kx}\right)^n\frac{1}{n!}\right)$$
$$= D_{k,c,x}^{\alpha_1-w,c_1,c_2}\left(x^{\frac{\alpha_1}{k}-1}(1-kx)^{\frac{-\beta_1}{k}-n} \sum_{n=0}^{\infty} \frac{(\beta_1)_{n,k}(\gamma_1)_{n,k}}{(\gamma_1-rk)_{n,k}}\right.$$
$$\left.\frac{B_{k,c}^{(c_1,c_2)}(\gamma_1-rk+nk,\delta_1-\gamma_1+rk)}{B_k(\gamma_1-rk,\delta_1-\gamma_1+rk)}\frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty} \frac{(\beta_1)_{n,k}(\gamma_1)_{n,k}}{(\gamma_1-rk)_{n,k}} \frac{B_{k,c}^{(c_1,c_2)}(\gamma_1-rk+nk,\delta_1-\gamma_1+rk)}{B_k(\gamma_1-rk,\delta_1-\gamma_1+rk)}\frac{t^n}{n!}$$
$$D_{k,c,x}^{\alpha_1-w,c_1,c_2}\left(x^{\frac{\alpha_1}{k}-1}(1-kx)^{\frac{-(\beta_1+nk)}{k}}\right).$$

Applying Theorem 5.3.5, we have:

$$= \frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1)}{\Gamma_k(r-\frac{\alpha_1-w}{k})\Gamma_k(w)} \frac{x^{\frac{\alpha_1-w-k}{k}}}{k^r} \sum_{n,u=0}^{\infty} \frac{(\alpha_1)_{u,k}(\beta_1)_{n,k}(\beta_1+nk)_{u,k}(\gamma_1)_{n,k}}{(\alpha_1-rk)_{u,k}(\gamma_1-rk)_{n,k}}$$
$$\frac{B_{k,c}^{(c_1,c_2)}(\alpha_1-rk+uk, rk-\alpha_1+w)}{B_k(\alpha_1-rk, rk-\alpha_1+w)}$$
$$\frac{B_{k,c}^{(c_1,c_2)}(\gamma_1-rk+nk, \delta_1-\gamma_1+rk)}{B_k(\gamma_1-rk, \delta_1-\gamma_1+rk)} \frac{t^n}{n!} \frac{x^u}{u!}.$$

Applying the property of the $k$-Pochhammer symbol, we have:

$$= \frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1)}{\Gamma_k(r-\frac{\alpha_1-w}{k})\Gamma_k(w)} \frac{x^{\frac{\alpha_1-w-k}{k}}}{k^r} \sum_{n,u=0}^{\infty} \frac{(\alpha_1)_{u,k}(\beta_1)_{n+u,k}(\gamma_1)_{n,k}}{(\alpha_1-rk)_{u,k}(\gamma_1-rk)_{n,k}}$$
$$\frac{B_{k,c}^{(c_1,c_2)}(\alpha_1-rk+uk, rk-\alpha_1+w)}{B_k(\alpha_1-rk, rk-\alpha_1+w)}$$
$$\frac{B_{k,c}^{(c_1,c_2)}(\gamma_1-rk+nk, \delta_1-\gamma_1+rk)}{B_k(\gamma_1-rk, \delta_1-\gamma_1+rk)} \frac{t^n}{n!} \frac{x^u}{u!}.$$

Applying Definition 5.2.3, we have:

$$= \frac{\Gamma_k(rk-\alpha_1+w)\Gamma_k(\alpha_1)}{\Gamma_k(r-\frac{\alpha_1-w}{k})\Gamma_k(w)} \frac{x^{\frac{\alpha_1-w-k}{k}}}{k^r}$$
$$F_{2,k,c}^{(c_1,c_2)}[(\beta_1,k),(\gamma_1,k),(\alpha_1,k);(\delta_1,k),(w,k);x,y].$$

Therefore, the desired result follows, and the proof is complete. □

## 5.4 Conclusion

We finish our analysis by noting that the conclusions provided in this work may be easily obtained by extending the classical Caputo $k$-fractional derivative operator and a few more special functions. Such findings are novel and extremely valuable for the expansion of other special functions in the field of fractional calculus. We are currently looking for possible applications of these findings in other fields of research.

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CHAPTER

# 6 An extension of $k$-hypergeometric functions

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## 6.1 Introduction and preliminaries

In the field of mathematics, special functions play a significant role. Although it is a vast discipline our focus is mainly on the generalizations of the special functions. In 2007, Diaz and Pariguan established a new concept of the $k$-gamma and $k$-beta functions, and the Pochhammer $k$-symbol [5]. This concept is well known in the literature as generalizations of special functions [1–4,6–9,14–23].

The $k$-gamma function is given in the following manner:

$$\Gamma_k(v_0) = \int_0^{\infty} w^{v_0-1} e^{-w^k k^{-1}} dw \tag{6.1.1}$$

and

$$\Gamma_k(v_0 + k) = v_0 \Gamma_k(v_0), \tag{6.1.2}$$

where $k > 0$ and $\Re(v_0) > 0$.

The Pochhammer $k$-symbol is defined in the following way:

$$(v_0)_{n,k} := \frac{\Gamma_k(v_0 + nk)}{\Gamma_k(v_0)} = \begin{cases} 1 & n = 0;\, v_0 \in \mathbb{C} \setminus \{0\}, \\ v_0(v_0 + k)\cdots(v_0 + (n-1)k) & n \in \mathbb{N};\, v_0 \in \mathbb{C}. \end{cases} \tag{6.1.3}$$

The $k$-beta function is given in the following manner:

$$B_k(v_0, v_1) = \frac{1}{k}\int_0^1 w^{\frac{v_0}{k}-1}(1-w)^{\frac{v_1}{k}-1} dw, \tag{6.1.4}$$

where $\Re(v_0)$ and $\Re(v_1) > 0$.

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The relationship between the $k$-gamma function and the $k$-beta function is:

$$B_k(v_0, v_1) = \frac{\Gamma_k(v_0)\Gamma_k(v_1)}{\Gamma_k(v_0+v_1)}, \tag{6.1.5}$$

where $\Re(v_0)$ and $\Re(v_1) > 0$.

For $k > 0$, the generalized $k$-hypergeometric function is defined as follows [12]:

$${}_pF_{q,k}[(u_1,k),(u_2,k)\ldots(u_p,k);(v_1,k),(v_2,k)\ldots(v_q,k);w] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(u_i)_{n,k}}{\prod_{i=1}^{q}(v_i)_{n,k}} \frac{w^n}{n!}, \tag{6.1.6}$$

where $|w| < \frac{1}{k}$, $\Re(u_i) > 0$, $\Re(v_i) > 0$ and $(v_0)_{n,k}$ is the $k$-Pochhammer symbol.

For $k > 0$, the $k$-Gauss hypergeometric function is defined as [11]:

$${}_2F_{1,k}[(v_0,k),(v_1,k);(v_2,k);w] = \sum_{n=0}^{\infty} \frac{(v_0)_{n,k}(v_1)_{n,k}}{(v_2)_{n,k}} \frac{w^n}{n!}, \tag{6.1.7}$$

where $\Re(v_0) > 0$, $\Re(v_1) > 0$, $\Re(v_2) > 0$, and $|w| < \frac{1}{k}$ and $(v_0)_{n,k}$ is the $k$-Pochhammer symbol.

For $k > 0$, the $k$-confluent hypergeometric function is defined as [11]:

$${}_1F_{1,k}[(v_1,k);(v_2,k);w] = \phi[(v_1,k);(v_2,k);w] = \sum_{n=0}^{\infty} \frac{(v_1)_{n,k}}{(v_2)_{n,k}} \frac{w^n}{n!}, \tag{6.1.8}$$

where $\Re(v_1) > 0$, $\Re(v_2) > 0$ and $(v_0)_{n,k}$ is the $k$-Pochhammer symbol.

Series representations of the $k$-Gauss hypergeometric and $k$-confluent hypergeometric functions are given as follows:

$${}_2F_{1,k}[(v_0,k),(v_1,k);(v_2,k);w] = \sum_{n=0}^{\infty} (v_0)_{n,k} \frac{B_k(v_1+nk, v_2-v_1)}{B_k(v_1, v_2-v_1)} \frac{w^n}{n!} \tag{6.1.9}$$

and

$${}_1F_{1,k}[(v_1,k);(v_2,k);w] = \sum_{n=0}^{\infty} \frac{B_k(v_1+nk, v_2-v_1)}{B_k(v_1, v_2-v_1)} \frac{w^n}{n!}, \tag{6.1.10}$$

where $\Re(v_0) > 0$, $\Re(v_1) > 0$, $\Re(v_2) > 0$, and $|w| < \frac{1}{k}$ and $B_k(v_1, v_2)$ is the $k$-beta function.

Mubeen and Habibullah [13] introduced integral forms of the $k$-hypergeometric and $k$-confluent hypergeometric functions as follows:

$$\begin{aligned} &{}_2F_{1,k}[(v_0,k),(v_1,k);(v_2,k);w] \\ &= \frac{1}{kB_k(v_1, v_2-v_1)} \int_0^1 t^{\frac{v_1}{k}-1}(1-t)^{\frac{v_2-v_1}{k}-1}(1-kwt)^{-\frac{v_0}{k}}\,dt \end{aligned} \tag{6.1.11}$$

and

$$ {}_1F_{1,k}[(v_1,k);(v_2,k);w] = \frac{1}{kB_k(v_1,v_2-v_1)}\int_0^1 t^{\frac{v_1}{k}-1}(1-t)^{\frac{v_2-v_1}{k}-1}e^{wt}\,dt, \tag{6.1.12} $$

where $\Re(v_0) > 0$, $\Re(v_1) > 0$, $\Re(v_2) > 0$, and $|w| < \frac{1}{k}$ and $B_k(v_1, v_2)$ is the $k$-beta function.

Very recently Laxmi et al. extended the $k$-gamma and $k$-beta functions by using the 2-parameter $k$-Mittag-Leffler function as kernel and defined as follows (see, [24]):

$$ \Gamma_{k,s}^{(s_1,s_2)}(v_0) = \int_0^\infty t^{v_0-1}E_{k,s_1,s_2}\left(-\frac{t^k}{k} - \frac{s^k t^{-k}}{k}\right)dt, \tag{6.1.13} $$

where $k > 0$, $\Re(v_0) > 0$, $s \geq 0$, $\Re(s_1) > 0$, $\Re(s_2) > 0$ and $E_{k,s_1,s_2}(z)$ is defined in (6.1.16) and

$$ B_{k,s}^{(s_1,s_2)}(v_0,v_1) = \frac{1}{k}\int_0^1 t^{\frac{v_0}{k}-1}(1-t)^{\frac{v_1}{k}-1}E_{k,s_1,s_2}\left(\frac{-s^k}{kt(1-t)}\right)dt, \tag{6.1.14} $$

where $k > 0$, $\min\{\Re(v_0), \Re(v_1)\} > 0$, $s \geq 0$, $\Re(s_1) > 0$, $\Re(s_2) > 0$ and $E_{k,s_1,s_2}(z)$ is defined in (6.1.16).

To obtain our main result we need to define the $k$-Mittag-Leffler functions [10] as follows:

$$ E_{k,s_1}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma_k(ns_1+1)}, \qquad k > 0, \Re(s_1) \geq 0, z \in C, \tag{6.1.15} $$

$$ E_{k,s_1,s_2}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma_k(ns_1+s_2)}, \qquad k > 0, \Re(s_1) \geq 0, \Re(s_2) \geq 0, z \in C. \tag{6.1.16} $$

**Note.** If we consider variable $s_1 = s_2 = 1$ in Eq. (6.1.16), then we obtain the $k$-exponential function $E_k(z)$.

## 6.2 Main results

Here, inspired and motivated by all the above generalizations and extensions we introduce an extension of $k$-hypergeometric functions by use of the 2-parameter $k$-Mittag-Leffler function.

**Definition 6.2.1.** A new extended $k$-hypergeometric function is defined as follows:

$$ F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w] = \sum_{n=0}^{\infty}\frac{B_{k,s}^{(s_1,s_2)}(v_1+nk,v_2-v_1)}{B_k(v_1,v_2-v_1)}(v_0)_{n,k}\frac{w^n}{n!}, \tag{6.2.1} $$

where $k > 0$, $\Re(v_0) > 0$, $\Re(v_1) > 0$, $\Re(v_2) > 0$, $\Re(s_1) > 0$, $\Re(s_2) > 0$, and $|w| < \frac{1}{k}$ and $B_{k,s}^{(s_1,s_2)}(v_0, v_1)$ is the extended $k$-beta function.

**Definition 6.2.2.** A new extended $k$-confluent hypergeometric function is defined as follows:

$$\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w] = \sum_{n=0}^{\infty} \frac{B_{k,s}^{(s_1,s_2)}(v_1+nk, v_2-v_1)}{B_k(v_1, v_2-v_1)} \frac{w^n}{n!}, \tag{6.2.2}$$

where $k > 0$, $\Re(v_1) > 0$, $\Re(v_2) > 0$, $\Re(s_1) > 0$, $\Re(s_2) > 0$ and $B_{k,s}^{(s_1,s_2)}(v_0, v_1)$ is the extended $k$-beta function.

**Note.** If we consider variable $s_2 = 1$ in Definitions 6.2.1 and 6.2.2, then we obtain other extensions of the $k$-hypergeometric and $k$-confluent hypergeometric functions as follows.

**Definition 6.2.3.** An extension of the $k$-hypergeometric function is defined as follows:

$$F_{k,s}^{(s_1)}[(v_0,k),(v_1,k);(v_2,k);w] = \sum_{n=0}^{\infty} \frac{B_{k,s}^{(s_1)}(v_1+nk, v_2-v_1)}{B_k(v_1, v_2-v_1)} (v_0)_{n,k} \frac{w^n}{n!}, \tag{6.2.3}$$

where $k > 0$, $\Re(v_0) > 0$, $\Re(v_1) > 0$, $\Re(v_2) > 0$, $\Re(s_1) > 0$, and $|w| < \frac{1}{k}$ and $B_{k,s}^{(s_1)}(v_0, v_1)$ is the extended $k$-beta function.

**Definition 6.2.4.** An extension of the $k$-confluent hypergeometric function is defined as follows:

$$\phi_{k,s}^{(s_1)}[(v_1,k);(v_2,k);w] = \sum_{n=0}^{\infty} \frac{B_{k,s}^{(s_1)}(v_1+nk, v_2-v_1)}{B_k(v_1, v_2-v_1)} \frac{w^n}{n!}, \tag{6.2.4}$$

where $k > 0$, $\Re(v_1) > 0$, $\Re(v_2) > 0$, $\Re(s_1) > 0$ and $B_{k,s}^{(s_1)}(v_0, v_1)$ is the extended $k$-beta function.

**Remark.** (i) If we put $s_1 = s_2 = 1$ and $s = 0$ in Eqs. (6.2.1) and (6.2.2), then we obtain $k$-Gauss hypergeometric and $k$-confluent hypergeometric functions (6.1.9) and (6.1.10), respectively:

$$F_{k,0}^{(1,1)}[(v_0,k),(v_1,k);(v_2,k);w] = F_k[(v_0,k),(v_1,k);(v_2,k);w] \tag{6.2.5}$$

and

$$\phi_{k,0}^{(1,1)}[(v_1,k);(v_2,k);w] = \phi_k[(v_1,k);(v_2,k);w]. \tag{6.2.6}$$

(ii) If we put $s_1 = s_2 = 1$, $k = 1$, and $s = 0$ in Eqs. (6.2.1) and (6.2.2), then we obtain the classical Gauss hypergeometric and confluent hypergeometric functions, respectively:

$$F_{1,0}^{(1,1)}[(v_0,k),(v_1,k);(v_2,k);w] = F[v_0,v_1;v_2;w] \tag{6.2.7}$$

and

$$\phi_{1,0}^{(1,1)}[(v_1,k);(v_2,k);w] = \phi[v_1;v_2;w]. \tag{6.2.8}$$

**Theorem 6.2.5.** *The extended k-hypergeometric function has the following integral representations:*

$$\begin{aligned} &F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w] \\ &= \frac{1}{kB_k(v_1,v_2-v_1)} \int_0^1 t^{\frac{v_1}{k}-1}(1-t)^{\frac{v_2-v_1}{k}-1}(1-kwt)^{-\frac{v_0}{k}} \\ &E_{k,s_1,s_2}\left(-s^k(kt(1-t))^{-1}\right)dt \end{aligned} \tag{6.2.9}$$

$$\begin{aligned} &F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w] \\ &= \frac{2}{kB_k(v_1,v_2-v_1)} \int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2v_1}{k}-1}(\sin\theta)^{\frac{2v_2-2v_1}{k}-1}(1-kw\cos^2\theta)^{-\frac{v_0}{k}} \\ &E_{k,s_1,s_2}\left(\frac{-s^k}{k}\sec^2\theta\csc^2\theta\right)d\theta \end{aligned} \tag{6.2.10}$$

$$\begin{aligned} &F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w] \\ &= \frac{z^{1+\frac{v_0-v_2}{k}}}{kB_k(v_1,v_2-v_1)} \int_0^z u^{\frac{v_1}{k}-1}(z-u)^{\frac{v_2-v_1}{k}-1}(z-kwu)^{-\frac{v_0}{k}} \\ &E_{k,s_1,s_2}\left(\frac{-s^k}{k}\frac{z^2}{u(z-u)}\right)du \end{aligned} \tag{6.2.11}$$

$$\begin{aligned} &F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w] \\ &= \frac{2}{kB_k(v_1,v_2-v_1)} \int_0^\infty \left[\frac{(\sinh\theta)^{\frac{2v_1}{k}-1}(\cosh\theta)^{1+\frac{2(v_0-v_2)}{k}}}{\left(\cosh^2\theta-kw\sinh^2\theta\right)^{\frac{v_0}{k}}}\right] \\ &E_{k,s_1,s_2}\left(\frac{-s^k}{k}\cosh^4\theta\,\mathrm{csch}^2\theta\right)d\theta, \end{aligned} \tag{6.2.12}$$

*where* $k > 0$, $\Re(v_0) > 0$, $\Re(v_1) > 0$, $\Re(v_2) > 0$, $\Re(s_1) > 0$, $\Re(s_2) > 0$, *and* $|w| < \frac{1}{k}$ *and* $B_{k,s}^{(s_1,s_2)}(v_0,v_1)$ *is the extended k-beta function.*

*Proof.* To prove (6.2.9), consider the definition of the extended $k$-hypergeometric function (6.2.1), we have:

$$F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]=\sum_{n=0}^{\infty}\frac{B_{k,s}^{(s_1,s_2)}(v_1+nk,v_2-v_1)}{B_k(v_1,v_2-v_1)}(v_0)_{n,k}\frac{w^n}{n!}. \tag{6.2.13}$$

Then, using the definition of the extended $k$-beta function (6.1.14), we have:

$$\begin{aligned}&F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]\\&=\frac{1}{B_k(v_1,v_2-v_1)}\sum_{n=0}^{\infty}\left[\frac{1}{k}\int_0^1 t^{\frac{v_1+nk}{k}-1}(1-t)^{\frac{v_2-v_1}{k}-1}\right.\\&\left.E_{k,s_1,s_2}\left(\frac{-s^k}{kt(1-t)}\right)dt\right](v_0)_{n,k}\frac{w^n}{n!}.\end{aligned} \tag{6.2.14}$$

On interchanging summation and integration with some re-arrangement of terms, we have:

$$\begin{aligned}&F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]\\&=\frac{1}{kB_k(v_1,v_2-v_1)}\int_0^1 t^{\frac{v_1}{k}-1}(1-t)^{\frac{v_2-v_1}{k}-1}\sum_{n=0}^{\infty}\left((v_0)_{n,k}\frac{(wt)^n}{n!}\right)\\&E_{k,s_1,s_2}\left(\frac{-s^k}{kt(1-t)}\right)dt.\end{aligned} \tag{6.2.15}$$

Now, using the power-series result in the above equation we obtain our desired result:

$$\sum_{n=0}^{\infty}(v_0)_{n,k}\frac{t^n}{n!}=(1-kt)^{\frac{-v_0}{k}} \tag{6.2.16}$$

$$\begin{aligned}&F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]\\&=\frac{1}{kB_k(v_1,v_2-v_1)}\int_0^1 t^{\frac{v_1}{k}-1}(1-t)^{\frac{v_2-v_1}{k}-1}(1-kwt)^{\frac{-v_0}{k}}\\&E_{k,s_1,s_2}\left(\frac{-s^k}{kt(1-t)}\right)dt.\end{aligned} \tag{6.2.17}$$

In order to prove (6.2.10), (6.2.11), and (6.2.12), we substitute $t=\cos^2\theta$, $t=\frac{u}{z}$, and $t=\tanh^2\theta$ into Eq. (6.2.9), respectively, and follow a similar process as above and we obtain all our desired results. □

**Corollary 6.2.6.** *The following results hold true:*

$$F_{k,s}^{(s_1)}[(v_0,k),(v_1,k);(v_2,k);w]$$

$$= \frac{1}{kB_k(v_1, v_2-v_1)} \int_0^1 t^{\frac{v_1}{k}-1}(1-t)^{\frac{v_2-v_1}{k}-1}(1-kwt)^{-\frac{v_0}{k}} E_{k,s_1}\left(-s^k(kt(1-t))^{-1}\right)dt$$

$$= \frac{2}{kB_k(v_1, v_2-v_1)} \int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2v_1}{k}-1}(\sin\theta)^{\frac{2v_2-2v_1}{k}-1}(1-kw\cos^2\theta)^{-\frac{v_0}{k}} E_{k,s_1}\left(\frac{-s^k}{k}\sec^2\theta\csc^2\theta\right)d\theta$$

$$= \frac{z^{1+\frac{v_0-v_2}{k}}}{kB_k(v_1, v_2-v_1)} \int_0^z u^{\frac{v_1}{k}-1}(z-u)^{\frac{v_2-v_1}{k}-1}(z-kwu)^{-\frac{v_0}{k}} E_{k,s_1}\left(\frac{-s^k}{k}\frac{z^2}{u(z-u)}\right)du$$

$$= \frac{2}{kB_k(v_1, v_2-v_1)} \int_0^\infty \left[\frac{(\sinh\theta)^{\frac{2v_1}{k}-1}(\cosh\theta)^{1+\frac{2(v_0-v_2)}{k}}}{\left(\cosh^2\theta-kw\sinh^2\theta\right)^{\frac{v_0}{k}}}\right] E_{k,s_1}\left(\frac{-s^k}{k}\cosh^4\theta\,\mathrm{csch}^2\theta\right)d\theta, \tag{6.2.18}$$

*where* $k>0$, $\Re(v_0)>0$, $\Re(v_1)>0$, $\Re(v_2)>0$, $\Re(s_1)>0$, *and* $|w|<\frac{1}{k}$ *and* $B_{k,s}^{(s_1)}(v_0, v_1)$ *is the extended k-beta function.*

*Proof.* Considering $s_2=1$ in Theorem 6.2.5, we obtain our results. ☐

**Theorem 6.2.7.** *The extended k-hypergeometric function has the following integral representations:*

$$F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]$$
$$= \frac{(c-d)^{1+\frac{v_0-v_2}{k}}}{kB_k(v_1, v_2-v_1)} \int_d^c (u-d)^{\frac{v_1}{k}-1}(c-u)^{\frac{v_2-v_1}{k}-1}(c-d-kw(u-d))^{-\frac{v_0}{k}} E_{k,s_1,s_2}\left(\frac{-s^k}{k}\frac{(c-d)^2}{(u-d)(c-u)}\right)du \tag{6.2.19}$$

$$F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]$$
$$= \frac{2^{1+\frac{v_0-v_2}{k}}}{kB_k(v_1, v_2-v_1)} \int_{-1}^1 (1+u)^{\frac{v_1}{k}-1}(1-u)^{\frac{v_2-v_1}{k}-1}(2-kw(1+u))^{-\frac{v_0}{k}} E_{k,s_1,s_2}\left(\frac{-s^k}{k}\frac{4}{(1-u^2)}\right)du \tag{6.2.20}$$

$$F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]$$
$$= \frac{2^{1-\frac{v_2}{k}}}{kB_k(v_1, v_2-v_1)} \int_{-\infty}^\infty e^{\theta\left(\frac{2v_1-v_2}{k}\right)}(\cosh\theta)^{\frac{v_0-v_2}{k}}\left(\cosh\theta-\frac{kw}{2}e^\theta\right)^{-\frac{v_0}{k}}$$

$$E_{k,s_1,s_2}\left(\frac{-4s^k}{k}\cosh^2\theta\right)d\theta, \tag{6.2.21}$$

*where* $k>0$, $\Re(v_0)>0$, $\Re(v_1)>0$, $\Re(v_2)>0$, $\Re(s_1)>0$, $\Re(s_2)>0$, *and* $|w|<\frac{1}{k}$ *and* $B_{k,s}^{(s_1,s_2)}(v_0,v_1)$ *is the extended $k$-beta function.*

*Proof.* To prove Eqs. (6.2.19) and (6.2.20), we substitute $t=\frac{u-d}{c-d}$ and $t=\frac{1+u}{2}$ in the integral form of the $k$-extended hypergeometric function (6.2.9), then we obtain our results. In order to prove (6.2.21), substitute $u=\tanh\theta$ in Eq. (6.2.20), and we obtain our desired result. □

**Corollary 6.2.8.** *The following results hold true:*

$$\begin{aligned}
&F_{k,s}^{(s_1)}[(v_0,k),(v_1,k);(v_2,k);w]\\
&=\frac{(c-d)^{1+\frac{v_0-v_2}{k}}}{kB_k(v_1,v_2-v_1)}\int_d^c(u-d)^{\frac{v_1}{k}-1}(c-u)^{\frac{v_2-v_1}{k}-1}(c-d-kw(u-d))^{-\frac{v_0}{k}}\\
&\quad E_{k,s_1}\left(\frac{-s^k}{k}\frac{(c-d)^2}{(u-d)(c-u)}\right)du\\
&=\frac{2^{1+\frac{v_0-v_2}{k}}}{kB_k(v_1,v_2-v_1)}\int_{-1}^1(1+u)^{\frac{v_1}{k}-1}(1-u)^{\frac{v_2-v_1}{k}-1}(2-kw(1+u))^{-\frac{v_0}{k}}\\
&\quad E_{k,s_1}\left(\frac{-s^k}{k}\frac{4}{(1-u^2)}\right)du\\
&=\frac{2^{1-\frac{v_2}{k}}}{kB_k(v_1,v_2-v_1)}\int_{-\infty}^{\infty}e^{\theta\left(\frac{2v_1-v_2}{k}\right)}(\cosh\theta)^{\frac{v_0-v_2}{k}}(\cosh\theta-\frac{kw}{2}e^{\theta})^{-\frac{v_0}{k}}\\
&\quad E_{k,s_1}\left(\frac{-4s^k}{k}\cosh^2\theta\right)d\theta.
\end{aligned}\tag{6.2.22}$$

*Proof.* Considering $s_2=1$ in Theorem 6.2.7, we obtain our results. □

**Theorem 6.2.9.** *The extended $k$-hypergeometric function has the following integral representations:*

$$\begin{aligned}
&F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]\\
&\quad=\frac{1}{kB_k(v_1,v_2-v_1)}\int_0^{\infty}u^{\frac{v_1}{k}-1}(1+u)^{\frac{v_0-v_2}{k}}(1+u(1-kw))^{-\frac{v_0}{k}}\\
&\quad E_{k,s_1,s_2}\left(\frac{-s^k}{k}\left(2+u+\frac{1}{u}\right)\right)du
\end{aligned}\tag{6.2.23}$$

$$F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]=\frac{1}{kB_k(v_1,v_2-v_1)}$$
$$\int_0^1\left[\frac{t^{\frac{v_1}{k}-1}(1+t(1-kw))^{-\frac{v_0}{k}}+t^{\frac{v_2-v_1}{k}-1}(1+t-kw))^{-\frac{v_0}{k}}}{(1+t)^{\frac{v_2-v_0}{k}}}\right]$$

$$E_{k,s_1,s_2}\left(\frac{-s^k}{k}\left(2+t+\frac{1}{t}\right)\right)dt, \tag{6.2.24}$$

*where* $k>0$, $\Re(v_0)>0$, $\Re(v_1)>0$, $\Re(v_2)>0$, $\Re(s_1)>0$, $\Re(s_2)>0$, *and* $|w|<\frac{1}{k}$ *and* $B_{k,s}^{(s_1,s_2)}(v_0,v_1)$ *is the extended k-beta function.*

*Proof.* To prove Eq. (6.2.23), substituting $t=\frac{u}{1+u}$ in (6.2.9) with some re-arrangement, we obtain our desired result. Then, in order to prove Eq. (6.2.24), consider (6.2.23), and we have:

$$\begin{aligned}
&F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]\\
&=\frac{1}{kB_k(v_1,v_2-v_1)}\int_0^\infty u^{\frac{v_1}{k}-1}(1+u)^{\frac{v_0-v_2}{k}}(1+u(1-kw))^{-\frac{v_0}{k}}\\
&E_{k,s_1,s_2}\left(\frac{-s^k}{k}\left(2+u+\frac{1}{u}\right)\right)du.
\end{aligned} \tag{6.2.25}$$

After using the properties of the integral we have:

$$\begin{aligned}
&F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]\\
&=\frac{1}{kB_k(v_1,v_2-v_1)}\int_0^1 u^{\frac{v_1}{k}-1}(1+u)^{\frac{v_0-v_2}{k}-1}(1+u(1-kw))^{-\frac{v_0}{k}}\\
&E_{k,s_1,s_2}\left(\frac{-s^k}{k}\left(2+u+\frac{1}{u}\right)\right)du\\
&+\frac{1}{kB_k(v_1,v_2-v_1)}\int_1^\infty u^{\frac{v_1}{k}-1}(1+u)^{\frac{v_0-v_2}{k}-1}(1+u(1-kw))^{-\frac{v_0}{k}}\\
&E_{k,s_1,s_2}\left(\frac{-s^k}{k}\left(2+u+\frac{1}{u}\right)\right)du.
\end{aligned} \tag{6.2.26}$$

Then, substituting $u=t$ in the first part of the integral and $u=\frac{1}{t}$ in the second part of the integral and simplifying, we obtain our desired result. □

**Corollary 6.2.10.** *The following results hold true:*

$$\begin{aligned}
&F_{k,s}^{(s_1)}[(v_0,k),(v_1,k);(v_2,k);w]\\
&=\frac{1}{kB_k(v_1,v_2-v_1)}\int_0^\infty u^{\frac{v_1}{k}-1}(1+u)^{\frac{v_0-v_2}{k}}(1+u(1-kw))^{-\frac{v_0}{k}}\\
&E_{k,s_1}\left(\frac{-s^k}{k}\left(2+u+\frac{1}{u}\right)\right)du\\
&=\frac{1}{kB_k(v_1,v_2-v_1)}\int_0^1\left[\frac{t^{\frac{v_1}{k}-1}(1+t(1-kw))^{-\frac{v_0}{k}}+t^{\frac{v_2-v_1}{k}-1}(1+t-kw)^{-\frac{v_0}{k}}}{(1+t)^{\frac{v_2-v_0}{k}}}\right]\\
&E_{k,s_1}\left(\frac{-s^k}{k}\left(2+t+\frac{1}{t}\right)\right)dt.
\end{aligned} \tag{6.2.27}$$

*Proof.* Considering $s_2 = 1$ in Theorem 6.2.9, we obtain our results. □

**Theorem 6.2.11.** *The extended k-confluent hypergeometric function has the following integral representations:*

$$\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w] = \frac{1}{kB_k(v_1,v_2-v_1)} \int_0^1 t^{\frac{v_1}{k}-1}(1-t)^{\frac{v_2-v_1}{k}-1}(e)^{wt} E_{k,s_1,s_2}\Big(-s^k(kt(1-t))^{-1}\Big)dt \tag{6.2.28}$$

$$\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w] = \frac{e^w}{kB_k(v_1,v_2-v_1)} \int_0^1 t^{\frac{v_2-v_1}{k}-1}(1-t)^{\frac{v_1}{k}-1}(e)^{-wt} E_{k,s_1,s_2}\Big(-s^k(kt(1-t))^{-1}\Big)dt \tag{6.2.29}$$

$$\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w] = \frac{2}{kB_k(v_1,v_2-v_1)} \int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{2v_1}{k}-1}(\cos\theta)^{\frac{2v_2-2v_1}{k}-1}(e)^{w\sin^2\theta} E_{k,s_1,s_2}\left(\frac{-s^k}{k}\sec^2\theta\csc^2\theta\right)d\theta \tag{6.2.30}$$

$$\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w] = \frac{2}{kB_k(v_1,v_2-v_1)} \int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2v_1}{k}-1}(\sin\theta)^{\frac{2v_2-2v_1}{k}-1}(e)^{w\cos^2\theta} E_{k,s_1,s_2}\left(\frac{-s^k}{k}\sec^2\theta\csc^2\theta\right)d\theta \tag{6.2.31}$$

$$\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w] = \frac{z^{1-\frac{v_2}{k}}}{kB_k(v_1,v_2-v_1)} \int_0^z u^{\frac{v_1}{k}-1}(z-u)^{\frac{v_2-v_1}{k}-1}(e)^{\frac{wu}{z}} E_{k,s_1,s_2}\left(\frac{-s^k}{k}\frac{z^2}{u(z-u)}\right)du \tag{6.2.32}$$

$$\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w] = \frac{2}{kB_k(v_1,v_2-v_1)} \int_0^{\infty} (\sinh\theta)^{\frac{2v_1}{k}-1}(\cosh\theta)^{1-\frac{2v_2}{k}}(e)^{w\tanh^2\theta} E_{k,s_1,s_2}\left(\frac{-s^k}{k}\cosh^4\theta\,\mathrm{csch}^2\theta\right)d\theta, \tag{6.2.33}$$

*where* $k>0$, $\Re(v_1)>0$, $\Re(v_2)>0$, $\Re(s_1)>0$, $\Re(s_2)>0$ *and* $B_{k,s}^{(s_1,s_2)}(v_0,v_1)$ *is the extended k-beta function.*

*Proof.* To prove (6.2.28), consider the definition of the extended $k$-confluent hypergeometric function (6.2.2), and we have:

$$\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w]=\sum_{n=0}^{\infty}\frac{B_{k,s}^{(s_1,s_2)}(v_1+nk,v_2-v_1)}{B_k(v_1,v_2-v_1)}\frac{w^n}{n!}. \tag{6.2.34}$$

Then, using the definition of the extended $k$-beta function (6.1.14), we have:

$$\begin{aligned}&\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w]\\&=\frac{1}{B_k(v_1,v_2-v_1)}\sum_{n=0}^{\infty}\left[\frac{1}{k}\int_0^1 t^{\frac{v_1+nk}{k}-1}(1-t)^{\frac{v_2-v_1}{k}-1}E_{k,s_1,s_2}\left(\frac{-s^k}{kt(1-t)}\right)dt\right]\frac{w^n}{n!}.\end{aligned} \tag{6.2.35}$$

On interchanging summation and integration with some re-arrangement of terms, we have:

$$\begin{aligned}&\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w]\\&=\frac{1}{kB_k(v_1,v_2-v_1)}\int_0^1 t^{\frac{v_1}{k}-1}(1-t)^{\frac{v_2-v_1}{k}-1}\sum_{n=0}^{\infty}\left(\frac{(wt)^n}{n!}\right)E_{k,s_1,s_2}\left(\frac{-s^k}{kt(1-t)}\right)dt.\end{aligned} \tag{6.2.36}$$

Now, using the power-series result in the above equation we obtain our desired result:

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}=(e)^t \tag{6.2.37}$$

$$\begin{aligned}&\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w]\\&=\frac{1}{kB_k(v_1,v_2-v_1)}\int_0^1 t^{\frac{v_1}{k}-1}(1-t)^{\frac{v_2-v_1}{k}-1}(e)^{wt}E_{k,s_1,s_2}\left(\frac{-s^k}{kt(1-t)}\right)dt.\end{aligned} \tag{6.2.38}$$

In order to prove (6.2.29), (6.2.30), (6.2.31), (6.2.32), and (6.2.33) we substitute, $t=1-t$, $t=\sin^2\theta$, $t=\cos^2\theta$, $t=\frac{u}{z}$, and $t=\tanh^2\theta$ into Eq. (6.2.28), respectively, and following a similar process as above we obtain all our desired results. □

**Corollary 6.2.12.** *The following results hold true:*

$$\begin{aligned}&\phi_{k,s}^{(s_1)}[(v_1,k);(v_2,k);w]\\&=\frac{1}{kB_k(v_1,v_2-v_1)}\int_0^1 t^{\frac{v_1}{k}-1}(1-t)^{\frac{v_2-v_1}{k}-1}(e)^{wt}E_{k,s_1}\left(-s^k(kt(1-t))^{-1}\right)dt\\&=\frac{e^w}{kB_k(v_1,v_2-v_1)}\int_0^1 t^{\frac{v_2-v_1}{k}-1}(1-t)^{\frac{v_1}{k}-1}(e)^{-wt}E_{k,s_1}\left(-s^k(kt(1-t))^{-1}\right)dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{kB_k(v_1, v_2 - v_1)} \int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{2v_1}{k}-1} (\cos\theta)^{\frac{2v_2-2v_1}{k}-1} (e)^{w\sin^2\theta} \\
&\quad E_{k,s_1}\left(\frac{-s^k}{k} \sec^2\theta \csc^2\theta\right) d\theta \\
&= \frac{2}{kB_k(v_1, v_2 - v_1)} \int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2v_1}{k}-1} (\sin\theta)^{\frac{2v_2-2v_1}{k}-1} (e)^{w\cos^2\theta} \\
&\quad E_{k,s_1}\left(\frac{-s^k}{k} \sec^2\theta \csc^2\theta\right) d\theta \\
&= \frac{z^{1-\frac{v_2}{k}}}{kB_k(v_1, v_2 - v_1)} \int_0^{z} u^{\frac{v_1}{k}-1} (z-u)^{\frac{v_2-v_1}{k}-1} (e)^{\frac{wu}{z}} E_{k,s_1}\left(\frac{-s^k}{k} \frac{z^2}{u(z-u)}\right) du \\
&= \frac{2}{kB_k(v_1, v_2 - v_1)} \int_0^{\infty} (\sinh\theta)^{\frac{2v_1}{k}-1} (\cosh\theta)^{1-\frac{2v_2}{k}} (e)^{w\tanh^2\theta} \\
&\quad E_{k,s_1}\left(\frac{-s^k}{k} \cosh^4\theta \,\mathrm{csch}^2\theta\right) d\theta. \qquad (6.2.39)
\end{aligned}
$$

*Proof.* Considering $s_2 = 1$ in Theorem 6.2.11, we obtain our results. □

**Theorem 6.2.13.** *The extended k-confluent hypergeometric function has the following integral representations:*

$$
\begin{aligned}
&\phi_{k,s}^{(s_1,s_2)}[(v_1, k); (v_2, k); w] \\
&\quad = \frac{1}{kB_k(v_1, v_2 - v_1)} \int_0^{\infty} u^{\frac{v_1}{k}-1} (1+u)^{\frac{-v_2}{k}} (e)^{\frac{uw}{1+u}} E_{k,s_1,s_2}\left(\frac{-s^k}{k}\left(2 + u + \frac{1}{u}\right)\right) du
\end{aligned}
\qquad (6.2.40)
$$

$$
\begin{aligned}
\phi_{k,s}^{(s_1,s_2)}[(v_1, k); (v_2, k); w] &= \frac{1}{kB_k(v_1, v_2 - v_1)} \int_0^{1} \left[\frac{t^{\frac{v_1}{k}-1}(e)^{\frac{wt}{1+t}} + t^{\frac{v_2-v_1}{k}-1}(e)^{\frac{w}{1+t}}}{(1+t)^{\frac{v_2}{k}}}\right] \\
&\quad E_{k,s_1,s_2}\left(\frac{-s^k}{k}\left(2 + t + \frac{1}{t}\right)\right) dt,
\end{aligned}
\qquad (6.2.41)
$$

*where $k > 0$, $\Re(v_1) > 0$, $\Re(v_2) > 0$, $\Re(s_1) > 0$, $\Re(s_2) > 0$ and $B_{k,s}^{(s_1,s_2)}(v_0, v_1)$ is the extended k-beta function.*

*Proof.* To prove Eq. (6.2.40), we substitute $t = \frac{u}{1+u}$ in (6.2.28) with some rearrangement, and we obtain our desired result. Then, in order to prove Eq. (6.2.41), consider (6.2.40), we have:

$$
\begin{aligned}
&\phi_{k,s}^{(s_1,s_2)}[(v_1, k); (v_2, k); w] \\
&\quad = \frac{1}{kB_k(v_1, v_2 - v_1)} \int_0^{\infty} u^{\frac{v_1}{k}-1} (1+u)^{\frac{-v_2}{k}} (e)^{\frac{uw}{1+u}} E_{k,s_1,s_2}\left(\frac{-s^k}{k}\left(2 + u + \frac{1}{u}\right)\right) du.
\end{aligned}
\qquad (6.2.42)
$$

After using the properties of the integral we have:

$$\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w]$$
$$= \frac{1}{kB_k(v_1,v_2-v_1)}\int_0^1 u^{\frac{v_1}{k}-1}(1+u)^{\frac{-v_2}{k}}(e)^{\frac{uw}{1+u}}E_{k,s_1,s_2}\left(\frac{-s^k}{k}\left(2+u+\frac{1}{u}\right)\right)du$$
$$+\frac{1}{kB_k(v_1,v_2-v_1)}\int_1^\infty u^{\frac{v_1}{k}-1}(1+u)^{\frac{-v_2}{k}}(e)^{\frac{uw}{1+u}}E_{k,s_1,s_2}\left(\frac{-s^k}{k}\left(2+u+\frac{1}{u}\right)\right)du. \tag{6.2.43}$$

Then, substituting $u=t$ in the first part of the integral and $u=\frac{1}{t}$ in the second part of the integral and simplifying, we obtain our desired result. □

**Corollary 6.2.14.** *The following results hold true:*

$$\phi_{k,s}^{(s_1)}[(v_1,k);(v_2,k);w]$$
$$= \frac{1}{kB_k(v_1,v_2-v_1)}\int_0^\infty u^{\frac{v_1}{k}-1}(1+u)^{\frac{-v_2}{k}}(e)^{\frac{uw}{1+u}}E_{k,s_1}\left(\frac{-s^k}{k}\left(2+u+\frac{1}{u}\right)\right)du$$
$$= \frac{1}{kB_k(v_1,v_2-v_1)}\int_0^1\left[\frac{t^{\frac{v_1}{k}-1}(e)^{\frac{wt}{1+t}}+t^{\frac{v_2-v_1}{k}-1}(e)^{\frac{w}{1+t}}}{(1+t)^{\frac{v_2}{k}}}\right]$$
$$E_{k,s_1}\left(\frac{-s^k}{k}\left(2+t+\frac{1}{t}\right)\right)dt. \tag{6.2.44}$$

*Proof.* Considering $s_2=1$ in Theorem 6.2.13, we obtain our results. □

**Theorem 6.2.15.** *The extended confluent k-hypergeometric function has the following integral representations:*

$$\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w]$$
$$= \frac{(c-d)^{1-\frac{v_2}{k}}}{kB_k(v_1,v_2-v_1)}\int_d^c (u-d)^{\frac{v_1}{k}-1}(c-u)^{\frac{v_2-v_1}{k}-1}(e)^{\frac{w(u-d)}{(c-d)}}$$
$$E_{k,s_1,s_2}\left(\frac{-s^k}{k}\frac{(c-d)^2}{(u-d)(c-u)}\right)du \tag{6.2.45}$$

$$\phi_{k,s}^{(s_1,s_2)}[v_1,k);(v_2,k);w]$$
$$= \frac{2^{1-\frac{v_2}{k}}}{kB_k(v_1,v_2-v_1)}\int_{-1}^1 (1+u)^{\frac{v_1}{k}-1}(1-u)^{\frac{v_2-v_1}{k}-1}(e)^{\frac{w(1+u)}{2}}$$
$$E_{k,s_1,s_2}\left(\frac{-s^k}{k}\frac{4}{(1-u^2)}\right)du \tag{6.2.46}$$

$$\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w]$$

$$= \frac{2^{1-\frac{v_2}{k}}}{kB_k(v_1, v_2 - v_1)} \int_{-\infty}^{\infty} e^{\left[\theta\left(\frac{2v_1-v_2}{k}\right)+\frac{ze^{\theta}}{2\cosh\theta}\right]} (cosh\theta)^{\frac{-v_2}{k}} E_{k,s_1,s_2}\left(\frac{-4s^k}{k}\cosh^2\theta\right) d\theta, \tag{6.2.47}$$

*where* $k > 0$, $\Re(v_1) > 0$, $\Re(v_2) > 0$, $\Re(s_1) > 0$, $\Re(s_2) > 0$ *and* $B_{k,s}^{(s_1,s_2)}(v_0, v_1)$ *is the extended k-beta function.*

*Proof.* To prove Eqs. (6.2.45) and (6.2.46), we substitute $t = \frac{u-d}{c-d}$ and $t = \frac{1+u}{2}$ in the integral form of the $k$-extended hypergeometric function (6.2.28), then we obtain our results. In order to prove (6.2.47), we substitute $u = \tanh\theta$ in Eq. (6.2.46), and we obtain our desired result. □

**Corollary 6.2.16.** *The following results hold true:*

$$\begin{aligned}
&\phi_{k,s}^{(s_1)}[(v_1, k); (v_2, k); w] \\
&= \frac{(c-d)^{1-\frac{v_2}{k}}}{kB_k(v_1, v_2 - v_1)} \int_d^c (u-d)^{\frac{v_1}{k}-1}(c-u)^{\frac{v_2-v_1}{k}-1}(e)^{\frac{w(u-d)}{c-d}} \\
&\quad E_{k,s_1}\left(\frac{-s^k}{k}\frac{(c-d)^2}{(u-d)(c-u)}\right) du \\
&= \frac{2^{1-\frac{v_2}{k}}}{kB_k(v_1, v_2 - v_1)} \int_{-1}^{1} (1+u)^{\frac{v_1}{k}-1}(1-u)^{\frac{v_2-v_1}{k}-1}(e)^{\frac{w(1+u)}{2}} \\
&\quad E_{k,s_1}\left(\frac{-s^k}{k}\frac{4}{(1-u^2)}\right) du \\
&= \frac{2^{1-\frac{v_2}{k}}}{kB_k(v_1, v_2 - v_1)} \int_{-\infty}^{\infty} e^{\left[\theta\left(\frac{2v_1-v_2}{k}\right)+\frac{ze^{\theta}}{2\cosh\theta}\right]} (cosh\theta)^{\frac{-v_2}{k}} E_{k,s_1}\left(\frac{-4s^k}{k}\cosh^2\theta\right) d\theta.
\end{aligned} \tag{6.2.48}$$

*Proof.* Considering $s_2 = 1$ in Theorem 6.2.15, we obtain our results. □

**Theorem 6.2.17.** *The extended k-hypergeometric function and confluent k-hypergeometric function satisfy the following functional relations:*

$$\begin{aligned}
&v_2 F_{k,s}^{(s_1,s_2)}[(v_0, k), (v_1, k); (v_2, k); w] \\
&= v_1 F_{k,s}^{(s_1,s_2)}[(v_0, k), (v_1 + k, k); (v_2 + k, k); w] \\
&\quad + (v_2 - v_1) F_{k,s}^{(s_1,s_2)}[(v_0, k), (v_1, k); (v_2 + k, k); w]
\end{aligned} \tag{6.2.49}$$

$$\begin{aligned}
&v_2\phi_{k,s}^{(s_1,s_2)}[(v_1, k); (v_2, k); w] \\
&= v_1\phi_{k,s}^{(s_1,s_2)}[(v_1 + k, k); (v_2 + k, k); w] + (v_2 - v_1)\phi_{k,s}^{(s_1,s_2)}[(v_1, k); (v_2 + k, k); w],
\end{aligned} \tag{6.2.50}$$

*where* $k > 0$, $\Re(v_0) > 0$ $\Re(v_1) > 0$, $\Re(v_2) > 0$, $\Re(s_1) > 0$, $\Re(s_2) > 0$ *and* $B_{k,s}^{(s_1,s_2)}(v_0, v_1)$ *is the extended* $k$*-beta function.*

*Proof.* In order to prove Eq. (6.2.49), we use the properties of the extended $k$-beta function, in the definition of the extended $k$-hypergeometric function, and we have:

$$B_{k,s}^{(s_1,s_2)}(v_1, v_2) = B_{k,s}^{(s_1,s_2)}(v_1 + k, v_2) + B_{k,s}^{(s_1,s_2)}(v_1, v_2 + k) \tag{6.2.51}$$

$$\begin{aligned} &F_{k,s}^{(s_1,s_2)}[(v_0, k), (v_1, k); (v_2, k); w] \\ &= \sum_{n=0}^{\infty} \frac{B_{k,s}^{(s_1,s_2)}(v_1 + nk + k, v_2 - v_1) + B_{k,s}^{(s_1,s_2)}(v_1 + nk, v_2 - v_1 + k)}{B_k(v_1, v_2 - v_1)} (v_0)_{n,k} \frac{w^n}{n!}. \end{aligned} \tag{6.2.52}$$

After re-arranging the terms, we obtain:

$$\begin{aligned} &F_{k,s}^{(s_1,s_2)}[(v_0, k), (v_1, k); (v_2, k); w] \\ &= \sum_{n=0}^{\infty} \frac{B_{k,s}^{(s_1,s_2)}(v_1 + nk + k, v_2 - v_1)}{B_k(v_1, v_2 - v_1)} (v_0)_{n,k} \frac{w^n}{n!} \\ &\quad + \sum_{n=0}^{\infty} \frac{B_{k,s}^{(s_1,s_2)}(v_1 + nk, v_2 - v_1 + k)}{B_k(v_1, v_2 - v_1)} (v_0)_{n,k} \frac{w^n}{n!}. \end{aligned} \tag{6.2.53}$$

Multiplying and dividing $B_k(v_1 + k, v_2 - v_1)$ in the first part and multiplying and dividing $B_k(v_1, v_2 - v_1 + k)$ in the second part of the above equations, we have:

$$\begin{aligned} &F_{k,s}^{(s_1,s_2)}[(v_0, k), (v_1, k); (v_2, k); w] \\ &= \frac{B_k(v_1 + k, v_2 - v_1)}{B_k(v_1, v_2 - v_1)} \sum_{n=0}^{\infty} \frac{B_{k,s}^{(s_1,s_2)}(v_1 + nk + k, v_2 - v_1)}{B_k(v_1 + k, v_2 - v_1)} (v_0)_{n,k} \frac{w^n}{n!} \\ &\quad + \frac{B_k(v_1, v_2 - v_1 + k)}{B_k(v_1, v_2 - v_1)} \sum_{n=0}^{\infty} \frac{B_{k,s}^{(s_1,s_2)}(v_1 + nk, v_2 - v_1 + k)}{B_k(v_1, v_2 - v_1 + k)} (v_0)_{n,k} \frac{w^n}{n!}. \end{aligned} \tag{6.2.54}$$

Then, using the properties of the $k$-beta and $k$-gamma functions $B_k(a, b) = \frac{\Gamma_k(a)\Gamma_k(b)}{\Gamma_k(a+b)}$ and $\Gamma_k(a+k) = a\Gamma_k(a)$ with mathematical calculations, we obtain our desired result:

$$\begin{aligned} &v_2 F_{k,s}^{(s_1,s_2)}[(v_0, k), (v_1, k); (v_2, k); w] \\ &= v_1 F_{k,s}^{(s_1,s_2)}[(v_0, k), (v_1 + k, k); (v_2 + k, k); w] \\ &\quad + (v_2 - v_1) F_{k,s}^{(s_1,s_2)}[(v_0, k), (v_1, k); (v_2 + k, k); w]. \end{aligned} \tag{6.2.55}$$

Similarly, with the same proof as for Eq. (6.2.49), we obtain Eq. (6.2.50). □

**Corollary 6.2.18.** *The following results hold true:*

$$\begin{aligned} v_2 F_{k,s}^{(s_1)}&[(v_0,k),(v_1,k);(v_2,k);w] \\ &= v_1 F_{k,s}^{(s_1)}[(v_0,k),(v_1+k,k);(v_2+k,k);w] \\ &\quad + (v_2-v_1) F_{k,s}^{(s_1)}[(v_0,k),(v_1,k);(v_2+k,k);w] \end{aligned} \tag{6.2.56}$$

$$\begin{aligned} v_2\phi_{k,s}^{(s_1)}&[(v_1,k);(v_2,k);w] \\ &= v_1\phi_{k,s}^{(s_1)}[(v_1+k,k);(v_2+k,k);w] + (v_2-v_1)\phi_{k,s}^{(s_1)}[(v_1,k);(v_2+k,k);w]. \end{aligned} \tag{6.2.57}$$

*Proof.* Considering $s_2 = 1$ in Theorem 6.2.17, we obtain our results. □

**Theorem 6.2.19.** *The extended k-hypergeometric function and k-confluent hypergeometric function satisfy the following infinite summation relations:*

$$\begin{aligned} F_{k,s}^{(s_1,s_2)}&[(v_0,k),(v_1,k);(v_2,k);w] \\ &= (v_2-v_1)\sum_{p=0}^{\infty}\frac{(v_1)_{p,k}}{(v_2)_{p+1,k}} F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1+pk,k);(v_2+pk+k,k);w] \end{aligned} \tag{6.2.58}$$

$$\begin{aligned} \phi_{k,s}^{(s_1,s_2)}&[(v_1,k);(v_2,k);w] \\ &= (v_2-v_1)\sum_{p=0}^{\infty}\frac{(v_1)_{p,k}}{(v_2)_{p+1,k}}\phi_{k,s}^{(s_1,s_2)}[(v_1+pk,k);(v_2+pk+k,k);w], \end{aligned} \tag{6.2.59}$$

*where $k>0$, $\Re(v_0)>0$ $\Re(v_1)>0$, $\Re(v_2)>0$, $\Re(s_1)>0$, $\Re(s_2)>0$ and $B_{k,s}^{(s_1,s_2)}(v_0,v_1)$ is the extended k-beta function.*

*Proof.* In order to prove Eq. (6.2.58), we use the properties of the extended $k$-beta function, in the definition of the extended $k$-hypergeometric function, and we have:

$$B_{k,s}^{(s_1,s_2)}(v_1,v_2) = \sum_{n=0}^{\infty} B_{k,s}^{(s_1,s_2)}(v_1+nk, v_2+k) \tag{6.2.60}$$

$$\begin{aligned} F_{k,s}^{(s_1,s_2)}&[(v_0,k),(v_1,k);(v_2,k);w] \\ &= \sum_{n=0}^{\infty}\frac{\sum_{p=0}^{\infty} B_{k,s}^{(s_1,s_2)}(v_1+pk+nk, v_2-v_1+k)}{B_k(v_1,v_2-v_1)}(v_0)_{n,k}\frac{w^n}{n!}. \end{aligned} \tag{6.2.61}$$

Multiplying and dividing $B_k(v_1+pk, v_2-v_1+k)$, and changing the order of summation, we have:

$$F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]$$

$$= \sum_{p=0}^{\infty} \frac{B_k(v_1 + pk, v_2 - v_1 + k)}{B_k(v_1, v_2 - v_1)}$$
$$\sum_{n=0}^{\infty} (v_0)_{n,k} \frac{B_{k,s}^{(s_1,s_2)}(v_1 + pk + nk, v_2 - v_1 + k)}{B_k(v_1 + pk, v_2 - v_1 + k)} \frac{w^n}{n!}. \tag{6.2.62}$$

Then, using the properties of the $k$-beta and $k$-gamma functions $B_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}$ and $\Gamma_k(a+k) = a\Gamma_k(a)$, $(a)_{n,k} = \frac{\Gamma_k(a+nk)}{\Gamma_k(a)}$, and $(a)_{n+1,k} = a(a+k)_{n,k}$, and using the definition of the extended $k$-hypergeometric function, we obtain our desired result:

$$F_{k,s}^{(s_1,s_2)}[(v_0, k), (v_1, k); (v_2, k); w]$$
$$= (v_2 - v_1) \sum_{p=0}^{\infty} \frac{(v_1)_{p,k}}{(v_2)_{p+1,k}} F_{k,s}^{(s_1,s_2)}[(v_0, k), (v_1 + pk, k); (v_2 + pk + k, k); w]. \tag{6.2.63}$$

Similarly, with the same proof as for Eq. (6.2.58), we obtain Eq. (6.2.59). □

**Corollary 6.2.20.** *The following results hold true:*

$$F_{k,s}^{(s_1)}[(v_0, k), (v_1, k); (v_2, k); w]$$
$$= (v_2 - v_1) \sum_{p=0}^{\infty} \frac{(v_1)_{p,k}}{(v_2)_{p+1,k}} F_{k,s}^{(s_1)}[(v_0, k), (v_1 + pk, k); (v_2 + pk + k, k); w] \tag{6.2.64}$$

$$\phi_{k,s}^{(s_1)}[(v_1, k); (v_2, k); w]$$
$$= (v_2 - v_1) \sum_{p=0}^{\infty} \frac{(v_1)_{p,k}}{(v_2)_{p+1,k}} \phi_{k,s}^{(s_1)}[(v_1 + pk, k); (v_2 + pk + k, k); w]. \tag{6.2.65}$$

*Proof.* Considering $s_2 = 1$ in Theorem 6.2.19, we obtain our results. □

**Theorem 6.2.21.** *The extended $k$-hypergeometric function and $k$-confluent hypergeometric function satisfy the following infinite summation relations:*

$$F_{k,s}^{(s_1,s_2)}[(v_0, k), (v_1, k); (v_2, k); w]$$
$$= \sum_{p=0}^{\infty} \frac{(k + v_1 - v_2)_{p,k}}{k^p\, p!} \frac{B_k(v_1 + pk, k)}{B_k(v_1, v_2 - v_1)}$$
$$F_{k,s}^{(s_1,s_2)}[(v_0, k), (v_1 + pk, k); (v_1 + pk + k, k); w] \tag{6.2.66}$$

$$\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w] = \sum_{p=0}^{\infty} \frac{(k+v_1-v_2)_{p,k}}{k^p p!} \frac{B_k(v_1+pk,k)}{B_k(v_1,v_2-v_1)} \phi_{k,s}^{(s_1,s_2)}[(v_1+pk,k);(v_1+pk+k,k);w], \tag{6.2.67}$$

*where* $k>0$, $\Re(v_0)>0$ $\Re(v_1)>0$, $\Re(v_2)>0$, $\Re(s_1)>0$, $\Re(s_2)>0$ *and* $B_{k,s}^{(s_1,s_2)}(v_0,v_1)$ *is the extended $k$-beta function.*

*Proof.* In order to prove Eq. (6.2.66), we use the properties of the extended $k$-beta function, in the definition of the extended $k$-hypergeometric function, and we have:

$$B_{k,s}^{(s_1,s_2)}(v_1,k-v_2) = \sum_{n=0}^{\infty} \frac{(v_2)_{n,k}}{k^n n!} B_{k,s}^{(s_1,s_2)}(v_1+nk,k) \tag{6.2.68}$$

$$F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w] = \sum_{n=0}^{\infty} \frac{\sum_{p=0}^{\infty} \frac{(k+v_1-v_2)_{p,k}}{k^p p!} B_{k,s}^{(s_1,s_2)}(v_1+nk+pk,k)}{B_k(v_1,v_2-v_1)} (v_0)_{n,k} \frac{w^n}{n!}. \tag{6.2.69}$$

Multiplying and dividing $B_k(v_1+pk,k)$, and changing the order of summation, we have:

$$F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w] = \sum_{p=0}^{\infty} \frac{(k+v_1-v_2)_{p,k}}{k^p p!} \frac{B_k(v_1+pk,k)}{B_k(v_1,v_2-v_1)} \sum_{n=0}^{\infty} (v_0)_{n,k} \frac{B_{k,s}^{(s_1,s_2)}(v_1+pk+nk,k)}{B_k(v_1+pk,k)} \frac{w^n}{n!}. \tag{6.2.70}$$

Then, using the definition of the extended $k$-hypergeometric function, we obtain our desired result:

$$F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w] = \sum_{p=0}^{\infty} \frac{(k+v_1-v_2)_{p,k}}{k^p p!} \frac{B_k(v_1+pk,k)}{B_k(v_1,v_2-v_1)} F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1+pk,k);(v_1+pk+k,k);w]. \tag{6.2.71}$$

Similarly, with the same proof as for Eq. (6.2.66), we obtain Eq. (6.2.67). □

**Corollary 6.2.22.** *The following results hold true:*

$$F_{k,s}^{(s_1)}[(v_0,k),(v_1,k);(v_2,k);w] = \sum_{p=0}^{\infty} \frac{(k+v_1-v_2)_{p,k}}{k^p p!} \frac{B_k(v_1+pk,k)}{B_k(v_1,v_2-v_1)}$$

$$F_{k,s}^{(s_1)}[(v_0,k),(v_1+pk,k);(v_1+pk+k,k);w] \tag{6.2.72}$$

$$\phi_{k,s}^{(s_1)}[(v_1,k);(v_2,k);w] = \sum_{p=0}^{\infty} \frac{(k+v_1-v_2)_{p,k}}{k^p\, p!} \frac{B_k(v_1+pk,k)}{B_k(v_1,v_2-v_1)} \phi_{k,s}^{(s_1)}[(v_1+pk,k);(v_1+pk+k,k);w]. \tag{6.2.73}$$

*Proof.* Considering $s_2 = 1$ in Theorem 6.2.21, we obtain our results. □

**Theorem 6.2.23.** *The extended k-hypergeometric function and k-confluent hypergeometric function satisfy the following relations:*

$$F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w] = \sum_{p=0}^{r} \binom{r}{p} \frac{B_k(v_1+pk,v_2-v_1-pk+rk)}{B_k(v_1,v_2-v_1)} F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1+pk,k);(v_2+rk,k);w] \tag{6.2.74}$$

$$\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w] = \sum_{p=0}^{r} \binom{r}{p} \frac{B_k(v_1+pk,v_2-v_1-pk+rk)}{B_k(v_1,v_2-v_1)} \phi_{k,s}^{(s_1,s_2)}[(v_1+pk,k);(v_2+rk,k);w], \tag{6.2.75}$$

*where* $k>0$, $\Re(v_0)>0$ $\Re(v_1)>0$, $\Re(v_2)>0$, $\Re(s_1)>0$, $\Re(s_2)>0$ *and* $B_{k,s}^{(s_1,s_2)}(v_0,v_1)$ *is the extended k-beta function.*

*Proof.* In order to prove Eq. (6.2.74), we use the properties of the extended $k$-beta function, in the definition of the extended $k$-hypergeometric function, and we have:

$$B_{k,s}^{(s_1,s_2)}(v_1,v_2) = \sum_{n=0}^{r} \binom{r}{n} B_{k,s}^{(s_1,s_2)}(v_1+nk, v_2+rk-nk) \tag{6.2.76}$$

$$F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w] = \sum_{n=0}^{\infty} \frac{(v_0)_{n,k}}{B_k(v_1,v_2-v_1)} \left\{ \sum_{p=0}^{r} \binom{r}{p} B_{k,s}^{(s_1,s_2)}(v_1+pk+nk, v_2-v_1-pk+rk) \frac{w^n}{n!} \right\}. \tag{6.2.77}$$

Multiplying and dividing $B_k(v_1+pk, v_2-v_1+rk-pk)$, and changing the order of summation, we have:

$$F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]$$

$$= \sum_{p=0}^{r} \binom{r}{p} \frac{B_k(v_1 + pk, v_2 - v_1 + rk - pk)}{B_k(v_1, v_2 - v_1)}$$
$$\sum_{n=0}^{\infty} (v_0)_{n,k} \frac{B_{k,s}^{(s_1,s_2)}(v_1 + pk + nk, v_2 - v_1 + rk - pk)}{B_k(v_1 + pk, v_2 - v_1 + rk - pk)} \frac{w^n}{n!}. \tag{6.2.78}$$

Then, using the definition of the extended $k$-hypergeometric function, we obtain our desired result:

$$F_{k,s}^{(s_1,s_2)}[(v_0, k), (v_1, k); (v_2, k); w]$$
$$= \sum_{p=0}^{r} \binom{r}{p} \frac{B_k(v_1 + pk, v_2 - v_1 - pk + rk)}{B_k(v_1, v_2 - v_1)}$$
$$F_{k,s}^{(s_1,s_2)}[(v_0, k), (v_1 + pk, k); (v_2 + rk, k); w]. \tag{6.2.79}$$

Similarly, with the same proof as for Eq. (6.2.74), we obtain Eq. (6.2.75). □

**Corollary 6.2.24.** *The following results hold true:*

$$F_{k,s}^{(s_1)}[(v_0, k), (v_1, k); (v_2, k); w]$$
$$= \sum_{p=0}^{r} \binom{r}{p} \frac{B_k(v_1 + pk, v_2 - v_1 - pk + rk)}{B_k(v_1, v_2 - v_1)}$$
$$F_{k,s}^{(s_1)}[(v_0, k), (v_1 + pk, k); (v_2 + rk, k); w] \tag{6.2.80}$$

$$\phi_{k,s}^{(s_1)}[(v_1, k); (v_2, k); w]$$
$$= \sum_{p=0}^{r} \binom{r}{p} \frac{B_k(v_1 + pk, v_2 - v_1 - pk + rk)}{B_k(v_1, v_2 - v_1)} \phi_{k,s}^{(s_1)}[(v_1 + pk, k); (v_2 + rk, k); w]. \tag{6.2.81}$$

*Proof.* Considering $s_2 = 1$ in Theorem 6.2.23, we obtain our results. □

**Theorem 6.2.25.** *The following derivative formulae for the extended $k$-hypergeometric and the $k$-confluent hypergeometric functions hold true:*

$$\frac{d^p}{dw^p}\left\{F_{k,s}^{(s_1,s_2)}[(v_0, k), (v_1, k); (v_2, k); w]\right\}$$
$$= \frac{(v_0)_{p,k}(v_1)_{p,k}}{(v_2)_{p,k}} F_{k,s}^{(s_1,s_2)}[(v_0 + pk, k), (v_1 + pk, k); (v_2 + pk, k); w] \tag{6.2.82}$$

$$\frac{d^p}{dw^p}\left\{\phi_{k,s}^{(s_1,s_2)}[(v_1, k); (v_2, k); w]\right\} = \frac{(v_1)_{p,k}}{(v_2)_{p,k}} \phi_{k,s}^{(s_1,s_2)}[(v_1 + pk, k); (v_2 + pk, k); w], \tag{6.2.83}$$

*where $k > 0$, $\Re(v_0) > 0$ $\Re(v_1) > 0$, $\Re(v_2) > 0$, $\Re(s_1) > 0$, and $\Re(s_2) > 0$.*

*Proof.* Differentiating the extended $k$-hypergeometric function with respect to variable $w$, $p$ times, we have:

$$\frac{d^p}{dw^p}\left\{F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]\right\} = \frac{d^p}{dw^p}\left\{\sum_{n=0}^{\infty}\frac{B_{k,s}^{(s_1,s_2)}(v_1+nk,v_2-v_1)}{B_k(v_1,v_2-v_1)}(v_0)_{n,k}\frac{w^n}{n!}\right\} \tag{6.2.84}$$

$$\frac{d^p}{dw^p}\left\{F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]\right\} = \sum_{n=p}^{\infty}\frac{B_{k,s}^{(s_1,s_2)}(v_1+nk,v_2-v_1)}{B_k(v_1,v_2-v_1)}(v_0)_{n,k}\frac{w^{n-p}}{n!}\frac{n!}{(n-p)!}. \tag{6.2.85}$$

Replacing $n-p$ with $p$, we have:

$$\frac{d^p}{dw^p}\left\{F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]\right\} = \sum_{p=0}^{\infty}\frac{B_{k,s}^{(s_1,s_2)}(v_1+2pk,v_2-v_1)}{B_k(v_1,v_2-v_1)}(v_0)_{2p,k}\frac{w^p}{p!}. \tag{6.2.86}$$

Then, using the properties $(a)_{p+p,k}=(a)_{p,k}(a+pk)_{p,k}$ and $B_k(b,c-b)=\frac{(c)_{p,k}}{(b)_{p,k}}B_k(b+pk,c-b)$ and using the definition of the extended $k$-hypergeometric function with some re-arrangement, we obtain our desired result:

$$\frac{d^p}{dw^p}\left\{F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]\right\} = \frac{(v_0)_{p,k}(v_1)_{p,k}}{(v_2)_{p,k}}F_{k,s}^{(s_1,s_2)}[(v_0+pk,k),(v_1+pk,k);(v_2+pk,k);w]. \tag{6.2.87}$$

Similarly, with the same proof as for Eq. (6.2.82), we obtain Eq. (6.2.83). □

**Corollary 6.2.26.** *The following results hold true:*

$$\frac{d^p}{dw^p}\left\{F_{k,s}^{(s_1)}[(v_0,k),(v_1,k);(v_2,k);w]\right\} = \frac{(v_0)_{p,k}(v_1)_{p,k}}{(v_2)_{p,k}}F_{k,s}^{(s_1)}[(v_0+pk,k),(v_1+pk,k);(v_2+pk,k);w] \tag{6.2.88}$$

$$\frac{d^p}{dw^p}\left\{\phi_{k,s}^{(s_1)}[(v_1,k);(v_2,k);w]\right\} = \frac{(v_1)_{p,k}}{(v_2)_{p,k}}\phi_{k,s}^{(s_1)}[(v_1+pk,k);(v_2+pk,k);w]. \tag{6.2.89}$$

*Proof.* Considering $s_2=1$ in Theorem 6.2.25, we obtain our results. □

**Theorem 6.2.27.** *The extended $k$-hypergeometric and the confluent $k$-hypergeometric functions have the following Mellin transforms that hold true:*

$$M\left\{F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]\right\} = \frac{\Gamma_k^{(s_1,s_2)}(p)B_k(v_1+p,v_2-v_1+p)}{B_k(v_1,v_2-v_1)}F_k[(v_0,k),(v_1+p,k);(v_2+2p,k);w] \tag{6.2.90}$$

$$M\left\{\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w]\right\} = \frac{\Gamma_k^{(s_1,s_2)}(p)B_k(v_1+p,v_2-v_1+p)}{B_k(v_1,v_2-v_1)}\phi_k[(v_1+p,k);(v_2+2p,k);w], \tag{6.2.91}$$

*where* $k>0$, $\Re(v_0)>0$ $\Re(v_1)>0$, $\Re(v_2)>0$, $\Re(s_1)>0$, *and* $\Re(s_2)>0$.

*Proof.* From the definition of the Mellin transform, we have:

$$M\left\{F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]\right\} = \int_0^\infty s^{p-1}F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]ds. \tag{6.2.92}$$

Using the integral form of the extended $k$-hypergeometric function, we have:

$$M\left\{F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]\right\} = \int_0^\infty s^{p-1}\left\{\frac{1}{kB_k(v_1,v_2-v_1)}\int_0^1 t^{\frac{v_1}{k}-1}(1-t)^{\frac{v_2-v_1}{k}-1}(1-kwt)^{-\frac{v_0}{k}} E_{k,s_1,s_2}\left(-s^k(kt(1-t))^{-1}\right)dt\right\}ds. \tag{6.2.93}$$

Interchanging the order of integration, we have:

$$M\left\{F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]\right\} = \frac{1}{kB_k(v_1,v_2-v_1)}\int_0^1 t^{\frac{v_1}{k}-1}(1-t)^{\frac{v_2-v_1}{k}-1}(1-kwt)^{-\frac{v_0}{k}} \left\{\int_0^\infty s^{p-1}E_{k,s_1,s_2}\left(-s^k(kt(1-t))^{-1}\right)ds\right\}dt. \tag{6.2.94}$$

On substituting $u=\frac{s}{t^{\frac{1}{k}}(1-t)^{\frac{1}{k}}}$ and with some re-arrangement we have:

$$M\left\{F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]\right\}$$

$$= \frac{1}{kB_k(v_1, v_2 - v_1)} \int_0^1 t^{\frac{v_1+p}{k}-1}(1-t)^{\frac{v_2-v_1+p}{k}-1}(1-kwt)^{-\frac{v_0}{k}}$$
$$\left\{ \int_0^\infty u^{p-1} E_{k,s_1,s_2}\left(\frac{-u^k}{k}\right) du \right\} dt. \tag{6.2.95}$$

On applying the integral form of the $k$-hypergeometric function and the extended $k$-gamma function (when $s = 0$), we obtain our desired result:

$$M\left\{F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]\right\}$$
$$= \frac{\Gamma_k^{(s_1,s_2)}(p)B_k(v_2+p, v_2-v_1+p)}{B_k(v_1, v_2-v_1)} F_k[(v_0,k),(v_1+p,k);(v_2+2p,k);w]. \tag{6.2.96}$$

Similarly, with the same proof as for Eq. (6.2.90), we obtain Eq. (6.2.91). □

**Corollary 6.2.28.** *The following results hold true:*

$$M\left\{F_{k,s}^{(s_1)}[(v_0,k),(v_1,k);(v_2,k);w]\right\}$$
$$= \frac{\Gamma_k^{(s_1)}(p)B_k(v_1+p, v_2-v_1+p)}{B_k(v_1, v_2-v_1)} F_k[(v_0,k),(v_1+p,k);(v_2+2p,k);w] \tag{6.2.97}$$

$$M\left\{\phi_{k,s}^{(s_1)}[(v_1,k);(v_2,k);w]\right\}$$
$$= \frac{\Gamma_k^{(s_1)}(p)B_k(v_1+p, v_2-v_1+p)}{B_k(v_1, v_2-v_1)} \phi_k[(v_1+p,k);(v_2+2p,k);w]. \tag{6.2.98}$$

*Proof.* Considering $s_2 = 1$ in Theorem 6.2.27, we obtain our results. □

**Theorem 6.2.29.** *The extended $k$-hypergeometric and $k$-confluent hypergeometric functions satisfy the following transformation formulas:*

$$F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]$$
$$= (1-kw)^{\frac{-v_0}{k}} F_{k,s}^{(s_1,s_2)}\left[(v_0,k),(v_2-v_1,k);(v_2,k);\frac{kw}{kw-1}\right] \tag{6.2.99}$$

$$F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);1-\frac{1}{w}]$$
$$= \left(\frac{w}{w+k(1-w)}\right)^{\frac{v_0}{k}} F_{k,s}^{(s_1,s_2)}\left[(v_0,k),(v_2-v_1,k);(v_2,k);\frac{k(1-w)}{w+k(1-w)}\right] \tag{6.2.100}$$

$$F_{k,s}^{(s_1,s_2)}\left[(v_0,k),(v_1,k);(v_2,k);\frac{w}{1+w}\right]$$
$$=\left(\frac{1+w}{1+w-kw)}\right)^{\frac{v_0}{k}}F_{k,s}^{(s_1,s_2)}\left[(v_0,k),(v_2-v_1,k);(v_2,k);\frac{-kw}{1+w-kw}\right] \tag{6.2.101}$$

$$\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w]=e^{w}\phi_{k,s}^{(s_1,s_2)}[(v_2-v_1,k);(v_2,k);-w], \tag{6.2.102}$$

*where* $k>0$, $\Re(v_0)>0$ $\Re(v_1)>0$, $\Re(v_2)>0$, $\Re(s_1)>0$, *and* $\Re(s_2)>0$.

*Proof.* From the integral form of the extended $k$-hypergeometric function, we have:

$$F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]$$
$$=\frac{1}{kB_k(v_1,v_2-v_1)}\int_0^1 t^{\frac{v_1}{k}-1}(1-t)^{\frac{v_2-v_1}{k}-1}(1-kwt)^{-\frac{v_0}{k}}$$
$$E_{k,s_1,s_2}\left(-s^k(kt(1-t))^{-1}\right)dt. \tag{6.2.103}$$

Replacing $t$ by $1-t$ in the above equation with some mathematical re-arrangement and using the integral form of the extended $k$-hypergeometric function, we obtain our desired result (6.2.99):

$$F_{k,s}^{(s_1,s_2)}[(v_0,k),(v_1,k);(v_2,k);w]$$
$$=(1-kw)^{\frac{-v_0}{k}}F_{k,s}^{(s_1,s_2)}\left[(v_0,k),(v_2-v_1,k);(v_2,k);\frac{kw}{kw-1}\right]. \tag{6.2.104}$$

Now, replacing $w$ by $1-\frac{1}{w}$ in Eq. (6.2.99), we obtain our desired result (6.2.100):

$$F_{k,s}^{(s_1,s_2)}\left[(v_0,k),(v_1,k);(v_2,k);1-\frac{1}{w}\right]$$
$$=\left(\frac{w}{w+k(1-w)}\right)^{\frac{v_0}{k}}F_{k,s}^{(s_1,s_2)}\left[(v_0,k),(v_2-v_1,k);(v_2,k);\frac{k(1-w)}{w+k(1-w)}\right]. \tag{6.2.105}$$

Then, replacing $w$ by $\frac{w}{1+w}$ in Eq. (6.2.99), we obtain our desired result (6.2.101):

$$F_{k,s}^{(s_1,s_2)}\left[(v_0,k),(v_1,k);(v_2,k);\frac{w}{1+w}\right]$$
$$=\left(\frac{1+w}{1+w-kw)}\right)^{\frac{v_0}{k}}F_{k,s}^{(s_1,s_2)}\left[(v_0,k),(v_2-v_1,k);(v_2,k);\frac{-kw}{1+w-kw}\right]. \tag{6.2.106}$$

From the integral form of the extended $k$-confluent hypergeometric function, we have:

$$\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w] = \frac{1}{kB_k(v_1,v_2-v_1)}\int_0^1 t^{\frac{v_1}{k}-1}(1-t)^{\frac{v_2-v_1}{k}-1}(e)^{wt}E_{k,s_1,s_2}\left(-s^k(kt(1-t))^{-1}\right)dt. \tag{6.2.107}$$

Replacing $t$ by $1-t$ in the above equation and multiplying and dividing by $B_k(v_2-v_1,v_1)$ with some mathematical re-arrangement and using the integral form of the extended $k$-confluent hypergeometric function, we obtain our desired result (6.2.102):

$$\phi_{k,s}^{(s_1,s_2)}[(v_1,k);(v_2,k);w] = e^w\phi_{k,s}^{(s_1,s_2)}[(v_2-v_1,k);(v_2,k);-w]. \tag{6.2.108}$$

□

**Corollary 6.2.30.** *The following results hold true:*

$$F_{k,s}^{(s_1)}[(v_0,k),(v_1,k);(v_2,k);w] = (1-kw)^{\frac{-v_0}{k}}F_{k,s}^{(s_1)}\left[(v_0,k),(v_2-v_1,k);(v_2,k);\frac{kw}{kw-1}\right] \tag{6.2.109}$$

$$F_{k,s}^{(s_1)}\left[(v_0,k),(v_1,k);(v_2,k);1-\frac{1}{w}\right] = \left(\frac{w}{w+k(1-w)}\right)^{\frac{v_0}{k}}F_{k,s}^{(s_1)}\left[(v_0,k),(v_2-v_1,k);(v_2,k);\frac{k(1-w)}{w+k(1-w)}\right] \tag{6.2.110}$$

$$F_{k,s}^{(s_1)}\left[(v_0,k),(v_1,k);(v_2,k);\frac{w}{1+w}\right] = \left(\frac{1+w}{1+w-kw)}\right)^{\frac{v_0}{k}}F_{k,s}^{(s_1)}\left[(v_0,k),(v_2-v_1,k);(v_2,k);\frac{-kw}{1+w-kw}\right] \tag{6.2.111}$$

$$\phi_{k,s}^{(s_1)}[(v_1,k);(v_2,k);w] = e^w\phi_{k,s}^{(s_1)}[(v_2-v_1,k);(v_2,k);-w]. \tag{6.2.112}$$

*Proof.* Considering $s_2=1$ in Theorem 6.2.29, we obtain our results. □

**Theorem 6.2.31.** *The following generating function for the extended k-hypergeometric function holds true:*

$$\sum_{p=0}^{\infty}(v_0)_{p,k}F_{k,s}^{(s_1,s_2)}[(v_0+p,k),(v_1,k);(v_2,k);w]\frac{z^p}{p!}$$

$$= (1-kz)^{\frac{-v_0}{k}} F_{k,s}^{(s_1,s_2)}\left[(v_0,k),(v_1,k);(v_2,k);\frac{w}{1-kz}\right], \tag{6.2.113}$$

*where* $k>0$, $\Re(v_0)>0$ $\Re(v_1)>0$, $\Re(v_2)>0$, $\Re(s_1)>0$, *and* $\Re(s_2)>0$.

*Proof.* Consider the left-hand side of the above equation and using the definition of the extended $k$-hypergeometric function we have:

$$\sum_{p=0}^{\infty}(v_0)_{p,k}F_{k,s}^{(s_1,s_2)}[(v_0+p,k),(v_1,k);(v_2,k);w]\frac{z^p}{p!}$$
$$=\sum_{p=0}^{\infty}(v_0)_{p,k}\left\{\sum_{n=0}^{\infty}\frac{B_{k,s}^{(s_1,s_2)}(v_1+nk,v_2-v_1)}{B_k(v_1,v_2-v_1)}(v_0+p)_{n,k}\frac{w^n}{n!},\right\}\frac{z^p}{p!}. \tag{6.2.114}$$

Then, using the identity $(a)_{p,k}(a+pk)_{n,k}=(a)_{n,k}(a+nk)_{p,k}$ and the power series $\sum_{n=0}^{\infty}(a)_{n,k}\frac{x^n}{n!}=(1-kx)^{\frac{-a}{k}}$ with some re-arrangement and using the extended $k$-hypergeometric function, we obtain our desired result:

$$\sum_{p=0}^{\infty}(v_0)_{p,k}F_{k,s}^{(s_1,s_2)}[(v_0+p,k),(v_1,k);(v_2,k);w]\frac{z^p}{p!}$$
$$= (1-kz)^{\frac{-v_0}{k}} F_{k,s}^{(s_1,s_2)}\left[(v_0,k),(v_1,k);(v_2,k);\frac{w}{1-kz}\right]. \tag{6.2.115}$$

□

**Corollary 6.2.32.** *The following result holds true:*

$$\sum_{p=0}^{\infty}(v_0)_{p,k}F_{k,s}^{(s_1)}[(v_0+p,k),(v_1,k);(v_2,k);w]\frac{z^p}{p!}$$
$$= (1-kz)^{\frac{-v_0}{k}} F_{k,s}^{(s_1)}\left[(v_0,k),(v_1,k);(v_2,k);\frac{w}{1-kz}\right]. \tag{6.2.116}$$

*Proof.* Considering $s_2=1$ in Theorem 6.2.29, we obtain our results. □

## 6.3 Conclusion

Finally, we conclude this chapter by remarking that here, we have introduced a new extension of $k$-hypergeometric functions by use of the 2-parameter $k$-Mittag-Leffler function as the kernel. Then, we have studied some basic properties of these extensions together with their special cases. Our results in this chapter are general in nature and are very useful for further study of the generalizations of the special functions.

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CHAPTER

# 7 Certain pathway fractional integral formulae involving extended $k$-hypergeometric functions

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## 7.1 Introduction and preliminaries

Fractional calculus has gained considerable attention in the last 5 decades owing to its effectiveness in modeling complex systems characterized by memory and nonlocal effects, with applications spanning mathematical physics, engineering, and applied sciences (for example, see [4–7,10,11,13,14,16,17]). Concurrently, special functions, particularly hypergeometric functions and their various extensions, have played a pivotal role in obtaining analytical solutions to fractional integral and differential equations (for example, see [1–3] and their cited works). Among these, extended k-hypergeometric functions provide a flexible and unifying framework that encompasses several classical and generalized special functions as particular cases (for example, see [8,9,12,15,18–27]). In this chapter, we establish certain pathway fractional integral formulae involving extended k-hypergeometric functions in the context of fractional calculus. The derived results generalize a number of existing integral identities and offer a systematic methodology for generating new fractional integral relations. These findings enhance the theoretical foundations of fractional integral operators and are expected to be useful in further analytical studies and potential applications.

The $k$-gamma function is expressed in the way shown below ([3] page no. 3 pro. 5):

$$\Gamma_k(q_1) = \int_0^\infty u^{q_1-1} e^{-u^k k^{-1}} du \qquad (7.1.1)$$

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and

$$\Gamma_k(q_1+k)=q_1\Gamma_k(q_1), \tag{7.1.2}$$

where, $k>0$ and $\Re(q_1)>0$.

The Pochhammer $k$-symbol is described as follows ([3] page no. 4 pro. 6):

$$(q_1)_{n,k}:=\frac{\Gamma_k(q_1+nk)}{\Gamma_k(q_1)}=\begin{cases}1 & n=0;\, q_1\in\mathbb{C}\setminus\{0\},\\ q_1(q_1+k)\cdots(q_1+(n-1)k) & n\in\mathbb{N};\, q_1\in\mathbb{C}.\end{cases} \tag{7.1.3}$$

The $k$-beta function is expressed in the way shown below ([3] page no. 7 pro. 14):

$$B_k(q_1,q_2)=\frac{1}{k}\int_0^1 u^{\frac{q_1}{k}-1}(1-u)^{\frac{q_2}{k}-1}du, \tag{7.1.4}$$

where $k>0$, $\Re(q_1)$ and $\Re(q_2)>0$.

For $k>0$, the generalized $k$-hypergeometric function is expressed in the way shown below ([15] page no. 381 Eq. (2.10)):

$${}_pF_{q,k}[(c_1,k),(c_2,k)...(c_p,k);(d_1,k),(d_2,k)...(d_q,k);w]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(c_i)_{n,k}}{\prod_{i=1}^{q}(d_i)_{n,k}}\frac{w^n}{n!}, \tag{7.1.5}$$

where $|w|<\frac{1}{k}$, $\Re(c_i)>0$, $\Re(d_i)>0$ and $(d_1)_{n,k}$ is the $k$-Pochhammer symbol.

When we put $p=2$, $q=1$, and $p=1$, $q=1$ in Eq. (7.1.5), we have the $k$-hypergeometric function and the confluent $k$-hypergeometric function defined as ([4] page no. 2 Eq. (1.4)):

$${}_2F_{1,k}[(c_0,k),(c_1,k);(c_2,k);x]=\sum_{n=0}^{\infty}\frac{(c_0)_{n,k}(c_1)_{n,k}}{(c_2)_{n,k}}\frac{w^n}{n!}, \tag{7.1.6}$$

where, $k>0$, $\Re(c_0)>0$, $\Re(c_1)>0$, $\Re(c_2)>0$, and $|w|<\frac{1}{k}$ and $(c_1)_{n,k}$ is the $k$-Pochhammer symbol:

$${}_1F_{1,k}[(c_1,k);(c_2,k);w]=\phi[(c_1,k);(c_2,k);w]=\sum_{n=0}^{\infty}\frac{(c_1)_{n,k}}{(c_2)_{n,k}}\frac{w^n}{n!}, \tag{7.1.7}$$

where, $k>0$, $\Re(c_1)>0$, $\Re(c_2)>0$.

Series representations of the $k$-hypergeometric and the confluent $k$-hypergeometric functions are given as follows:

$${}_2F_{1,k}[(c_0,k),(c_1,k);(c_2,k);w]=\sum_{n=0}^{\infty}(c_0)_{n,k}\frac{B_k(c_1+nk,c_2-c_1)}{B_k(c_1,c_2-c_1)}\frac{w^n}{n!} \tag{7.1.8}$$

and

$$ {}_1F_{1,k}[(c_1,k);(c_2,k);w] = \sum_{n=0}^{\infty} \frac{B_k(c_1+nk, c_2-c_1)}{B_k(c_1,c_2-c_1)} \frac{w^n}{n!}, \tag{7.1.9} $$

where, $k>0$, $\Re(c_0)>0$, $\Re(c_1)>0$, $\Re(c_2)>0$, and $|w|<\frac{1}{k}$ and $B_k(c_1,c_2)$ is the $k$-beta function.

Very recently, Laxmi et al. investigated the extended $k$-gamma and the $k$-beta functions by using the 2-parameter $k$-Mittag-Leffler function as kernel and defined as follows (see,[17]):

$$ \Gamma_{k,b}^{(b_1,b_2)}(q_1) = \int_0^{\infty} t^{q_1-1} E_{k,b_1,b_2}\left(-\frac{t^k}{k} - \frac{b^k t^{-k}}{k}\right) dt, \tag{7.1.10} $$

where, $k>0$, $\Re(q_1)>0$, $b\geq 0$, $\Re(b_1)>0$, $\Re(b_2)>0$ and $E_{k,b_1,b_2}(z)$ is defined in (7.1.13) and

$$ B_{k,b}^{(b_1,b_2)}(q_1,q_2) = \frac{1}{k}\int_0^1 t^{\frac{q_1}{k}-1}(1-t)^{\frac{q_2}{k}-1} E_{k,b_1,b_2}\left(\frac{-b^k}{kt(1-t)}\right) dt, \tag{7.1.11} $$

where, $k>0$, $\min\{\Re(q_1), \Re(q_2)\}>0$, $b\geq 0$, $\Re(b_1)>0$, $\Re(b_2)>0$ and $E_{k,b_1,b_2}(z)$ is defined in (7.1.13).

To obtain our main result we need to define the $k$-Mittag-Leffler functions ([5] page no. 6 Eqs. (2.1) and (2.2)) as follows:

$$ E_{k,b_1}(w) = \sum_{n=0}^{\infty} \frac{w^n}{\Gamma_k(nb_1+1)}, \qquad k>0, \Re(b_1)\geq 0, w\in C, \tag{7.1.12} $$

$$ E_{k,b_1,b_2}(w) = \sum_{n=0}^{\infty} \frac{w^n}{\Gamma_k(nb_1+b_2)}, \qquad k>0, \Re(b_1)\geq 0, \Re(b_2)\geq 0, w\in C. \tag{7.1.13} $$

**Note.** If we consider the variable $b_1=b_2=1$ in Eq. (7.1.13), then we obtain the $k$-exponential function $E_k(z)$.

Very recently, Laxmi et al. investigated the new extended $k$-hypergeometric function and the confluent $k$-hypergeometric function by using the extended $k$-beta function (7.1.11) as kernel and defined as follows.

A new extended $k$-hypergeometric function is defined as follows:

$$ F_{k,b}^{(b_1,b_2)}[(q_1,k),(q_2,k);(q_3,k);w] = \sum_{n=0}^{\infty} \frac{B_{k,b}^{(b_1,b_2)}(q_2+nk, q_3-q_2)}{B_k(q_2,q_3-q_2)} (q_1)_{n,k} \frac{w^n}{n!}, \tag{7.1.14} $$

where, $k>0$, $\Re(q_1)>0$, $\Re(q_2)>0$, $\Re(q_3)>0$, $\Re(b_1)>0$, $\Re(b_2)>0$, and $|w|<\frac{1}{k}$ and $B_{k,b}^{(b_1,b_2)}(q_1,q_2)$ is the extended $k$-beta function.

A new extended $k$-confluent hypergeometric function is defined as follows:

$$\phi_{k,b}^{(b_1,b_2)}[(q_2,k);(q_3,k);w]=\sum_{n=0}^{\infty}\frac{B_{k,b}^{(b_1,b_2)}(q_2+nk,q_3-q_2)}{B_k(q_2,q_3-q_2)}\frac{w^n}{n!}, \tag{7.1.15}$$

where, $k>0$, $\Re(q_2)>0$, $\Re(q_3)>0$, $\Re(b_1)>0$, $\Re(b_2)>0$ and $B_{k,b}^{(b_1,b_2)}(q_1,q_2)$ is the extended $k$-beta function.

In recent decades, several scholars have studied the applications and different generalizations of various hypergeometric operators of fractional integration.

Using the pathway concept developed by Mathai and Haubold [5,6], Nair [10] built a pathway fractional integral operator in 2009.

Let $h(w)\in L(c,d)$, $\nu\in\mathbb{C}$ such that $\Re(\nu)>0$, $e>0$ and consider a pathway parameter $\delta<1$, then the pathway fractional integration operator is given as follows ([16] page no. 2 Eq. (12)):

$$\left(P_{0+}^{\nu,\delta}h\right)(x)=x^{\nu}\int_0^{\left[\frac{x}{e(1-\delta)}\right]}\left[1-\frac{e(1-\delta)t}{x}\right]^{\frac{\nu}{1-\delta}}h(t)dt. \tag{7.1.16}$$

The subsequent probability density function (p.d.f.) for the pathway model for scalar random variables is given as follows:

$$h(x)=k|x|^{\delta-1}\left[1-e(1-\delta)|x|^{\gamma}\right]^{\frac{\alpha}{(1-\delta)}}, \tag{7.1.17}$$

given that, $\delta\in\mathbb{R}$ and $-\infty<x<\infty$, $\gamma>0$, $\alpha>0$, $\left[1-e(1-\delta)|x|^{\gamma}\right]>0$, $\delta>0$, where $\delta$ is called the pathway parameter and $k$ is the normalizing constant.

When $\delta\to 1_{-}$:

$$\left[1-\frac{e(1-\delta)t}{x}\right]^{\frac{\nu}{(1-\delta)}}\to exp(\frac{-e\nu t}{x}). \tag{7.1.18}$$

Then, we observe that operator (7.1.16) reduces to the Laplace integral transform of $h$ with parameter $\frac{e\nu}{x}$ ([16] page no. 2 Eq. (18)):

$$\left(P_{0+}^{\nu,1}h\right)(x)=x^{\nu}\int_0^{\infty}exp(\frac{-e\nu t}{x})h(t)dt=L_h(\frac{e\nu}{x}). \tag{7.1.19}$$

If, we put $\delta=0$, $e=1$, and changing $\nu$ by $\nu-1$ in (7.1.16) the integral operator becomes the Riemann–Liouville fractional integral operator (see, for example, [11–14]).

## 7.2 Main results

Here, we find the image formula for the extended $k$-hypergeometric function (7.1.14) and the confluent $k$-hypergeometric function (7.1.15) involving pathway fractional integral formulae.

**Theorem 7.2.1.** *Assume* $k>0$, $\Re(b_2)>\Re(b_1)>0$, $\Re(c_1)>0$, $\Re(c_2)>0$, $b\geq 0$, $|wt|<\frac{1}{k}$, *and* $0<\delta<1$, *then the following result holds true ([16] page no. 3 Eq. (20)):*

$$\left(P_{0+}^{\nu,\delta}t^{\frac{\gamma}{k}-1}F_{k,b}^{(b_1,b_2)}[(c_0,k),(c_1,k);(c_2,k);wt]\right)(x)=kx^{\nu}\left[\frac{x}{e(1-\delta)}\right]^{\frac{\gamma}{k}}$$
$$B_k\left(\frac{\nu k}{1-\delta}+k,\gamma\right){}_1F_{k,b,1}^{(b_1,b_2)}\left[\begin{array}{cc}(c_0,k),(c_1,k),(\gamma,k); & \frac{wx}{e(1-\delta)}\\ (c_2,k),(\gamma+\frac{\nu k}{1-\delta}+k,k); & \end{array}\right]. \tag{7.2.1}$$

*Proof.* Consider the left-hand side of Eq. (7.2.1) and denote it by $L$, then using the definition given in Eq. (7.1.16), we obtain:

$$L=x^{\nu}\int_0^{\left[\frac{x}{e(1-\delta)}\right]}\left[1-\frac{e(1-\delta)t}{x}\right]^{\frac{\nu}{1-\delta}}t^{\frac{\gamma}{k}-1}\left\{F_{k,b}^{(b_1,b_2)}[(c_0,k),(c_1,k);(c_2,k);wt]\right\}dt. \tag{7.2.2}$$

Then, using the definition given in Eq. (7.1.14), we have:

$$L=x^{\nu}\int_0^{\left[\frac{x}{e(1-\delta)}\right]}\left[1-\frac{e(1-\delta)t}{x}\right]^{\frac{\nu}{1-\delta}}t^{\frac{\gamma}{k}-1}$$
$$\left\{\sum_{n=0}^{\infty}\frac{B_{k,b}^{(b_1,b_2)}(c_1+nk,c_2-c_1)}{B(c_1,c_2-c_1)}(c_0)_{n.k}\frac{(wt)^n}{n!}\right\}dt. \tag{7.2.3}$$

By transposing the sequence of summation and integration, we obtain:

$$L=x^{\nu}\sum_{n=0}^{\infty}\frac{B_{k,b}^{(b_1,b_2)}(c_1+nk,c_2-c_1)}{B(c_1,c_2-c_1)}(c_0)_{n,k}\frac{w^n}{n!}$$
$$\left\{\int_0^{\left[\frac{x}{e(1-\delta)}\right]}\left[1-\frac{e(1-\delta)t}{x}\right]^{\frac{\nu}{1-\delta}}t^{\frac{\gamma}{k}+n-1}\right\}dt. \tag{7.2.4}$$

Substituting $v=\frac{e(1-\delta)t}{x}$, in Eq. (7.2.4), we have:

$$L=x^{\nu}\sum_{n=0}^{\infty}\frac{B_{k,b}^{(b_1,b_2)}(c_1+nk,c_2-c_1)}{B(c_1,c_2-c_1)}(c_0)_{n,k}\frac{w^n}{n!}\left[\frac{x}{e(1-\delta)}\right]^{\frac{\gamma}{k}+n}$$
$$\left\{\int_0^1(1-v)^{\frac{\nu}{1-\delta}}v^{\frac{\gamma}{k}+n-1}dv\right\}. \tag{7.2.5}$$

By using Eq. (7.1.4), the properties of the $k$-Pochhammer Symbol $(a)_{n,k}=\frac{\Gamma_k(a+nk)}{\Gamma_k(a)}$, and the properties of the $k$-beta function $B_k(a,b)=\frac{\Gamma_k(a)\Gamma_k(b)}{\Gamma_k(a+b)}$, simplifying the above equation we obtain:

$$L = kx^{\nu}\left[\frac{x}{e(1-\delta)}\right]^{\frac{\gamma}{k}} B_k\left(\frac{\nu k}{1-\delta}+k,\gamma\right) \times \sum_{n=0}^{\infty} \frac{B_{k,b}^{(b_1,b_2)}(c_1+nk, c_2-c_1)}{B_k(c_1,c_2-c_1)}(c_0)_{n,k} \frac{(\gamma)_{n,k}}{\left(\frac{\nu k}{1-\delta}+k+\gamma\right)_{n,k}} \frac{\left(\frac{wx}{e(1-\delta)}\right)^n}{n!}. \tag{7.2.6}$$

Then, in view of the generalized $k$-hypergeometric function (7.1.5), we obtain our main result of Theorem 7.2.1:

$$\left(P_{0+}^{\nu,\delta} t^{\frac{\gamma}{k}-1} F_{k,b}^{(b_1,b_2)}[(c_0,k),(c_1,k);(c_2,k);wt]\right)(x) = kx^{\nu}\left[\frac{x}{e(1-\delta)}\right]^{\frac{\gamma}{k}} B_k\left(\frac{\nu k}{1-\delta}+k,\gamma\right) {}_1F_{k,b,1}^{(b_1,b_2)}\left[\begin{matrix}(c_0,k),(c_1,k),(\gamma,k); \\ (c_2,k),(\gamma+\frac{\nu k}{1-\delta}+k,k);\end{matrix}\; \frac{wx}{e(1-\delta)}\right]. \tag{7.2.7}$$

□

**Theorem 7.2.2.** *Assume $k>0$, $\Re(b_2)>\Re(b_1)>0$, $\Re(c_1)>0$, $\Re(c_2)>0$, $b\geq 0$, $|wt|<\frac{1}{k}$, and $0<\delta<1$, then the following result holds true ([16] page no. 3 Eq. (27)):*

$$\left(P_{0+}^{\nu,\delta} t^{\frac{\gamma}{k}-1} \phi_{k,b}^{(b_1,b_2)}[(c_1,k);(c_2,k);wt]\right)(x) = kx^{\nu}\left[\frac{x}{e(1-\delta)}\right]^{\frac{\gamma}{k}} B_k\left(\frac{\nu k}{1-\delta}+k,\gamma\right) {}_1\phi_{k,b,1}^{(b_1,b_2)}\left[\begin{matrix}(c_1,k),(\gamma,k); \\ (c_2,k),(\gamma+\frac{\nu k}{1-\delta}+k,k);\end{matrix}\; \frac{wx}{e(1-\delta)}\right]. \tag{7.2.8}$$

*Proof.* Consider the left-hand side of Eq. (7.2.8) and denote it by $M$, then using the definition given in Eq. (7.1.16), we obtain:

$$M = x^{\nu}\int_0^{\left[\frac{x}{e(1-\delta)}\right]}\left[1-\frac{e(1-\delta)t}{x}\right]^{\frac{\nu}{1-\delta}} t^{\frac{\gamma}{k}-1}\left\{\phi_{k,b}^{(b_1,b_2)}[(c_1,k);(c_2,k);wt]\right\}dt. \tag{7.2.9}$$

Then, using the definition given in Eq. (7.1.15), we have:

$$M = x^{\nu}\int_0^{\left[\frac{x}{e(1-\delta)}\right]}\left[1-\frac{e(1-\delta)t}{x}\right]^{\frac{\nu}{1-\delta}} t^{\frac{\gamma}{k}-1} \left\{\sum_{n=0}^{\infty}\frac{B_{k,b}^{(b_1,b_2)}(c_1+nk,c_2-c_1)}{B(c_1,c_2-c_1)}\frac{(wt)^n}{n!}\right\}dt. \tag{7.2.10}$$

By transposing the sequence of summation and integration, we obtain:

$$M = x^{\nu} \sum_{n=0}^{\infty} \frac{B_{k,b}^{(b_1,b_2)}(c_1+nk, c_2-c_1)}{B(c_1, c_2-c_1)} \frac{w^n}{n!}$$
$$\left\{ \int_0^{\left[\frac{x}{e(1-\delta)}\right]} \left[1 - \frac{e(1-\delta)t}{x}\right]^{\frac{\nu}{1-\delta}} t^{\frac{\gamma}{k}+n-1} dt \right\}. \quad (7.2.11)$$

Substituting $v = \frac{e(1-\delta)t}{x}$, in Eq. (7.2.11), we have:

$$M = x^{\nu} \sum_{n=0}^{\infty} \frac{B_{k,b}^{(b_1,b_2)}(c_1+nk, c_2-c_1)}{B(c_1, c_2-c_1)} \frac{w^n}{n!} \left[\frac{x}{e(1-\delta)}\right]^{\frac{\gamma}{k}+n}$$
$$\left\{ \int_0^1 (1-v)^{\frac{\nu}{1-\delta}} v^{\frac{\gamma}{k}+n-1} dv \right\}. \quad (7.2.12)$$

By using Eq. (7.1.4), the properties of the $k$-Pochhammer's Symbol $(a)_{n,k} = \frac{\Gamma_k(a+nk)}{\Gamma_k(a)}$, and the properties of the $k$-beta function $B_k(a,b) = \frac{\Gamma_k(a)\Gamma_k(b)}{\Gamma_k(a+b)}$, simplifying the above equation we obtain:

$$M = kx^{\nu} \left[\frac{x}{e(1-\delta)}\right]^{\frac{\gamma}{k}} B_k\left(\frac{\nu k}{1-\delta}+k, \gamma\right) \times \sum_{n=0}^{\infty} \frac{B_{k,b}^{(b_1,b_2)}(c_1+nk, c_2-c_1)}{B(c_1, c_2-c_1)}$$
$$\frac{(\gamma)_{n,k}}{\left(\frac{\nu k}{1-\delta}+k+\gamma\right)_{n,k}} \frac{\left(\frac{wx}{e(1-\delta)}\right)^n}{n!}. \quad (7.2.13)$$

Then, in view of the generalized $k$-hypergeometric function (7.1.5), we obtain our main result of Theorem 7.2.2:

$$\left(P_{0+}^{\nu,\delta} t^{\frac{\gamma}{k}-1} \phi_{k,b}^{(b_1,b_2)}[(c_1,k);(c_2,k);wt]\right)(x) = kx^{\nu} \left[\frac{x}{e(1-\delta)}\right]^{\frac{\gamma}{k}}$$
$$B_k\left(\frac{\nu k}{1-\delta}+k, \gamma\right) {}_1\phi_{k,b,1}^{(b_1,b_2)} \left[\begin{matrix} (c_1,k),(\gamma,k); \\ (c_2,k),(\gamma+\frac{\nu k}{1-\delta}+k,k); \end{matrix} \ \frac{wx}{e(1-\delta)}\right]. \quad (7.2.14)$$

□

**Theorem 7.2.3.** *If $k>0$, $\Re(b_2) > \Re(b_1) > 0$, $\Re(c_1) > 0$, $\Re(c_2) > 0$, $b \geq 0$, $|wt| < \frac{1}{k}$, and $\delta = 1$, then following result holds true ([16] page no. 5 Eq. (38)):*

$$\left(P_{0+}^{\nu,1} t^{\frac{\gamma}{k}-1} F_{k,b}^{(b_1,b_2)}[(c_0,k),(c_1,k);(c_2,k);wt]\right)(x) = kx^{\nu}\Gamma_k(\gamma)\left(\frac{x}{ke\nu}\right)^{\frac{\gamma}{k}}$$

$$\times {}_1F_{k,b}^{(b_1,b_2)}\left[\begin{array}{cc}(c_0,k),(c_1,k),(\gamma,k); & \frac{wx}{kev}\\ (c_2,k); & \end{array}\right]. \tag{7.2.15}$$

*Proof.* To prove Theorem 7.2.3, consider the left-hand side of the above equation and denote it by $N$:

$$N=\left(P_{0+}^{\nu,1}t^{\frac{\gamma}{k}-1}F_{k,b}^{(b_1,b_2)}[(c_0,k),(c_1,k);(c_2,k);wt]\right)(x). \tag{7.2.16}$$

By using the result (7.1.19), we have

$$N=x^{\nu}\int_0^{\infty}exp\left(\frac{-evt}{x}\right)t^{\frac{\gamma}{k}-1}\left\{F_{k,b}^{(b_1,b_2)}[(c_0,k),(c_1,k);(c_2,k);wt]\right\}dt. \tag{7.2.17}$$

From use of Eq. (7.1.14), we obtain:

$$N=x^{\nu}\int_0^{\infty}exp\left(\frac{-evt}{x}\right)t^{\frac{\gamma}{k}-1}\left\{\sum_{n=0}^{\infty}\frac{B_{k,b}^{(b_1,b_2)}(c_1+nk,c_2-c_1)}{B_k(c_1,c_2-c_1)}(c_0)_{n,k}\frac{(wt)^n}{n!}\right\}dt. \tag{7.2.18}$$

By transposing the sequence of summation and integration, we obtain:

$$N=x^{\nu}\sum_{n=0}^{\infty}\frac{B_{k,b}^{(b_1,b_2)}(c_1+nk,c_2-c_1)}{B_k(c_1,c_2-c_1)}(c_0)_{n,k}\frac{w^n}{n!}\left\{\int_0^{\infty}exp\left(\frac{-evt}{x}\right)t^{n+\frac{\gamma}{k}-1}dt\right\}. \tag{7.2.19}$$

Putting $l=\frac{evt}{x}$ and re-arranging the terms, we obtain:

$$N=x^{\nu}\left(\frac{x}{ev}\right)^{\frac{\gamma}{k}}\sum_{n=0}^{\infty}\frac{B_{k,b}^{(b_1,b_2)}(c_1+nk,c_2-c_1)}{B_k(c_1,c_2-c_1)}(c_0)_{n,k}\frac{(\frac{wx}{ev})^n}{n!}\left\{\int_0^{\infty}e^{(-l)}l^{n+\frac{\gamma}{k}-1}dl\right\}. \tag{7.2.20}$$

By the definition of the gamma function, the $k$-Pochhammer Symbol $(a)_{n,k}=\frac{\Gamma_k(a+nk)}{\Gamma_k(a)}$, and the property of gamma function $\Gamma_k(ak)=k^{a-1}\Gamma(a)$, we have:

$$N=x^{\nu}\left(\frac{x}{ev}\right)^{\frac{\gamma}{k}}\sum_{n=0}^{\infty}\frac{B_{k,b}^{(b_1,b_2)}(c_1+nk,c_2-c_1)}{B_k(c_1,c_2-c_1)}(c_0)_{n,k}\frac{(\frac{wx}{ev})^n}{n!}\Gamma_k\left(\frac{\gamma}{k}+n\right), \tag{7.2.21}$$

$$N=k\Gamma_k(\gamma)x^{\nu}\left(\frac{x}{kev}\right)^{\frac{\gamma}{k}}\left\{\sum_{n=0}^{\infty}\frac{B_{k,b}^{(b_1,b_2)}(c_1+nk,c_2-c_1)}{B_k(c_1,c_2-c_1)}(c_0)_{n,k}(\gamma)_{n,k}\frac{(\frac{wx}{kev})^n}{n!}\right\}. \tag{7.2.22}$$

Then, in view of the generalized $k$-hypergeometric function (7.1.5), we obtain our main result of Theorem 7.2.3:

$$\left(P_{0+}^{\nu,1}t^{\frac{\gamma}{k}-1}F_{k,b}^{(b_1,b_2)}[(c_0,k),(c_1,k);(c_2,k);wt]\right)(x)=kx^{\nu}\Gamma_k(\gamma)\left(\frac{x}{k e\nu}\right)^{\frac{\gamma}{k}}$$
$$\times{}_1F_{k,b}^{(b_1,b_2)}\left[\begin{array}{cc}(c_0,k),(c_1,k),(\gamma,k); & \frac{wx}{ke\nu}\\ (c_2,k); & \end{array}\right]. \tag{7.2.23}$$

□

**Theorem 7.2.4.** *If* $k>0$*,* $\Re(b_2)>\Re(b_1)>0$*,* $\Re(c_1)>0$*,* $\Re(c_2)>0$*,* $b\geq 0$*,* $|wt|<\frac{1}{k}$*, and* $\delta=1$*, then the following result holds true ([16] page no. 5 Eq. (46)):*

$$\left(P_{0+}^{\nu,1}t^{\frac{\gamma}{k}-1}\phi_{k,b}^{(b_1,b_2)}[(c_1,k);(c_2,k);wt]\right)(x)=kx^{\nu}\Gamma_k(\gamma)\left(\frac{x}{k e\nu}\right)^{\frac{\gamma}{k}}$$
$$\times{}_1\phi_{k,b}^{(b_1,b_2)}\left[\begin{array}{cc}(c_1,k),(\gamma,k); & \frac{wx}{k\nu e}\\ (c_2,k); & \end{array}\right]. \tag{7.2.24}$$

*Proof.* To prove Theorem 7.2.4, consider the left-hand side of the above equation and denote it by $R$:

$$R=\left(P_{0+}^{\nu,1}t^{\frac{\gamma}{k}-1}\phi_{k,b}^{(b_1,b_2)}[(c_1,k);(c_2,k);wt]\right)(x). \tag{7.2.25}$$

By using the result (7.1.19), we have:

$$R=x^{\nu}\int_0^{\infty}exp\left(\frac{-e\nu t}{x}\right)t^{\frac{\gamma}{k}-1}\phi_{k,b}^{(b_1,b_2)}[(c_1,k);(c_2,k);wt]dt. \tag{7.2.26}$$

From the use of Eq. (7.1.15), we obtain:

$$R=x^{\nu}\int_0^{\infty}exp\left(\frac{-e\nu t}{x}\right)t^{\frac{\gamma}{k}-1}\left\{\sum_{n=0}^{\infty}\frac{B_{k,b}^{(b_1,b_2)}(c_1+nk,c_2-c_1)}{B_k(c_1,c_2-c_1)}\frac{(wt)^n}{n!}\right\}dt. \tag{7.2.27}$$

By transposing the sequence of summation and integration, we obtain:

$$R=x^{\nu}\sum_{n=0}^{\infty}\frac{B_{k,b}^{(b_1,b_2)}(c_1+nk,c_2-c_1)}{B_k(c_1,c_2-c_1)}\frac{w^n}{n!}\left\{\int_0^{\infty}exp\left(\frac{-e\nu t}{x}\right)t^{n+\frac{\gamma}{k}-1}dt\right\}. \tag{7.2.28}$$

Putting $l=\frac{e\nu t}{x}$ and re-arranging the terms, we obtain:

$$R=x^{\nu}\left(\frac{x}{e\nu}\right)^{\frac{\gamma}{k}}\sum_{n=0}^{\infty}\frac{B_{k,b}^{(b_1,b_2)}(c_1+nk,c_2-c_1)}{B_k(c_1,c_2-c_1)}\frac{(\frac{wx}{e\nu})^n}{n!}\left\{\int_0^{\infty}e^{(-l)}l^{n+\frac{\gamma}{k}-1}dl\right\}. \tag{7.2.29}$$

By the definition of the gamma function, the $k$-Pochhammer Symbol $(a)_{n,k} = \frac{\Gamma_k(a+nk)}{\Gamma_k(a)}$, and the property of the gamma function $\Gamma_k(ak) = k^{a-1}\Gamma(a)$, we have:

$$R = x^{\nu}\left(\frac{x}{e\nu}\right)^{\frac{\gamma}{k}} \sum_{n=0}^{\infty} \frac{B_{k,b}^{(b_1,b_2)}(c_1+nk, c_2-c_1)}{B_k(c_1, c_2-c_1)} \frac{(\frac{wx}{e\nu})^n}{n!} \Gamma_k\left(\frac{\gamma}{k}+n\right), \tag{7.2.30}$$

$$R = k\Gamma_k(\gamma)x^{\nu}\left(\frac{x}{ke\nu}\right)^{\frac{\gamma}{k}} \left\{ \sum_{n=0}^{\infty} \frac{B_{k,b}^{(b_1,b_2)}(c_1+nk, c_2-c_1)}{B_k(c_1, c_2-c_1)} (\gamma)_{n,k} \frac{(\frac{wx}{ke\nu})^n}{n!} \right\}. \tag{7.2.31}$$

Then, in view of the generalized $k$-hypergeometric function (7.1.5), we obtain our main result of Theorem 7.2.4:

$$\left(P_{0+}^{\nu,1} t^{\frac{\gamma}{k}-1} \phi_{k,b}^{(b_1,b_2)}[(c_1,k);(c_2,k);wt]\right)(x) = kx^{\nu}\Gamma_k(\gamma)\left(\frac{x}{ke\nu}\right)^{\frac{\gamma}{k}} \times {}_1\phi_{k,b}^{(b_1,b_2)}\left[\begin{matrix}(c_1,k),(\gamma,k); & \frac{wx}{ke\nu} \\ (c_2,k); & \end{matrix}\right]. \tag{7.2.32}$$

□

## 7.3 Special cases

In this section, we discuss some special cases by substituting particular values into the parameters.

**Corollary 7.3.1.** *Assume* $k>0$, $\Re(b_1)>0$, $\Re(c_1)>0$, $\Re(c_2)>0$, $b \geq 0$, $|w| < \frac{1}{k}$, *and* $0<\delta<1$, *then the following result holds true:*

$$\left(P_{0+}^{\nu,\delta} t^{\frac{\gamma}{k}-1} F_{k,b}^{(b_1)}[(c_0,k),(c_1,k);(c_2,k);wt]\right)(x) = kx^{\nu}\left[\frac{x}{e(1-\delta)}\right]^{\frac{\gamma}{k}} B_k\left(\frac{\nu k}{1-\delta}+k,\gamma\right) {}_1F_{k,b,1}^{(b_1)}\left[\begin{matrix}(c_0,k),(c_1,k),(\gamma,k); & \frac{wx}{e(1-\delta)} \\ (c_2,k),(\gamma+\frac{\nu k}{1-\delta}+k,k); & \end{matrix}\right]. \tag{7.3.1}$$

**Corollary 7.3.2.** *Assume* $k>0$, $\Re(b_1)>0$, $\Re(c_1)>0$, $\Re(c_2)>0$, $b \geq 0$, *and* $0 < \delta < 1$, *then the following result holds true:*

$$\left(P_{0+}^{\nu,\delta} t^{\frac{\gamma}{k}-1} \phi_{k,b}^{(b_1)}[(c_1,k);(c_2,k);wt]\right)(x) = kx^{\nu}\left[\frac{x}{e(1-\delta)}\right]^{\frac{\gamma}{k}} B_k\left(\frac{\nu k}{1-\delta}+k,\gamma\right) {}_1\phi_{k,b,1}^{(b_1)}\left[\begin{matrix}(c_1,k),(\gamma,k); & \frac{wx}{e(1-\delta)} \\ (c_2,k),(\gamma+\frac{\nu k}{1-\delta}+k,k); & \end{matrix}\right]. \tag{7.3.2}$$

If we consider $b_2 = 1$, in Theorem 7.2.1 and Theorem 7.2.2, we obtain Corollary 7.3.1 and Corollary 7.3.2.

**Corollary 7.3.3.** *Assume* $k > 0$, $\Re(c_1) > 0$, $\Re(c_2) > 0$, $b \geq 0$, $|w| < \frac{1}{k}$, *and* $0 < \delta < 1$, *then the following result holds true:*

$$\left(P_{0+}^{\nu,\delta} t^{\frac{\gamma}{k}-1} F_{k,b}[(c_0,k),(c_1,k);(c_2,k);wt]\right)(x) = kx^{\nu}\left[\frac{x}{e(1-\delta)}\right]^{\frac{\gamma}{k}} B_k\left(\frac{\nu k}{1-\delta}+k,\gamma\right) {}_1F_{k,b,1}\left[\begin{array}{c} (c_0,k),(c_1,k),(\gamma,k); \\ (c_2,k),(\gamma+\frac{\nu k}{1-\delta}+k,k); \end{array} \frac{wx}{e(1-\delta)}\right]. \tag{7.3.3}$$

**Corollary 7.3.4.** *Assume* $k > 0$, $\Re(c_1) > 0$, $\Re(c_2) > 0$, $b \geq 0$, *and* $0 < \delta < 1$, *then the following result holds true:*

$$\left(P_{0+}^{\nu,\delta} t^{\frac{\gamma}{k}-1} \phi_{k,b}[(c_1,k);(c_2,k);wt]\right)(x) = kx^{\nu}\left[\frac{x}{e(1-\delta)}\right]^{\frac{\gamma}{k}} B_k\left(\frac{\nu k}{1-\delta}+k,\gamma\right) {}_1\phi_{k,b,1}\left[\begin{array}{c} (c_1,k),(\gamma,k); \\ (c_2,k),(\gamma+\frac{\nu k}{1-\delta}+k,k); \end{array} \frac{wx}{e(1-\delta)}\right]. \tag{7.3.4}$$

If we consider $b_1 = b_2 = 1$, in Theorem 7.2.1 and Theorem 7.2.2, we obtain Corollary 7.3.3 and Corollary 7.3.4.

**Corollary 7.3.5.** *Assume* $k > 0$, $\Re(c_1) > 0$, $\Re(c_2) > 0$, $|w| < \frac{1}{k}$, *and* $0 < \delta < 1$, *then the following result holds true:*

$$\left(P_{0+}^{\nu,\delta} t^{\frac{\gamma}{k}-1} {}_2F_{1,k}[(c_0,k),(c_1,k);(c_2,k);wt]\right)(x) = kx^{\nu}\left[\frac{x}{e(1-\delta)}\right]^{\frac{\gamma}{k}} B_k\left(\frac{\nu k}{1-\delta}+k,\gamma\right) {}_3F_{2,k}\left[\begin{array}{c} (c_0,k),(c_1,k),(\gamma,k); \\ (c_2,k),(\gamma+\frac{\nu k}{1-\delta}+k,k); \end{array} \frac{wx}{e(1-\delta)}\right]. \tag{7.3.5}$$

**Corollary 7.3.6.** *Assume* $k > 0$, $\Re(c_1) > 0$, $\Re(c_2) > 0$, $b \geq 0$, *and* $0 < \delta < 1$, *then the following result holds true:*

$$\left(P_{0+}^{\nu,\delta} t^{\frac{\gamma}{k}-1} {}_1F_{1,k}[(c_1,k);(c_2,k);wt]\right)(x) = kx^{\nu}\left[\frac{x}{e(1-\delta)}\right]^{\frac{\gamma}{k}} B_k\left(\frac{\nu k}{1-\delta}+k,\gamma\right) {}_2\phi_{2,k}\left[\begin{array}{c} (c_1,k),(\gamma,k); \\ (c_2,k),(\gamma+\frac{\nu k}{1-\delta}+k,k); \end{array} \frac{wx}{e(1-\delta)}\right]. \tag{7.3.6}$$

If we consider $b_1 = b_2 = 1$ and $b = 0$, in Theorem 7.2.1 and Theorem 7.2.2, we obtain Corollary 7.3.5 and Corollary 7.3.6.

**Corollary 7.3.7.** *If $k > 0$, $\Re(b_1) > 0$, $\Re(c_1) > 0$, $\Re(c_2) > 0$, $b \geq 0$, $|w| < \frac{1}{k}$, and $\delta = 1$, then the following result holds true:*

$$\left(P_{0+}^{\nu,1} t^{\frac{\gamma}{k}-1} F_{k,b}^{(b_1)}[(c_0,k),(c_1,k);(c_2,k);wt]\right)(x) = k x^{\nu} \Gamma_k(\gamma) \left(\frac{x}{k e \nu}\right)^{\frac{\gamma}{k}} \times {}_1F_{k,b}^{(b_1)}\left[\begin{matrix} (c_0,k),(c_1,k),(\gamma,k); & \frac{wx}{k\nu e} \\ (c_2,k); & \end{matrix}\right]. \tag{7.3.7}$$

**Corollary 7.3.8.** *If $k > 0$, $\Re(b_1) > 0$, $\Re(c_1) > 0$, $\Re(c_2) > 0$, $b \geq 0$, and $\delta = 1$, then the following result holds true:*

$$\left(P_{0+}^{\nu,1} t^{\frac{\gamma}{k}-1} \phi_{k,b}^{(b_1)}[(c_1,k);(c_2,k);wt]\right)(x) = k x^{\nu} \Gamma_k(\gamma) \left(\frac{x}{k e \nu}\right)^{\frac{\gamma}{k}} \times {}_1\phi_{k,b}^{(b_1)}\left[\begin{matrix} (c_1,k),(\gamma,k); & \frac{wx}{k\nu e} \\ (c_2,k); & \end{matrix}\right]. \tag{7.3.8}$$

If we consider $b_2 = 1$, in Theorem 7.2.3 and Theorem 7.2.4, we obtain Corollary 7.3.7 and Corollary 7.3.8.

**Corollary 7.3.9.** *If $k > 0$, $\Re(c_1) > 0$, $\Re(c_2) > 0$, $b \geq 0$, $|w| < \frac{1}{k}$, and $\delta = 1$, then the following result holds true:*

$$\left(P_{0+}^{\nu,1} t^{\frac{\gamma}{k}-1} F_{k,b}[(c_0,k),(c_1,k);(c_2,k);wt]\right)(x) = k x^{\nu} \Gamma_k(\gamma) \left(\frac{x}{k e \nu}\right)^{\frac{\gamma}{k}} \times {}_1F_{k,b}\left[\begin{matrix} (c_0,k),(c_1,k),(\gamma,k); & \frac{wx}{k\nu e} \\ (c_2,k); & \end{matrix}\right]. \tag{7.3.9}$$

**Corollary 7.3.10.** *If $k > 0$, $\Re(c_1) > 0$, $\Re(c_2) > 0$, $b \geq 0$, and $\delta = 1$, then the following result holds true:*

$$\left(P_{0+}^{\nu,1} t^{\frac{\gamma}{k}-1} \phi_{k,b}[(c_1,k);(c_2,k);wt]\right)(x) = k x^{\nu} \Gamma_k(\gamma) \left(\frac{x}{k e \nu}\right)^{\frac{\gamma}{k}} \times {}_1\phi_{k,b}\left[\begin{matrix} (c_1,k),(\gamma,k); & \frac{wx}{k\nu e} \\ (c_2,k); & \end{matrix}\right]. \tag{7.3.10}$$

If we consider $b_1 = b_2 = 1$, in Theorem 7.2.3 and Theorem 7.2.4, we obtain Corollary 7.3.9 and Corollary 7.3.10.

**Corollary 7.3.11.** *If* $k>0$, $\Re(c_1)>0$, $\Re(c_2)>0$, $|w|<\frac{1}{k}$, *and* $\delta=1$, *then the following result holds true:*

$$\left(P_{0+}^{\nu,1}t^{\frac{\gamma}{k}-1}{}_2F_{1,k}[(c_0,k),(c_1,k);(c_2,k);wt]\right)(x)=kx^{\nu}\Gamma_k(\gamma)\left(\frac{x}{ke\nu}\right)^{\frac{\gamma}{k}} \times {}_3F_{1,k}\left[\begin{matrix}(c_0,k),(c_1,k),(\gamma,k); & \frac{wx}{k\nu e}\\ (c_2,k); & \end{matrix}\right]. \tag{7.3.11}$$

**Corollary 7.3.12.** *If* $k>0$, $\Re(c_1)>0$, $\Re(c_2)>0$, *and* $\delta=1$, *then the following result holds true:*

$$\left(P_{0+}^{\nu,1}t^{\frac{\gamma}{k}-1}{}_1F_{1,k}[(c_1,k);(c_2,k);wt]\right)(x)=kx^{\nu}\Gamma_k(\gamma)\left(\frac{x}{ke\nu}\right)^{\frac{\gamma}{k}} \times {}_2F_{1,k}\left[\begin{matrix}(c_1,k),(\gamma,k); & \frac{wx}{k\nu e}\\ (c_2,k); & \end{matrix}\right]. \tag{7.3.12}$$

If we consider $b_1=b_2=1$ and $b=0$ in Theorem 7.2.3 and Theorem 7.2.4, we obtain Corollary 7.3.11 and Corollary 7.3.12.

## 7.4 Conclusion

In conclusion, this chapter has successfully developed new pathway fractional integral formulae associated with extended and confluent k-hypergeometric functions. The results obtained provide natural extensions of several known fractional integral relations. Furthermore, by assigning specific values to the involved parameters, a variety of meaningful special cases and unique conditions have been derived, demonstrating the versatility and generality of the proposed approach. We believe that the framework presented in this chapter is expected to facilitate further research on fractional integral and differential operators involving other classes of special functions, as well as their applications in mathematical physics, engineering models, and related areas.

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CHAPTER

# 8

# Certain integral transforms for the $k$-hypergeometric functions

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## 8.1 Introduction and preliminaries

In mathematics, hypergeometric functions are extremely valuable. These Functions could be very beneficial in clearing up many interesting problems in engineering and the sciences [1,5,6,9,15–24]. In 2005, Diaz et al. [3,4] studied the $k$-gamma, $k$-beta functions and additionally showed their various properties. In addition, they also defined $k$-hypergeometric functions that are based on the $k$-Pochhammer symbol. Later, in 2007, Diaz et al. [2] established integral representations of the $k$-gamma and $k$-beta functions. The integral representations of some generalized $k$-hypergeometric functions also defined by Mubeen and Habibullah [5]. Some important generalizations of these functions are also being continuously investigated.

For $k > 0$, the $k$-gamma function is defined as follows [7]:

$$\Gamma_k(v_1) = \int_0^\infty t^{v_1-1} e^{(-\frac{t^k}{k})} dt, \qquad \Re(v_1) > 0. \tag{8.1.1}$$

Here, we note that when $k = 1$, $\Gamma_k(v_1) = \Gamma(v_1)$, where $\Gamma(v_1)$ is the classical Euler gamma function:

$$(v_1)_{n,k} = \frac{\Gamma_k(v_1 + nk)}{\Gamma_k(v_1)}, \tag{8.1.2}$$

$$\Gamma_k(v_1 + k) = v_1 \Gamma_k(v_1). \tag{8.1.3}$$

The relation between the $k$-gamma function $\Gamma_k(v_1)$ and the gamma function $\Gamma(v_1)$ is defined as follows [10]:

$$\Gamma_k(v_1) = k^{\frac{v_1}{k}-1} \Gamma\left(\frac{v_1}{k}\right) \qquad or \qquad \Gamma_k(ak) = k^{a-1}\Gamma(a). \tag{8.1.4}$$

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For $k > 0$, the $k$-beta function is defined as follows [7]:

$$B_k(v_1, v_2) = \frac{1}{k}\int_0^1 t^{\frac{v_1}{k}-1}(1-t)^{\frac{v_2}{k}-1}dt, \qquad k > 0, \Re(v_1), \Re(v_2) > 0, \tag{8.1.5}$$

$$B_k(v_1, v_2) = \frac{\Gamma_k(v_1)\Gamma_k(v_2)}{\Gamma_k(v_1+v_2)}, \qquad k > 0, \Re(v_1), \Re(v_2) > 0, \tag{8.1.6}$$

when $k = 1$, $B_k(v_1, v_2) = B(v_1, v_2)$, where $B(v_1, v_2)$ is the classical Euler beta function.

The relation between the $k$-beta function $B_k(v_1, v_2)$ and the beta function $B(v_1, v_2)$ is defined as follows:

$$B_k(v_1, v_2) = \frac{1}{k}B\left(\frac{v_1}{k}, \frac{v_2}{k}\right). \qquad or \qquad B_k(ak, bk) = \frac{1}{k}B(a, b). \tag{8.1.7}$$

For $k > 0$, the $k$-hypergeometric function is defined as follows [7]:

$${}_2F_{1,k}[(\gamma_1, k), (\gamma_2, k); (\gamma_3, k); v] = \sum_{n=0}^{\infty}\frac{(\gamma_1)_{n,k}(\gamma_2)_{n,k}}{(\gamma_3)_{n,k}}\frac{v^n}{n!}, \qquad |v| < 1, \tag{8.1.8}$$

where $(\gamma_1)_{n,k}$ is the $k$-Pochhammer symbol. When $k = 1$, (8.1.8) is reduced to the Gauss hypergeometric function ${}_2F_1(.)$:

$${}_2F_1[\gamma_1, \gamma_2; \gamma_3; v] = \sum_{n=0}^{\infty}\frac{(\gamma_1)_n(\gamma_2)_n}{(\gamma_3)_n}\frac{v^n}{n!}, \qquad |v| < 1. \tag{8.1.9}$$

For $k > 0$, the generalized $k$-hypergeometric function is defined as follows [9]:

$${}_pF_{q,k}[(c_1, k), (c_2, k)...(c_p, k); (d_1, k), (d_2, k)...(d_q, k); v] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(c_i)_{n,k}}{\prod_{i=1}^{q}(d_i)_{n,k}}\frac{v^n}{n!}, \quad |v| < 1. \tag{8.1.10}$$

In 2023, Parik et al. [13,14] studied some well-known $k$-type gamma, beta, and hypergeometric functions and discussed many important properties and results:

$$\Gamma_{k,s}^{(s_1,s_2)}(v_1) = \int_0^{\infty} t^{v_1-1}E_{k,s_1,s_2}\left(-\frac{t^k}{k} - \frac{s^k t^{-k}}{k}\right)dt, \tag{8.1.11}$$

where, $k > 0$, $\Re(v_1) > 0$, $\Re(s) > 0$, $\Re(s_1) > 0$, $\Re(s_2) > 0$ and $E_{k,s_1,s_2}(z)$ is the $k$-Mittag-Leffler function defined in [13]:

$$B_{k,s}^{(s_1,s_2)}(v_1, v_2) = \frac{1}{k}\int_0^1 t^{\frac{v_1}{k}-1}(1-t)^{\frac{v_2}{k}-1}E_{k,s_1,s_2}\left(\frac{-s^k}{kt(1-t)}\right)dt, \tag{8.1.12}$$

where, $k>0$, $\min\{\Re(v_1),\Re(v_2)\}>0$, $\Re(s)>0$, $\Re(s_1)>0$, $\Re(s_2)>0$ and $E_{k,s_1,s_2}(z)$ is the $k$-Mittag-Leffler function defined in [13]:

$$F_{k,s}^{(s_1,s_2)}[(v_1,k),(v_2,k);(v_3,k);w]=\sum_{n=0}^{\infty}\frac{B_{k,s}^{(s_1,s_2)}(v_2+nk,v_3-v_2)}{B_k(v_2,v_3-v_2)}(v_1)_{n,k}\frac{w^n}{n!}, \tag{8.1.13}$$

where, $k>0$, $\Re(v_1)>0$, $\Re(v_2)>0$, $\Re(v_3)>0$, $\Re(s)>0$, $\Re(s_1)>0$, $\Re(s_2)>0$, and $|w|<\frac{1}{k}$ and $B_{k,s}^{(s_1,s_2)}(v_1,v_2)$ is the extended $k$-beta function defined in [13]:

$$\phi_{k,s}^{(s_1,s_2)}[(v_2,k);(v_3,k);w]=\sum_{n=0}^{\infty}\frac{B_{k,s}^{(s_1,s_2)}(v_2+nk,v_3-v_2)}{B_k(v_2,v_3-v_2)}\frac{w^n}{n!}, \tag{8.1.14}$$

where, $k>0$, $\Re(v_2)>0$, $\Re(v_3)>0$, $\Re(s)>0$, $\Re(s_1)>0$, $\Re(s_2)>0$ and $B_{k,s}^{(s_1,s_2)}(v_1,v_2)$ is extended $k$-beta function defined in [13].

To obtain our main result, we need to define the $k$-Mittag-Leffler functions [11] as follows:

$$E_{k,s_1}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma_k(ns_1+1)}, \qquad k>0,\Re(s_1)\geq 0, z\in C, \tag{8.1.15}$$

$$E_{k,s_1,s_2}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma_k(ns_1+s_2)}, \qquad k>0,\Re(s_1)\geq 0,\Re(s_2)\geq 0, z\in C. \tag{8.1.16}$$

A remarkably large number of integral transforms involving various special functions have been investigated by many authors. Very recently, Agarwal [8] gave some integral transforms and fractional integral formulas involving the $F_p^{s_1,s_2}(.)$. In the following, using the same technique, we establish certain integral transforms for the extended $k$-hypergeometric function $F_{k,s}^{(s_1,s_2)}[(v_1,k),(v_2,k);(v_3,k);w]$.

The $k$-beta transform of function $f(v)$ is defined as follows [8]:

$$B_k(f(v):a,b)=\frac{1}{k}\int_0^1 v^{(\frac{a}{k}-1)}(1-v)^{(\frac{b}{k}-1)}f(v)dv. \tag{8.1.17}$$

The Laplace transform of function $f(v)$ is defined as follows [8]:

$$L(f(v):s)=\int_0^{\infty}e^{-sv}f(v)dv. \tag{8.1.18}$$

For a non-integer complex number $y$ and positive real numbers $k,r$, the following identities hold [10]:

$$(y)_{n,r}=\left(\frac{r}{k}\right)^n\left(\frac{ky}{r}\right)_{n,k}, \tag{8.1.19}$$

when $r = 1$:

$$(y)_n = \left(\frac{1}{k}\right)^n (ky)_{n,k}. \tag{8.1.20}$$

The Varma transform of function $f(v)$ is defined by the following integral equation [8]:

$$V(f, x, y; s) = \int_0^\infty e^{-\frac{1}{2}sv}(sv)^{y-\frac{1}{2}} W_{x,y}(sv) f(v) dv, \qquad \Re(s) > 0, \tag{8.1.21}$$

where $W_{x,y}(sv)$ is the Whittaker function defined by:

$$W_{x,y}(v) = \sum_{y,-y} \frac{\Gamma(-2y)}{\Gamma(\frac{1}{2} - x - y)} \; M_{x,y}(v), \tag{8.1.22}$$

where

$$M_{x,y}(v) = v^{y+\frac{1}{2}} e^{\frac{-v}{2}} {}_1F_1\left[\frac{1}{2} - x + y; 2y + 1; v\right]. \tag{8.1.23}$$

The following formula will be used to prove our main results:

$$\int_0^\infty e^{-\frac{1}{2}sv}(v)^{\rho-1} W_{x,y}(sv) dv = s^{-\rho} \; \frac{\Gamma(\rho + y + \frac{1}{2})\Gamma(\rho - y + \frac{1}{2})}{\Gamma(1 - x + \rho)}, \tag{8.1.24}$$

where $\Re(s) > 0$, $\Re(\rho + y) > \frac{-1}{2}$.

The Elzaki transform of function $f(v)$ is defined by [12]:

$$E[f(v)] = w^2 \int_0^\infty f(vw) e^{-v} dv. \tag{8.1.25}$$

The Elzaki transform of the function $f(v) = v^n$ is:

$$E[v^n] = n! u^{n+2}. \tag{8.1.26}$$

## 8.2 Main results

In this section, we will prove various integral transforms for extended $k$-hypergeometric functions.

**Theorem 8.2.1.** *Suppose* $k > 0$, $q_0, q_1, q_2, w, t \in \mathbb{C}$, $q_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $\Re(q_0) > 0$, $\Re(q_1) > 0$, $\Re(q_2) > 0$, $|w| < \frac{1}{k}$, *and* $\Re(s) > 0$, $\Re(s_1) > 0$, $\Re(s_2) > 0$. *Then, the subsequent k-beta transform for the extended k-hypergeometric function is defined as follows:*

$$B_k\left[F_{k,s}^{(s_1,s_2)}\left[\begin{array}{c} (q_0 + q_1, k), (q_2, k); \\ (q_3, k); \end{array} \; wt\right] : q_0, q_1\right]$$

$$= B_k(q_0,q_1)\ F_{k,s}^{(s_1,s_2)}\left[\begin{array}{c}(q_0,k),(q_2,k);\\(q_3,k);\end{array}\ w\right]. \tag{8.2.1}$$

*Proof.* Consider the left-hand side of Eq. (8.2.1) and denote it by $L$, applying the $k$-beta transform (8.1.17) to the extended $k$-hypergeometric function (8.1.13), we have:

$$L = \frac{1}{k}\int_0^1 t^{\frac{q_0}{k}-1}(1-t)^{\frac{q_1}{k}-1}\left[\sum_{n=0}^{\infty}(q_0+q_1)_{n,k}\frac{B_{k,s}^{(s_1,s_2)}(q_2+nk,q_3-q_2)}{B_k(q_2,q_3-q_2)}\frac{(wt)^n}{n!}\right]dt. \tag{8.2.2}$$

By changing the order of integration and summation and using the definition of the $k$-beta function (8.1.5), we obtain:

$$L = \sum_{n=0}^{\infty}(q_0+q_1)_{n,k}\frac{B_{k,s}^{(s_1,s_2)}(q_2+nk,q_3-q_2)}{B_k(q_2,q_3-q_2)}\frac{w^n}{n!}B_k(q_0+nk,q_1). \tag{8.2.3}$$

By using Eq. (8.1.6) and the properties of the $k$-Pochhammer's Symbol (8.1.2) and simplifying the above equation, we obtain:

$$L = \frac{\Gamma_k(q_0+nk)\Gamma_k(q_1)}{\Gamma_k(q_0+q_1+nk)}\ \sum_{n=0}^{\infty}(q_0+q_1)_{n,k}\frac{B_{k,s}^{(s_1,s_2)}(q_2+nk,q_3-q_2)}{B_k(q_2,q_3-q_2)}\frac{w^n}{n!}. \tag{8.2.4}$$

Then, after applying the generalized $k$-hypergeometric function (8.1.10), we have:

$$L = B_k(q_0,q_1)\ F_{k,s}^{(s_1,s_2)}\left[\begin{array}{c}(q_0,k),(q_2,k);\\(q_3,k);\end{array}\ w\right]. \tag{8.2.5}$$

□

**Corollary 8.2.2.** *The following result holds true:*

$$B_k\left[F_{k,s}^{(s_1)}\left[\begin{array}{c}(q_0+q_1,k),(q_2,k);\\(q_3,k);\end{array}\ wt\right]:q_0,q_1\right]$$
$$= B_k(q_0,q_1)\ F_{k,s}^{(s_1)}\left[\begin{array}{c}(q_0,k),(q_2,k);\\(q_3,k);\end{array}\ w\right]. \tag{8.2.6}$$

*Proof.* Considering $s_2=1$ in Theorem 8.2.1, we obtain our results. □

**Theorem 8.2.3.** *Let* $k>0$, $q_1,q_2,w,l,t\in\mathbb{C}$, $q_3\in\mathbb{C}\setminus\mathbb{Z}_0^-$, $\Re(q_1)>0$, $\Re(q_2)>0$, $|\frac{w}{ks}|<\frac{1}{k}$, *and* $\Re(s)>0$, $\Re(s_1)>0$, $\Re(s_2)>0$. *Then, the subsequent Laplace transform for the extended k-hypergeometric function is defined as follows:*

$$L\left[t^{l-1}F_{k,s}^{(s_1,s_2)}\left[\begin{array}{c}(q_1,k),(q_2,k);\\(q_3,k);\end{array}\ wt\right]\right]$$
$$= \frac{k\Gamma_k(lk)}{(sk)^l}\,{}_1F_{k,s}^{(s_1,s_2)}\left[\begin{array}{c}(q_1,k),(q_2,k),(lk,k);\\(q_3,k);\end{array}\ \frac{w}{ks}\right]. \tag{8.2.7}$$

*Proof.* Consider the left-hand side of Eq. (8.2.7) and denote it by $L$, applying the Laplace transform (8.1.18) to the extended $k$-hypergeometric function (8.1.13), we have:

$$L = \int_0^\infty t^{l-1} e^{-st} \left[ \sum_{n=0}^{\infty} (q_1)_{n,k} \frac{B_{k,s}^{(s_1,s_2)}(q_2+nk, q_3-q_2)}{B_k(q_2, q_3-q_2)} \frac{(wt)^n}{n!} \right] dt. \tag{8.2.8}$$

By changing the order of integration and summation, we obtain:

$$L = \sum_{n=0}^{\infty} (q_1)_{n,k} \frac{B_{k,s}^{(s_1,s_2)}(q_2+nk, q_3-q_2)}{B_k(q_2, q_3-q_2)} \frac{(w)^n}{n!} \left[ \int_0^\infty t^{l+n-1} e^{-st} dt \right]. \tag{8.2.9}$$

Putting $st = \frac{x^k}{k} \Rightarrow t = \frac{x^k}{ks} \Rightarrow dt = \frac{x^{k-1}}{s} dx$:

$$L = \sum_{n=0}^{\infty} (q_1)_{n,k} \frac{B_{k,s}^{(s_1,s_2)}(q_2+nk, q_3-q_2)}{B_k(q_2, q_3-q_2)} \frac{(w)^n}{n!} \left[ \int_0^\infty \frac{e^{(-\frac{x^k}{k})} \; x^{(lk+nk-1)}}{s^{l+n} \; k^{l+n-1}} dx \right]. \tag{8.2.10}$$

By using the definition of the $k$-gamma function and the properties of the $k$-Pochhammer Symbol (8.1.2) and then after applying the generalized $k$-hypergeometric function (8.1.10), we have:

$$L = \frac{k\Gamma_k(lk)}{(sk)^l} {}_1F_{k,s}^{(s_1,s_2)} \left[ \begin{matrix} (q_1,k), (q_2,k), (lk,k); \\ (q_3,k); \end{matrix} \; \frac{w}{ks} \right]. \tag{8.2.11}$$

□

**Corollary 8.2.4.** *The following result holds true:*

$$L\left[ t^{l-1} F_{k,s}^{(s_1)} \left[ \begin{matrix} (q_1,k), (q_2,k); \\ (q_3,k); \end{matrix} \; wt \right] \right] = \frac{k\Gamma_k(lk)}{(sk)^l} {}_1F_{k,s}^{(s_1)} \left[ \begin{matrix} (q_1,k), (q_2,k), (lk,k); \\ (q_3,k); \end{matrix} \; \frac{w}{sk} \right]. \tag{8.2.12}$$

*Proof.* Considering $s_2 = 1$ in Theorem 8.2.3, we obtain our results. □

**Theorem 8.2.5.** *Suppose $k > 0$, $q_1, q_2, l, \omega, t, \in \mathbb{C}$, $q_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $\Re(q_1) > 0$, $\Re(q_2) > 0$, $|\frac{\omega}{ks}| < \frac{1}{k}$, and $\Re(s) > 0$, $\Re(s_1) > 0$, $\Re(s_2) > 0$. Then, the subsequent Varma transform for the extended $k$-hypergeometric function is defined as follows:*

$$V\left[ t^{l-1} F_{k,s}^{(s_1,s_2)} \left[ \begin{matrix} (q_1,k), (q_2,k); \\ (q_3,k); \end{matrix} \; \omega t \right] \right] = \frac{k^{\frac{3}{2}-y-x}}{(ks)^l} \frac{\Gamma_k(2yk+lk)\Gamma_k(lk)}{\Gamma_k(yk+lk-xk+\frac{k}{2})} {}_2F_{k,s,1}^{(s_1,s_2)} \left[ \begin{matrix} (q_1,k), (q_2,k), (lk,k), (2yk+lk,k); \\ (q_3,k), (yk+lk-xk+\frac{k}{2},k); \end{matrix} \; \frac{\omega}{ks} \right]. \tag{8.2.13}$$

*Proof.* Consider the left-hand side of Eq. (8.2.13) and denote it by $L$, applying the Laplace transform (8.1.18) to the extended $k$-hypergeometric function (8.1.13), we have:

$$L = \int_0^\infty t^{l-1} \; e^{(-\frac{st}{2})} \; (st)^{y-\frac{1}{2}} \; W_{x,y}(st) \left[\sum_{n=0}^{\infty} (q_1)_{n,k} \frac{B_{k,s}^{(s_1,s_2)}(q_2+nk, q_3-q_2)}{B_k(q_2, q_3-q_2)} \frac{(wt)^n}{n!}\right] dt. \tag{8.2.14}$$

By changing the order of integration and summation, we obtain:

$$L = \sum_{n=0}^{\infty} (q_1)_{n,k} \frac{B_{k,s}^{(s_1,s_2)}(q_2+nk, q_3-q_2)}{B_k(q_2, q_3-q_2)} \frac{(w)^n}{n!} \int_0^\infty t^{l+n+y-\frac{1}{2}-1} e^{(-\frac{1}{2}st)} (s)^{y-\frac{1}{2}} \; W_{x,y}(st) \; dt. \tag{8.2.15}$$

By using the definition of the $k$-gamma function (8.1.4) and the properties of the Pochhammer Symbol and the $k$-Pochhammer Symbol (8.1.2), and simplifying the above equation we obtain:

$$L = s^{-l} \; \frac{\Gamma_k(lk+2yk)\Gamma_k(lk)k^{(l+y+\frac{1}{2}-1-x)}}{\Gamma_k(lk+yk+\frac{k}{2}-xk)k^{2y+l-1}k^{l-1}} \sum_{n=0}^{\infty} \frac{(q_1)_{n,k}(2yk+lk)_{n,k}(lk)_{n,k}}{(lk+yk+\frac{k}{2}-xk)_{n,k}} \frac{B_{k,s}^{(s_1,s_2)}(q_2+nk, q_3-q_2)}{B_k(q_2, q_3-q_2)} \left(\frac{\omega}{ks}\right)^n \frac{1}{n!}. \tag{8.2.16}$$

Applying the generalized $k$-hypergeometric function (8.1.10), we have:

$$L = \frac{k^{\frac{3}{2}-y-x}}{(ks)^l} \; \frac{\Gamma_k(2yk+lk)\Gamma_k(lk)}{\Gamma_k(yk+lk-xk+\frac{k}{2})} \; {}_2F_{k,s,1}^{(s_1,s_2)}\left[\begin{array}{cc} (q_1,k),(q_2,k),(lk,k),(2yk+lk,k); & \frac{\omega}{ks} \\ (q_3,k),(yk+lk-xk+\frac{k}{2},k); & \end{array}\right]. \tag{8.2.17}$$

□

**Corollary 8.2.6.** *The following result holds true:*

$$V\left[t^{l-1}{}_1F_{k,s}^{(s_1)}\left[\begin{array}{cc} (q_1,k),(q_2,k); & \omega t \\ (q_3,k); & \end{array}\right]\right] = \frac{k^{\frac{3}{2}-y-x}}{(ks)^l} \; \frac{\Gamma_k(2yk+lk)\Gamma_k(lk)}{\Gamma_k(yk+lk-xk+\frac{k}{2})} \; {}_2F_{k,s,1}^{(s_1)}\left[\begin{array}{cc} (q_1,k),(q_2,k),(lk,k),(2yk+lk,k); & \frac{\omega}{ks} \\ (q_3,k),(yk+lk-xk+\frac{k}{2},k); & \end{array}\right]. \tag{8.2.18}$$

*Proof.* Considering $s_2 = 1$ in Theorem 8.2.5, we obtain our results. □

**Theorem 8.2.7.** *Suppose* $k > 0$, $q_1, q_2, u, t, l, y \in \mathbb{C}$, $q_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $\Re(q_1) > 0$, $\Re(q_2) > 0$, $|\frac{y}{uk}| < 1$, *and* $\Re(s) > 0$, $\Re(s_1) > 0$, $\Re(s_2) > 0$. *Then, the subsequent Whittaker transform for the extended $k$-hypergeometric function is defined as follows:*

$$\int_0^\infty t^{l-1}\, e^{(-\frac{ut}{2})}\, W_{\lambda,\mu}(ut)\, F_{k,s}^{(s_1,s_2)}\left[\begin{array}{c}(q_1,k),(q_2,k);\\ (q_3,k);\end{array}\ yt\right]dt$$
$$= \frac{k^{\frac{1}{2}-\lambda}}{(ku)^l}\ \frac{\Gamma_k(\frac{k}{2}+\mu k+lk)\Gamma_k(\frac{k}{2}-\mu k+lk)}{\Gamma_k(\frac{k}{2}-\lambda k+lk)}$$
$${}_2F_{k,s,1}^{(s_1,s_2)}\left[\begin{array}{c}(q_1,k),(q_2,k),(\frac{k}{2}+\mu k+lk,k),(\frac{k}{2}-\mu k+lk,k);\\ (q_3,k),(\frac{k}{2}-\lambda k+lk,k);\end{array}\ \frac{y}{uk}\right]. \tag{8.2.19}$$

*Proof.* Consider the left-hand side of Eq. (8.2.19) and denote it by $L$, using the definition of the extended $k$-hypergeometric function (8.1.13) and setting $ut = v$, we have:

$$L = \int_0^\infty \left(\frac{v}{u}\right)^{l-1} e^{(-\frac{v}{2})}\ W_{\lambda,\mu}(v)$$
$$\left\{\sum_{n=0}^\infty (q_1)_{n,k}\frac{B_{k,s}^{(s_1,s_2)}(q_2+nk,q_3-q_2)}{B_k(q_2,q_3-q_2)}\left(\frac{(yv)^n}{u^{n+1}}\right)\frac{dv}{n!}\right\}. \tag{8.2.20}$$

By changing the order of integration and summation, we obtain:

$$L = u^{-l}\ \sum_{n=0}^\infty (q_1)_{n,k}\frac{B_{k,s}^{(s_1,s_2)}(q_2+nk,q_3-q_2)}{B_k(q_2,q_3-q_2)}\left(\frac{y}{u}\right)^n\frac{1}{n!}$$
$$\int_0^\infty v^{l+n-1}e^{(-\frac{v}{2})}\ W_{\lambda,\mu}(v)dv. \tag{8.2.21}$$

Here, we use the following integral formula involving the Whittaker function [8]:

$$\int_0^\infty t^{v-1}\ e^{(-\frac{t}{2})}\ W_{\lambda,\mu}(t)dt = \frac{\Gamma(\frac{1}{2}+\mu+v)\Gamma(\frac{1}{2}-\mu+v)}{\Gamma(\frac{1}{2}-\lambda+v)}, \tag{8.2.22}$$

$$L = u^{-l}\ \sum_{n=0}^\infty (q_1)_{n,k}\frac{B_{k,s}^{(s_1,s_2)}(q_2+nk,q_3-q_2)}{B_k(q_2,q_3-q_2)}\left(\frac{y}{u}\right)^n\frac{1}{n!}$$
$$\frac{\Gamma(\frac{1}{2}+\mu+l+n)\Gamma(\frac{1}{2}-\mu+l+n)}{\Gamma(\frac{1}{2}-\lambda+l+n)}. \tag{8.2.23}$$

By using the definition of the $k$-gamma function (8.1.1) and the properties of the Pochhammer Symbol and the $k$-Pochhammer Symbol (8.1.2), and simplifying the above equation we obtain:

$$L = u^{-l}\, k^{(\frac{1}{2}-\lambda-l)} \frac{\Gamma_k(\frac{k}{2}+\mu k+lk)\Gamma_k(\frac{k}{2}-\mu k+lk)}{\Gamma_k(\frac{k}{2}-\lambda k+lk)} \sum_{n=0}^{\infty} \frac{(\frac{k}{2}+\mu k+lk)_{n,k}(\frac{k}{2}-\mu k+lk)_{n,k}}{(\frac{k}{2}-\lambda k+lk)_{n,k}} (q_1)_{n,k} \frac{B_{k,s}^{(s_1,s_2)}(q_2+nk, q_3-q_2)}{B_k(q_2,q_3-q_2)} \left(\frac{y}{uk}\right)^n \frac{1}{n!}. \tag{8.2.24}$$

Applying the generalized $k$-hypergeometric function (8.1.10), we have:

$$L = \frac{k^{\frac{1}{2}-\lambda}}{(ku)^l} \frac{\Gamma_k(\frac{k}{2}+\mu k+lk)\Gamma_k(\frac{k}{2}-\mu k+lk)}{\Gamma_k(\frac{k}{2}-\lambda k+lk)} {}_2F_{k,s,1}^{(s_1,s_2)}\left[\begin{array}{c} (q_1,k),(q_2,k),(\frac{k}{2}+\mu k+lk,k),(\frac{k}{2}-\mu k+lk,k); \\ (q_3,k),(\frac{k}{2}-\lambda k+lk,k); \end{array} \frac{y}{uk}\right]. \tag{8.2.25}$$

□

**Corollary 8.2.8.** *The following result holds true:*

$$\int_0^{\infty} t^{l-1}\, e^{(-\frac{ut}{2})}\, W_{\lambda,\mu}(ut)\; F_{k,s}^{(s_1)}\left[\begin{array}{c} (q_1,k),(q_2,k); \\ (q_3,k); \end{array} yt\right] dt = \frac{k^{\frac{1}{2}-\lambda}}{(ku)^l} \frac{\Gamma_k(\frac{k}{2}+\mu k+lk)\Gamma_k(\frac{k}{2}-\mu k+lk)}{\Gamma_k(\frac{k}{2}-\lambda k+lk)} {}_2F_{k,s,1}^{(s_1)}\left[\begin{array}{c} (q_1,k),(q_2,k),(\frac{k}{2}+\mu k+lk,k),(\frac{k}{2}-\mu k+lk,k); \\ (q_3,k),(\frac{k}{2}-\lambda k+lk,k); \end{array} \frac{y}{uk}\right]. \tag{8.2.26}$$

*Proof.* Considering $s_2 = 1$ in Theorem 8.2.7, we obtain our results. □

**Theorem 8.2.9.** *Suppose* $k > 0$, $q_0, q_1, q_2, w, t \in \mathbb{C}$, $q_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $\Re(q_0) > 0$, $\Re(q_1) > 0$, $\Re(q_2) > 0$, $|wu| < \frac{1}{k}$, *and* $\Re(s) > 0$, $\Re(s_1) > 0$, $\Re(s_2) > 0$. *Then, the subsequent Elzaki transform for the extended $k$-hypergeometric function is defined as follows:*

$$E\left[t\, F_{k,s}^{(s_1,s_2)}\left[\begin{array}{c} (q_0,k),(q_1,k); \\ (q_2,k); \end{array} wt\right]\right] = u^2\Gamma(2)\,{}_1F_{k,s}^{(s_1,s_2)}\left[\begin{array}{c} (q_0,k),(q_1,k),(2,k); \\ (q_2,k); \end{array} wu\right]. \tag{8.2.27}$$

*Proof.* Consider the left-hand side of Eq. (8.2.27) and denote it by $L$, applying the Elzaki transform (8.1.25) to the extended $k$-hypergeometric function (8.1.13), we have:

$$L = E\left\{t\left[\sum_{n=0}^{\infty}(q_0)_{n,k}\frac{B_{k,s}^{(s_1,s_2)}(q_1+nk,q_2-q_1)}{B_k(q_1,q_2-q_1)}\frac{(wt)^n}{n!}\right]\right\}, \tag{8.2.28}$$

$$L = \sum_{n=0}^{\infty}(q_0)_{n,k}\frac{B_{k,s}^{(s_1,s_2)}(q_1+nk,q_2-q_1)}{B_k(q_1,q_2-q_1)}\frac{w^n}{n!}E(t^{n+1}). \tag{8.2.29}$$

By using Eq. (8.1.26) and the properties of the Pochhammer Symbol and simplifying the above equation, we obtain:

$$L = \sum_{n=0}^{\infty}(q_0)_{n,k}\frac{B_{k,s}^{(s_1,s_2)}(q_1+nk,q_2-q_1)}{B_k(q_1,q_2-q_1)}\frac{w^n}{n!}(n+1)!u^{n+2} \tag{8.2.30}$$

$$L = \sum_{n=0}^{\infty}(q_0)_{n,k}\frac{B_{k,s}^{(s_1,s_2)}(q_1+nk,q_2-q_1)}{B_k(q_1,q_2-q_1)}\frac{w^n}{n!}(2)_n\Gamma(2)u^{n+2} \tag{8.2.31}$$

$$L = \Gamma(2)\sum_{n=0}^{\infty}(q_0)_{n,k}(2)_n\frac{B_{k,s}^{(s_1,s_2)}(q_1+nk,q_2-q_1)}{B_k(q_1,q_2-q_1)}\frac{w^n}{n!}u^{n+2}. \tag{8.2.32}$$

Then, after applying the generalized $k$-hypergeometric function (8.1.10), we have:

$$L = u^2\Gamma(2)\,{}_1F_{k,s}^{(s_1,s_2)}\left[\begin{matrix}(q_0,k),(q_1,k),(2,k);\\(q_2,k);\end{matrix}\ wu\right]. \tag{8.2.33}$$

□

**Corollary 8.2.10.** *The following result holds true:*

$$E\left[t\,F_{k,s}^{(s_1)}\left[\begin{matrix}(q_0,k),(q_1,k);\\(q_2,k);\end{matrix}\ wt\right]\right] = u^2\Gamma(2)\,{}_1F_{k,s}^{(s_1)}\left[\begin{matrix}(q_0,k),(q_1,k),(2,k);\\(q_2,k);\end{matrix}\ wu\right]. \tag{8.2.34}$$

*Proof.* Considering $s_2 = 1$ in Theorem 8.2.9, we obtain our results. □

## 8.3 Conclusion

In the present chapter, we investigated the $k$-beta transform, Laplace transform, Varma transform, Whittaker transform, and Elzaki transform of the extended $k$-hypergeometric functions. We also investigated another fractional integral formula of the extended $k$-hypergeometric function, which is in progress.

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CHAPTER

# 9 The generalized Riemann–Liouville k-fractional derivative and its properties

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## 9.1 Introduction and preliminaries

The classical Riemann–Liouville fractional derivative of order $\alpha$ is defined as ([2, page-3, Eqs. (2.2), (2.2)]):

$$D_y^{\alpha} f(y) = \frac{1}{\Gamma(-\alpha)} \int_0^y (y-t)^{-\alpha-1} f(t)dt. \tag{9.1.1}$$

For the case $s-1 < \Re(\alpha) < s, n \in \mathbb{N}$:

$$\begin{aligned} D_y^{\alpha} f(y) &= \frac{d^s}{dt^s} D_y^{\alpha-s}\left\{f(y)\right\} \\ &= \frac{1}{\Gamma(-\alpha+s)} \int_0^y (y-t)^{-\alpha+s-1} f(t)dt, \Re(\alpha) > 0. \end{aligned} \tag{9.1.2}$$

The Riemann–Liouville $k$-fractional derivative of order $\alpha$ is defined as ([2, page-3, Eqs. (2.3), (2.4)]):

$$D_{k,y}^{\alpha} f(y) = \frac{1}{k\Gamma_k(-\alpha)} \int_0^y (y-t)^{-\frac{\alpha}{k}-1} f(t)dt. \tag{9.1.3}$$

For the case $s-1 < \Re(\alpha) < s, n \in \mathbb{N}$:

$$\begin{aligned} D_{k,y}^{\alpha} f(y) &= \frac{d^s}{dt^s} D_{k,y}^{\alpha-sk}\left\{f(y)\right\} \\ &= \frac{1}{k\Gamma_k(-\alpha+sk)} \int_0^y (y-t)^{-\frac{\alpha}{k}+s-1} f(t)dt, \Re(\alpha) > 0. \end{aligned} \tag{9.1.4}$$

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In 2025, Laxmi et al. extended the $k$-gamma, $k$-beta, $k$-hypergeometric, and confluent $k$-hypergeometric functions by using a 2-parameter $k$-Mittag-Leffler function as kernel and defined as follows (see, [4] and their cited works):

$$\Gamma_{k,w}^{(w_1,w_2)}(p_1) = \int_0^\infty x^{p_1-1} E_{k,w_1,w_2}\left(-\frac{x^k}{k} - \frac{w^k x^{-k}}{k}\right) dx, \tag{9.1.5}$$

where, $k > 0$, $\Re(p_1) > 0$, $w \geq 0$, $\Re(w_1) > 0$, $\Re(w_2) > 0$ and $E_{k,w_1,w_2}(z)$ is defined in (see, [4] and their cited works) and

$$B_{k,w}^{(w_1,w_2)}(p_1, p_2) = \frac{1}{k} \int_0^1 x^{\frac{p_1}{k}-1} (1-x)^{\frac{p_2}{k}-1} E_{k,w_1,w_2}\left(\frac{-w^k}{kx(1-x)}\right) dx, \tag{9.1.6}$$

where, $k > 0$, $\min\{\Re(w_1), \Re(w_2)\} > 0$, $w \geq 0$, $\Re(w_1) > 0$, $\Re(w_2) > 0$ and $E_{k,w_1,w_2}(z)$ is defined in (see, [4] and their cited works) and:

$$F_{k,w}^{(w_1,w_2)}[(p_1,k),(p_2,k);(p_3,k);z] = \sum_{n=0}^{\infty} \frac{B_{k,w}^{(w_1,w_2)}(p_2+nk, p_3-p_2)}{B_k(p_2, p_3-p_2)} (p_1)_{n,k} \frac{z^n}{n!}, \tag{9.1.7}$$

where, $k > 0$, $\Re(p_1) > 0$, $\Re(p_2) > 0$, $\Re(p_3) > 0$, $\Re(w_1) > 0$, $\Re(w_2) > 0$, and $|z| < \frac{1}{k}$ and $B_{k,w}^{(w_1,w_2)}(p_1, p_2)$ is the extended $k$-beta function:

$$\phi_{k,w}^{(w_1,w_2)}[(p_2,k);(p_3,k);z] = \sum_{n=0}^{\infty} \frac{B_{k,w}^{(w_1,w_2)}(p_2+nk, p_3-p_2)}{B_k(p_2, p_3-p_2)} \frac{z^n}{n!}, \tag{9.1.8}$$

where, $k > 0$, $\Re(p_2) > 0$, $\Re(p_3) > 0$, $\Re(w_1) > 0$, $\Re(w_2) > 0$ and $B_{k,w}^{(w_1,w_2)}(p_1, p_2)$ is the extended $k$-beta function.

The extended $k$-hypergeometric function has the following integral representations:

$$\begin{aligned} &F_{k,w}^{(w_1,w_2)}[(p_1,k),(p_2,k);(p_3,k);z] \\ &\quad = \frac{1}{kB_k(p_2, p_3-p_2)} \int_0^1 t^{\frac{p_2}{k}-1} (1-t)^{\frac{p_3-p_2}{k}-1} (1-kzt)^{-\frac{p_1}{k}} \\ &\quad\quad E_{k,w_1,w_2}\left(-w^k (kt(1-t))^{-1}\right) dt. \end{aligned} \tag{9.1.9}$$

To obtain our main result we need to define $k$-Mittag-Leffler functions ([1, page-6, Eqs. (2.1), (2.2)]) as follows:

$$E_{k,w_1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(nw_1+1)}, \qquad k > 0, \Re(w_1) \geq 0, z \in C, \tag{9.1.10}$$

$$E_{k,w_1,w_2}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(nw_1 + w_2)}, \qquad k > 0, \Re(w_1) \geq 0, \Re(w_2) \geq 0, z \in C. \tag{9.1.11}$$

**Note.** If we consider variable $w_1 = w_2 = 1$ in Eq. (9.1.11), then we obtain the $k$-exponential function $E_k(z)$.

## 9.2 Extension of the Riemann–Liouville $k$-fractional derivative operator

In this section, we define a new extension of the Riemann–Liouville $k$-fractional derivative operator by using a two-parameter $k$-Mittag-Leffler function.

**Definition 9.2.1.**

$$D_{k,y,w}^{\alpha,w_1,w_2} f(y) = \frac{1}{k\Gamma_k(-\alpha)} \int_0^y (y-t)^{-\frac{\alpha}{k}-1} E_{k,w_1,w_2}\left(\frac{-w^k y^2}{kt(y-t)}\right) f(t)dt. \tag{9.2.1}$$

For the case $s - 1 < \Re(\alpha) < s, n \in \mathbb{N}$:

$$\begin{aligned} D_{k,y,w}^{\alpha,w_1,w_2} f(y) &= D_{k,y,w}^{\alpha-sk,w_1,w_2}\left\{f(y)\right\} \\ &= \frac{1}{k\Gamma_k(-\alpha+sk)} \int_0^y (y-t)^{-\frac{\alpha}{k}+s-1} E_{k,w_1,w_2}\left(\frac{-w^k y^2}{kt(y-t)}\right) f(t)dt, \end{aligned} \tag{9.2.2}$$

where $\Re(w) > 0$, $\Re(\alpha) > 0$, and $s - 1 < \Re(\alpha) < s, n \in \mathbb{N}$.

In the case $w_1 = w_2 = 1$, $w = 0$, $k = 1$, the extended Riemann–Liouville $k$-fractional derivative reduces to the classical Riemann–Liouville fractional derivative: $D_{y,0,1}^{\alpha,1,1} f(y) = D_y^{\alpha} f(y)$.

**Theorem 9.2.2.** *Let* $\Re(w) > 0$ *and* $s - 1 < \Re(\alpha) < s$, *then*

$$D_{k,y,w}^{\alpha,w_1,w_2}(y^{\frac{\mu}{k}}) = \frac{B_{k,w}^{(w_1,w_2)}(\mu+k,-\alpha)}{\Gamma_k(-\alpha)} y^{\frac{\mu-\alpha}{k}}. \tag{9.2.3}$$

*Proof.* Applying Definition 9.2.1 to the left-hand side of (9.2.3), we have:

$$D_{k,y,w}^{\alpha,w_1,w_2}(y^{\frac{\mu}{k}}) = \frac{1}{k\Gamma_k(-\alpha)} \int_0^y (y-t)^{-\frac{\alpha}{k}-1} E_{k,w_1,w_2}\left(\frac{-w^k y^2}{kt(y-t)}\right) t^{\frac{\mu}{k}} dt.$$

Putting $t = yu$ then $dt = ydu$ in the above equation, we have:

$$D_{k,y,w}^{\alpha,w_1,w_2}(y^{\frac{\mu}{k}}) = \frac{1}{k\Gamma_k(-\alpha)} \int_0^1 (y-yu)^{-\frac{\alpha}{k}-1} E_{k,w_1,w_2}\left(\frac{-w^k}{ku(1-u)}\right)(yu)^{\frac{\mu}{k}} ydu$$

$$D_{k,y,w}^{\alpha,w_1,w_2}(y^{\frac{\mu}{k}}) = \frac{y^{\frac{\mu-\alpha}{k}}}{k\Gamma_k(-\alpha)}\int_0^1 u^{\frac{\mu}{k}}(1-u)^{-\frac{\alpha}{k}-1}E_{k,w_1,w_2}\left(\frac{-w^k}{ku(1-u)}\right)du.$$

Applying the definition of the extended $k$-beta function (9.1.6), we have:

$$D_{k,y,w}^{\alpha,w_1,w_2}(y^{\frac{\mu}{k}}) = \frac{B_{k,w}^{(w_1,w_2)}(\mu+k,-\alpha)}{\Gamma_k(-\alpha)}y^{\frac{\mu-\alpha}{k}}.$$

□

**Theorem 9.2.3.** *Let $\Re(\alpha)>0$ and suppose that the function $f(y)$ is analytic at the origin with its Maclaurian expansion given by $f(y)=\sum_{n=0}^{\infty}a_n y^n$, where $|y|<\rho$ for some $n\in\mathbb{R}^+$, then*

$$D_{k,y,w}^{\alpha,w_1,w_2}f(y) = D_{k,y,w}^{\alpha,w_1,w_2}\left(\sum_{n=0}^{\infty}a_n y^n\right) = \sum_{n=0}^{\infty}a_n D_{k,y,w}^{\alpha,w_1,w_2}(y^n). \tag{9.2.4}$$

*Proof.* Using the power-series expansion of $f(y)$, we have:

$$D_{k,y,w}^{\alpha,w_1,w_2}(f(y)) = \frac{1}{k\Gamma_k(-\alpha)}\int_0^y (y-t)^{-\frac{\alpha}{k}-1}E_{k,w_1,w_2}\left(\frac{-w^k y^2}{kt(y-t)}\right)\sum_{n=0}^{\infty}a_n t^n dt.$$

By changing the order of integration and summation, we have:

$$D_{k,y,w}^{\alpha,w_1,w_2}(f(y)) = \sum_{n=0}^{\infty}a_n\left(\frac{1}{k\Gamma_k(-\alpha)}\int_0^y t^n(y-t)^{-\frac{\alpha}{k}-1}E_{k,w_1,w_2}\left(\frac{-w^k y^2}{kt(y-t)}\right)dt\right).$$

Applying Definition 9.2.1, we have:

$$D_{k,y,w}^{\alpha,w_1,w_2}(f(y)) = \sum_{n=0}^{\infty}a_n D_{k,y,w}^{\alpha,w_1,w_2}(y^n).$$

□

**Theorem 9.2.4.** *Let $f(y)$ be the analytic function on the disk $|y|<\rho$ that has a power-series expansion given by $f(y)=\sum_{n=0}^{\infty}a_n y^n$, then*

$$D_{k,y,w}^{\alpha,w_1,w_2}(y^{\frac{\mu}{k}-1}f(y)) = \frac{y^{\frac{\mu-\alpha}{k}-1}}{\Gamma_k(-\alpha)}\sum_{n=0}^{\infty}a_n B_{k,w}^{(w_1,w_2)}(\mu+nk,-\alpha)y^n. \tag{9.2.5}$$

*Proof.* Applying Definition 9.2.1 to the left-hand side of (9.2.5) and using the power-series expansion of $f(y)$, we have:

$$D_{k,y,w}^{\alpha,w_1,w_2}(y^{\frac{\mu}{k}-1}f(y))$$

$$= \frac{1}{k\Gamma_k(-\alpha)} \int_0^y (y-t)^{-\frac{\alpha}{k}-1} E_{k,w_1,w_2}\left(\frac{-w^k y^2}{kt(y-t)}\right) t^{\frac{\mu}{k}-1} \sum_{n=0}^{\infty} a_n t^n dt.$$

Changing the order of summation and integration, we have:

$$D_{k,y,w}^{\alpha,w_1,w_2}(y^{\frac{\mu}{k}-1} f(y)) = \sum_{n=0}^{\infty} a_n \left\{ \frac{1}{k\Gamma_k(-\alpha)} \int_0^y (y-t)^{-\frac{\alpha}{k}-1} E_{k,w_1,w_2}\left(\frac{-w^k y^2}{kt(y-t)}\right) t^{\frac{\mu}{k}+n-1} dt \right\}.$$

Applying Definition 9.2.1, we have:

$$D_{k,y,w}^{\alpha,w_1,w_2}(y^{\frac{\mu}{k}-1} f(y)) = \sum_{n=0}^{\infty} a_n D_{k,y,w}^{\alpha,w_1,w_2}(y^{\frac{\mu+nk-k}{k}}).$$

Applying Theorem 9.2.2, we have:

$$D_{k,y,w}^{\alpha,w_1,w_2}(y^{\frac{\mu}{k}-1} f(y)) = \sum_{n=0}^{\infty} a_n \frac{B_{k,w}^{(w_1,w_2)}(\mu+nk-k+k,-\alpha)}{\Gamma_k(-\alpha)} y^{\frac{\mu+nk-k-\alpha}{k}}$$

$$D_{k,y,w}^{\alpha,w_1,w_2}(y^{\frac{\mu}{k}-1} f(y)) = \frac{y^{\frac{\mu-\alpha}{k}-1}}{\Gamma_k(-\alpha)} \sum_{n=0}^{\infty} a_n B_{k,w}^{(w_1,w_2)}(\mu+nk,-\alpha) y^n.$$

□

**Theorem 9.2.5.** *Let* $\Re(\mu) > \Re(\alpha) > 0$ *and* $|y| < \frac{1}{k}$, *then*

$$D_{k,w,y}^{\mu-\alpha,w_1,w_2}(y^{\frac{\mu}{k}-1}(1-ky)^{\frac{-\beta}{k}}) = y^{\frac{\alpha}{k}-1} \frac{\Gamma_k(\mu)}{\Gamma_k(\alpha)} F_{k,w}^{(w_1,w_2)}[(\beta,k),(\mu,k);(\alpha,k);z]. \tag{9.2.6}$$

*Proof.* Applying Definition 9.2.1 to the left-hand side of (9.2.6), we have:

$$D_{k,y,w}^{\mu-\alpha,w_1,w_2}(y^{\frac{\mu}{k}-1}(1-ky)^{\frac{-\beta}{k}}) = \frac{1}{k\Gamma_k(\alpha-\mu)} \int_0^y (y-t)^{\frac{\alpha-\mu}{k}-1} t^{\frac{\mu}{k}-1}(1-kt)^{\frac{-\beta}{k}} E_{k,w_1,w_2}\left(\frac{-w^k y^2}{kt(y-t)}\right) dt$$

$$= \frac{y^{\frac{\alpha-\mu}{k}-1}}{k\Gamma_k(\alpha-\mu)} \int_0^y \left(1-\frac{t}{y}\right)^{\frac{\alpha-\mu}{k}-1} t^{\frac{\mu}{k}-1}(1-kt)^{\frac{-\beta}{k}} E_{k,w_1,w_2}\left(\frac{-w^k y^2}{kt(y-t)}\right) dt.$$

Putting $t = yu$ then $dt = ydu$ in the above equation, we have:

$$= \frac{y^{\frac{\alpha-\mu}{k}-1}}{k\Gamma_k(\alpha-\mu)} \int_0^1 \left(1-\frac{yu}{y}\right)^{\frac{\alpha-\mu}{k}-1} (yu)^{\frac{\mu}{k}-1}(1-kyu)^{\frac{-\beta}{k}} E_{k,w_1,w_2}\left(\frac{-w^k}{ku(1-u)}\right) ydu$$

$$= \frac{y^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha-\mu)} \int_0^1 u^{\frac{\mu}{k}-1}(1-u)^{\frac{\alpha-\mu}{k}-1}(1-kyu)^{\frac{-\beta}{k}} E_{k,w_1,w_2}\left(\frac{-w^k}{ku(1-u)}\right)du.$$

Multiplying and dividing by $B_k(\mu, \alpha-\mu)$, we have:

$$= \frac{y^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha-\mu)} \frac{B_k(\mu,\alpha-\mu)}{B_k(\mu,\alpha-\mu)} \int_0^1 u^{\frac{\mu}{k}-1}(1-u)^{\frac{\alpha-\mu}{k}-1}(1-kyu)^{\frac{-\beta}{k}} E_{k,w_1,w_2}\left(\frac{-w^k}{ku(1-u)}\right)du.$$

Applying the result of the extended $k$-beta function $B_k(v_1, v_2) = \frac{\Gamma_k(v_1)\Gamma_k(v_2)}{\Gamma_k(v_1+v_2)}$ and definition (9.1.9) we have:

$$= y^{\frac{\alpha}{k}-1} \frac{\Gamma_k(\mu)}{\Gamma_k(\alpha)} F_{k,w}^{(w_1,w_2)}[(\beta,k),(\mu,k);(\alpha,k);z].$$

□

**Theorem 9.2.6.** *Let* $\Re(\mu) > \Re(\alpha) > 0$, $\Re(\beta_1) > 0$, $\Re(\beta_2) > 0$, $max\left\{|ay|, |by|\right\} < \frac{1}{k}$, *then*

$$D_{k,w,y}^{\mu-\alpha,w_1,w_2}(y^{\frac{\mu}{k}-1}(1-kay)^{\frac{-\beta_1}{k}}(1-kby)^{\frac{-\beta_2}{k}}) = \sum_{m,n=0}^{\infty} \frac{(\beta_1)_{m,k}}{m!} \frac{(\beta_2)_{n,k}}{n!} a^m b^n \frac{B_{k,w}^{(w_1,w_2)}(\mu+mk+nk,\alpha-\mu)}{\Gamma_k(\alpha-\mu)} y^{\frac{\alpha}{k}+m+n-1}. \tag{9.2.7}$$

*Proof.* To prove Theorem 9.2.6, we use the power-series expansion and Theorem 9.2.3, and we have:

$$(1-kay)^{\frac{-\beta_1}{k}}(1-kby)^{\frac{-\beta_2}{k}} = \sum_{m,n=0}^{\infty} \frac{(\beta_1)_{m,k}}{m!} \frac{(\beta_2)_{n,k}}{n!} (ay)^m (by)^n$$

$$D_{k,w,y}^{\mu-\alpha,w_1,w_2}(y^{\frac{\mu}{k}-1}(1-kay)^{\frac{-\beta_1}{k}}(1-kby)^{\frac{-\beta_2}{k}}) = \sum_{m,n=0}^{\infty} \frac{(\beta_1)_{m,k}}{m!} \frac{(\beta_2)_{n,k}}{n!} (a)^m (b)^n D_{k,w,y}^{\mu-\alpha,w_1,w_2}(y^{\frac{\mu}{k}+m+n-1}).$$

Applying Theorem 9.2.2, we have:

$$= \sum_{m,n=0}^{\infty} \frac{(\beta_1)_{m,k}}{m!} \frac{(\beta_2)_{n,k}}{n!} a^m b^n \frac{B_{k,w}^{(w_1,w_2)}(\mu+mk+nk,\alpha-\mu)}{\Gamma_k(\alpha-\mu)} y^{\frac{\alpha}{k}+m+n-1}.$$

□

## 9.3 More results

**Theorem 9.3.1.** *The extended Riemann–Liouville k-fractional derivative of $f(y) = e^y$ is:*

$$D_{k,w,y}^{\alpha,w_1,w_2}(e^y) = \frac{y^{\frac{-\alpha}{k}}}{\Gamma_k(-\alpha)} \sum_{n=0}^{\infty} B_{k,w}^{(w_1,w_2)}(nk+k,-\alpha)\frac{y^n}{n!}. \tag{9.3.1}$$

*Proof.* Using the power-series expansion of $e^y$ in Theorem 9.3.1, we have:

$$D_{k,w,y}^{\alpha,w_1,w_2}(e^y) = D_{k,w,y}^{\alpha,w_1,w_2}\left(\sum_{n=0}^{\infty}\frac{y^n}{n!}\right)$$

$$D_{k,w,y}^{\alpha,w_1,w_2}(e^y) = \sum_{n=0}^{\infty}\frac{1}{n!}D_{k,w,y}^{\alpha,w_1,w_2}(y^{\frac{nk}{k}}).$$

Applying Theorem 9.2.2, we have:

$$D_{k,w,y}^{\alpha,w_1,w_2}(e^y) = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{B_{k,w}^{(w_1,w_2)}(nk+k,-\alpha)}{\Gamma_k(-\alpha)}(y^{\frac{nk-\alpha}{k}})$$

$$D_{k,w,y}^{\alpha,w_1,w_2}(e^y) = \frac{y^{\frac{-\alpha}{k}}}{\Gamma_k(-\alpha)} \sum_{n=0}^{\infty} B_{k,w}^{(w_1,w_2)}(nk+k,-\alpha)\frac{y^n}{n!}.$$

□

**Theorem 9.3.2.** *Let $\Re(\mu) > \Re(\alpha) > 0$ and $|y| < \frac{1}{k}$, then*

$$D_{k,y,w}^{\mu-\alpha,w_1,w_2}(y^{\frac{\mu}{k}-1}E_{k,s_1,s_2}(y)) = \frac{y^{\frac{\alpha}{k}-1}}{\Gamma_k(\alpha-\mu)} \sum_{n=0}^{\infty} \frac{B_{k,w}^{(w_1,w_2)}(\mu+nk,\alpha-\mu)}{\Gamma_k(ns_1+s_2)}y^n. \tag{9.3.2}$$

*Proof.* Applying the definition of the 2-parameter $k$-Mittag-Leffler function (9.1.11) to the left-hand side of (9.3.2), we have:

$$D_{k,y,w}^{\mu-\alpha,w_1,w_2}(y^{\frac{\mu}{k}-1}E_{k,s_1,s_2}(y)) = D_{k,y,w}^{\mu-\alpha,w_1,w_2}\left(y^{\frac{\mu}{k}-1}\left\{\sum_{n=0}^{\infty}\frac{y^n}{\Gamma_k(ns_1+s_2)}\right\}\right).$$

Applying Theorem 9.2.3, we have:

$$D_{k,y,w}^{\mu-\alpha,w_1,w_2}(y^{\frac{\mu}{k}-1}E_{k,s_1,s_2}(y)) = \sum_{n=0}^{\infty}\frac{1}{\Gamma_k(ns_1+s_2)}D_{k,y,w}^{\mu-\alpha,w_1,w_2}\left(y^{\frac{\mu}{k}+n-1}\right)$$

$$D_{k,y,w}^{\mu-\alpha,w_1,w_2}(y^{\frac{\mu}{k}-1}E_{k,s_1,s_2}(y)) = \sum_{n=0}^{\infty}\frac{1}{\Gamma_k(ns_1+s_2)}\frac{B_{k,w}^{(w_1,w_2)}(\mu+nk,\alpha-\mu)}{\Gamma_k(\alpha-\mu)}y^{\frac{\alpha}{k}+n-1}$$

$$D_{k,y,w}^{\mu-\alpha,w_1,w_2}(y^{\frac{\mu}{k}-1}E_{k,s_1,s_2}(y)) = \frac{y^{\frac{\alpha}{k}-1}}{\Gamma_k(\alpha-\mu)}\sum_{n=0}^{\infty}\frac{B_{k,w}^{(w_1,w_2)}(\mu+nk,\alpha-\mu)}{\Gamma_k(ns_1+s_2)}y^{n}.$$

□

**Corollary 9.3.3.** *The following result holds true:*

$$D_{k,y,w}^{\mu-\alpha,w_1,w_2}(y^{\frac{\mu}{k}-1}E_{k,s_1}(y)) = \frac{y^{\frac{\alpha}{k}-1}}{\Gamma_k(\alpha-\mu)}\sum_{n=0}^{\infty}\frac{B_{k,w}^{(w_1,w_2)}(\mu+nk,\alpha-\mu)}{\Gamma_k(ns_1+1)}y^{n}. \quad (9.3.3)$$

*Proof.* By setting $s_2 = 1$, we obtain our desired result. □

**Theorem 9.3.4.** *Let* $\Re(\beta) > 0$, $\Re(\alpha) > 0$, $|y| < \frac{1}{k}$, *and* $u > 0$, *then*

$$M\left(D_{k,w,y}^{\alpha,w_1,w_2}(y^{\frac{\beta}{k}}) : u\right) = \frac{y^{\frac{\beta-\alpha}{k}}\Gamma_{k,0}^{w_1,w_2}(u)}{\Gamma_k(-\alpha)}B_k(\beta+u+k,u-\alpha). \quad (9.3.4)$$

*Proof.* Applying the Mellin transform, we have:

$$M\left(D_{k,w,y}^{\alpha,w_1,w_2}(y^{\frac{\beta}{k}}) : u\right) = \int_0^{\infty} w^{u-1}D_{k,w,y}^{\alpha,w_1,w_2}(y^{\frac{\beta}{k}})dw.$$

Applying the definition of the Riemann–Liouville $k$-fractional derivative, we have:

$$M\left(D_{k,w,y}^{\alpha,w_1,w_2}(y^{\frac{\beta}{k}}) : u\right) = \int_0^{\infty} w^{u-1}\left(\frac{1}{k\Gamma_k(-\alpha)}\int_0^{y}(y-t)^{-\frac{\alpha}{k}-1} E_{k,w_1,w_2}\left(\frac{-w^k y^2}{kt(y-t)}\right)t^{\frac{\beta}{k}}dt\right)dw$$

$$M\left(D_{k,w,y}^{\alpha,w_1,w_2}(y^{\frac{\beta}{k}}) : u\right) = \int_0^{\infty} w^{u-1}\left(\frac{1}{k\Gamma_k(-\alpha)}\int_0^{y}y^{-\frac{\alpha}{k}-1}\left(1-\frac{t}{y}\right)^{-\frac{\alpha}{k}-1} E_{k,w_1,w_2}\left(\frac{-w^k y^2}{kt(y-t)}\right)t^{\frac{\beta}{k}}dt\right)dw.$$

Putting $t = yx$ in the above equation, we have:

$$M\left(D_{k,w,y}^{\alpha,w_1,w_2}(y^{\frac{\beta}{k}}) : u\right) = \frac{y^{\frac{\beta-\alpha}{k}}}{k\Gamma_k(-\alpha)}\int_0^{\infty} w^{u-1}\left(\int_0^{1}x^{\frac{\beta}{k}}\left(1-x\right)^{-\frac{\alpha}{k}-1} E_{k,w_1,w_2}\left(\frac{-w^k}{kx(1-x)}\right)dx\right)dw.$$

Interchanging the order of integration, we have:

$$M\left(D_{k,w,y}^{\alpha,w_1,w_2}(y^{\frac{\beta}{k}}):u\right)=\frac{y^{\frac{\beta-\alpha}{k}}}{k\Gamma_k(-\alpha)}\int_0^1 x^{\frac{\beta}{k}}\left(1-x\right)^{-\frac{\alpha}{k}-1}\left(\int_0^\infty w^{u-1}\right.$$
$$\left.E_{k,w_1,w_2}\left(\frac{-w^k}{kx(1-x)}\right)dw\right)dx.$$

Substituting $v=\frac{w}{x^{\frac{1}{k}}(1-x)^{\frac{1}{k}}}$ then $dw=x^{\frac{1}{k}}(1-x)^{\frac{1}{k}}dv$ and applying the definition of the $k$-gamma function, we have:

$$M\left(D_{k,w,y}^{\alpha,w_1,w_2}(y^{\frac{\beta}{k}}):u\right)=\frac{y^{\frac{\beta-\alpha}{k}}}{k\Gamma_k(-\alpha)}\int_0^1 x^{\frac{\beta+u}{k}}\left(1-x\right)^{\frac{u-\alpha}{k}-1}\Gamma_{k,0}^{w_1,w_2}(u)dx$$
$$M\left(D_{k,w,y}^{\alpha,w_1,w_2}(y^{\frac{\beta}{k}}):u\right)=\frac{y^{\frac{\beta-\alpha}{k}}\Gamma_{k,0}^{w_1,w_2}(u)}{\Gamma_k(-\alpha)}B_k(\beta+u+k,u-\alpha).$$

□

**Theorem 9.3.5.** *Let* $\Re(\beta)>0$, $\Re(\alpha)>0$, $|y|<\frac{1}{k}$, *and* $u>0$, *then*

$$M\left(D_{k,w,y}^{\alpha,w_1,w_2}(1-ky)^{-\frac{\beta}{k}}:u\right)=\frac{y^{\frac{\beta-\alpha}{k}}\Gamma_{k,0}^{w_1,w_2}(u)}{\Gamma_k(-\alpha)}B_k(\beta+u+k,u-\alpha). \tag{9.3.5}$$

*Proof.* From the definition of the Mellin transform, we have

$$M\left(D_{k,w,y}^{\alpha,w_1,w_2}(1-ky)^{-\frac{\beta}{k}}:u\right)=\int_0^\infty w^{u-1}D_{k,w,y}^{\alpha,w_1,w_2}((1-ky)^{-\frac{\beta}{k}})dw.$$

Applying the definition of the Riemann–Liouville fractional derivative operator, we have:

$$=\int_0^\infty w^{u-1}\left(\frac{1}{k\Gamma_k(-\alpha)}\int_0^y (y-t)^{-\frac{\alpha}{k}-1}E_{k,w_1,w_2}\left(\frac{-w^k y^2}{kt(y-t)}\right)(1-kt)^{-\frac{\beta}{k}}dt\right)dw.$$

The binomial expansion is defined as follows:

$$(1-kt)^{\frac{-\beta}{k}}=\sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{n!}t^n.$$

Using the above result, we have:

$$=\int_0^\infty w^{u-1}\left(\frac{1}{k\Gamma_k(-\alpha)}\int_0^y (y-t)^{-\frac{\alpha}{k}-1}\right.$$

$$E_{k,w_1,w_2}\left(\frac{-w^k y^2}{kt(y-t)}\right)\left(\sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{n!}t^n\right)dt\Bigg)dw.$$

Interchanging the order of integration and summation, we have:

$$=\sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{n!}\int_0^{\infty}w^{u-1}\left(\frac{1}{k\Gamma_k(-\alpha)}\int_0^y t^n(y-t)^{-\frac{\alpha}{k}-1}E_{k,w_1,w_2}\left(\frac{-w^k y^2}{kt(y-t)}\right)dt\right)dw.$$

From the definition of the Mellin transform, we have:

$$=\sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{n!}M\left(D^{\alpha,w_1,w_2}_{k,w,y}(y^{\frac{nk}{k}}):u\right).$$

Applying Theorem 9.3.4, we have:

$$=\sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{n!}\frac{y^{\frac{nk-\alpha}{k}}\Gamma^{w_1,w_2}_{k,0}(u)}{\Gamma_k(-\alpha)}B_k(nk+u+k,u-\alpha)$$

$$=\frac{y^{\frac{-\alpha}{k}}}{\Gamma_k(-\alpha)}\Gamma^{w_1,w_2}_{k,0}(u)\sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{n!}B_k(nk+u+k,u-\alpha)y^n.$$

□

**Theorem 9.3.6.** *Let $\Re(\beta)>0$, $\Re(\alpha)>0$, $|y|<\frac{1}{k}$, and $u>0$, then*

$$M\left(e^{-w}D^{\alpha,w_1,w_2}_{k,w,y}(y^{\frac{\mu}{k}}):u\right)=\frac{\Gamma(u)}{\Gamma_k(-\alpha)}B^{(w_1,w_2)}_{k,w}(\mu+k,-\alpha)y^{\frac{\mu-\alpha}{k}}. \tag{9.3.6}$$

*Proof.* Applying the Mellin transform, we have:

$$M\left(e^{-w}D^{\alpha,w_1,w_2}_{k,w,y}(y^{\frac{\mu}{k}}):u\right)=\int_0^{\infty}w^{u-1}e^{-w}D^{\alpha,w_1,w_2}_{k,w,y}(y^{\frac{\mu}{k}})dw.$$

Applying Theorem 9.2.2, we have:

$$M\left(e^{-w}D^{\alpha,w_1,w_2}_{k,w,y}(y^{\frac{\mu}{k}}):u\right)=\int_0^{\infty}w^{u-1}e^{-w}\frac{B^{(w_1,w_2)}_{k,w}(\mu+k,-\alpha)}{\Gamma_k(-\alpha)}y^{\frac{\mu-\alpha}{k}}dw$$

$$M\left(e^{-w}D^{\alpha,w_1,w_2}_{k,w,y}(y^{\frac{\mu}{k}}):u\right)=\frac{B^{(w_1,w_2)}_{k,w}(\mu+k,-\alpha)}{\Gamma_k(-\alpha)}y^{\frac{\mu-\alpha}{k}}\left\{\int_0^{\infty}w^{u-1}e^{-w}dw\right\}.$$

Applying the definition of the $k$-gamma function, we have:

$$M\left(e^{-w}D^{\alpha,w_1,w_2}_{k,w,y}(y^{\frac{\mu}{k}}):u\right)=\frac{B^{(w_1,w_2)}_{k,w}(\mu+k,-\alpha)}{\Gamma_k(-\alpha)}y^{\frac{\mu-\alpha}{k}}\Gamma(u)$$

$$M\left(e^{-w}D_{k,w,y}^{\alpha,w_1,w_2}(y^{\frac{\mu}{k}}):u\right)=\frac{\Gamma(u)}{\Gamma_k(-\alpha)}B_{k,w}^{(w_1,w_2)}(\mu+k,-\alpha)y^{\frac{\mu-\alpha}{k}}.$$

□

**Theorem 9.3.7.** *Let* $\Re(\beta)>0$, $\Re(\alpha)>0$, $|y|<\frac{1}{k}$, *and* $u>0$, *then*

$$M\left(e^{-w}D_{k,w,y}^{\alpha,w_1,w_2}((1-ky)^{-\frac{\beta}{k}}):u\right)=\frac{\Gamma(u)y^{-\frac{\alpha}{k}}}{\Gamma_k(-\alpha)}\sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{n!}B_{k,w}^{(w_1,w_2)}(nk+k,-\alpha)y^n. \tag{9.3.7}$$

*Proof.* From the binomial expansion defined as follows:

$$(1-kt)^{\frac{-\beta}{k}}=\sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{n!}t^n.$$

Applying Theorem 9.2.3, we have:

$$M\left(e^{-w}D_{k,w,y}^{\alpha,w_1,w_2}((1-ky)^{-\frac{\beta}{k}}):u\right)=M\left(e^{-w}D_{k,w,y}^{\alpha,w_1,w_2}\left(\sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{n!}t^n\right):u\right)$$

$$M\left(e^{-w}D_{k,w,y}^{\alpha,w_1,w_2}((1-ky)^{-\frac{\beta}{k}}):u\right)=\sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{n!}M\left(e^{-w}D_{k,w,y}^{\alpha,w_1,w_2}(t^{\frac{nk}{k}}):u\right).$$

Applying Theorem 9.3.6, we have:

$$M\left(e^{-w}D_{k,w,y}^{\alpha,w_1,w_2}((1-ky)^{-\frac{\beta}{k}}):u\right)=\frac{\Gamma(u)y^{-\frac{\alpha}{k}}}{\Gamma_k(-\alpha)}\sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{n!}B_{k,w}^{(w_1,w_2)}(nk+k,-\alpha)y^n.$$

□

## 9.4 Generating function

**Theorem 9.4.1.** *Let* $\Re(\beta)>0$, $\Re(\alpha)>0$, $|\frac{y}{(1-t)}|<\frac{1}{k}$, *and* $\Re(\nu)>0$, *then*

$$\sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{k^n n!}F_{k,w}^{(w_1,w_2)}[(\beta+nk,k),(\alpha,k);(\nu,k);y]t^n$$
$$=(1-t)^{-\frac{\beta}{k}}F_{k,w}^{(w_1,w_2)}\left[(\beta,k),(\alpha,k);(\nu,k);\frac{y}{1-t}\right]. \tag{9.4.1}$$

*Proof.* Taking the identity

$$\left((1-ky)-t\right)^{-\frac{\beta}{k}}=(1-t)^{-\frac{\beta}{k}}\left(1-\frac{ky}{1-t}\right)^{-\frac{\beta}{k}}$$

$$(1-ky)^{-\frac{\beta}{k}}\left(1-\frac{t}{(1-ky)}\right)^{-\frac{\beta}{k}}=(1-t)^{-\frac{\beta}{k}}\left(1-\frac{ky}{1-t}\right)^{-\frac{\beta}{k}}$$

and applying the power-series expansion of $(1-ky)^{\frac{-\beta}{k}}$, we have:

$$\sum_{n=0}^{\infty}(1-ky)^{-\frac{\beta}{k}}\frac{(\frac{\beta}{k})_n}{n!}\left(\frac{t}{(1-ky)}\right)^n=(1-t)^{-\frac{\beta}{k}}\left(1-\frac{ky}{1-t}\right)^{-\frac{\beta}{k}}.$$

Applying the properties of the $k$-Pochhammer symbol $(\frac{y}{k})_n=\frac{(y)_{n,k}}{k^n}$, we have:

$$\sum_{n=0}^{\infty}(1-ky)^{-\frac{\beta}{k}}\frac{(\beta)_{n,k}}{k^n n!}\left(\frac{t}{(1-ky)}\right)^n=(1-t)^{-\frac{\beta}{k}}\left(1-\frac{ky}{1-t}\right)^{-\frac{\beta}{k}},$$

when $|t|<|1-y|$. If we multiply both sides with $y^{\frac{\alpha}{k}-1}$ and apply the extended Caputo $k$-fractional derivative operator $D_{k,w,y}^{\alpha-\nu,w_1,w_2}$, we have:

$$\sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{k^n n!}t^n D_{k,w,y}^{\alpha-\nu,w_1,w_2}\left(y^{\frac{\alpha}{k}-1}(1-ky)^{-\frac{(\beta+nk)}{k}}\right)$$
$$=(1-t)^{-\frac{\beta}{k}}D_{k,w,y}^{\alpha-\nu,w_1,w_2}\left(y^{\frac{\alpha}{k}-1}\left(1-\frac{ky}{1-t}\right)^{-\frac{\beta}{k}}\right)$$
$$\sum_{n=0}^{\infty}\frac{(\beta)_{n,k}}{k^n n!}F_{k,w}^{(w_1,w_2)}[(\beta+nk,k),(\alpha,k);(\nu,k);y]t^n$$
$$=(1-t)^{-\frac{\beta}{k}}F_{k,w}^{(w_1,w_2)}\left[(\beta,k),(\alpha,k);(\nu,k);\frac{y}{1-t}\right].\qquad\square$$

## 9.5 Conclusion

We conclude our investigation by remarking that the extension of the Riemann-Liouville k-fractional derivative operator presented in this chapter makes a novel contribution and offers significant potential for the further development and expansion of other classes of special functions within the framework of fractional calculus in the last decades (see, for example, [3–13]). Future research may explore the application of the proposed operator to fractional differential equations, boundary value problems, and mathematical models arising in physics and engineering, as well as its extension to other generalized fractional operators and function spaces.

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CHAPTER

# 10 Main results on $U$-Bernoulli–Korobov-type polynomials and their approximate roots

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## 10.1 Introduction and background

The study of discrete Appell polynomials holds great importance in mathematics due to their distinctive properties and broad spectrum of applications. Similar to their continuous counterparts, these polynomials are characterized by a discrete shift operator acting as their fundamental differential operator. They play a key role in the development of special functions, which have practical uses in various areas such as approximation theory, numerical analysis, quantum mechanics, and other fields in mathematics, physics, engineering, and statistics (see [9,11]).

In this context, let $f : \mathbb{Z} \to \mathbb{R}$ be any function of the integers, and consider the discrete operator $\Delta f(x) = f(x+1) - f(x)$. This operator plays a crucial role in the definition and analysis of discrete Appell polynomials, further highlighting their importance in both theoretical and applied mathematics.

A discrete Appell sequence $\{p_n(x)\}_{n=0}^{\infty}$ is a sequence of polynomials such that (see [5]):

$$\Delta p_k(x) = p_k(x+1) - p_k(x) = k p_{k-1}(x), \quad k \geq 1.$$

It is known that a Taylor-series expansion can define Appell sequences (see [1]):

$$A(z)e^{xz} = \sum_{n=0}^{\infty} P_n(x) \frac{z^n}{n!}, \tag{10.1}$$

where $A(z)$ is an analytic function at $z = 0$ with $A(0) \neq 0$. Similarly, discrete Appell sequences can be defined by the Taylor-generating expansion:

$$A(z)(1+z)^x = \sum_{n=0}^{\infty} p_n(x) \frac{z^n}{n!}, \tag{10.2}$$

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where again $A(0) \neq 0$.

Typical examples include the elementary case $\{x^k\}_{k=0}^{\infty}$, whose generating function corresponds to (10.1) with $A(z) = 1$, as well as the classical Bernoulli polynomials, employed by Euler in 1740 to evaluate the series $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$. For the Bernoulli polynomials, the generating function (10.1) becomes $A(z) = \frac{z}{e^z - 1}$.

The discrete analogs of the Bernoulli polynomials are known as the Bernoulli polynomials of the second kind (see [3]), denoted by $b_k(x)$. Introduced independently by Jordan [10] and Rey Pastor [15] in 1929, they are also referred to as Rey–Pastor polynomials (see [2]). Their generating function can be written as:

$$\frac{z}{\log(1+z)}(1+z)^x = \sum_{k=0}^{\infty} b_k(x)\frac{z^k}{k!}.$$

Throughout this chapter, the following notations are used: $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{N}_0$ the set of all non-negative integers, $\mathbb{Z}$ the set of all integers, $\mathbb{R}$ the set of all real numbers, and $\mathbb{C}$ the set of all complex numbers.

The Korobov polynomials $K_n(x;\lambda)$ of the first kind are defined by the generating function (cf. [12]):

$$\frac{\lambda z}{(1+z)^{\lambda} - 1}(1+z)^x = \sum_{n=0}^{\infty} K_n(x;\lambda)\frac{z^n}{n!}.$$

When $x = 0$, the numbers $K_n(\lambda) = K_n(0;\lambda)$ are called the Korobov numbers.

In [4], Carlitz introduced the degenerate Bernoulli polynomials, given by the generating function:

$$\frac{z}{(1+\lambda z)^{\frac{1}{\lambda}} - 1}(1+\lambda z)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathscr{B}_n(x;\lambda)\frac{z^n}{n!}. \tag{10.3}$$

From (10.3), it follows that

$$\lim_{\lambda \to 0} \mathscr{B}_n(x;\lambda) = B_n(x), \quad (n \geq 0),$$

where $B_n(x)$ are the classical Bernoulli polynomials.

Additionally, for $n \in \mathbb{N}_0$, we define a new family called the $U$-Bernoulli polynomials $M_n(x)$ of degree $n$ by the power-series expansion at 0 of the following generating function (see [13]):

$$f(x;z) = \frac{z}{e^{-z} - 1}e^{-xz} = \sum_{n=0}^{\infty} M_n(x)\frac{z^n}{n!}, \quad |z| < 2\pi. \tag{10.4}$$

We have, for the first few $U$-Bernoulli polynomials $M_n(x)$:

$$M_0(x) = -1, \qquad M_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x,$$
$$M_1(x) = x - \tfrac{1}{2}, \qquad M_4(x) = -x^4 + 2x^3 - x^2 + \tfrac{1}{30},$$
$$M_2(x) = -x^2 + x - \tfrac{1}{6}, \qquad M_5(x) = x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{3}x^3 - \tfrac{1}{6}x.$$

When $x = 0$ in (10.4), the $U$-Bernoulli numbers $M_n$ are defined by the generating function:

$$f(z) = \frac{z}{e^{-z} - 1} = \sum_{n=0}^{\infty} M_n \frac{z^n}{n!}, \qquad |z| < 2\pi.$$

Some of the first $U$-Bernoulli numbers are:

$$M_0 = -1, \quad M_1 = -\frac{1}{2}, \quad M_2 = -\frac{1}{6}, \quad M_3 = 0, \quad M_4 = \frac{1}{30}, \quad M_5 = 0.$$

On the topic of polynomial families and their various extensions, a remarkably large amount of research has appeared in the literature (see, for example, [6–8,17, 18]).

Given this context, the main objective of this work is to define and study discrete $U$-Bernoulli–Korobov polynomials. We explore their algebraic and differential properties, and present graphical representations of their zeros, computed using a Python program.

## 10.2 Main results on $U$-Bernoulli–Korobov discrete polynomials

This section explores the properties of $U$-Bernoulli–Korobov discrete polynomials. We include schematic proofs to highlight the main methods and results; further details can be found in [14,16].

**Definition 10.2.1.** The new family of $U$-Bernoulli–Korobov discrete polynomials $\mathscr{P}_n(x)$ of degree $n$ in $x$ are defined by the generating function:

$$\left(\frac{z}{e^{-z} - 1}\right)(1+z)^x = \sum_{n=0}^{\infty} \mathscr{P}_n(x) \frac{z^n}{n!}, \qquad |z| < 2\pi. \tag{10.5}$$

The first six $U$-Bernoulli–Korobov discrete polynomials $\mathscr{P}_n(x)$, are:

$$\mathscr{P}_0(x) = -1, \qquad \mathscr{P}_3(x) = -x^3 + \frac{3}{2}x^2 - x,$$
$$\mathscr{P}_1(x) = -x - \frac{1}{2}, \qquad \mathscr{P}_4(x) = -x^4 + 4x^3 - 4x^2 + 3x + \frac{1}{30},$$
$$\mathscr{P}_2(x) = -x^2 - \frac{1}{6}, \qquad \mathscr{P}_5(x) = -x^5 + \frac{15}{2}x^4 - \frac{65}{3}x^3 + \frac{55}{2}x^2 - \frac{33}{6}x.$$

For $x = 0$ in (10.5) the $U$-Bernoulli–Korobov discrete numbers $\mathscr{P}_n(0)$ are defined by the generating function:

$$\frac{z}{e^{-z}-1} = \sum_{n=0}^{\infty} \mathscr{P}_n \frac{z^n}{n!}, \quad |z| < 2\pi. \tag{10.6}$$

Some of these numbers are:

$$\mathscr{P}_0 = -1; \quad \mathscr{P}_1 = -\frac{1}{2}; \quad \mathscr{P}_2 = -\frac{1}{6}; \quad \mathscr{P}_3 = 0; \quad \mathscr{P}_4 = \frac{1}{30}; \quad \mathscr{P}_5 = 0.$$

A consequence of (10.5) and (10.6) is the following proposition.

**Proposition 10.2.1.** *For $n \in \mathbb{N}$, let $\{\mathscr{P}_n(x)\}_{n\geq 0}$ be the sequences of $U$-Bernoulli–Korobov discrete polynomials in the variable $x$. Then, the following statement holds:*

$$\mathscr{P}_n(x) = \sum_{k=0}^{\infty} \binom{x}{k} \frac{n!}{(n-k)!} \mathscr{P}_{n-k}.$$

**Proof.** It is sufficient to use the generatriz function given in Definition 10.2.1:

$$\begin{aligned}\sum_{n=0}^{\infty} \mathscr{P}_n(x) \frac{z^n}{n!} &= \left(\frac{z}{e^{-z}-1}\right)(1+z)^x \\ &= \sum_{n=0}^{\infty} \mathscr{P}_n \frac{z^n}{n!} \sum_{m=0}^{\infty} \binom{x}{m} z^m \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{x}{k} \frac{\mathscr{P}_n - k}{(n-k)!} z^n.\end{aligned}$$

Comparing the coefficients of $z^n$ on both sides of the equation, we have

$$\begin{aligned}\frac{\mathscr{P}_n(x)}{n!} &= \sum_{k=0}^{n} \binom{x}{k} \frac{\mathscr{P}_n - k}{(n-k)!} \\ \mathscr{P}_n(x) &= \sum_{k=0}^{n} \binom{x}{k} \frac{n!}{(n-k)!} \mathscr{P}_{n-k}.\end{aligned}$$

This completes the proof.

**Theorem 10.2.1.** *For $n \in \mathbb{N}$, let $\{\mathscr{P}_n(x)\}_{n\geq 0}$ be the sequences of $U$-Bernoulli–Korobov discrete polynomials in the variable $x$ that satisfy the following relation:*

$$\mathscr{P}_n(x) = \mathscr{P}_n(x+1) - n\mathscr{P}_{n-1}(x).$$

**Proof.** Of the generating function (10.5), we have:

$$\left(\frac{z}{e^z-1}\right)(1+z)^x=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}. \tag{10.7}$$

Multiplying by $(1+z)$ both sides of (10.7), we have:

$$\begin{aligned}
\left(\frac{z}{e^{-z}-1}\right)(1+z)^{(x+1)}&=(1+z)\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}\\
\sum_{n=0}^{\infty}\mathscr{P}_n(x+1)\frac{z^n}{n!}&=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}+z\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}\\
&=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}+\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^{n+1}}{n!}\\
&=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}+\sum_{n=1}^{\infty}\mathscr{P}_{n-1}(x)\frac{z^n}{(n-1)!}\\
&=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}+\sum_{n=0}^{\infty}n\,\mathscr{P}_{n-1}(x)\frac{z^n}{n!}\\
\sum_{n=0}^{\infty}\mathscr{P}_n(x+1)\frac{z^n}{n!}&=\sum_{n=0}^{\infty}[\mathscr{P}_n(x)+n\mathscr{P}_{n-1}(x)]\frac{z^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{n!}$ on both sides of the equation, the result is:

$$\mathscr{P}_n(x)=\mathscr{P}_n(x+1)-n\,\mathscr{P}_{n-1}(x).$$

**Theorem 10.2.2** (Differential expressions). *For $n\in\mathbb{N}$, let $\{\mathscr{P}_n(x)\}_{n\geq 0}$ be the sequences of $U$-Bernoulli–Korobov discrete polynomials in the variable $x$, which satisfy the following relations:*

**1.**

$$(n-1)\mathscr{P}_n(x)-n\psi(x;n;z)\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x)=0,$$

*where*

$$\psi(x;n;z)=\left[\frac{x}{(z+1)\log(z+1)}+\frac{e^{-z}}{(e^{-z}-1)\log(z+1)}\right].$$

**2.**

$$\frac{\partial\mathscr{P}_n(x)}{\partial x}=\sum_{k=0}^{n-1}n\binom{n-1}{k}(-1)^k\frac{k!}{k+1}\mathscr{P}_{n-k-1}(x).$$

**Proof.** For the proof of 1. Consider the following equations:

$$L(x;z)=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}, \tag{10.8}$$

$$L(x;z)=\frac{z}{e^{-z}-1}(1+z)^x. \tag{10.9}$$

Partially differentiating with respect to $z$ in (10.8) and (10.9), the result is:

$$\frac{\partial L(x;z)}{\partial z}=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{nz^{n-1}}{n!}$$

and

$$\frac{\partial L(x;z)}{\partial z}=\frac{(1+z)^x}{e^{-z}-1}+\left[\frac{z(1+z)^x}{e^{-z}-1}\right]\frac{x}{1+z}+\left[\frac{z(1+z)^x}{e^{-z}-1}\right]\frac{e^{-z}}{e^{-z}-1}. \tag{10.10}$$

Partially differentiating with respect to $x$ in (10.9), we have:

$$\frac{\partial L(x;z)}{\partial x}=\frac{z\log(z+1)(1+z)^x}{e^{-z}-1}.$$

Of (10.10), we have

$$\begin{aligned}
0&=\frac{\partial L(x;z)}{\partial z}-\frac{(1+z)^x}{e^{-z}-1}-\left[\frac{z\log(z+1)(1+z)^x}{e^{-z}-1}\right]\frac{x}{(1+z)\log(z+1)}\\
&\quad-\left[\frac{z\log(z+1)(1+z)^x}{e^{-z}-1}\right]\frac{e^{-z}}{(e^{-z}-1)\log(z+1)}\\
0&=\frac{\partial L(x;z)}{\partial z}-\frac{(1+z)^x}{e^{-z}-1}-\frac{x}{(1+z)\log(z+1)}\frac{\partial L(x;z)}{\partial x}\\
&\quad-\frac{e^{-z}}{(e^{-z}-1)\log(z+1)}\frac{\partial L(x;z)}{\partial x}\\
0&=\frac{z\partial L(x;z)}{\partial z}-\left[\frac{zx}{(1+z)\log(z+1)}+\frac{ze^{-z}}{(e^{-z}-1)\log(z+1)}\right]\frac{\partial L(x;z)}{\partial x}-\frac{z(1+z)^x}{e^{-z}-1}\\
0&=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{nz^{n-1}}{n!}z-\sum_{n=0}^{\infty}\left[\frac{zx}{(1+z)\log(z+1)}+\frac{ze^{-z}}{(e^{-z}-1)\log(z+1)}\right]\\
&\quad\frac{\partial}{\partial x}\mathscr{P}_n(x)\frac{z^n}{n!}-\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}\\
0&=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{nz^n}{n!}-\sum_{n=0}^{\infty}\left[\frac{x}{(1+z)\log(z+1)}+\frac{e^{-z}}{(e^{-z}-1)\log(z+1)}\right]
\end{aligned}$$

$$\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x)\frac{nz^n}{n!}-\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}$$

$$0=(n-1)\mathscr{P}_n(x)-\left[\frac{x}{(1+z)\log(z+1)}+\frac{e^{-z}}{(e^{-z}-1)\log(z+1)}\right]n\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x).$$

This completes the proof.

**Proof.** For the proof of (10.1). Partially differentiating with respect to $x$ in 10.2.1, we have:

$$\left(\frac{z}{e^{-z}-1}\right)\frac{\partial}{\partial x}\left[(1+z)^x\right]=\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\mathscr{P}_n(x)\frac{z^n}{n!}$$

$$\left(\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}z^{n+1}\right)=\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\mathscr{P}_n(x)\frac{z^n}{n!}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\mathscr{P}_{n-1-k}(x)(-1)^k\binom{n-1}{k}\frac{k!}{(k+1)}n\frac{z^n}{n!}=\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\mathscr{P}_n(x)\frac{z^n}{n!}.$$

Comparing the coefficients of $\frac{z^n}{n!}$ on both sides of the equation, the result is:

$$\frac{\partial}{\partial x}\mathscr{P}_n(x)=\sum_{k=0}^{n-1}n\binom{n-1}{k}(-1)^k\frac{k!}{k+1}\mathscr{P}_{n-k-1}(x).$$

**Proposition 10.2.2.** *The $U$-Bernoulli–Korobov discrete polynomials in the variable $x$ satisfy the following relations:*

$(i)$ $\mathscr{P}_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}(y)_k\mathscr{P}_{n-k}(x).$

$(ii)$ $\mathscr{P}_n(x)=\sum_{k=0}^{n-1}n\binom{n-1}{k}(x)_k+\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_k(x).$

**Proof.** For the proof of $(i)$, it is sufficient to use the generatriz function given in (10.5):

$$\sum_{n=0}^{\infty}\mathscr{P}_n(x+y)\frac{z^n}{n!}=\left(\frac{z}{e^{-z}-1}\right)(1+z)^{x+y}$$

$$=\left(\frac{z}{e^{-z}-1}\right)(1+z)^x(1+z)^y$$

$$=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}\sum_{k=0}^{n}\binom{y}{k}z^n$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(y)_k \mathscr{P}_{n-k}(x)\frac{z^n}{n!}.$$

Comparing the coefficients of $\frac{z^n}{n!}$ on both sides of the equation, the result is:

$$\mathscr{P}_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}(y)_k \mathscr{P}_{n-k}(x).$$

To prove $(ii)$. We will examine the generating function (10.5) as follows:

$$\left(\frac{z}{e^{-z}-1}\right)(1+z)^x = \sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}$$

$$e^z z(1+z)^x = \sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!} - e^z\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}$$

$$z\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{z^{n-k}}{(n-k)!}\binom{x}{k}z^k = \sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!} - \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{z^{n-k}}{(n-k)!}\mathscr{P}_k(x)\frac{z^k}{k!}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\binom{n-1}{k}(x)_k n\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left[\mathscr{P}_n(x) - \sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_k(x)\right]\frac{z^n}{n!}.$$

Comparing the coefficients of $\frac{z^n}{n!}$ on both sides of the equation, the result is:

$$\mathscr{P}_n(x) = \sum_{k=0}^{n-1} n\binom{n-1}{k}(x)_k + \sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_k(x).$$

**Theorem 10.2.3.** *For $n \geq 0$, let $\{\mathscr{P}_n(x)\}_{n\geq 0}$ be the sequences of $U$-Bernoulli–Korobov discrete polynomials in the variable $x$ that satisfy the following relation:*

$$\sum_{k=0}^{n}\binom{n}{k}[\mathscr{P}_k(x+y)\mathscr{P}_{n-k} - \mathscr{P}_{n-k}(x)\,\mathscr{P}_k(y)] = 0.$$

**Proof.** Let's consider the following expressions:

$$\left(\frac{z}{e^{-z}-1}\right)(1+z)^x = \sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!} \tag{10.11}$$

and

$$\left(\frac{z}{e^{-z}-1}\right)(1+z)^y = \sum_{n=0}^{\infty}\mathscr{P}_n(y)\frac{z^n}{n!}. \tag{10.12}$$

Of (10.11) and (10.12), we have

$$\left[\frac{z}{e^{-z}-1}\right]^2 (1+z)^{x+y} = \left(\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(y)\frac{z^n}{n!}\right)$$

$$\left(\sum_{n=0}^{\infty}\mathscr{P}_n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(x+y)\frac{z^n}{n!}\right) = \left(\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(y)\frac{z^n}{n!}\right)$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}\mathscr{P}_k(x+y)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}(x)\mathscr{P}_k(y)\frac{z^n}{n!}$$

$$\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}\mathscr{P}_k(x+y) = \sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}(x)\mathscr{P}_k(y).$$

Therefore

$$\sum_{k=0}^{n}\binom{n}{k}[\mathscr{P}_k(x+y)\mathscr{P}_{n-k} - \mathscr{P}_{n-k}(x)\,\mathscr{P}_k(y)] = 0.$$

**Theorem 10.2.4.** *For $n\in\mathbb{N}$, let $\{\mathscr{P}_n(x)\}_{n\geq 0}$ be the sequences of $U$-Bernoulli–Korobov discrete polynomials in the variable $x$ that satisfy the following relation:*

$$\mathscr{P}_n(x) = \mathscr{P}_n + \sum_{k=0}^{n-1}\frac{n}{(k+1)}\binom{n-1}{k}(x)_{k+1}\mathscr{P}_{n-1-k}.$$

**Proof.** Of (10.5) and (10.6), we obtain:

$$\left(\frac{z}{e^{-z}-1}\right)(1+z)^x = \sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}$$

and

$$\left(\frac{z}{e^{-z}-1}\right) = \sum_{n=0}^{\infty}\mathscr{P}_n\frac{z^n}{n!},$$

then

$$\begin{aligned}\sum_{n=0}^{\infty}[\mathscr{P}_n(x)-\mathscr{P}_n]\frac{z^n}{n!} &= \sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!} - \sum_{n=0}^{\infty}\mathscr{P}_n\frac{z^n}{n!}\\ &= \frac{z}{e^{-}z-1}\left[(1+z^x)-1\right]\\ &= \sum_{n=0}^{\infty}\mathscr{P}_n\frac{z^n}{n!}\sum_{n=0}^{\infty}\binom{x}{n+1}z^{n+1}\end{aligned}$$

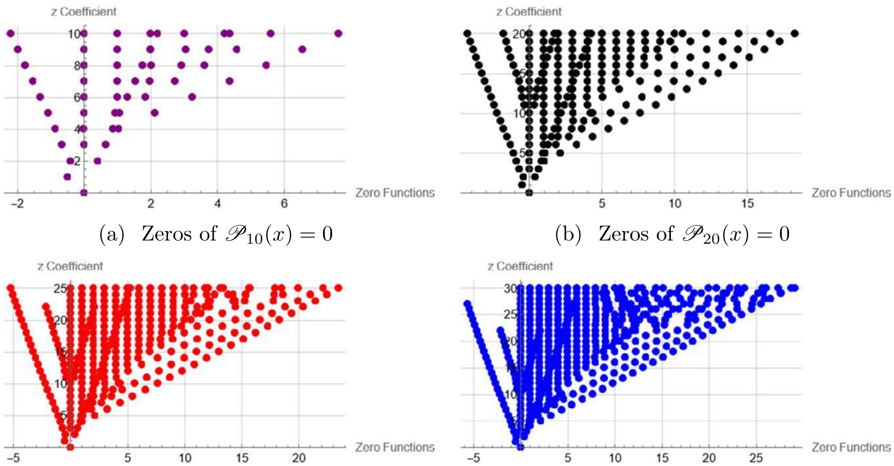
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{x}{k+1} \frac{\mathscr{P}_{n-k}}{(n-k)!} z^{n+1}$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \frac{n}{(k+1)} \binom{n-1}{k} (x)_{k+1} \mathscr{P}_{n-k-1} \frac{z^n}{n!}.$$

Comparing the coefficients of $\frac{z^n}{n!}$ on both sides of the equation, we obtain:

$$\mathscr{P}_n(x) - \mathscr{P}_n = \sum_{k=0}^{n-1} \frac{n}{(k+1)} \binom{n-1}{k} (x)_{k+1} \mathscr{P}_{n-k-1}$$
$$\mathscr{P}_n(x) = \mathscr{P}_n + \sum_{k=0}^{n-1} \frac{n}{(k+1)} \binom{n-1}{k} (x)_{k+1} \mathscr{P}_{n-k-1}.$$

## 10.3 Approximate roots of $U$-Bernoulli–Korobov-type polynomials and their applications

In this section, we investigate certain zero distributions of $U$-Bernoulli–Korobov-type polynomials. The graph plots using the Mathematica program show the zero distribution patterns.




(a) Zeros of $\mathscr{P}_{10}(x) = 0$

(b) Zeros of $\mathscr{P}_{20}(x) = 0$

(c) Zeros of $\mathscr{P}_{25}(x) = 0$

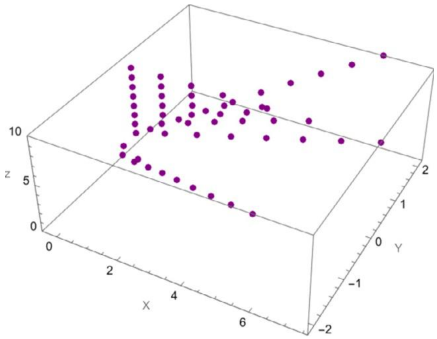
(d) Zeros of $\mathscr{P}_{30}(x) = 0$

**FIGURE 10.1**

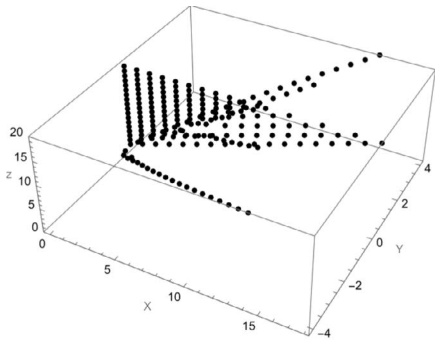
Roots of $U$-Bernoulli–Korobov-type polynomials, plotted considering 10, 20, 25, and 30 points.

In Fig. 10.1 we show four different plots, each representing an amount of zero distribution. In 10.1(a) purple dots for 10 points, in 10.1(b) black dots for 20 points, in 10.1(c) red dots for 25 points, and 1(d) blue dots for 30 points, and we can see that all the roots are located along the $x$-axes, except in the zero distribution for 10.1(b).

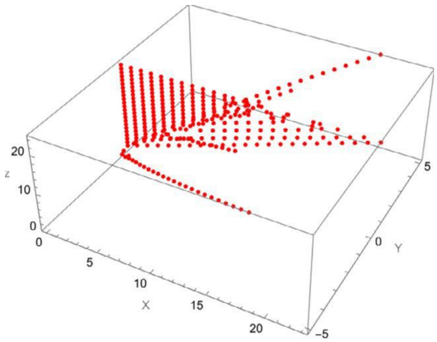
Finally, we will show the induced mesh of $U$-Bernoulli–Korobov-type polynomials for different values of $n$ (10.5) $\mathscr{P}_n(x) = 0$.



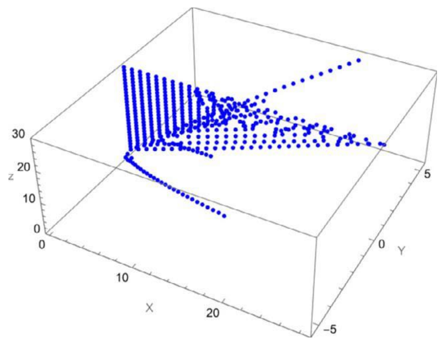

(a) Data visualization of zeros of $\mathscr{P}_{10}(x)$.




(b) Induced meshes of $\mathscr{P}_{20}(x)$.




(c) Data visualization of zeros of $\mathscr{P}_{25}(x)$.




(d) Data visualization of zeros of $\mathscr{P}_{30}(x)$.

**FIGURE 10.2**

Data visualization of zeros of $U$-Bernoulli–Korobov-type polynomials, plotted considering 10, 20, 25, and 30 points.

In Fig. 10.2, we present four 3D plots illustrating the distribution of the zeros of the polynomials $\mathscr{P}_n(x)$ for degrees $n = 10$, 20, 25, and 30, respectively. The plots reveal that the zeros tend to cluster toward the upper left region of each mesh. As the degree increases, the number of zeros naturally grows, and the mesh becomes denser and more structured. Notably, a slight perturbation appears around $x \approx 10$, becoming especially noticeable in the blue mesh of Fig. 10.2(d), corresponding to $\mathscr{P}_{30}(x)$.

Next, we calculated an approximate solution satisfying the $U$-Bernoulli–Korobov-type polynomials $\mathscr{P}_n(x)$ for $n = 2, 3, \ldots, 15$. The results are presented in Table 10.1.

**Table 10.1** Approximate solutions for $\mathscr{R}_n(x) = 0$.

| Grade $n$ | $x$ |
|---|---|
| 0 | — |
| 1 | $-0.5000$ |
| 2 | $-0.4082i$, $0.4082i$ |
| 3 | 0, $0.75 - 0.6614i$, $0.75 + 0.6614i$ |
| 4 | $-0.01087$, 1.033, $1.489 \pm 0.8662i$ |
| 5 | 0, 0.9438, 2.132, $2.212 \pm 1.074i$ |
| 6 | 0.0003813, 0.9977, 1.850, 3.253, $2.949 \pm 1.303i$ |
| 7 | 0, 1.003, 1.982, 2.728, 4.363, $3.712 \pm 1.537i$ |
| 8 | $-0.00001231$, 1.000, 2.014, 2.918, 3.620, 5.462<br>$4.493 \pm 1.763i$ |
| 9 | 0, 0.9999, 2.001, 3.048, 3.751, 4.577<br>6.553, $5.285 \pm 1.983i$ |
| 10 | $3.821 \times 10^{-7}$, 1.000, 1.999, 3.005, 4.218, 4.366<br>5.605, 7.639, $6.084 \pm 2.2i$ |

## 10.4 Conclusions

In this work, the algebraic and differential properties of a new family of polynomials called discrete $U$-Bernoulli–Korobov polynomials were defined and studied. They are constructed from a generating function that combines characteristics of Bernoulli and Korobov polynomials. Recurrence formulas, explicit expressions, and functional relations describing the behavior of these polynomials were presented, including their expansion in power series, derivatives, and discrete shifts. Additionally, the approximate roots of these polynomials were analyzed using graphical representations generated with computer programs, revealing structured patterns and a characteristic concentration of zeros in certain regions of the plane. These observations allow for a deeper study of the behavior of their zeros and open up new possibilities for their application in mathematical contexts related to discrete analysis and special polynomial theory.

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CHAPTER

# 11 Solvability of the Cauchy problem for a fractionally loaded equation with variable coefficients

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## 11.1 Introduction and problem statement

At present, special attention is being paid to the most rapidly developing direction of mathematical physics on a global scale – methods of studying direct and inverse problems [1], [2], [3], [4], [5]. These areas have become one of the most important problems in mathematical physics and engineering. Due to the wide application of this problem and the novelty and complexity of its theory, it has attracted the attention of many scientists. In recent years, the control of heat generation processes has been rapidly developing, since the thermal conductivity and relaxation function of each medium are different, and these quantities are closely related to the initial state and properties of the medium. For this reason, the study of initial, initial-boundary value problems, and the problems posed for loaded integro-differential equations of fractional heat transfer is a targeted scientific study.

We should note that the scientific studies [6], [7], and [8] were dedicated to the Cauchy problem for loaded equations, while works [9] and [10] focused on the heat equation without loading. These studies formulated the problem and proposed methods that were utilized in our research.

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In this chapter, we investigate the Cauchy problem for a heat equation involving fractional derivatives. We establish the unique solvability of the direct problem for a multidimensional loaded heat equation with variable coefficients in Hölder spaces. The loaded term is formulated using the fractional Riemann–Liouville derivative. The findings of this study are applicable to the analysis of heat equations with fractional loads, particularly in cases where the order is higher and the loaded term is represented as a fractional function.

Before proceeding to the formulation of the problem, we give some definitions and propositions in Holder spaces [11].

We introduce the following notations:

Let $R^n$ denote the $n$-dimensional Euclidean space, where $x = (x^{(1)}, ..., x^{(n)}) \in R^n$;

Let $R_T^n$ represent the $(n+1)$-dimensional Euclidean space, with points denoted as $(x, y)$, where $x \in R^n$ and $y \in (0, T]$, and

$$R_T^{n-1} = \{(x', y) | x' \in R^{n-1},\ 0 < y < T\},$$
$$\left(\overline{R}_T^{n-1}\right) = \left\{\left(x', y\right) x' \in R^{n-1}, 0 \le y \le T\right\}.$$

Let $f(x)$ be a function defined on $R^n$

**Definition 11.1.1.** If for any two points $x^{(1)}, x^{(2)} \in R^n$:

$$|f(x^{(1)}) - f(x^{(2)})| \le A|x^{(1)} - x^{(2)}|^l,\ A = const > 0,\ l \in (0,1), \tag{11.1.1}$$

the inequality holds, then, the function $f(x)$ is said to satisfy the Holder condition with $l$ exponent in $R^n$, and the class of functions satisfying the condition (11.1.1) is denoted as $H^l(R^n)$.

**Definition 11.1.2.** If for any given pair of values $(x_1, y_1), (x_2, y_2) \in R_T^n$, $x_1 = x_1^{(1)}, x_1^{(2)}, ..., x_1^{(n)}, x_2 = x_2^{(1)}, x_2^{(2)}, ..., x_2^{(n)}$ holds,

$$|f(x_1, y_1) - f(x_2, y_2)| \le \sum_{i=1}^{n} A_i |x_1^{(i)} - x_2^{(i)}|^l + A_{i+1}|y_1 - y_2|^{l/2}, \tag{11.1.2}$$
$$A_i = const > 0,\ \ l \in (0,1),$$

then, the function $f(x, y)$ is said to satisfy the Holder condition with exponent $l, l/2$ on $R_T^n$, and the class of functions satisfying the condition with $l, l/2$, is denoted as $H^{l,l/2}(R_T^n)$.

If $\varphi(x) \in H^l(R^n)$, $f(x, y) \in H^{l,l/2}(R_T^n)$, then $H^{l+m}(R^n)$, $H^{l+m,(l+m)/2}(R_T^n)$ the norms in the spaces are defined as [11].

**Cauchy problem.** Find a solution $u(x, y)$ in the domain $(x, y) \in R_T^n$ of the following loaded heat equation:

$$u_y - k(y)\,\Delta u = \lambda D_{0y}^{-\alpha}(\alpha_0 u_x(0, y) + \beta_0 u(0, y)),\ \ (x, y) \in R_T^n \tag{11.1.3}$$

that satisfies the condition:

$$u\left(x,0\right)=\phi\left(x\right),\ x\in R^{n}, \tag{11.1.4}$$

where $k(y)$, $\phi\left(x\right)$ are given real-valued sufficiently smooth functions, $\lambda,\ \alpha_0,\ \beta_0\in R$, $D_{0y}^{-\alpha}$ is the Riemann–Liouville fractional integral operator of order $\alpha$ is defined by:

$$D_{0y}^{-\alpha}f(y)=\frac{1}{\Gamma(\alpha)}\int\limits_{0}^{y}\frac{f(z)dz}{(y-z)^{1-\alpha}},\ \ if\ \ \alpha>0,$$

$D_{0y}^{0}f(y)=f(y)$, if $\alpha=0$; $f(y)\in L_1(a,b)(a<b<+\infty)$.

The inhomogeneous problem (11.1.3) and (11.1.4) in the case of $\lambda\equiv 0$ is well-known, and has been studied in many works. This equation, considered as a heat equation, is investigated with various kinds of conditions such as initial Cauchy problems [12], mixed, nonlocal, and other types. At the beginning of the study of the Cauchy problem (11.1.3) and (11.1.4), we use the following formula:

$$\begin{aligned}W(x,y)&=\int_{R^n}\chi(\xi)G(x,\sigma(y),\xi,0)d\xi+\\&+\int_0^{\sigma(y)}\frac{d\eta}{k(\sigma^{-1}(\eta))}\int_{R^n}F(\xi,\sigma^{-1}(\eta))G(x,\sigma(y),\xi,\eta)d\xi,\end{aligned} \tag{11.1.5}$$

which is the solution of the following Cauchy problem for the heat equation with variable coefficients:

$$W_y-k\left(y\right)\Delta W=F\left(x,y\right),\ (x,y)\in R_T^n,$$
$$W\left(x,0\right)=\chi\left(x\right),\ x\in R^n.$$

In (11.1.5), the functions $\sigma\left(y\right)=\int_0^y k\left(\eta\right)d\eta$ and $\sigma^{-1}\left(y\right)$ are the inverse functions of $\sigma\left(y\right)$, $G(x,\sigma\left(y\right),\xi,\eta)=G_1(x-\xi,\sigma\left(y\right)-\eta)$ is a fundamental solution of a differential operator with a variable coefficient $\partial/\partial y-k\left(y\right)\Delta$, where:

$$\xi=\left(\xi^{(1)},\xi^{(2)},...,\xi^{(n)}\right),\ \ \xi'=\left(\xi^{(1)},\xi^{(2)},...,\xi^{(n-1)}\right),\ \ d\xi=d\xi^{(1)}...d\xi^{(n)}.$$

We note that the Cauchy problem can also be studied if the $\alpha_0$ and $\beta_0$ coefficients in Eq. (11.1.3) are functions of $x$ and $y$.

It should be noted that in the work of Dikinov et al. [13], the problem related to a one-dimensional bounded medium was considered at one of the ends of which there is a heat source in this medium, the heat conduction equation. Later, in [14], studies were conducted on cases where the loaded term involved a combination (differential) of traces of the function $u(x,t)$. However, the loaded heat equation with a fractional operator was investigated in [10], [15], [16], [17], and [18].

We note that in this work, we investigated the Cauchy problem for the multidimensional fractionally loaded heat equation in Holder spaces. If $\alpha_0=0$ in Eq. (11.1.3), the problem is analyzed analogously to [19]. Hence, in this study, we focus on the case $\alpha_0\neq 0$ and $\beta_0=0$.

## 11.2 Investigation of the problem

Our main result is stated as follows:

**Theorem 11.2.1.** *If* $k(y) \in E := \left\{k(y) \in C^1[0,T], 0 < k_0 < k(y) \le k_1 < \infty\right\}$, *and*

$$\phi(x) \in H^{l+3}\left(R^n\right), \tag{11.2.1}$$

$$\phi'(x) \le \phi_0 = const > 0, \tag{11.2.2}$$

*then the Cauchy problem admits a unique solution in the function space* $u(x,y) \in H^{l+2,(l+2)/2}\left(R^n_T\right)$.

Before proceeding with the proof of Theorem 11.2.1, we first outline the method used to establish the unique solvability of the problem.

To prove Theorem 11.2.1, i.e., to solve the Cauchy problem, we transform it into an equivalent Volterra-type integral equation. By employing the theory of integral equations and imposing suitable conditions on the given functions, in accordance with the theorem's assumptions, we establish the existence of a unique solution to the derived equation. This, in turn, ensures the unique solvability of the original problem. Following this approach, we first reduce problem (11.1.3) and (11.1.4) to an integral equation of the Volterra type.

Using formula (11.1.5) and considering the properties of the fundamental solution, the Cauchy problem (11.1.3) and (11.1.4) is equivalently rewritten as follows:

$$\begin{aligned} u(x,y) = \int_{R^n} G(x,\sigma(y);\xi,0)\phi(\xi)d\xi + \frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))} \times \\ \times \int_{R^n}\int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} u_x(0,s)\, G(x,\sigma(y);\xi,\eta)\, ds d\xi. \end{aligned} \tag{11.2.3}$$

Then, we compute the derivative $u(x,y)$ with respect to $x$ for the case $n = 1$:

$$\begin{aligned} u_x(x,y) = \int_{R^1} G_x(x,\sigma(y);\xi,0)\,\phi(\xi)d\xi + \frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))} \times \\ \times \int_{R^1}\int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} u_x(0,s)\, G_x(x,\sigma(y);\xi,\eta)\, ds d\xi \end{aligned}$$

and we consider the point where $x = 0$:

$$\begin{aligned} u_x(0,y) = \int_{R^1} G_x(x,\sigma(y);\xi,0)|_{x=0}\,\phi(\xi)d\xi + \frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))} \times \\ \times \int_{R^1}\int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} u_x(0,s)\, G_x(x,\sigma(y);\xi,\eta)|_{x=0}\, ds d\xi. \end{aligned} \tag{11.2.4}$$

Thus by denoting $u_x(0, y) = v(0, y)$, we obtain the following Volterra-type integral equation of the second kind:

$$v(0, y) - \frac{\lambda}{\Gamma(\alpha)} \int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))} \int_{R^1} \int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} v(0, s) \frac{\xi}{2(\sigma(y) - \eta)}$$
$$\times G_x(x, \sigma(y); \xi, \eta))|_{x=0} ds d\xi = \int_{R^1} G_x(x, \sigma(y); \xi, 0)|_{x=0} \phi(\xi) d\xi. \qquad (11.2.5)$$

Similarly, for $n = 2, 3, \ldots$, taking the derivative with respect to $x_i$ $(i = 2, 3, \ldots)$, we obtain the corresponding two-dimensional, three-dimensional, and subsequently integral equations of the Volterra type (11.2.5), which are studied in a similar manner.

**Lemma 11.2.2.** *Let $k(t) \in E$, $\phi(x) \in H^{l+3}\left(R^1\right)$, and suppose that $|\phi'(x)| \le \phi_1$, with conditions (11.2.1) and (11.2.2) satisfied. Then, there exists a unique solution to the integral equation (11.2.5) in the function space $H^{l+3,(l+2)/2} \in \left(R_T^0\right)$.*

*Proof.* We prove this lemma using the method of successive approximations, constructing the following functional sequence:

$$v_0(0, y) = \int_{R^1} G_x(x, \sigma(y); \xi, 0)|_{x=0} \phi(\xi) d\xi$$

$$v_1(0, y) = \frac{\lambda}{\Gamma(\alpha)} \int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))} \int_{R^1} \int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} v_0(0, s) \times$$
$$\times G_x(x, \sigma(y); \xi, \eta)|_{x=0} ds d\xi,$$

$$v_2(0, y) = \frac{\lambda}{\Gamma(\alpha)} \int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))} \int_{R^1} \int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} v_1(0, s) \times$$
$$\times G_x(x, \sigma(y); \xi, \eta)|_{x=0} ds d\xi,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (11.2.6)$$

$$v_n(0, y) = \frac{\lambda}{\Gamma(\alpha)} \int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))} \int_{R^1} \int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} \times$$
$$\times v_{n-1}(0, s) G_x(x, \sigma(y); \xi, \eta)|_{x=0} ds d\xi,$$

$$(x, y) \in R_T^1, \quad n = 1, 2, \ldots\ldots$$

Therefore taking into account (11.2.1) and (11.2.2), along with the properties of the fundamental solution of the heat equation [12], the equality $\int_{R^n} G_1(x - \xi, \sigma(y) - \eta) d\xi = 1$, and introducing the notation

$$\phi_1 = max|\phi'(x)|, \qquad (11.2.7)$$

we estimate the functional sequence $\{v_n(0, y)\}$ defined with the integral Eq. (11.2.5), modulo in the class $R_T^0$, according to $G_{1x} = -G_{1\xi}$:

$$|v_0(0, y)|_T^{l+2,(l+2)/2} \le \int_{R^1} |\phi(\xi)|_T^{l+3,(l+2)/2} G_{1x}(x - \xi, \sigma(y))|_{x=0} d\xi$$

$$\le -\int_{R^1} |\phi(\xi)|_T^{l+3,(l+2)/2} G_{1\xi}(x-\xi,\sigma(y))|_{x=0} d\xi$$
$$\le \int_{R^1} \left|\phi'(\xi)\right|_T^{l+3,(l+2)/2} G_1(x-\xi,\sigma(y))|_{x=0} d\xi \le \phi_1.$$

Similarly, we evaluate

$$|v_1(0,y)|_T^{l+2,(l+2)/2} \le \frac{|\lambda|}{\Gamma(\alpha)} \int_0^{\sigma(y)} \frac{d\eta}{\left|k(\sigma^{-1}(\eta))\right|} \int_{R^1}\int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta)-s\right)^{\alpha-1} \times$$
$$\times |v_0(0,s)|_T^{l+2,(l+2)/2} G_1(-\xi,\sigma(y)-\eta) ds d\xi$$
$$\le \frac{|\lambda|\phi_1 2\sqrt{k_1 T}}{k_0\sqrt{\pi}} \frac{y^\alpha}{\alpha!},$$
$$|v_2(0,y)|_T^{l+2,(l+2)/2} \le \frac{|\lambda|\,\phi_1 2\sqrt{k_1 T}}{k_0\Gamma(\alpha)\,\Gamma(\alpha+1)} \int_0^{\sigma(y)} \frac{d\eta}{\left|k(\sigma^{-1}(\eta))\right|}$$
$$\times \int_{R^1}\int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta)-s\right)^{\alpha-1} \times$$
$$\times s^\alpha |v_1(0,s)|_T^{l+2,(l+2)/2} G_1(-\xi,\sigma(y)-\eta) ds d\xi$$
$$\le \frac{\phi_1\left(2|\lambda|\sqrt{k_1 T}\right)^2}{\left(k_0\sqrt{\pi}\right)^2} \frac{y^{2\alpha}}{(2\alpha)!}$$

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$$|v_n(0,y)|_T^{l+2,(l+2)/2} \le \frac{\phi_1\left(2\,|\lambda|\,\sqrt{k_1 T}\right)^{n-1}}{\sqrt{\pi}^{n-1} k_0^{n-1}\Gamma(\alpha)\,\Gamma((n-1)\alpha+1)} \int_0^{\sigma(y)} \frac{d\eta}{\left|k(\sigma^{-1}(\eta))\right|} \times$$
$$\times \int_{R^1}\int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta)-s\right)^{\alpha-1} s^\alpha |v_{n-1}(0,s)|_T^{l+2,(l+2)/2} \times$$
$$\times G_1(-\xi,\sigma(y)-\eta) ds d\xi \le \frac{\phi_1\left(2|\lambda|\sqrt{k_1 T}\right)^n}{\left(k_0\sqrt{\pi}\right)^n} \frac{y^{n\alpha}}{(n\alpha)!}$$

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Thus we have constructed the following functional series:

$$\sum_{n=0}^{\infty} v_n(0,y). \tag{11.2.8}$$

Using the above estimates, we estimate (majorize) the obtained functional series with a numerical in the domain $(0,t) \in R_T^0$, i.e.,

$$\sum_{n=0}^{\infty} |v_n(0,y)| \le \sum_{n=0}^{\infty} \phi_1 A^n \frac{y^{n\alpha}}{(n\alpha)!} \le \phi_1 \sum_{n=0}^{\infty} \frac{\left(A^{\frac{1}{\alpha}}\right)^{n\alpha} T^{n\alpha}}{(n\alpha)!} \le$$

$$\leq \phi_1 \sum_{n=0}^{\infty} \frac{\left(A^{\frac{1}{\alpha}} T\right)^{n\alpha}}{(n\alpha)!} \leq \phi_1 exp\left(A^{\frac{1}{\alpha}} T\right).$$

Thus according to the latest estimates, according to the Weierstrass theorem, one can easily verify that the resulting functional series (11.2.8) converges absolutely and uniformly on the set $H$. Therefore the sequence of functions $v_k(0,t)$ defined by the integral Eq. (11.2.5), converges uniformly for some function $v(0, y)$ in the space $H^{l+2,(l+2)/2}\left(R_T^0\right)$.

Thus we have established the existence of a solution to (11.2.5). By substituting the obtained function into (11.2.3), we uniquely determine the solution to the Cauchy problem (11.1.3) and (11.1.4) in the class $H^{l+2,(l+2)/2}\left(R_T^1\right)$.

Now, we will prove that the integral Eq. (11.2.5) and thus the Cauchy problems (11.1.3) and (11.1.4) have a unique solution. To do this, first assume the opposite, i.e., let there be two different solutions $v_1(0, y)$ and $v_2(0, y)$ of the integral Eq. (11.2.5):

$$v_1(0, y) = \int_{R^1} G_1(-\xi, \sigma(y))\phi(\xi)d\xi + \frac{\lambda}{\Gamma(\alpha)} \int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))} \times$$
$$\times \int_{R^1} \int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} v_1(0, s) G_{1x}(-\xi, \sigma(y) - \eta)dsd\xi. \qquad (11.2.9)$$

$$v_2(0, y) = \int_{R^1} G_1(-\xi, \sigma(y))\phi(\xi)d\xi + \frac{\lambda}{\Gamma(\alpha)} \int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))} \times$$
$$\times \int_{R^1} \int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} v_2(0, s) G_{1x}(-\xi, \sigma(y) - \eta)dsd\xi. \qquad (11.2.10)$$

The difference of these functions $v_1$ and $v_2$, is denoted by $w(0, y)$:

$$w(0, y) = v_1(0, y) - v_2(0, y).$$

As a result, we obtain a homogeneous integral equation of the second kind:

$$w(0, y) = \int_{R^1} G_1(-\xi, \sigma(y))\, \phi(\xi)\, d\xi + \frac{\lambda}{\Gamma(\alpha)} \int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))} \times$$
$$\times \int_{R^1} \int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} w(0, s)\, G_{1x}(-\xi, \sigma(y) - \eta)\, dsd\xi. \qquad (11.2.11)$$

For each fixed $y \in [0, T]$, we define the supremum of the modulus of the function $w(0, y)$ as:

$$\overline{w} = \sup |w\,(0, y)|\,,\; y \in [0, T]\,.$$

From the integral Eq. (11.2.11), we derive the following inequality:

$$|w(0, y)| \leq \frac{|\lambda|}{|\Gamma(\alpha)|} \int_0^{\sigma(y)} \frac{d\eta}{|k(\sigma^{-1}(\eta))|} \int_{R^1} \int_0^{\sigma^{-1}(\eta)} |\left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} w(0, s)|\cdot$$

$$\cdot G_{1x}(-\xi, \sigma(y) - \eta)dsd\xi \le \overline{w}\frac{2}{k_0}\sqrt{\frac{k_1 T}{\pi}}\frac{y^\alpha}{\alpha!}.$$

By continuing this process, we obtain the following result for any arbitrary natural number $n$:

$$|w(0, y)| \le \overline{w}\left(\frac{2}{k_0}\sqrt{\frac{k_1 T}{\pi}}\right)^n \frac{y^{n\alpha}}{(n\alpha)!} \le \frac{c^n}{(n\alpha)!},$$

where $c = \frac{2}{k_0}\sqrt{\frac{k_1 T}{\pi}}T^\alpha$. This inequality implies that either $w(0, y) = 0$ or $v_1(0, y) = v_2(0, y)$ as $n \to \infty$. Hence, the integral Eq. (11.2.5) has a unique solution.

Indeed, considering the theory of integral equations and expressing the solutions of (11.2.5) through the resolvent, it is straightforward to verify that for $\phi(x) = 0$, we obtain $v(0, t) \equiv 0$. This completes the proof of the lemma.

As a consequence, under conditions (11.2.1) and (11.2.2), from (11.2.3) it follows that $u(x, y) \equiv 0$, for all $(x, y) \in R^n_T$. Thus we conclude that the Cauchy problem formulated at the beginning has a unique solution. *Theorem 11.2.1 is proved.* □

## Declaration of competing interest

The authors declare that there is no conflict of interest regarding this work.

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CHAPTER

# 12 Inverse problem for the loaded heat conductivity equation with variable coefficients

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## 12.1 Introduction and formulation of the problem

Thermal conduction refers to the transfer of internal energy, in the form of heat, between neighboring molecules within a solid, liquid, or gas, as well as between different materials in close contact, without requiring any bulk motion of the material itself [1–3]. On this basis, heat conduction has been studied for more than two centuries and remains a topic of continuing scientific interest, playing a fundamental role in both natural and engineered systems.

From both physical and mathematical viewpoints, the heat conduction equation occupies a universal position. It arises naturally in diverse contexts, including the modeling of mass diffusion processes and the description of vorticity diffusion in viscous fluids. Heat conduction problems are also of considerable interest in the theory of partial differential equations. The one-dimensional heat equation is the most thoroughly studied case and has found extensive analytical and practical applications. In contrast, heat conduction problems in three-dimensional and multi dimensional do-

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mains remain an active area of research, particularly when nonstandard operators are involved.

In recent years, substantial attention has been devoted to fractional differential equations, in which derivatives of non-integer order are employed. This growing interest is closely linked to advances in fractional calculus, including the systematic development of fractional integrals and derivatives and their successful application across numerous scientific disciplines [4–8].

Fractional diffusion models extend classical diffusion equations by incorporating derivatives of fractional order, thereby providing a more accurate description of anomalous diffusion processes. Such models have proven effective in representing complex transport phenomena encountered in physics, medicine, and biology [9–13].

The analysis of two-dimensional and multidimensional space-fractional diffusion equations with variable coefficients presents significant challenges from both theoretical and computational perspectives [14]. In many cases, classical methods are insufficient to establish well-posedness or to construct efficient numerical schemes. Consequently, the study of fractional diffusion equations with variable coefficients often relies on a combination of analytical techniques and advanced integration or approximation methods.

The study presented in [15] explores numerical approximation techniques for solving fractional diffusion equations with variable coefficients. The application of the method of prior estimates to boundary value problems for fractional diffusion equations, following an approach analogous to that in the classical case, is examined in [17]. In [16], a finite difference scheme is proposed for approximating solutions to spatial fractional convection-diffusion models governed by equations with variable coefficients. The homotopy analysis method and the Adomian decomposition method are utilized in [18] to address high-order time-fractional partial differential equations. Furthermore, initial and boundary value problems for fractional diffusion equations with variable coefficients have been extensively studied in [19–23].

The increased interest in the study of loaded differential equations [24] is attributed to their numerous applications and the fact that they form a distinct class of partial differential equations [25,26]. Notably, the pioneering works of Nakhushev provided one of the first generalized definitions of loaded equations, along with their classification and applications to various problems in mathematical physics and biology [27–34]. Boundary value problems for heat conduction equations with loaded terms have been extensively studied in both bounded and unbounded domains [28–31]. These investigations are particularly focused on cases where the order of the derivative in the loaded term is greater than or equal to the order of the differential operator in the equation.

In this article, we focus on both aspects, specifically examining an analogue of the fractional diffusion equation with variable coefficients under a fractional load.

Inverse problems for parabolic equations with variable coefficients have been studied extensively, and fundamental results on existence and uniqueness for cores ponding initial-boundary value problems are well established. Motivated by these developments, we consider an inverse problem for a loaded integro-differential heat conduction equation with variable coefficients. In particular, following recent progress [35–39], the present work addresses the problem of identifying an unknown coefficient in the loaded integro-differential heat conduction equation.

We introduce the following notations:

Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Let $\mathbb{R}^2_T$ represent a subset of three-dimensional Euclidean space, specified by a point $(x, y, z)$, where $(x, y) \in \mathbb{R}^2$ and $z \in (0, T]$, with $T > 0$:

$$R^2_T = \left\{ (x, y, z) \middle| (x, y) \in R^2, 0 \leq z \leq T \right\}.$$

Let $f(x, y)$ be a function defined on $\mathbb{R}^2$.

**Definition 12.1.1.** If for any $x^{(1)}, x^{(2)} \in \mathbb{R}^2$, the inequality

$$|f(x^{(1)}) - f(x^{(2)})| \leq k|x^{(1)} - x^{(2)}|^l, \quad k > 0,\ l \in (0, 1), \tag{12.1.1}$$

holds, then the function $f(x)$ is said to satisfy the Holder condition with exponent $l$ in $\mathbb{R}^2$. The class of functions satisfying condition (12.1.1) is denoted by $H^l(\mathbb{R}^2)$.

**Definition 12.1.2.** If for any given pair of values

$$\left(x^{(1)}, y^{(1)}, z^{(1)}\right), \left(x^{(2)}, y^{(2)}, z^{(2)}\right) \in \mathbb{R}^2_T,$$

the inequality

$$\begin{aligned} &\left| f\left(x^{(1)}, y^{(1)}, z^{(1)}\right) - f\left(x^{(2)}, y^{(2)}, z^{(2)}\right) \right| \\ &\quad \leq k_1 \left| x^{(1)} - x^{(2)} \right|^l + k_2 \left| y^{(1)} - y^{(2)} \right|^l + k_3 \left| z^{(1)} - z^{(2)} \right|^{l/2}, \end{aligned} \tag{12.1.2}$$

where

$$k_i > 0 \,(\text{constants}), \ i = 1, 2, 3, \quad l \in (0, 1),$$

holds, then the function $f(x, y, t)$ is said to satisfy the Holder condition with exponents $l$ and $l/2$ in $\mathbb{R}^2_T$. The class of functions satisfying condition (12.1.2) is denoted by $H^{l,l/2}(\mathbb{R}^2_T)$.

**Inverse problem.** Find a pair of functions $u(x, y, z)$ and $k(x, z)$ in $(x, y, z) \in \mathbb{R}^2_T$, satisfying the following properties:

$$\begin{aligned} u_z - \gamma(z)\left(u_{xx} + u_{yy}\right) = \lambda[D^{-\alpha}_{0z} u(0, y, z) + u(0, y, z)] \\ + \int_0^z k(x, \tau)\, u(x, y, z - \tau)\, d\tau, \end{aligned} \tag{12.1.3}$$

$$u(x, y, z)|_{z=0} = \varphi(x, y), \quad (x, y) \in R^2, \tag{12.1.4}$$

$$u(x, y, z)|_{y=0} = \chi(x, z), \quad (x, z) \in R^1_T, \tag{12.1.5}$$

where $\gamma(z)$, $\varphi(x,y)$, $\chi(x,z)$ are given functions and

$$\gamma(z) \in I := \left\{\gamma(z)\,|\gamma(z) > 0,\ \gamma(z) \in C^1[0,T] \cap C(0,T)\right\}, \tag{12.1.6}$$

$$\varphi(x,y) \in H^{l+2}\left(R^2\right),\ \varphi(x,y) \le \varphi_0 = const > 0,\ \ \varphi(x,0) = \chi(x,0), \tag{12.1.7}$$

$$\chi(x,z) \in H^{l+4,(l+4)/2}\left(\bar{R}_T^1\right),\ \ l \in (0,1),\ \lambda \in R. \tag{12.1.8}$$

$D_{0z}^{-\alpha}$ is the Riemann–Liouville fractional integral operator [4] of order $\alpha$ and $\alpha > 0$. The inverse problem consists of determining the unknown functions $u(x,y,z)$ and $k(x,0,z)$ from the equalities (12.1.3)–(12.1.5).

## 12.2 Investigation of the problem

First, we will construct auxiliary problems equivalent to the inverse problem (12.1.3), (12.1.4), (12.1.5).

Let us introduce the following replacement in the problem (12.1.3)–(12.1.5):

$$\vartheta(x,y,z) = u_{yy}(x,y,z),\ \ (x,y,z) \in R_T^2. \tag{12.2.1}$$

Using the change of variable (12.2.1), the inverse problem (12.1.3), (12.1.4), (12.1.5), is equivalently reduced to the following problem.

**Auxiliary problem:** Find functions $\vartheta(x,y,z)$ and $k(x,z)$ in $(x,y,z) \in R_T^2$, possessing the following properties:

$$\vartheta_z - \gamma(z)\,\Delta\vartheta = \lambda[D_{0z}^{-\alpha}\vartheta(0,y,z) + \vartheta(0,y,z)] + \int_0^z k(x,\tau)\,\vartheta(x,y,z-\tau)\,d\tau, \tag{12.2.2}$$

$$\vartheta(x,y,z)|_{t=0} = \varphi_{yy}(x,y),\ \ (x,y) \in R^2, \tag{12.2.3}$$

$$\vartheta(x,y,z)|_{y=0} = \frac{1}{\gamma(z)}\chi_z(x,z) - \chi_{xx}(x,z) -$$
$$-\frac{\lambda}{\gamma(z)}[D_{0z}^{-\alpha}\chi(0,0,z) + \chi(0,0,z)] - \frac{1}{\gamma(z)}\int_0^z k(x,\tau)\,\chi(x,0,z-\tau)\,d\tau, \tag{12.2.4}$$

moreover, we assume that $\chi(0,0,0) = 0$, from the initial condition (12.2.3) and (12.2.4) the following condition of agreement is satisfied:

$$\varphi_{yy}(x,0) = \frac{1}{\gamma(0)}\chi_z(x,0) - \chi_{xx}(x,0). \tag{12.2.5}$$

Indeed, if the compatibility conditions (12.1.5) and (12.2.5) are satisfied, and the functions $\varphi$ and $\chi$ are sufficiently smooth, it can be shown that the problems (12.2.2)–(12.2.4) are equivalent to the inverse problem (12.1.3)–(12.1.5):

First, integrating twice from the (12.2.1) substitution above from 0 to $y$, we get:

$$u(x,y,z)=\chi(x,z)+y\varphi(x,0)+\int_0^y(y-\eta)\vartheta(x,\eta,z)d\eta \tag{12.2.6}$$

and, consequently, in (12.2.1) for the function $u(x,y,t)$, taking into account the agreement condition (12.2.5) for $z=0$, we have:

$$\begin{aligned}
u(x,y,0)|_{z=0}&=\chi(x,0)+yu_y(x,0,0)+\int_0^y(y-\xi)\varphi_{\xi\xi}(x,\xi)d\xi=\\
&=\chi(x,0)+yu_y(x,0,0)+\int_0^y(y-\xi)d\varphi_\xi=\chi(x,0)+yu_y(x,0,0)+\\
&+(y-\xi)\varphi_\xi(x,\xi)|_0^y+\int_0^y\varphi_\xi(x,\xi)d\xi=\chi(x,0)+yu_y(x,0,0)-\\
&-y\varphi_y(x,0)+\varphi(x,y)-\varphi(x,0)=y\left(u_y(x,0,0)-\varphi_y(x,0)\right)+\varphi(x,y)=\varphi(x,y).
\end{aligned}$$

As can be seen from (12.2.6), on $y=0$ the additional condition in (12.1.5) follows.

The sequence of obtaining Eq. (12.1.3) from Eq. (12.2.2) is as follows: first, we integrate both sides of Eq. (12.2.2) twice from 0 to $y$:

$$\begin{aligned}
&\int_0^y(y-\xi)\vartheta_z(x,\xi,z)d\xi-\gamma(z)\int_0^y(y-\xi)\left(\vartheta_{xx}(x,\xi,z)+\vartheta_{\xi\xi}(x,\xi,z)\right)d\xi=\\
&=\int_0^y(y-\xi)[\lambda\vartheta(0,\xi,z)+\int_0^z k(x,\tau)\vartheta(x,\xi,z-\tau)d\tau]d\xi+\\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^y(y-\xi)d\xi\int_0^z(z-\tau)^{\alpha-1}\vartheta(0,\xi,\tau)d\tau
\end{aligned}$$

and taking into account equality (12.2.6), i.e.,

$$\int_0^y(y-\xi)\vartheta(x,\xi,z)d\xi=u(x,y,z)-\chi(x,z)-y\varphi(x,0).$$

Accordingly, we establish the following relations:

$$\begin{aligned}
&u_z(x,y,z)-\chi_z(x,z)-\gamma(z)u_{xx}(x,y,z)+\gamma(z)\chi(x,z)+\gamma(z)y\vartheta_y(x,0,z)-\\
&-\gamma(z)\vartheta(x,y,z)+\gamma(z)\vartheta(x,0,z)=\\
&=\lambda\left(u(0,y,z)-\chi(0,z)-y\varphi_y(0,0)\right)+\\
&+\int_0^z k(x,\tau)\left(u(x,y,z-\tau)-\chi(x,z-\tau)-y\varphi_y(x,0)\right)d\tau+\\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^z(z-\tau)^{\alpha-1}\left(u(0,y,\tau)-\chi(0,z-\tau)-y\varphi_y(0,0)\right)d\tau,
\end{aligned}$$

Hence, taking into account the condition (12.2.4) for $\vartheta(x, y, z)$, it is easy to see that Eq. (12.1.3) has been obtained. Thus the (12.1.3), (12.1.4), (12.1.5) inverse problem of finding functions $u(x, y, z)$ and $k(x, z)$ is equivalent to the inverse problem for finding functions $\vartheta(x, y, z)$ and $k(x, 0, z)$ from (12.2.2)–(12.2.4).

**Auxiliary problem**

In the second step, if we differentiate the resulting equations to t and make the replacement $\vartheta_z(x, y, z) = \rho(x, y, z)$ in (12.2.2)–(12.2.4), we obtain the following auxiliary problem for finding the functions $\vartheta(x, y, z)$, $k(x, z)$, $\rho(x, y, z)$:

$$\rho_z - \gamma(z)\left(\rho_{xx} + \rho_{yy}\right) = (\ln\gamma(z))'(\rho - \int_0^z k(x,\tau)\,\vartheta(x, y, z-\tau)\,d\tau) +$$
$$+\int_0^z k(x,\tau)\,\rho(x, y, z-\tau)\,d\tau - \lambda(\ln\gamma(z))'(\vartheta(0, y, z) + D_{0z}^{-\alpha}\vartheta(0, y, z) +$$
$$+\lambda(\rho(0, y, z) + (D_{0z}^{-\alpha}\rho(0, y, z)) + k(x, z)\,\varphi_{yy}(x, y) + \frac{\lambda}{\Gamma(\alpha)} z^{\alpha-1}\varphi_{yy}(0, y, 0), \tag{12.2.7}$$

$$\rho|_{z=0} = \gamma(0)\,\Delta\varphi_{yy}(x, y), \tag{12.2.8}$$

$$\rho|_{y=0} = F_z(x, z) + \frac{\gamma'(z)}{\gamma^2(z)}\int_0^z k(x,\tau)\,\chi(x, z-\tau)\,d\tau -$$
$$-\frac{1}{\gamma(z)}\int_0^z k(x,\tau)\,\chi_z(x, z-\tau)\,d\tau - \frac{1}{\gamma(z)} k(x, z)\,\varphi\left(x', 0\right), \tag{12.2.9}$$

where

$$F(x, z) = \frac{1}{\gamma(z)}\chi_z(x, z) - \chi_{xx}(x, z) - \frac{\lambda}{\gamma(z)\,\Gamma(\alpha)}\int_0^z (z-\tau)^{\alpha-1}\,\chi(0, 0, \tau)\,d\tau.$$

As a result, we obtained an auxiliary problem for finding functions $\vartheta(x, y, z)$, $k(x, 0, z)$, $\rho(x, y, z)$.

In the next step, integrating both parts of the last change of variable from 0 to $t$, we obtain the following equality:

$$\vartheta(x, y, z) = \varphi_{yy}(x, y) + \int_0^z \rho(x, y, \tau)\,d\tau. \tag{12.2.10}$$

If the function $\rho(x, y, z)$ is known, then the function $\vartheta(x, y, z)$ is found from (12.2.10). Thus the problem (12.2.7)–(12.2.9) leads to problems (12.2.2)–(12.2.4) and problems (12.2.2)–(12.2.4) lead to the inverse problems (12.1.3), (12.1.4), (12.1.5). Hence, finding the functions $\vartheta(x, y, z)$, $k(x, 0, z)$, $\rho(x, y, z)$ from problems (12.2.2)–(12.2.4) and (12.2.7)–(12.2.9) is equivalent to finding the functions $u(x, y, z)$, $k(x, 0, z)$, from the inverse problem (12.1.3), (12.1.4), (12.1.5).

Thus we have proved the following lemma:

**Lemma 12.2.1.** *Suppose that* $\gamma(z) \in I$, $\varphi(x,y) \in H^{l+6}\left(R^2\right)$, $\chi(x,z) \in H^{l+4,(l+4)/2}\left(\bar{R}^1_T\right)$, *and the matching conditions*

$$\chi(x,0) = \varphi(x,0)\,,\ \varphi_{yy}(x,0) = \frac{1}{\gamma(0)}\chi_z(x,0) - \chi_{xx}(x,0)$$

*are met. Then, the problem* (12.1.3), (12.1.4), (12.1.5) *is equivalent to the problem of determining the functions* $\vartheta(x,y,z)$, $k(x,0,z)$, $\rho(x,y,z)$ *from Eqs.* (12.2.2)–(12.2.4) *and* (12.2.7)–(12.2.9).

## 12.3 Uniqueness of solvability

**Lemma 12.3.1.** *The auxiliary problem* (12.2.2), (12.2.3), *and* (12.2.7)–(12.2.9) *is equivalent to finding the functions* $\vartheta(x,y,z)$, $k(x,y,z)$, $\rho(x,y,z)$ *from the following system of integral equations:*

$$\vartheta(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_{\eta\eta}(\xi,\eta)\,Gd\xi d\eta + \int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\times$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\sigma^{-1}(\tau)}k(\xi,0,\alpha)\vartheta\left(\xi,\eta,\sigma^{-1}(\tau)-\alpha\right)Gd\alpha d\xi d\eta + \qquad (12.3.1)$$
$$\lambda\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\vartheta(0,\eta,\beta)\,Gd\beta d\eta$$
$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\sigma^{-1}(\tau)}\left(\sigma^{-1}(\tau)-\beta\right)^{\alpha-1}\vartheta(0,\eta,\beta)\,Gd\beta d\xi d\eta,$$

$$\rho(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\gamma(0)\,\Delta\varphi_{\eta\eta}(\xi,\eta)\,Gd\xi d\eta + \int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\times$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\left(\ln\gamma\left(\sigma^{-1}(\tau)\right)\right)'\rho\left(\xi,\eta,\sigma^{-1}(\tau)\right)-\right.$$
$$\left.-\left(\ln\gamma\left(\sigma^{-1}(\tau)\right)\right)'\int_0^{\sigma^{-1}(\tau)}k(\xi,0,\alpha)\vartheta\left(\xi,\eta,\sigma^{-1}(\tau)-\alpha\right)d\alpha\right)Gd\xi d\eta +$$
$$+\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\sigma^{-1}(\tau)}k(\xi,0,\alpha)\rho\left(\xi,\eta,\sigma^{-1}(\tau)-\alpha\right)Gd\alpha d\xi d\eta +$$
$$+\int_0^{\sigma(t)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}k\left(\xi,0,\sigma^{-1}(\tau)\right)\varphi_{\eta\eta}(\xi,\eta)\,Gd\xi d\eta + \qquad (12.3.2)$$

$$
\begin{aligned}
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\\
&\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\sigma^{-1}(\tau)}\left(\sigma^{-1}(\tau)-\beta\right)^{\alpha-1}\rho(0,\eta,\beta)\,Gd\beta d\xi d\eta-\\
&-\frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\\
&\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left(\ln\gamma\left(\sigma^{-1}(\tau)\right)\right)'\int_0^{\sigma^{-1}(\tau)}\left(\sigma^{-1}(\tau)-\beta\right)^{\alpha-1}\times\right.\\
&\left.\times\vartheta(0,\eta,\beta)\,d\beta-\left(\sigma^{-1}(\tau)\right)^{\alpha-1}\varphi_{\eta\eta}(\xi,\eta)\right]Gd\xi d\eta,
\end{aligned}
$$

$$
\begin{aligned}
k(x,0,z)=&\frac{\gamma(t)}{\varphi(x,0)}\left(F_z(x,z)-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\gamma(0)\,\Delta\varphi_{\eta\eta}(\xi,\eta)\,Gd\xi d\eta-\right.\\
&-\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\ln\gamma\left(\sigma^{-1}(\tau)\right)\right)'\rho\left(\xi,\eta,\sigma^{-1}(\tau)\right)Gd\xi d\eta+\\
&+\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\times\\
&\times\int_{-\infty}^{\infty}\left(\left(\ln\gamma\left(\sigma^{-1}(\tau)\right)\right)'\int_0^{\sigma^{-1}(\tau)}k(\xi,0,\alpha)\vartheta\left(\xi,\eta,\sigma^{-1}(\tau)-\alpha\right)d\alpha\right)Gd\xi d\eta-\\
&-\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\sigma^{-1}(\tau)}k(\xi,0,\alpha)\rho\left(\xi,\eta,\sigma^{-1}(\tau)-\alpha\right)Gd\alpha d\xi d\eta-\\
&-\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}k\left(\xi,0,\sigma^{-1}(\tau)\right)\varphi_{\eta\eta}(\xi,\eta)\,Gd\xi d\eta- \qquad (12.3.3)\\
&-\frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\times\\
&\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\sigma^{-1}(\tau)}\left(\sigma^{-1}(\tau)-\beta\right)^{\alpha-1}\rho(0,\eta,\beta)\,Gd\beta d\xi d\eta+\\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\times\\
&\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\left(\ln\gamma\left(\sigma^{-1}(\tau)\right)\right)'\int_0^{\theta^{-1}(\tau)}\left(\sigma^{-1}(\tau)-\beta\right)^{\alpha-1}\vartheta(0,\eta,\beta)\,d\beta\right)Gd\xi d\eta-\\
&\left.-\frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sigma^{-1}(\tau)\right)^{\alpha-1}\varphi_{\eta\eta}(\xi,\eta)\,Gd\xi d\eta\right)+
\end{aligned}
$$

$$+\frac{1}{\varphi(x,0)}\int_0^z \big((\ln\gamma(z))'\chi(x,z-\tau)-\chi_z(x,z-\tau)\big)k(x,0,\tau)\,d\tau,$$

*where* $G = G(x-\xi, y-\eta, \sigma(z))$ *and, respectively, from* $\sigma(z)-\tau$.

*Proof.* In problems (12.2.2) and (12.2.3) taking into account formula (12.2.8) as in the correct problem from [23]:

$$F(x,y,z) = \lambda D_{0z}^{-\alpha}\vartheta(0,y,z) + \int_0^z k(x,0,\tau)\vartheta(x,y,z-\tau)\,d\tau$$

in the above form, we have the integral Eq. (12.3.1) correspondingly equivalent. In the same way, taking into account (12.2.7) and (12.2.8), we obtain formula (12.3.2). Then, taking into account the resulting integral equations and using (12.2.9), we obtain the integral Eq. (12.3.3).

From problems (12.2.3), (12.2.4) and (12.2.7), (12.2.8), integral Eqs. (12.3.1) and (12.3.2) are obtained analogously to Eq. (12.1.5). Eq. (12.3.3) follows from Eqs. (12.2.9) and (12.3.2). □

Now, we will prove the existence and uniqueness solution of problem (12.2.1)–(12.2.3). The proof is given using the contraction mapping principle.

**Theorem 12.3.2.** *If conditions (12.1.6), (12.1.7), (12.1.8), and (12.2.5) are satisfied, then there exists* $T_0 > 0$ *a sufficiently small number such that for* $T \in (0, T_0]$, *there exists a unique solution to the integral Eqs.* (12.3.1)–(12.3.3) *belonging to the classes:*

$$\{\vartheta(x,y,z),\ \rho(x,y,z)\} \in H^{l+2,(l+2)/2}\left(\bar{R}_T^2\right),\ k(x,0,z) \in H^{l,l/2}\left(\bar{R}_T^1\right).$$

*Proof.* To prove the theorem using the contraction mapping principle, we write the system of Eqs. (12.3.1)–(12.3.3) as a nonlinear operator:

$$\theta = L\theta,\ \ \theta = (\theta_1,\theta_2,\theta_3)^* = (\vartheta,\rho,k(x,0,z))^*, \tag{12.3.4}$$

where $*$ is the transposition symbol, $L\theta = \left[(L\theta)_1, (L\theta)_2, (L\theta)_3\right]^*$. Thus according to the right sides of Eqs. (12.3.1)–(12.3.3), $(L\theta)_i$ $(i=1,2,3)$, we have:

$$(L\theta)_1 = \theta_{01}(x,y,z) + \int_0^{\sigma(z)} \frac{d\tau}{\gamma(\widehat{\tau})}\times$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}} \theta_3(\xi,0,\alpha)\theta_1(\xi,\eta,\widehat{\tau}-\alpha)\,G\,d\alpha\, d\xi\, d\eta +$$

$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}}(\widehat{\tau}-\beta)^{\alpha-1}\theta_1(0,\eta,\beta)\,G\,d\beta\, d\xi\, d\eta,$$

where

$$(L\theta)_2 = \theta_{02}(x,y,z) + \int_0^{\sigma(z)}\frac{d\tau}{\gamma(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[(\ln\gamma(\widehat{\tau}))'\theta_2(\xi,\eta,\widehat{\tau}) - \right.$$

$$- (\ln \gamma (\widehat{\tau}))' \int_0^{\widehat{\tau}} \theta_3(\xi, 0, \alpha)\theta_1 (\xi, \eta, \widehat{\tau} - \alpha)\, d\alpha \Bigg] G d\xi d\eta +$$

$$+ \int_0^{\sigma(z)} \frac{d\tau}{\gamma (\widehat{\tau})} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\widehat{\tau}} \theta_3(\xi, 0, \alpha)\theta_2 (\xi, \eta, \widehat{\tau} - \alpha)\, G d\alpha d\xi d\eta +$$

$$+ \int_0^{\sigma(z)} \frac{d\tau}{\gamma (\widehat{\tau})} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_3 (\xi, 0, \widehat{\tau})\, \varphi_{\eta\eta} (\xi, \eta)\, G d\xi d\eta +$$

$$+ \frac{\lambda}{\Gamma (\alpha)} \int_0^{\sigma(z)} \frac{d\tau}{\gamma (\widehat{\tau})} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\widehat{\tau}} (\widehat{\tau} - \beta)^{\alpha - 1}\, \theta_2 (0, \eta, \beta)\, G d\beta d\xi d\eta -$$

$$- \frac{\lambda}{\Gamma (\alpha)} \int_0^{\sigma(z)} \frac{d\tau}{\gamma (\widehat{\tau})} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ (\ln \gamma (\widehat{\tau}))' \int_0^{\widehat{\tau}} (\widehat{\tau} - \beta)^{\alpha - 1}\, \theta_1 (0, \eta, \beta)\, d\beta \right] G d\xi d\eta.$$

$$(L\theta)_3 = \theta_{03} (x, y, z) -$$

$$- \frac{\gamma (z)}{\varphi (x, 0)} \int_0^{\sigma(z)} \frac{d\tau}{\gamma (\widehat{\tau})} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\ln \gamma (\widehat{\tau}))' \, [\theta_2 (\xi, \eta, \widehat{\tau}) -$$

$$- \int_0^{\widehat{\tau}} \theta_3(\xi, 0, \alpha)\theta_1 (\xi, \eta, \widehat{\tau} - \alpha)\, d\alpha \Bigg] G d\xi d\eta -$$

$$- \frac{\gamma (z)}{\varphi (x, 0)} \int_0^{\sigma(z)} \frac{d\tau}{\gamma (\widehat{\tau})} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\widehat{\tau}} \theta_3(\xi, 0, \alpha)\theta_2 (\xi, \eta, \widehat{\tau} - \alpha)\, G d\alpha d\xi d\eta -$$

$$- \frac{\gamma (z)}{\varphi (x, 0)} \int_0^{\sigma(z)} \frac{d\tau}{\gamma (\widehat{\tau})} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_3 (\xi, 0, \widehat{\tau})\, \varphi_{\eta\eta} (\xi, \eta)\, G d\xi d\eta -$$

$$- \frac{\lambda \gamma (z)}{\Gamma (\alpha) \varphi (x, 0)} \int_0^{\sigma(z)} \frac{d\tau}{\gamma (\widehat{\tau})} \Bigg( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\widehat{\tau}} (\widehat{\tau} - \beta)^{\alpha - 1}\, \theta_2 (0, \eta, \beta)\, G d\beta d\xi d\eta -$$

$$- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\ln \gamma (\widehat{\tau}))' \int_0^{\widehat{\tau}} (\widehat{\tau} - \beta)^{\alpha - 1}\, \theta_1 (0, \eta, \beta)\, d\beta G d\xi d\eta \Bigg) +$$

$$\frac{\lambda \gamma (z)}{\varphi (x, 0)} \int_0^{\sigma(z)} \frac{d\tau}{\gamma (\widehat{\tau})} \Bigg( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Big( \theta_2 (0, \eta, \beta) + (\ln \gamma (\widehat{\tau}))' \, \theta_1 (0, \eta, \beta) \Big) G d\beta d\xi d\eta$$

$$+ \frac{1}{\varphi (x, 0)} \int_0^{z} \Big( (\ln \gamma (z))' \, \chi (x, z - \tau) - \chi_z (x, z - \tau) \Big) \theta_3 (x, 0, \tau)\, d\tau,$$

where $\sigma^{-1}(\tau) = \widehat{\tau}$, $\theta_{01} (x, y, z)$, $\theta_{02} (x, y, z)$ and $\theta_{03} (x, y, z)$ depends on the given functions, i.e.,

$$\theta_{01} (x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_{\eta\eta} (\xi, \eta)\, G d\xi d\eta,$$

$$\theta_{02} (x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\gamma (0)\, \Delta \varphi_{\eta\eta} (\xi, \eta) + \lambda \varphi (\xi, \eta)) G d\xi d\eta +$$

$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\widehat{\tau})^{\alpha-1}\varphi_{\eta\eta}(0,\eta)\,Gd\xi d\eta,$$

$$\theta_{03}(x,y,z)=\frac{\gamma(z)}{\varphi(x,0)}\left(F_z(x,z)-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\gamma(0)\,\Delta\varphi_{\eta\eta}(\xi,\eta)+\varphi(\xi,\eta))Gd\xi d\eta-\right.$$

$$\left.-\frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(z)}\frac{d\tau}{\gamma(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\widehat{\tau})^{\alpha-1}\varphi_{\eta\eta}(0,\eta)\,Gd\xi d\eta\right).$$

Introducing the notation $|\theta|_T^{l,l/2}=\max\left(|\theta_1|_{T_0}^{l,l/2},|\theta_2|_{T_0}^{l,l/2},|\theta_3|_{T_0}^{l,l/2}\right)$ in $H^{l,l/2}\left(R_T^2\right)$, we give the following condition:

$$S(T)=|\theta_1-\theta_0|_T^{l,l/2}\leq|\theta_0|_{T_0}^{l,l/2}, \tag{12.3.5}$$

where $\theta_0=(\theta_{01},\theta_{02},\theta_{03})$ and $|\theta_0|_{T_0}^{l,l/2}=\max\left(|\theta_{01}|_{T_0}^{l,l/2},|\theta_{02}|_{T_0}^{l,l/2},|\theta_{03}|_{T_0}^{l,l/2}\right)$. Thus for any function $\sigma$ from $S(T)$, $T<T_0$, when (12.3.5) is executed, the following inequality is true:

$$|\theta_i|_T^{l,l/2}\leq 2\,|\theta_0|_{T_0}^{l,l/2},\; i=1,2,3.$$

As is known, for $\varphi(x,y)\in H^{l+2}\left(R^2\right)$, the Cauchy problem for the classical heat conduction equation has a solution. In problem (12.1.3)–(12.1.5), taking into account the auxiliary problem, the initial conditions must belong to $\varphi(x,y)\in H^{l+6}$ (since the auxiliary problem involves fourth-order derivatives).

If the Cauchy condition $\varphi(x,y)\in H^{l+2}\left(\mathbb{R}^2\right)$ holds for the classical heat conduction equation, then the Cauchy problem for the classical heat diffusion equation has a solution [39]. Since we are considering the class $\varphi(x,y)\in H^{l+6}$, the fourth-order derivatives are also involved in the obtained auxiliary problem. Based on this, we introduce the following notation:

$$\gamma_0:=\max_{z\in[0,T]}\left|(\ln\gamma(t))'\right|,\;\varphi_1:=|\varphi|^{l+6},\;\chi_0:=|\chi|_T^{l+4,(l+4)/2}.$$

**Contraction Mapping Principle.** Any contraction mapping defined in a complete metric space has a unique fixed point; that is, the equation $x=Ax$ has a unique solution $x_0\in S$.

First, using estimates of thermal volume potentials ([38], pages 318–325), it is easy to obtain the following inequalities:

$$|(L\theta)_1-\theta_{01}|_T^{l,l/2}=$$

$$=\left|\int_0^z\frac{d\tau}{\gamma(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}}\theta_3(\xi,0,\alpha)\theta_1(\xi,\eta,\widehat{\tau}-\alpha)\,Gd\alpha d\xi d\eta\right|_T^{l,l/2}+$$

$$\left|\lambda\int_0^z\frac{d\tau}{\gamma(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta_1(0,\eta,\beta)\,Gd\xi d\eta\right|_T^{l,l/2}$$

$$+\left|\frac{\lambda}{\Gamma(\alpha)}\int_0^z \frac{d\tau}{\gamma(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}}(\widehat{\tau}-\beta)^{\alpha-1}\theta_1(0,\eta,\beta)\,Gd\beta d\xi d\eta\right|_T^{l,l/2}\leq$$

$$\leq\bar{\beta}_0(T)\,|(\theta_3(\xi,\eta,z_0)\,\theta_1(\xi,\eta,z-z_0))|_T^{l,l/2}+\bar{\beta}_1(T)\,|(\theta_1(0,\eta,z_0))|_T^{l,l/2}\leq$$

$$\leq 4\beta_0(T)\left(|\theta_0|_{T_0}^{l,l/2}\right)^2+2\beta_1(T)\left(|\theta_0|_{T_0}^{l,l/2}\right),$$

$$|(L\theta)_2-\theta_{02}|_T^{l,l/2}\leq$$

$$\leq 4\beta_1(T)(\gamma_0+1)\left(|\theta_0|_{T_0}^{l,l/2}\right)^2+2\beta_2(T)(2\gamma_0+\varphi_1+1)\left(|\theta_0|_{T_0}^{l,l/2}\right),$$

$$\left|(L\theta)_3-\theta_{03}\right|_T^{l,l/2}\leq$$

$$\leq 2\left(\beta_1(T)\,\gamma_1\varphi_0^{-1}(2\gamma_0+\varphi_1+1)+\chi_0\varphi_0^{-1}T_0(\gamma_0+1)\right)|\theta_0|_{T_0}^{l,l/2}+$$

$$+4\beta_2(T)\,\gamma_1\varphi_0^{-1}(\gamma_0+1)\left(|\theta_0|_{T_0}^{l,l/2}\right)^2.$$

As $T\to 0$, $\beta_i(T)\to 0$, $(i=0,1,2)$. Therefore if we choose $T_0$ so that the following inequalities should be satisfied:

$$4\beta_0(T_0)\left(|\theta_0|_{T_0}^{l,l/2}\right)^2+2\beta_1(T_0)\left(|\theta_0|_{T_0}^{l,l/2}\right)\leq 1,$$

$$4\beta_1(T_0)(\gamma_0+1)\left(|\theta_0|_{T_0}^{l,l/2}\right)^2+2\beta_2(T_0)(2\gamma_0+\varphi_1+1)\left(|\theta_0|_{T_0}^{l,l/2}\right)\leq 1, \quad (12.3.6)$$

$$2\left(\beta_1(T_0)\,\gamma_1\varphi_0^{-1}(2\gamma_0+\varphi_1+1)+\chi_0\varphi_0^{-1}T_0(\gamma_0+1)\right)|\theta_0|_{T_0}^{l,l/2}+$$

$$+4\beta_2(T_0)\,\gamma_1\varphi_0^{-1}(\gamma_0+1)\left(|\theta_0|_{T_0}^{l,l/2}\right)^2\leq 1,$$

then the operator $L$ for $T<T_0$ has the first property of a contraction mapping operator, that is, $L\theta\in S(T)$.

Now, consider the second property of the contraction mapping for the operator $L$. Let $\theta^{(1)}=\left(\theta_1^{(1)},\theta_2^{(1)},\theta_3^{(1)}\right)\in S(T)$, $\theta^{(2)}=\left(\theta_1^{(2)},\theta_2^{(2)},\theta_3^{(2)}\right)\in S(T)$, then, following evaluation

$$\left|\theta_2^{(1)}\theta_1^{(1)}-\theta_2^{(2)}\theta_1^{(2)}\right|_T^{l,l/2}=\left|\left(\theta_2^{(1)}-\theta_2^{(2)}\right)\theta_1^{(1)}+\theta_2^{(2)}\left(\theta_1^{(1)}-\theta_1^{(2)}\right)\right|_T^{l,l/2}\leq$$

$$\leq 2\left|\theta^{(1)}-\theta^{(2)}\right|_T^{l,l/2}\max\left(\left|\theta_1^{(1)}\right|_T^{l,l/2},\left|\theta_2^{(2)}\right|_T^{l,l/2}\right)\leq 4\,|\theta_0|_T^{l,l/2}\left|\theta^{(1)}-\theta^{(2)}\right|_T^{l,l/2},$$

we have

$$\left|\left((L\theta)^{(1)}-(L\theta)^{(2)}\right)_1\right|_T^{l,l/2}=\left|\int_0^{\sigma(z)}\frac{d\tau}{\gamma(\widehat{\tau})}\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\right.$$

$$\int_0^{\widehat{\tau}} \left[\theta_3^{(1)}(\xi,0,\alpha)\theta_1^{(1)}(\xi,\eta,\widehat{\tau}-\alpha) - \theta_3^{(2)}(\xi,0,\alpha)\theta_1^{(2)}(\xi,\eta,\widehat{\tau}-\alpha)\right] G d\alpha \Big|_T^{l,l/2} +$$

$$+ \left| \int_0^{\sigma(z)} \frac{d\tau}{\gamma\left(\sigma^{-1}(\tau)\right)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\widehat{\tau}} \frac{\lambda\left(\widehat{\tau}-\beta\right)^{\alpha-1}}{\Gamma(\alpha)} \left[\theta_1^{(1)}(0,\eta,\beta) - \theta_1^{(2)}(0,\eta,\beta)\right] \times \right.$$

$$\times\, G d\beta d\xi d\eta \Big|_T^{l,l/2} \le \left[8\beta_0(T)\left|\theta_0\right|_{T_0}^{l,l/2} + 4\beta_1(T)\right] \left|\theta^{(1)} - \theta^{(2)}\right|_T^{l,l/2}.$$

Similarly, estimating the second and third components of $L\theta$ we have:

$$\left|\left((L\theta)^{(1)} - (L\theta)^{(2)}\right)_2\right|_T^{l,l/2} \le$$

$$\le \left[2\beta_1(T)(2\gamma_0+\varphi_1+1) + 8\beta_2(T)(\gamma_0+1)\left|\theta_0\right|_{T_0}^{l,l/2}\right] \left|\theta^{(1)} - \theta^{(2)}\right|_{T_0}^{l,l/2},$$

$$\left|\left((L\theta)^{(1)} - (L\theta)^{(2)}\right)_3\right|_T^{l,l/2} \le$$

$$\le \left[2\left(\beta_1(T)\gamma_1\varphi_0^{-1}(2\gamma_0+\varphi_1+1) + \chi_0\varphi_0^{-1}T_0(\gamma_0+1)\right)\right] \left|\theta^{(1)} - \theta^{(2)}\right|_{T_0}^{l,l/2} +$$

$$+ \left[8\beta_2(T)\gamma_1\varphi_0^{-1}(\gamma_0+1)\left(\left|\theta_0\right|_{T_0}^{l,l/2}\right)\right] \left|\theta^{(1)} - \theta^{(2)}\right|_{T_0}^{l,l/2}.$$

Therefore $\left|\left(L\theta^{(1)} - L\theta^{(2)}\right)\right|_T^{l,l/2} < \rho \left|\theta^{(1)} - \theta^{(2)}\right|_T^{l,l/2}$, where $\rho \le 1$, if satisfied

$$\begin{gathered}
\left[8\beta_0(T)\left|\theta_0\right|_{T_0}^{l,l/2} + 2\beta_1(T)\right] \le \rho < 1, \\
\left[2\beta_1(T)(2\gamma_0+\varphi_1+1) + 8\beta_2(T)(\gamma_0+1)\left|\theta_0\right|_{T_0}^{l,l/2}\right] \le \rho < 1, \\
\left[2\left(\beta_1(T)\gamma_1\varphi_0^{-1}(2\gamma_0+\varphi_1+1) + f_0\varphi_0^{-1}T_0(\gamma_0+1)\right)\right] + \\
+ \left[8\beta_2(T)\gamma_1\varphi_0^{-1}(\gamma_0+1)\left(\left|\theta_0\right|_{T_0}^{l,l/2}\right)\right] \le \rho < 1,
\end{gathered} \tag{12.3.7}$$

then the operator $L$ is also a contraction on $S(T)$.

From the satisfaction of inequality (12.3.7), it directly follows that (12.3.6) also holds. Furthermore, since $T_0$ satisfies $T < T_0$ and condition (12.3.7), the properties of a contraction mapping operator are fully satisfied. Consequently, by the Banach fixed-point theorem, Eq. (12.3.4) has a unique solution. Using the method of successive approximations for the system of Eqs. (12.3.1)–(12.3.3), we determine a unique solution within the function space $H^{l+2,(l+2)/2}\left(\bar{R}_T^2\right)$.

Thus the existence and uniqueness of a solution to the system of integral Eqs. (12.3.1)–(12.3.3) imply the existence and uniqueness of the solution to the equivalent problems (12.1.3)–(12.1.5). □

## Declaration of competing interest

This work does not have any conflicts of interest.

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CHAPTER

# 13

# Generalized arbitrary order Mittag-Leffler-type function and Marichev–Saigo–Maeda fractional operators

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## 13.1 Introduction

Fractional calculus is a branch of mathematical analysis focused on the properties of functions and operators defined by arbitrary non-integer order derivatives and integrals. This field has gained increased attention across various fields of science and engineering, including physics, chemistry, biology, fluid dynamics, astrophysics, electrical engineering, image processing, and others. For more details about fractal calculus and its applications, interested readers can refer to [5,7,8,18]. Recently, researchers have extensively studied fractional calculus, developing new fractional integral and derivative operators that have gained significant attention due to their wide applications in diverse fields. In particular, various fractional operators such as Riemann–Liouville, Erdélyi–Kober, Caputo, Saigo, Hilfer and Marichev–Saigo–Maeda fractional operators are present, see, for example, [9,21,22].

We recall here the generalized hypergeometric fractional integral and fractional derivative operators involving Appell's function $F_3$, introduced by Marichev [11] and later extended and studied by Saigo and Maeda [16]. These operators are known as the Marichev–Saigo–Maeda operators. The generalized fractional calculus operators with the Appell function $F_3$ in their kernel are defined as follows.

Let $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta \in \mathbb{C}$ with $\Re(\eta) > 0$, $x \in \mathbb{R}^+$, then the left and right fractional integral operators are defined as follows (see [16]):

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta} f\right)(x) = \frac{x^{-\nu}}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} t^{-\acute{\nu}} F_3\left(\nu, \acute{\nu}, \mu, \acute{\mu}; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt \tag{13.1.1}$$

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and

$$\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta} f\right)(x)$$
$$= \frac{x^{-\acute{\nu}}}{\Gamma(\eta)} \int_x^{\infty} (t-x)^{\eta-1} t^{-\nu} F_3\left(\nu, \acute{\nu}, \mu, \acute{\mu}; \eta; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)dt, \quad (13.1.2)$$

where $F_3$ is defined as follows (see [20]):

$$F_3\left(\nu, \acute{\nu}, \mu, \acute{\mu}; \eta; x, t\right) = \sum_{m,n=0}^{\infty} \frac{(\nu)_m (\acute{\nu})_n (\mu)_m (\acute{\mu})_n}{(\eta)_{m+n}} \frac{x^m}{m!} \frac{t^n}{n!}, \quad (max\{|x|, |t|\} < 1). \quad (13.1.3)$$

These operators are reduced to the following Saigo fractional integral operators (see [15]):

$$\left(I_{0+}^{\nu+\mu,0,-\rho,0,\nu} f\right)(x)$$
$$= \left(I_{0+}^{\nu,\mu,\rho} f\right)(x)$$
$$= \frac{x^{-\nu-\mu}}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} {}_2F_1\left(\nu+\mu, -\rho; \nu; 1-\frac{t}{x}\right) f(t)dt, \quad \rho \in \mathbb{C} \quad (13.1.4)$$

and

$$\left(I_{-}^{\nu+\mu,0,-\rho,0,\nu} f\right)(x)$$
$$= \left(I_{-}^{\nu,\mu,\rho} f\right)(x)$$
$$= \frac{1}{\Gamma(\nu)} \int_x^{\infty} (t-x)^{\nu-1} t^{-\nu-\mu} {}_2F_1\left(\nu+\mu, -\rho; \nu; 1-\frac{x}{t}\right) f(t)dt, \quad \rho \in \mathbb{C}, \quad (13.1.5)$$

where ${}_2F_1$ is the Gauss hypergeometric series defined by (see [20]):

$${}_2F_1(\nu, \mu; \rho; x) = \sum_{m=0}^{\infty} \frac{(\nu)_m (\mu)_m}{(\rho)_m} \frac{x^m}{m!}, \quad |x| < 1. \quad (13.1.6)$$

Let $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta \in \mathbb{C}$ with $\Re(\eta) > 0$, $x \in \mathbb{R}^+$, then the left and right generalized fractional differentiation operators involving the Appell function $F_3$ as a kernel are defined by (see [16]):

$$\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta} f\right)(x) = \left(I_{0+}^{-\acute{\nu},-\nu,-\acute{\mu},-\mu,-\eta} f\right)(x)$$
$$= \left(\frac{d}{dx}\right)^m \left(I_{0+}^{-\acute{\nu},-\nu,-\acute{\mu}+m,-\mu,-\eta+m} f\right)(x) \quad (13.1.7)$$

and

$$\left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta} f\right)(x) = \left(I_{-}^{-\acute{\nu},-\nu,-\acute{\mu},-\mu,-\eta} f\right)(x)$$
$$= \left(-\frac{d}{dx}\right)^m \left(I_{-}^{-\acute{\nu},-\nu,-\acute{\mu},-\mu+m,-\eta+m} f\right)(x), \tag{13.1.8}$$

where $m = [\Re(\eta)] + 1$ and $[\Re(\eta)]$ denotes the integer part of $\Re(\eta)$.

These operators are reduced to the following Saigo fractional derivative operators (see [15]):

$$\left(D_{0+}^{\nu+\mu,0,-\rho,0,\nu} f\right)(x) = \left(D_{0+}^{\nu,\mu,\rho} f\right)(x), \quad \rho \in \mathbb{C} \tag{13.1.9}$$

and

$$\left(D_{-}^{\nu+\mu,0,-\rho,0,\nu} f\right)(x) = \left(D_{-}^{\nu,\mu,\rho} f\right)(x), \quad \rho \in \mathbb{C}. \tag{13.1.10}$$

If we set $\mu = -\nu$ in (13.1.4), (13.1.5), (13.1.9), and (13.1.10), the Saigo fractional integral and derivative operators reduce to the Riemann–Liouville fractional integral and derivative operators, which are defined as follows (see [7]):

$$\left(I_{0+}^{\nu,-\nu,\rho} f\right) = \left(I_{0+}^{\nu} f\right)(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt, \tag{13.1.11}$$

$$\left(I_{-}^{\nu,-\nu,\rho} f\right) = \left(I_{-}^{\nu} f\right)(x) = \frac{1}{\Gamma(\nu)} \int_x^{\infty} (t-x)^{\nu-1} f(t)dt, \tag{13.1.12}$$

$$\left(D_{0+}^{\nu,-\nu,\rho} f\right) = \left(D_{0+}^{\nu} f\right)(x) = \left(\frac{d}{dx}\right)^m \left(I_{0+}^{m-\nu} f\right)(x) \tag{13.1.13}$$

and

$$\left(D_{-}^{\nu,-\nu,\rho} f\right) = \left(D_{-}^{\nu} f\right)(x) = \left(-\frac{d}{dx}\right)^m \left(I_{-}^{m-\nu} f\right)(x), \tag{13.1.14}$$

where $\nu \in \mathbb{C}$, $\Re(\nu) > 0$, $m = [\Re(\nu)] + 1$, and $x \in \mathbb{R}^+$.

When $\mu = 0$ in (13.1.4), (13.1.5), (13.1.9), and (13.1.10), the Saigo fractional integral and derivative operators reduce to the Erdélyi–Kober fractional integral and derivative operators, which are defined as follows (see [8]):

$$\left(I_{0+}^{\nu,0,\rho} f\right)(x) = \left(I_{\rho,\nu}^{+} f\right)(x) = \frac{x^{-\nu-\rho}}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} t^{\rho} f(t)dt, \tag{13.1.15}$$

$$\left(I_{-}^{\nu,0,\rho,} f\right)(x) = \left(K_{\rho,\nu}^{-} f\right)(x)) = \frac{x^{\rho}}{\Gamma(\nu)} \int_x^{\infty} (t-x)^{\nu-1} t^{-\nu-\rho} f(t)dt, \tag{13.1.16}$$

$$\left(D_{0+}^{\nu,0,\rho} f\right)(x) = \left(D_{\rho,\nu}^{+} f\right)(x) = \left(\frac{d}{dx}\right)^m \left(I_{0+}^{-\nu+m,-\nu,\nu+\rho-m} f\right)(x) \tag{13.1.17}$$

and

$$\left(D_{-}^{\nu,0,\rho}f\right)(x)=\left(D_{\rho,\nu}^{-}f\right)(x)=\left(-\frac{d}{dx}\right)^{m}\left(I_{-}^{-\nu+m,-\nu,\nu+\rho}f\right)(x), \tag{13.1.18}$$

where $\nu\in\mathbb{C}$, $\Re(\nu)>0$, $m=[\Re(\nu)]+1$, and $x\in\mathbb{R}^{+}$.

Further, the image formulas for a power function, under operators (13.1.1), (13.1.2), (13.1.7), and (13.1.8) are given by [16]:

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda-1}\right)(x)=\frac{\Gamma(\lambda)\Gamma\left(\lambda+\eta-\nu-\acute{\nu}-\mu\right)\Gamma\left(\lambda+\acute{\mu}-\acute{\nu}\right)}{\Gamma\left(\lambda+\acute{\mu}\right)\Gamma\left(\lambda+\eta-\nu-\acute{\nu}\right)\Gamma\left(\lambda+\eta-\acute{\nu}-\mu\right)}x^{\lambda-\nu-\acute{\nu}+\eta-1}, \tag{13.1.19}$$

where $\nu,\acute{\nu},\mu,\acute{\mu},\eta\in\mathbb{C}$ and $\Re(\eta)>0$, $\Re(\lambda)>max\left\{0,\Re(\nu+\acute{\nu}+\mu-\eta),\Re(\acute{\nu}-\acute{\mu})\right\}$,

$$\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda-1}\right)(x)=\frac{\Gamma(1-\lambda-\mu)\Gamma\left(1-\lambda-\eta+\nu+\acute{\nu}\right)\Gamma\left(1-\lambda+\nu-\acute{\mu}-\eta\right)}{\Gamma(1-\lambda)\Gamma(1-\lambda+\nu-\mu)\Gamma\left(1-\lambda+\nu+\acute{\nu}+\acute{\mu}-\eta\right)}\times x^{\lambda-\nu-\acute{\nu}+\eta-1}, \tag{13.1.20}$$

where $\nu,\acute{\nu},\mu,\acute{\mu},\eta\in\mathbb{C}$ and $\Re(\eta)>0$, $\Re(\lambda)<1+max\left\{\Re(-\mu),\Re(\nu+\acute{\nu}-\eta),\Re(\nu+\acute{\mu}-\eta)\right\}$,

$$\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda-1}\right)(x)=\frac{\Gamma(\lambda)\Gamma\left(\lambda-\eta+\nu+\acute{\nu}+\acute{\mu}\right)\Gamma(\lambda-\mu+\nu)}{\Gamma(\lambda-\mu)\Gamma\left(\lambda-\eta+\nu+\acute{\nu}\right)\Gamma\left(\lambda-\eta+\nu+\acute{\mu}\right)}x^{\lambda+\nu+\acute{\nu}-\eta-1}, \tag{13.1.21}$$

where $\nu,\acute{\nu},\mu,\acute{\mu},\eta\in\mathbb{C}$ and $\Re(\eta)>0$, $\Re(\lambda)>max\left\{0,\Re(\eta-\nu-\acute{\nu}-\acute{\mu}),\Re(\mu-\nu)\right\}$ and

$$\left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda-1}\right)(x)=\frac{\Gamma\left(1-\lambda+\acute{\mu}\right)\Gamma\left(1-\lambda-\acute{\nu}-\mu+\eta\right)\Gamma\left(1-\lambda-\nu-\acute{\nu}+\eta\right)}{\Gamma(1-\lambda)\Gamma\left(1-\lambda-\acute{\nu}+\acute{\mu}\right)\Gamma\left(1-\lambda-\nu-\acute{\nu}-\mu+\eta\right)}\times x^{\lambda+\nu+\acute{\nu}-\eta-1}, \tag{13.1.22}$$

where $\nu,\acute{\nu},\mu,\acute{\mu},\eta\in\mathbb{C}$ and $\Re(\eta)>0$, $\Re(\lambda)<1+max\left\{\Re(\acute{\mu}),\Re(\eta-\nu-\acute{\nu}),\Re(\eta-\acute{\nu}-\mu)\right\}$.

The Fox–Wright function ${}_p\Psi_q(z)$, which is a generalization of a hypergeometric function, is defined as follows (see, e.g., [7,20]):

$${}_p\Psi_q\left[\begin{array}{c}(d_1,D_1),\ldots,(d_p,D_p)\\(e_1,E_1),\ldots,(e_q,E_q)\end{array}\middle|z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma\left(d_i+D_in\right)}{\prod_{j=1}^{q}\Gamma\left(e_j+E_jn\right)}\frac{z^n}{n!}, \tag{13.1.23}$$

where $d_i,D_i,e_j,E_j,z\in\mathbb{C}$, $\Re(d_i)>0$, $\Re(D_i)>0$, $i=1,\ldots,p$, $\Re(e_i)>0$, $\Re(E_i)>0$, $j=1,\ldots,q$, and $1+\Re\left(\sum_{j=1}^{q}E_j-\sum_{i=1}^{p}D_i\right)\geq 0$.

The Mittag-Leffler function and its generalization appear in special functions as a solution of fractional integro-differential equations having arbitrary order. The importance of such functions in applied mathematics and engineering sciences is steadily increasing. Some interesting applications of the Mittag-Leffler function are considered in the study of quantum mechanics, electric networks, random walks, Levy flights, and kinetic equations, interested readers can refer to the recent work of researchers [2–4,10] and the references cited therein. In addition to fractional calculus the Mittag-Leffler function also plays an important role in several branches of science and engineering like applied physics, statistics, quantum mechanics, mechanics, thermodynamics, telecommunications, electrical engineering, and more.

In recent years, Mittag-Leffler functions have garnered significant attention in the field of special functions, prompting numerous researchers to explore their generalizations and applications. Pathan and Bin-Saad [12] introduced and studied the function $E_{\alpha,\beta}^{j,k}(z)$, which is defined as:

$$E_{\alpha,\beta}^{j,k}(z) = \sum_{n=0}^{\infty} \frac{z^{nj+k}}{\Gamma\left(\beta + \alpha(nj+k)\right)}, \quad \alpha, \beta, z \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, j \geq 1, k \geq 0. \tag{13.1.24}$$

Very recently, Bin-Saad and Younis [1] investigated a new generalization of the arbitrary order Mittag-Leffler-type function $E_{\alpha,\beta}^{j,k}(z)$, which is defined as:

$$E_{\alpha,\beta,\gamma,\delta}^{j,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n \Gamma\left(\beta + \alpha(nj+k)\right)} z^{nj+k}, \tag{13.1.25}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$ and $j \geq 1$, $k \geq 0$. Here, $(\gamma)_n$ denotes the Pochhammer symbol defined in terms of the familiar Gamma function $\Gamma$ by (see, e.g., [19]):

$$(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \begin{cases} 1 & (n=0), \\ \gamma(\gamma+1)\ldots(\gamma+n-1) & (n \in \mathbb{N} := \{1, 2, \ldots\}). \end{cases}$$

Clearly, when $j = 1$ and $k = 0$ in (13.1.25) yields the generalized Mittag-Leffler function given by Salim [17]:

$$E_{\alpha,\beta}^{\gamma,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma\left(\alpha n + \beta\right)(\delta)_n} z^n, \tag{13.1.26}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$.

For our present investigation, we also need the Hadamard product that helps to decompose a newly emerged function into two known analytical functions. Let $f$

and $g$ be two functions represented by the following power series:

$$f(z) = \sum_{m=0}^{\infty} a_m z^m \quad \text{and} \quad g(z) = \sum_{m=0}^{\infty} b_m z^m .$$

Then, the Hadamard product of the power series is defined by (see [13]):

$$(f * g)(z) := \sum_{m=0}^{\infty} a_m b_m z^m =: (g * f)(z). \tag{13.1.27}$$

In this chapter, we aim to investigate the Marichev–Saigo–Maeda fractional calculus operators and Caputo-type Marichev–Saigo–Maeda fractional differential operators of the generalized arbitrary order Mittag-Leffler-type function. All the results are expressed in terms of the Hadamard product of the generalized Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,\delta}(z)$ and the Fox–Wright function ${}_p\Psi_q(z)$. We also include some special cases of our main results as corresponding image formulas for the Saigo, Erdélyi–Kober, and Riemann–Liouville fractional integral and derivative operators.

## 13.2 Marichev–Saigo–Maeda fractional integrals with the function $E_{\alpha,\beta,\gamma,\delta}^{j,k}(z)$

In this section, we establish the composition formulas of Marichev–Saigo–Maeda fractional integrals involving the generalized arbitrary order Mittag-Leffler-type function $E_{\alpha,\beta,\gamma,\delta}^{j,k}(z)$ in terms of the Hadamard product of the generalized Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,\delta}(z)$ and the Fox–Wright function ${}_p\Psi_q(z)$.

**Theorem 13.2.1.** *Let $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta, \lambda, \alpha, \beta, \gamma, \delta, \sigma, \omega \in \mathbb{C}$ with $\Re(\lambda + \sigma k) > max\{0, \Re(\nu + \acute{\nu} + \mu - \eta), \Re(\acute{\nu} - \acute{\mu})\}$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, and $j \geq 1$, $k \geq 0$. Let $I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}$ be the left-sided operator of the Marichev–Saigo–Maeda fractional integral. Then,*

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}(\omega t^{\sigma})\right]\right)(x) = \omega^k x^{-\nu-\acute{\nu}+\eta+\lambda+\sigma k-1} E_{\alpha j,\beta+\alpha k}^{\gamma,\delta}(\omega^j x^{\sigma j})$$
$$* {}_4\Psi_3\left[\begin{matrix} (1,1), (\lambda+\sigma k, \sigma j), (-\nu-\acute{\nu}-\mu+\eta+\lambda+\sigma k, \sigma j), \\ (\acute{\mu}+\lambda+\sigma k, \sigma j), (-\nu-\acute{\nu}+\eta+\lambda+\sigma k, \sigma j), \end{matrix}\right.$$
$$\left.\begin{matrix} (-\acute{\nu}+\acute{\mu}+\lambda+\sigma k, \sigma j) \\ (-\nu-\mu+\eta+\lambda+\sigma k, \sigma j) \end{matrix}\,\middle|\, (\omega x^{\sigma})^j\right]. \tag{13.2.1}$$

*Proof.* By using the series form definition of the generalized arbitrary order Mittag-Leffler-type function (13.1.25) and the left-sided Saigo–Maeda fractional integration power function formula (13.1.19), we have:

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}\left(\omega t^{\sigma}\right)\right]\right)(x)$$
$$=\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\delta)_n\,\Gamma\left(\beta+\alpha(nj+k)\right)}\left(\omega t^{\sigma}\right)^{nj+k}\right]\right)(x).$$

By interchanging the order of integration and summation, we obtain:

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}(\omega t^{\sigma})\right]\right)(x)$$
$$=\omega^k\sum_{n=0}^{\infty}\frac{(\gamma)_n\,\omega^{nj}}{(\delta)_n\,\Gamma\left(\beta+\alpha(nj+k)\right)}\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda+\sigma(nj+k)-1}\right)(x)$$
$$=\omega^k x^{-\nu-\acute{\nu}+\eta+\lambda+\sigma k-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n\,\omega^{nj}}{(\delta)_n\,\Gamma\left(\beta+\alpha(nj+k)\right)}$$
$$\times\frac{\Gamma\left(\lambda+\sigma k+\sigma nj\right)\Gamma\left(-\acute{\nu}+\acute{\mu}+\lambda+\sigma k+\sigma nj\right)\Gamma\left(-\nu-\acute{\nu}-\mu+\eta+\lambda+\sigma k+\sigma nj\right)}{\Gamma\left(\acute{\mu}+\lambda+\sigma k+\sigma nj\right)\Gamma\left(-\nu-\acute{\nu}+\eta+\lambda+\sigma k+\sigma nj\right)\Gamma\left(-\nu-\mu+\eta+\lambda+\sigma k+\sigma nj\right)}x^{\sigma nj}$$
$$=\omega^k x^{-\nu-\acute{\nu}+\eta+\lambda+\sigma k-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\delta)_n\,\Gamma\left(\alpha nj+\beta+\alpha k\right)}$$
$$\times\left[\frac{\Gamma(1+n)\,\Gamma\left(\lambda+\sigma k+\sigma nj\right)\Gamma\left(-\acute{\nu}+\acute{\mu}+\lambda+\sigma k+\sigma nj\right)}{\Gamma\left(\acute{\mu}+\lambda+\sigma k+\sigma nj\right)\Gamma\left(-\nu-\acute{\nu}+\eta+\lambda+\sigma k+\sigma nj\right)}\right.$$
$$\left.\frac{\Gamma\left(-\nu-\acute{\nu}-\mu+\eta+\lambda+\sigma k+\sigma nj\right)}{\Gamma\left(-\nu-\mu+\eta+\lambda+\sigma k+\sigma nj\right)n!}\right](\omega^j x^{\sigma j})^n.$$

Finally, by expressing the above equation as the Hadamard product of the generalized Mittag-Leffler function (13.1.26) and Fox–Wright function (13.1.23), we arrive at the desired formula (13.2.1). □

**Corollary 13.2.1.** *Let $\nu,\mu,\rho,\lambda,\alpha,\beta,\gamma,\delta,\sigma,\omega\in\mathbb{C}$ such that $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\nu)>0$, $\Re(\lambda+\sigma k)>max\,\{0,\Re(\mu-\rho)\}$, and $j\geq 1$, $k\geq 0$. Then, the following left fractional integral formula holds true:*

$$\left(I_{0+}^{\nu,\mu,\rho}\left[t^{\lambda-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}(\omega t^{\sigma})\right]\right)(x)=\omega^k x^{-\mu+\lambda+\sigma k-1}E_{\alpha j,\beta+\alpha k}^{\gamma,\delta}(\omega^j x^{\sigma j})$$
$$*\,{}_3\Psi_2\left[\begin{array}{c}(1,1),(\lambda+\sigma k,\sigma j),(-\mu+\rho+\lambda+\sigma k,\sigma j)\\(-\mu+\lambda+\sigma k,\sigma j),(\nu+\rho+\lambda+\sigma k,\sigma j)\end{array}\middle|\left(\omega x^{\sigma}\right)^j\right]. \qquad (13.2.2)$$

**Corollary 13.2.2.** *Let $\nu,\rho,\lambda,\alpha,\beta,\gamma,\delta,\omega,\sigma\in\mathbb{C}$ such that $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\nu)>0$, $\Re(\lambda+\sigma k)>-\Re(\rho)$, and $j\geq 1$, $k\geq 0$. Then, the following left fractional integral formula holds true:*

$$\left(I_{\rho,\nu}^{+}\left[t^{\lambda-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}(\omega t^{\sigma})\right]\right)(x)=\omega^k x^{\lambda+\sigma k-1}E_{\alpha j,\beta+\alpha k}^{\gamma,\delta}(\omega^j x^{\sigma j})$$

$$*\,{}_2\Psi_1\left[\begin{array}{c}(1,1),(\rho+\lambda+\sigma k,\sigma j)\\(\rho+\nu+\lambda+\sigma k,\sigma j)\end{array}\middle|(\omega x^{\sigma})^{j}\right]. \tag{13.2.3}$$

**Corollary 13.2.3.** *Let* $\nu,\lambda,\alpha,\beta,\gamma,\delta,\omega,\sigma\in\mathbb{C}$ *such that* $\Re(\nu)>0$, $\Re(\lambda)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\sigma)>0$, *and* $j\geq 1$, $k\geq 0$. *Then, the following left fractional integral formula holds true:*

$$\left(I^{\nu}_{0+}\left[t^{\lambda-1}E^{j,k}_{\alpha,\beta,\gamma,\delta}(\omega t^{\sigma})\right]\right)(x)=\omega^{k}x^{\nu+\lambda+\sigma k-1}E^{\gamma,\delta}_{\alpha j,\beta+\alpha k}(\omega^{j}x^{\sigma j})$$
$$*\,{}_2\Psi_1\left[\begin{array}{c}(1,1),(\lambda+\sigma k,\sigma j)\\(\nu+\lambda+\sigma k,\sigma j)\end{array}\middle|(\omega x^{\sigma})^{j}\right]. \tag{13.2.4}$$

**Corollary 13.2.4.** *For* $\gamma=\delta=1$, *Eq.* (13.2.1) *reduces to the following form:*

$$\left(I^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}_{0+}\left[t^{\lambda-1}E^{j,k}_{\alpha,\beta}(\omega t^{\sigma})\right]\right)(x)=\omega^{k}x^{-\nu-\acute{\nu}+\eta+\lambda+\sigma k-1}E_{\alpha j,\beta+\alpha k}(\omega^{j}x^{\sigma j})$$
$$*\,{}_4\Psi_3\left[\begin{array}{c}(1,1),(\lambda+\sigma k,\sigma j),(-\nu-\acute{\nu}-\mu+\eta+\lambda+\sigma k,\sigma j),\\(\acute{\mu}+\lambda+\sigma k,\sigma j),(-\nu-\acute{\nu}+\eta+\lambda+\sigma k,\sigma j),\end{array}\right.$$
$$\left.\begin{array}{c}(-\acute{\nu}+\acute{\mu}+\lambda+\sigma k,\sigma j)\\(-\nu-\mu+\eta+\lambda+\sigma k,\sigma j)\end{array}\middle|\left(\omega x^{\sigma}\right)^{j}\right]. \tag{13.2.5}$$

**Corollary 13.2.5.** *For* $j=1$ *and* $k=0$, *Eq.* (13.2.1) *reduces to the following form:*

$$\left(I^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}_{0+}\left[t^{\lambda-1}E^{\gamma,\delta}_{\alpha,\beta}(\omega t^{\sigma})\right]\right)(x)=x^{-\nu-\acute{\nu}+\eta+\lambda-1}E^{\gamma,\delta}_{\alpha,\beta}(\omega x^{\sigma})$$
$$*\,{}_4\Psi_3\left[\begin{array}{c}(1,1),(\lambda,\sigma),(-\nu-\acute{\nu}-\mu+\eta+\lambda,\sigma),(-\acute{\nu}+\acute{\mu}+\lambda,\sigma)\\(\acute{\mu}+\lambda,\sigma),(-\nu-\acute{\nu}+\eta+\lambda,\sigma),(-\nu-\mu+\eta+\lambda,\sigma)\end{array}\middle|\omega x^{\sigma}\right]. \tag{13.2.6}$$

**Theorem 13.2.2.** *Let* $\nu,\acute{\nu},\mu,\acute{\mu},\eta,\lambda,\alpha,\beta,\gamma,\delta,\omega,\sigma\in\mathbb{C}$ *with* $\Re(\lambda-\sigma k)<1+min\left\{\Re(-\mu),\Re(\nu+\acute{\nu}-\eta),\Re(\nu+\acute{\mu}-\eta)\right\}$, $\Re(\eta)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, *and* $j\geq 1$, $k\geq 0$. *Let* $I^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}_{-}$ *be the right-sided operator of the Marichev–Saigo–Maeda fractional integral. Then,*

$$\left(I^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}_{-}\left[t^{\lambda-1}E^{j,k}_{\alpha,\beta,\gamma,\delta}(\omega t^{-\sigma})\right]\right)(x)=\omega^{k}x^{-\nu-\acute{\nu}+\eta+\lambda-\sigma k-1}E^{\gamma,\delta}_{\alpha j,\beta+\alpha k}(\omega^{j}x^{-\sigma j})$$
$$*\,{}_4\Psi_3\left[\begin{array}{c}(1,1),(-\mu-\lambda+\sigma k+1,\sigma j),(\nu+\acute{\nu}-\eta-\lambda+\sigma k+1,\sigma j),\\(-\lambda+\sigma k+1,\sigma j),(\nu-\mu-\lambda+\sigma k+1,\sigma j),\end{array}\right.$$
$$\left.\begin{array}{c}(\nu+\acute{\mu}-\eta-\lambda+\sigma k+1,\sigma j)\\(\nu+\acute{\nu}+\acute{\mu}-\eta-\lambda+\sigma k+1,\sigma j)\end{array}\middle|\left(\omega x^{-\sigma}\right)^{j}\right]. \tag{13.2.7}$$

*Proof.* By using the series form definition of the generalized arbitrary order Mittag-Leffler-type function (13.1.25) and right-sided Saigo–Maeda fractional integration power function formula (13.1.20), we obtain:

$$\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}\left(\omega t^{-\sigma}\right)\right]\right)(x)$$
$$=\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\delta)_n\,\Gamma\left(\beta+\alpha(nj+k)\right)}\left(\omega t^{-\sigma}\right)^{nj+k}\right]\right)(x).$$

By interchanging the order of integration and summation, we obtain:

$$\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}(\omega t^{-\sigma})\right]\right)(x)$$
$$=\omega^k\sum_{n=0}^{\infty}\frac{(\gamma)_n\,\omega^{nj}}{(\delta)_n\,\Gamma\left(\beta+\alpha(nj+k)\right)}\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda-\sigma(nj+k)-1}\right)(x)$$
$$=\omega^k x^{-\nu-\acute{\nu}+\eta+\lambda-\sigma k-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n\,\omega^{nj}}{(\delta)_n\,\Gamma\left(\beta+\alpha(nj+k)\right)}$$
$$\times\frac{\Gamma\left(-\mu-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(\nu+\acute{\nu}-\eta-\lambda+\sigma k+\sigma nj+1\right)}{\Gamma\left(-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(\nu-\mu-\lambda+\sigma k+\sigma nj+1\right)}$$
$$\frac{\Gamma\left(\nu+\acute{\mu}-\eta-\lambda+\sigma k+\sigma nj+1\right)}{\Gamma\left(\nu+\acute{\nu}+\acute{\mu}-\eta-\lambda+\sigma k+\sigma nj+1\right)}x^{-\sigma nj}$$
$$=\omega^k x^{-\nu-\acute{\nu}+\eta+\lambda+\sigma k-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\delta)_n\,\Gamma\left(\beta+\alpha(nj+k)\right)}$$
$$\times\left[\frac{\Gamma\left(1+n\right)\Gamma\left(-\mu-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(\nu+\acute{\nu}-\eta-\lambda+\sigma k+\sigma nj+1\right)}{\Gamma\left(-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(\nu-\mu-\lambda+\sigma k+\sigma nj+1\right)}\right.$$
$$\left.\frac{\Gamma\left(\nu+\acute{\mu}-\eta-\lambda+\sigma k+\sigma nj+1\right)}{\Gamma\left(\nu+\acute{\nu}+\acute{\mu}-\eta-\lambda+\sigma k+\sigma nj+1\right)n!}\right](\omega^j x^{-\sigma j})^n.$$

By expressing the above equation as the Hadamard product of the generalized Mittag-Leffler function (13.1.26) and Fox–Wright function (13.1.23), we arrive at the desired formula (13.2.7). □

**Corollary 13.2.6.** *Let $\nu,\mu,\rho,\lambda,\alpha,\beta,\gamma,\delta,\omega,\sigma\in\mathbb{C}$ such that $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\nu)>0$, $\Re(\lambda-\sigma k)<1+min\{\Re(\mu),\Re(\rho)\}$, and $j\geq 1$, $k\geq 0$. Then, the following right fractional integral formula holds true:*

$$\left(I_{-}^{\nu,\mu,\rho}\left[t^{\lambda-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}(\omega t^{-\sigma})\right]\right)(x)=\omega^k x^{-\mu+\lambda-\sigma k-1}E_{\alpha j,\beta+\alpha k}^{\gamma,\delta}(\omega^j x^{-\sigma j})$$
$$*\,{}_3\Psi_2\left[\begin{array}{c}(1,1),(\mu-\lambda+\sigma k+1,\sigma j),(\rho-\lambda+\sigma k+1,\sigma j)\\(-\lambda+\sigma k+1,\sigma j),(\nu+\mu+\rho-\lambda+\sigma k+1,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^j\right].\quad(13.2.8)$$

**Corollary 13.2.7.** *Let $\rho,\nu,\lambda,\alpha,\beta,\gamma,\delta,\omega,\sigma\in\mathbb{C}$ such that $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\nu)>0$, $\Re(\lambda-\sigma k)<1+\Re(\rho)$, and $j\geq 1$, $k\geq 0$. Then, the* *following right fractional integral formula holds true:*

$$\left(K_{\rho,\nu}^{-}\left[t^{\lambda-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}(\omega t^{-\sigma})\right]\right)(x)=\omega^{k}x^{\lambda-\sigma k-1}E_{\alpha j,\beta+\alpha k}^{\gamma,\delta}(\omega^{j}x^{-\sigma j})$$
$$*\,{}_{2}\Psi_{1}\left[\begin{array}{c}(1,1),(\rho-\lambda+\sigma k+1,\sigma j)\\(\rho+\nu-\lambda+\sigma k+1,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^{j}\right]. \tag{13.2.9}$$

**Corollary 13.2.8.** *Let $\nu,\lambda,\alpha,\beta,\gamma,\delta,\omega,\sigma\in\mathbb{C}$ such that $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $0<\Re(\nu)<1-\Re(\lambda-\sigma k)$, and $j\geq 1$, $k\geq 0$. Then, the following right fractional integral formula holds true:*

$$\left(I_{-}^{\nu}\left[t^{\lambda-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}(\omega t^{-\sigma})\right]\right)(x)=\omega^{k}x^{\nu+\lambda-\sigma k-1}E_{\alpha j,\beta+\alpha k}^{\gamma,\delta}(\omega^{j}x^{-\sigma j})$$
$$*\,{}_{2}\Psi_{1}\left[\begin{array}{c}(1,1),(-\nu-\lambda+\sigma k+1,\sigma j)\\(-\lambda+\sigma k+1,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^{j}\right]. \tag{13.2.10}$$

**Corollary 13.2.9.** *For $\gamma=\delta=1$, Eq. (13.2.7) reduces to the following form:*

$$\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E_{\alpha,\beta}^{j,k}(\omega t^{-\sigma})\right]\right)(x)=\omega^{k}x^{-\nu-\acute{\nu}+\eta+\lambda-\sigma k-1}E_{\alpha j,\beta+\alpha k}(\omega^{j}x^{-\sigma j})$$
$$*\,{}_{4}\Psi_{3}\left[\begin{array}{c}(1,1),(-\mu-\lambda+\sigma k+1,\sigma j),(\nu+\acute{\nu}-\eta-\lambda+\sigma k+1,\sigma j),\\(-\lambda+\sigma k+1,\sigma j),(\nu-\mu-\lambda+\sigma k+1,\sigma j),\end{array}\right.$$
$$\left.\begin{array}{c}(\nu+\acute{\mu}-\eta-\lambda+\sigma k+1,\sigma j)\\(\nu+\acute{\nu}+\acute{\mu}-\eta-\lambda+\sigma k+1,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^{j}\right]. \tag{13.2.11}$$

**Corollary 13.2.10.** *For $j=1$ and $k=0$, Eq. (13.2.7) reduces to the following form:*

$$\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E_{\alpha,\beta}^{\gamma,\delta}(\omega t^{-\sigma})\right]\right)(x)=x^{-\nu-\acute{\nu}+\eta+\lambda-1}E_{\alpha,\beta}^{\gamma,\delta}\left(\omega x^{-\sigma}\right)$$
$$*\,{}_{4}\Psi_{3}\left[\begin{array}{c}(1,1),(-\mu-\lambda+1,\sigma),(\nu+\acute{\nu}-\eta-\lambda+1,\sigma),\\(-\lambda+1,\sigma),(\nu-\mu-\lambda+1,\sigma),\end{array}\right.$$
$$\left.\begin{array}{c}(\nu+\acute{\mu}-\eta-\lambda+1,\sigma)\\(\nu+\acute{\nu}+\acute{\mu}-\eta-\lambda+1,\sigma)\end{array}\middle|\omega x^{-\sigma}\right]. \tag{13.2.12}$$

## 13.3 Marichev–Saigo–Maeda fractional derivatives with the function $E_{\alpha,\beta,\gamma,\delta}^{j,k}(z)$

In this section, we establish the composition formulas of Marichev–Saigo–Maeda fractional derivatives involving the generalized arbitrary order Mittag-Leffler-type function $E_{\alpha,\beta,\gamma,\delta}^{j,k}(z)$ in terms of the Hadamard product of the generalized Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,\delta}(z)$ and the Fox–Wright function ${}_{p}\Psi_{q}(z)$.

**Theorem 13.3.1.** *Let* $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta, \lambda, \alpha, \beta, \gamma, \delta, \sigma, \omega \in \mathbb{C}$ *with* $\Re(\lambda + \sigma k) > max\{0, \Re(-\nu - \acute{\nu} - \acute{\mu} + \eta), \Re(\mu - \nu)\}$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, *and* $j \geq 1$, $k \geq 0$. *Let* $D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}$ *be the left-sided operator of the Marichev–Saigo–Maeda fractional derivative. Then,*

$$\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}(\omega t^{\sigma})\right]\right)(x) = \omega^{k}x^{\nu+\acute{\nu}-\eta+\lambda+\sigma k-1}E_{\alpha j,\beta+\alpha k}^{\gamma,\delta}(\omega^{j}x^{\sigma j})$$
$$* \, {}_4\Psi_3\left[\begin{matrix}(1,1),(\lambda+\sigma k,\sigma j),(\nu-\mu+\lambda+\sigma k,\sigma j),\\(-\mu+\lambda+\sigma k,\sigma j),(\nu+\acute{\nu}-\eta+\lambda+\sigma k,\sigma j),\end{matrix}\right.$$
$$\left.\begin{matrix}(\nu+\acute{\nu}+\acute{\mu}-\eta+\lambda+\sigma k,\sigma j)\\(\nu+\acute{\mu}-\eta+\lambda+\sigma k,\sigma j)\end{matrix}\,\middle|\,(\omega x^{\sigma})^{j}\right]. \tag{13.3.1}$$

*Proof.* By using (13.1.25), the left-hand side of (13.3.1), leads to:

$$\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}(\omega t^{\sigma})\right]\right)(x)$$
$$= \omega^{k}\sum_{n=0}^{\infty}\frac{(\gamma)_n\,\omega^{nj}}{(\delta)_n\,\Gamma(\beta+\alpha(nj+k))}\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda+\sigma(nj+k)-1}\right)(x),$$

which on using the result (13.1.21), we obtain:

$$\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}(\omega t^{\sigma})\right]\right)(x)$$
$$= \omega^{k}x^{\nu+\acute{\nu}-\eta+\lambda+\sigma k-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n\,\omega^{nj}}{(\delta)_n\,\Gamma(\beta+\alpha(nj+k))}$$
$$\times\frac{\Gamma(\lambda+\sigma k+\sigma nj)\,\Gamma(\nu-\mu+\lambda+\sigma k+\sigma nj)\,\Gamma\left(\nu+\acute{\nu}+\acute{\mu}-\eta+\lambda+\sigma k+\sigma nj\right)}{\Gamma(-\mu+\lambda+\sigma k+\sigma nj)\,\Gamma\left(\nu+\acute{\nu}-\eta+\lambda+\sigma k+\sigma nj\right)\Gamma\left(\nu+\acute{\mu}-\eta+\lambda+\sigma k+\sigma nj\right)}x^{\sigma nj}$$
$$= \omega^{k}x^{\nu+\acute{\nu}-\eta+\lambda+\sigma k-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\delta)_n\,\Gamma(\alpha nj+\beta+\alpha k)}$$
$$\times\left[\frac{\Gamma(1+n)\,\Gamma(\lambda+\sigma k+\sigma nj)\,\Gamma(\nu-\mu+\lambda+\sigma k+\sigma nj)}{\Gamma(-\mu+\lambda+\sigma k+\sigma nj)\,\Gamma\left(\nu+\acute{\nu}-\eta+\lambda+\sigma k+\sigma nj\right)}\right.$$
$$\left.\frac{\Gamma\left(\nu+\acute{\nu}+\acute{\mu}-\eta+\lambda+\sigma k+\sigma nj\right)}{\Gamma\left(\nu+\acute{\mu}-\eta+\lambda+\sigma k+\sigma nj\right)n!}\right](\omega^{j}x^{\sigma j})^{n}. \tag{13.3.2}$$

By applying the Hadamard product (13.1.27) in (13.3.2), in view of (13.1.23) and (13.1.26), we obtain the right-hand side of the result (13.3.1). □

**Corollary 13.3.1.** *Let* $\nu, \mu, \rho, \lambda, \alpha, \beta, \gamma, \delta, \omega, \sigma \in \mathbb{C}$ *such that* $\Re(\lambda + \sigma k) > -min\{0, \Re(\nu + \mu + \rho)\}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, *and* $j \geq 1$, $k \geq 0$. *Then, the following left fractional derivative formula holds true:*

$$\left(D_{0+}^{\nu,\mu,\rho}\left[t^{\lambda-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}(\omega t^{\sigma})\right]\right)(x) = \omega^{k}x^{\mu+\lambda+\sigma k-1}E_{\alpha j,\beta+\alpha k}^{\gamma,\delta}(\omega^{j}x^{\sigma j})$$

$$* {}_3\Psi_2\left[\begin{array}{c}(1,1),(\lambda+\sigma k,\sigma j),(\nu+\mu+\rho+\lambda+\sigma k,\sigma j)\\(\mu+\lambda+\sigma k,\sigma j),(\rho+\lambda+\sigma k,\sigma j)\end{array}\middle|(\omega x^{\sigma})^{j}\right]. \tag{13.3.3}$$

**Corollary 13.3.2.** *Let $\rho,\nu,\lambda,\alpha,\beta,\gamma,\delta,\omega,\sigma\in\mathbb{C}$ such that $\Re(\lambda+\sigma k)>-\Re(\rho+\nu)$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, and $j\geq 1$, $k\geq 0$. Then, the following left fractional derivative formula holds true:*

$$\left(D^{+}_{\rho,\nu}\left[t^{\lambda-1}E^{j,k}_{\alpha,\beta,\gamma,\delta}(\omega t^{\sigma})\right]\right)(x)=\omega^{k}x^{\lambda+\sigma k-1}E^{\gamma,\delta}_{\alpha j,\beta+\alpha k}(\omega^{j}x^{\sigma j})$$
$$* {}_2\Psi_1\left[\begin{array}{c}(1,1),(\rho+\nu+\lambda+\sigma k,\sigma j)\\(\rho+\lambda+\sigma k,\sigma j)\end{array}\middle|(\omega x^{\sigma})^{j}\right]. \tag{13.3.4}$$

**Corollary 13.3.3.** *Let $\nu,\lambda,\alpha,\beta,\gamma,\delta,\omega,\sigma\in\mathbb{C}$ such that $\Re(\nu)>0$, $\Re(\lambda)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\sigma)>0$, and $j\geq 1$, $k\geq 0$. Then, the following left fractional derivative formula holds true:*

$$\left(D^{\nu}_{0+}\left[t^{\lambda-1}E^{j,k}_{\alpha,\beta,\gamma,\delta}(\omega t^{\sigma})\right]\right)(x)=\omega^{k}x^{-\nu+\lambda+\sigma k-1}E^{\gamma,\delta}_{\alpha j,\beta+\alpha k}(\omega^{j}x^{\sigma j})$$
$$* {}_2\Psi_1\left[\begin{array}{c}(1,1),(\lambda+\sigma k,\sigma j)\\(-\nu+\lambda+\sigma k,\sigma j)\end{array}\middle|(\omega x^{\sigma})^{j}\right]. \tag{13.3.5}$$

**Corollary 13.3.4.** *For $\gamma=\delta=1$, Eq. (13.3.1) reduces to the following form:*

$$\left(D^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}_{0+}\left[t^{\lambda-1}E^{j,k}_{\alpha,\beta}(\omega t^{\sigma})\right]\right)(x)=\omega^{k}x^{\nu+\acute{\nu}-\eta+\lambda+\sigma k-1}E_{\alpha j,\beta+\alpha k}(\omega^{j}x^{\sigma j})$$
$$* {}_4\Psi_3\left[\begin{array}{c}(1,1),(\lambda+\sigma k,\sigma j),(\nu-\mu+\lambda+\sigma k,\sigma j),\\(-\mu+\lambda+\sigma k,\sigma j),(\nu+\acute{\nu}-\eta+\lambda+\sigma k,\sigma j),\end{array}\right.$$
$$\left.\begin{array}{c}(\nu+\acute{\nu}+\acute{\mu}-\eta+\lambda+\sigma k,\sigma j)\\(\nu+\acute{\mu}-\eta+\lambda+\sigma k,\sigma j)\end{array}\middle|\left(\omega x^{\sigma}\right)^{j}\right]. \tag{13.3.6}$$

**Corollary 13.3.5.** *For $j=1$ and $k=0$, Eq. (13.3.1) reduces to the following form:*

$$\left(D^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}_{0+}\left[t^{\lambda-1}E^{\gamma,\delta}_{\alpha,\beta}(\omega t^{\sigma})\right]\right)(x)=x^{\nu+\acute{\nu}-\eta+\lambda-1}E^{\gamma,\delta}_{\alpha,\beta}(\omega x^{\sigma})$$
$$* {}_4\Psi_3\left[\begin{array}{c}(1,1),(\lambda,\sigma),(\nu-\mu+\lambda,\sigma),(\nu+\acute{\nu}+\acute{\mu}-\eta+\lambda,\sigma)\\(-\mu+\lambda,\sigma),(\nu+\acute{\nu}-\eta+\lambda,\sigma),(\nu+\acute{\mu}-\eta+\lambda,\sigma)\end{array}\middle|\omega x^{\sigma}\right]. \tag{13.3.7}$$

**Theorem 13.3.2.** *Let $\nu,\acute{\nu},\mu,\acute{\mu},\eta,\lambda,\alpha,\beta,\gamma,\delta,\omega,\sigma\in\mathbb{C}$ with $\Re(\lambda-\sigma k)<1+min\left\{\Re(\acute{\mu}),\Re(-\nu-\acute{\nu}+\eta),\Re(-\acute{\nu}-\mu+\eta)\right\}$, $\Re(\eta)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, and $j\geq 1$, $k\geq 0$. Let $D^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}_{-}$ be the right-sided operator of the Marichev–Saigo–Maeda fractional derivative. Then,*

$$\left(D^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}_{-}\left[t^{\lambda-1}E^{j,k}_{\alpha,\beta,\gamma,\delta}(\omega t^{-\sigma})\right]\right)(x)=\omega^{k}x^{\nu+\acute{\nu}-\eta+\lambda-\sigma k-1}E^{\gamma,\delta}_{\alpha j,\beta+\alpha k}(\omega^{j}x^{-\sigma j})$$
$$* {}_4\Psi_3\left[\begin{array}{c}(1,1),(\acute{\mu}-\lambda+\sigma k+1,\sigma j),(-\nu-\acute{\nu}+\eta-\lambda+\sigma k+1,\sigma j),\\(-\lambda+\sigma k+1,\sigma j),(-\acute{\nu}+\acute{\mu}-\lambda+\sigma k+1,\sigma j),\end{array}\right.$$

$$\left.\begin{matrix} (-\acute{\nu}-\mu+\eta-\lambda+\sigma k+1,\sigma j) \\ (-\nu-\acute{\nu}-\mu+\eta-\lambda+\sigma k+1,\sigma j) \end{matrix}\right| \left(\omega x^{-\sigma}\right)^j \Bigg]. \tag{13.3.8}$$

*Proof.* Using (13.1.25), the left-hand side of (13.3.8), leads to:

$$\begin{aligned} &\left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E^{j,k}_{\alpha,\beta,\gamma,\delta}(\omega t^{-\sigma})\right]\right)(x) \\ &= \omega^k \sum_{n=0}^{\infty} \frac{(\gamma)_n\,\omega^{nj}}{(\delta)_n\,\Gamma\left(\beta+\alpha(nj+k)\right)} \left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta} t^{\lambda-\sigma(nj+k)-1}\right)(x), \end{aligned}$$

which on using (13.1.22), we obtain:

$$\begin{aligned} &\left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E^{j,k}_{\alpha,\beta,\gamma,\delta}(\omega t^{-\sigma})\right]\right)(x) \\ &= \omega^k x^{\nu+\acute{\nu}-\eta+\lambda-\sigma k-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n\,\omega^{nj}}{(\delta)_n\,\Gamma\left(\beta+\alpha(nj+k)\right)} \\ &\times \frac{\Gamma\left(\acute{\mu}-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(-\nu-\acute{\nu}+\eta-\lambda+\sigma k+\sigma nj+1\right)}{\Gamma\left(-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(-\acute{\nu}+\acute{\mu}-\lambda+\sigma k+\sigma nj+1\right)} \\ &\frac{\Gamma\left(-\acute{\nu}-\mu+\eta-\lambda+\sigma k+\sigma nj+1\right)}{\Gamma\left(-\nu-\acute{\nu}-\mu+\eta-\lambda+\sigma k+\sigma nj+1\right)} x^{-\sigma nj} \\ &= \omega^k x^{\nu+\acute{\nu}-\eta+\lambda+\sigma k-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n\,\Gamma\left(\alpha nj+\beta+\alpha k\right)} \\ &\times \left[\frac{\Gamma(1+n)\,\Gamma\left(\acute{\mu}-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(-\nu-\acute{\nu}+\eta-\lambda+\sigma k+\sigma nj+1\right)}{\Gamma\left(-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(-\acute{\nu}+\acute{\mu}-\lambda+\sigma k+\sigma nj+1\right)}\right. \\ &\left.\frac{\Gamma\left(-\acute{\nu}-\mu+\eta-\lambda+\sigma k+\sigma nj+1\right)}{\Gamma\left(-\nu-\acute{\nu}-\mu+\eta-\lambda+\sigma k+\sigma nj+1\right)n!}\right] (\omega^j x^{-\sigma j})^n. \end{aligned} \tag{13.3.9}$$

By applying the Hadamard product (13.1.27) in (13.3.9), in view of (13.1.23) and (13.1.26), we have the right-hand side of the result (13.3.8). □

**Corollary 13.3.6.** *Let* $\nu,\mu,\rho,\lambda,\alpha,\beta,\gamma,\delta,\omega,\sigma \in \mathbb{C}$ *such that* $\Re(\lambda-\sigma k) < 1 + min\{\Re(-\mu)-m, \Re(\nu+\rho)\}$, $m = [\Re(\nu)]+1$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, *and* $j \geq 1$, $k \geq 0$. *Then, the following right fractional derivative formula holds true:*

$$\begin{aligned} &\left(D_{-}^{\nu,\mu,\rho}\left[t^{\lambda-1}E^{j,k}_{\alpha,\beta,\gamma,\delta}(\omega t^{-\sigma})\right]\right)(x) = \omega^k x^{\mu+\lambda-\sigma k-1} E^{\gamma,\delta}_{\alpha j,\beta+\alpha k}(\omega^j x^{-\sigma j}) \\ &* {}_3\Psi_2\left[\left.\begin{matrix} (1,1),(-\mu-\lambda+\sigma k+1,\sigma j),(\nu+\rho-\lambda+\sigma k+1,\sigma j) \\ (-\lambda+\sigma k+1,\sigma j),(-\mu+\rho-\lambda+\sigma k+1,\sigma j) \end{matrix}\right| (\omega x^{-\sigma})^j\right]. \end{aligned} \tag{13.3.10}$$

**Corollary 13.3.7.** *Let $\rho, \nu, \lambda, \alpha, \beta, \gamma, \delta, \omega, \sigma \in \mathbb{C}$ such that $\Re(\nu) > 0$, $\Re(\lambda - \sigma k) < \Re(\rho + \nu) - [\Re(\nu)]$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, and $j \geq 1$, $k \geq 0$. Then, the following right fractional derivative formula holds true:*

$$\left(D^{-}_{\rho,\nu}\left[t^{\lambda-1}E^{j,k}_{\alpha,\beta,\gamma,\delta}(\omega t^{-\sigma})\right]\right)(x) = \omega^k x^{\lambda-\sigma k-1}E^{\gamma,\delta}_{\alpha j,\beta+\alpha k}(\omega^j x^{-\sigma j}) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(\rho+\nu-\lambda+\sigma k+1,\sigma j)\\(\rho-\lambda+\sigma k+1,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^j\right]. \tag{13.3.11}$$

**Corollary 13.3.8.** *Let $\nu, \lambda, \alpha, \beta, \gamma, \delta, \omega, \sigma \in \mathbb{C}$ such that $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\nu) > 0$, $\Re(\lambda - \sigma k) < \Re(\nu) - [\Re(\nu)]$, and $j \geq 1$, $k \geq 0$. Then, the following right fractional derivative formula holds true:*

$$\left(D^{\nu}_{-}\left[t^{\lambda-1}E^{j,k}_{\alpha,\beta,\gamma,\delta}(\omega t^{-\sigma})\right]\right)(x) = \omega^k x^{-\nu+\lambda-\sigma k-1}E^{\gamma,\delta}_{\alpha j,\beta+\alpha k}(\omega^j x^{-\sigma j}) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(\nu-\lambda+\sigma k+1,\sigma j)\\(-\lambda+\sigma k+1,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^j\right]. \tag{13.3.12}$$

**Corollary 13.3.9.** *For $\gamma = \delta = 1$, Eq. (13.3.8) reduces to the following form:*

$$\left(D^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}_{-}\left[t^{\lambda-1}E^{j,k}_{\alpha,\beta}(\omega t^{-\sigma})\right]\right)(x) = \omega^k x^{\nu+\acute{\nu}-\eta+\lambda-\sigma k-1}E_{\alpha j,\beta+\alpha k}(\omega^j x^{-\sigma j}) * {}_4\Psi_3\left[\begin{array}{c}(1,1),(\acute{\mu}-\lambda+\sigma k+1,\sigma j),(-\nu-\acute{\nu}+\eta-\lambda+\sigma k+1,\sigma j),\\(-\lambda+\sigma k+1,\sigma j),(-\acute{\nu}+\acute{\mu}-\lambda+\sigma k+1,\sigma j),\end{array}\right.$$
$$\left.\begin{array}{c}(-\acute{\nu}-\mu+\eta-\lambda+\sigma k+1,\sigma j)\\(-\nu-\acute{\nu}-\mu+\eta-\lambda+\sigma k+1,\sigma j)\end{array}\middle|\left(\omega x^{-\sigma}\right)^j\right]. \tag{13.3.13}$$

**Corollary 13.3.10.** *For $j = 1$ and $k = 0$, Eq. (13.3.8) reduces to the following form:*

$$\left(D^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}_{-}\left[t^{\lambda-1}E^{\gamma,\delta}_{\alpha,\beta}(\omega t^{-\sigma})\right]\right)(x) = x^{\nu+\acute{\nu}-\eta+\lambda-1}E^{\gamma,\delta}_{\alpha,\beta}(\omega x^{-\sigma}) * {}_4\Psi_3\left[\begin{array}{c}(1,1),(\acute{\mu}-\lambda+1,\sigma),(-\nu-\acute{\nu}+\eta-\lambda+1,\sigma),\\(-\lambda+1,\sigma),(-\acute{\nu}+\acute{\mu}-\lambda+1,\sigma),\end{array}\right.$$
$$\left.\begin{array}{c}(-\acute{\nu}-\mu+\eta-\lambda+1,\sigma)\\(-\nu-\acute{\nu}-\mu+\eta-\lambda+1,\sigma)\end{array}\middle|\omega x^{-\sigma}\right]. \tag{13.3.14}$$

## 13.4 Caputo-type Marichev–Saigo–Maeda fractional derivatives with the function $E^{j,k}_{\alpha,\beta,\gamma,\delta}(z)$

Here, we present the Caputo-type Marichev–Saigo–Maeda fractional differentiation of the generalized Mittag-Leffler-type function of arbitrary order. Rao et al. [14] introduced the Caputo-type fractional derivatives that have the Gauss hypergeometric function in the kernel. Let $\nu, \mu, \rho \in \mathbb{C}$ with $\Re(\nu) > 0$, $x \in \mathbb{R}^+$, then the left and right Caputo-type fractional differential operators associated with the Gauss hypergeometric function are defined as follows:

$$\left({}^c D_{0+}^{\nu,\mu,\rho} f\right)(x) = \left(I_{0+}^{-\nu+[\Re(\nu)]+1,-\mu-[\Re(\nu)]-1,\nu+\rho-[\Re(\nu)]-1} f^{([\Re(\nu)]+1)}\right)(x) \tag{13.4.1}$$

and

$$\left({}^c D_{-}^{\nu,\mu,\rho} f\right)(x) = (-1)^{[\Re(\nu)]+1}\left(I_{0+}^{-\nu+[\Re(\nu)]+1,-\mu-[\Re(\nu)]-1,\nu+\rho} f^{([\Re(\nu)]+1)}\right)(x). \tag{13.4.2}$$

The corresponding Caputo-type Marichev–Saigo–Maeda fractional differential operators are given by Kataria and Vellaisamy [6]. Let $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta \in \mathbb{C}$ with $\Re(\eta) > 0$, $x \in \mathbb{R}^+$, then the left and right Caputo-type Marichev–Saigo–Maeda fractional differential operators are defined as follows:

$$\left({}^c D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta} f\right)(x) = \left(I_{0+}^{-\acute{\nu},-\nu,-\acute{\mu}+[\Re(\nu)]+1,-\mu,-\eta+[\Re(\nu)]+1} f^{([\Re(\nu)]+1)}\right)(x) \tag{13.4.3}$$

and

$$\begin{aligned}&\left({}^c D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta} f\right)(x)\\ &\quad = (-1)^{[\Re(\nu)]+1}\left(I_{-}^{-\acute{\nu},-\nu,-\acute{\mu},-\mu+[\Re(\nu)]+1,-\eta+[\Re(\nu)]+1} f^{([\Re(\nu)]+1)}\right)(x).\end{aligned} \tag{13.4.4}$$

The following are known results for the Caputo-type Marichev–Saigo–Maeda fractional differential operators of the power functions (see [6]):

$$\begin{aligned}\left({}^c D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta} t^{\lambda-1}\right)(x) &= \frac{\Gamma(\lambda)\,\Gamma\left(\lambda-\eta+\nu+\acute{\nu}+\acute{\mu}-m\right)\Gamma(\lambda-\mu+\nu-m)}{\Gamma(\lambda-\mu-m)\,\Gamma\left(\lambda-\eta+\nu+\acute{\nu}\right)\Gamma\left(\lambda-\eta+\nu+\acute{\mu}-m\right)}\\ &\quad\times x^{\lambda+\nu+\acute{\nu}-\eta-1},\end{aligned} \tag{13.4.5}$$

where $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta \in \mathbb{C}$, $\Re(\lambda)-m > max\left\{0, \Re(\eta-\nu-\acute{\nu}-\acute{\mu}), \Re(\mu-\nu)\right\}$ and $m = [\Re(\eta)]+1$ and

$$\begin{aligned}\left({}^c D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta} t^{-\lambda}\right)(x) &= \frac{\Gamma\left(\lambda+\acute{\mu}+m\right)\Gamma\left(\lambda-\acute{\nu}-\mu+\eta+m\right)\Gamma\left(\lambda-\nu-\acute{\nu}+\eta\right)}{\Gamma(\lambda)\,\Gamma\left(\lambda-\acute{\nu}+\acute{\mu}+m\right)\Gamma\left(\lambda-\nu-\acute{\nu}-\mu+\eta+m\right)}\\ &\quad\times x^{\nu+\acute{\nu}-\eta-\lambda},\end{aligned} \tag{13.4.6}$$

where $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta \in \mathbb{C}$, $\Re(\lambda)+m > max\left\{\Re(-\acute{\mu}), \Re(\acute{\nu}+\mu-\eta), \Re(\nu+\acute{\nu}-\eta)+m\right\}$, and $m = [\Re(\eta)]+1$.

Now, the left-hand sided Caputo-type Marichev–Saigo–Maeda fractional derivative of the function $E_{\alpha,\beta,\gamma,\delta}^{j,k}(z)$ is given as follows.

**Theorem 13.4.1.** *Let* $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta, \lambda, \alpha, \beta, \gamma, \delta, \sigma, \omega \in \mathbb{C}$ *with* $\Re(\lambda + \sigma k) - m > max\left\{0, \Re(-\nu - \acute{\nu} - \acute{\mu} + \eta), \Re(\mu - \nu)\right\}$, $m = [\Re(\eta)] + 1$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, *and* $j \geq 1$, $k \geq 0$. *Let* ${}^{c}D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}$ *be the left-sided operator of the Caputo-type Marichev–Saigo–Maeda fractional derivative. Then,*

$$\left({}^{c}D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}(\omega t^{\sigma})\right]\right)(x) = \omega^{k}x^{\nu+\acute{\nu}-\eta+\lambda+\sigma k-1}E_{\alpha j,\beta+\alpha k}^{\gamma,\delta}(\omega^{j}x^{\sigma j})$$
$$* {}_4\Psi_3\left[\begin{array}{c}(1,1),(\lambda+\sigma k,\sigma j),(\nu-\mu+\lambda+\sigma k-m,\sigma j),\\(-\mu+\lambda+\sigma k-m,\sigma j),(\nu+\acute{\nu}-\eta+\lambda+\sigma k,\sigma j),\end{array}\right.$$
$$\left.\begin{array}{c}(\nu+\acute{\nu}+\acute{\mu}-\eta+\lambda+\sigma k-m,\sigma j)\\(\nu+\acute{\mu}-\eta+\lambda+\sigma k-m,\sigma j)\end{array}\right|\left(\omega x^{\sigma}\right)^{j}\Bigg]. \tag{13.4.7}$$

*Proof.* By using (13.1.25) and (13.4.5), we obtain:

$$\begin{aligned}
&\left({}^{c}D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}(\omega t^{\sigma})\right]\right)(x)\\
&=\left({}^{c}D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n\,\omega^{nj+k}}{(\delta)_n\,\Gamma\left(\beta+\alpha(nj+k)\right)}t^{\sigma(nj+k)}\right)(x)\\
&=\sum_{n=0}^{\infty}\frac{(\gamma)_n\,\omega^{nj+k}}{(\delta)_n\,\Gamma\left(\beta+\alpha(nj+k)\right)}\left({}^{c}D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda+\sigma(nj+k)-1}\right)(x).\\
&=\omega^{k}x^{\nu+\acute{\nu}-\eta+\lambda+\sigma k-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n\,\omega^{nj+k}}{(\delta)_n\,\Gamma\left(\beta+\alpha(nj+k)\right)}\\
&\times\frac{\Gamma\left(\lambda+\sigma k+\sigma nj\right)\Gamma\left(\nu-\mu+\lambda+\sigma k-m+\sigma nj\right)}{\Gamma\left(-\mu+\lambda+\sigma k-m+\sigma nj\right)\Gamma\left(\nu+\acute{\nu}-\eta+\lambda+\sigma k+\sigma nj\right)}\\
&\times\frac{\Gamma\left(\nu+\acute{\nu}+\acute{\mu}-\eta+\lambda+\sigma k-m+\sigma nj\right)}{\Gamma\left(\nu+\acute{\mu}-\eta+\lambda+\sigma k-m+\sigma nj\right)}x^{\sigma nj}\\
&=\omega^{k}x^{\nu+\acute{\nu}-\eta+\lambda+\sigma k-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\delta)_n\,\Gamma\left(\alpha nj+\beta+\alpha k\right)}\\
&\times\left[\frac{\Gamma\left(1+n\right)\Gamma\left(\lambda+\sigma k+\sigma nj\right)\Gamma\left(\nu-\mu+\lambda+\sigma k-m+\sigma nj\right)}{\Gamma\left(-\mu+\lambda+\sigma k-m+\sigma nj\right)\Gamma\left(\nu+\acute{\nu}-\eta+\lambda+\sigma k+\sigma nj\right)}\right.\\
&\times\left.\frac{\Gamma\left(\nu+\acute{\nu}+\acute{\mu}-\eta+\lambda+\sigma k-m+\sigma nj\right)}{\Gamma\left(\nu+\acute{\mu}-\eta+\lambda+\sigma k-m+\sigma nj\right)}\right]\frac{(\omega^{j}x^{\sigma j})^{n}}{n!}.
\end{aligned}$$

By using (13.1.27) and in view of (13.1.23) and (13.1.26), we obtain the result. □

**Corollary 13.4.1.** *Let* $\nu, \mu, \rho, \lambda, \alpha, \beta, \gamma, \delta, \omega, \sigma \in \mathbb{C}$ *such that* $\Re(\lambda + \sigma k) - m > max\{0, -\Re(\nu + \mu + \rho)\}$, $m = [\Re(\nu)]+1$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, *and* $j \geq 1$, $k \geq 0$. *Let* ${}^cD_{0+}^{\nu,\mu,\rho}$ *be the left-sided operator of the Caputo-type Saigo fractional derivative. Then, the following formula holds true:*

$$\left({}^cD_{0+}^{\nu,\mu,\rho}\left[t^{\lambda-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}(\omega t^{\sigma})\right]\right)(x) = \omega^k x^{\mu+\lambda+\sigma k-1}E_{\alpha j,\beta+\alpha k}^{\gamma,\delta}(\omega^j x^{\sigma j}) * {}_3\Psi_2\left[\begin{array}{c}(1,1),(\lambda+\sigma k,\sigma j),(\nu+\mu+\rho+\lambda+\sigma k-m,\sigma j)\\(\mu+\lambda+\sigma k,\sigma j),(\rho+\lambda+\sigma k-m,\sigma j)\end{array}\middle|(\omega x^{\sigma})^j\right]. \quad (13.4.8)$$

**Corollary 13.4.2.** *Let* $\rho,\nu,\lambda,\alpha,\beta,\gamma,\delta,\omega,\sigma \in \mathbb{C}$ *such that* $\Re(\lambda+\sigma k)-m > max\,\{0,-\Re(\rho+\nu)\}$, $m = [\Re(\nu)]+1$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, *and* $j \geq 1$, $k \geq 0$. *Let* ${}^cD_{\rho,\nu}^{+}$ *be the left-sided operator of the Caputo-type Erdélyi–Kober fractional derivative. Then, the following formula holds true:*

$$\left({}^cD_{\rho,\nu}^{+}\left[t^{\lambda-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}(\omega t^{\sigma})\right]\right)(x) = \omega^k x^{\lambda+\sigma k-1}E_{\alpha j,\beta+\alpha k}^{\gamma,\delta}(\omega^j x^{\sigma j}) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(\rho+\nu+\lambda+\sigma k-m,\sigma j)\\(\rho+\lambda+\sigma k-m,\sigma j)\end{array}\middle|(\omega x^{\sigma})^j\right]. \quad (13.4.9)$$

**Corollary 13.4.3.** *When* $\gamma=\delta=1$, *Eq. (13.4.7) reduces to the following form:*

$$\left({}^cD_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E_{\alpha,\beta}^{j,k}(\omega t^{\sigma})\right]\right)(x) = \omega^k x^{\nu+\acute{\nu}-\eta+\lambda+\sigma k-1}E_{\alpha j,\beta+\alpha k}(\omega^j x^{\sigma j}) * {}_4\Psi_3\left[\begin{array}{c}(1,1),(\lambda+\sigma k,\sigma j),(\nu-\mu+\lambda+\sigma k-m,\sigma j),\\(-\mu+\lambda+\sigma k-m,\sigma j),(\nu+\acute{\nu}-\eta+\lambda+\sigma k,\sigma j),\end{array}\right.$$
$$\left.\begin{array}{c}(\nu+\acute{\nu}+\acute{\mu}-\eta+\lambda+\sigma k-m,\sigma j)\\(\nu+\acute{\mu}-\eta+\lambda+\sigma k-m,\sigma j)\end{array}\middle|(\omega x^{\sigma})^j\right]. \quad (13.4.10)$$

**Corollary 13.4.4.** *When* $j=1$ *and* $k=0$, *Eq. (13.4.7) reduces to the following form:*

$$\left({}^cD_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E_{\alpha,\beta}^{\gamma,\delta}(\omega t^{\sigma})\right]\right)(x) = x^{\nu+\acute{\nu}-\eta+\lambda-1}E_{\alpha,\beta}^{\gamma,\delta}(\omega x^{\sigma}) * {}_4\Psi_3\left[\begin{array}{c}(1,1),(\lambda,\sigma),(\nu-\mu+\lambda-m,\sigma),(\nu+\acute{\nu}+\acute{\mu}-\eta+\lambda-m,\sigma)\\(-\mu+\lambda-m,\sigma),(\nu+\acute{\nu}-\eta+\lambda,\sigma),(\nu+\acute{\mu}-\eta+\lambda-m,\sigma)\end{array}\middle|\omega x^{\sigma}\right]. \quad (13.4.11)$$

Further, the right-hand sided Caputo-type Marichev–Saigo–Maeda fractional derivatives of the function $E_{\alpha,\beta,\gamma,\delta}^{j,k}(z)$ are given as follows.

**Theorem 13.4.2.** *Let* $\nu,\acute{\nu},\mu,\acute{\mu},\eta,\lambda,\alpha,\beta,\gamma,\delta,\sigma,\omega \in \mathbb{C}$ *with* $\Re(\lambda+\sigma k)+m > max\left\{-\Re(\acute{\mu}),\Re(\acute{\nu}+\mu-\eta),\Re(\nu+\acute{\nu}-\eta)+m\right\}$, $m=[\Re(\eta)]+1$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, *and* $j \geq 1$, $k \geq 0$. *Let* ${}^cD_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}$ *be the right-sided operator of the Caputo-type Marichev–Saigo–Maeda fractional derivative. Then,*

$$\left({}^cD_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{-\lambda}E_{\alpha,\beta,\gamma,\delta}^{j,k}(\omega t^{-\sigma})\right]\right)(x) = \omega^k x^{\nu+\acute{\nu}-\eta-\lambda-\sigma k}E_{\alpha j,\beta+\alpha k}^{\gamma,\delta}(\omega^j x^{-\sigma j})$$

$$
*\,{}_4\Psi_3\left[\begin{array}{c}(1,1),(\acute{\mu}+\lambda+\sigma k+m,\sigma j),(-\nu-\acute{\nu}+\eta+\lambda+\sigma k,\sigma j),\\(\lambda+\sigma k,\sigma j),(-\acute{\nu}+\acute{\mu}+\lambda+\sigma k+m,\sigma j),\end{array}\right.
$$
$$
\left.\begin{array}{c}(-\acute{\nu}-\mu+\eta+\lambda+\sigma k+m,\sigma j)\\(-\nu+\acute{\nu}-\mu+\eta+\lambda+\sigma k+m,\sigma j)\end{array}\middle|\left(\omega x^{-\sigma}\right)^j\right]. \tag{13.4.12}
$$

*Proof.* By using (13.1.25) and (13.4.6), we obtain:

$$
\begin{aligned}
&\left({}^cD_-^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{-\lambda}E^{j,k}_{\alpha,\beta,\gamma,\delta}(\omega t^{-\sigma})\right]\right)(x)\\
&=\left({}^cD_-^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{-\lambda}\sum_{n=0}^{\infty}\frac{(\gamma)_n\,\omega^{nj+k}}{(\delta)_n\,\Gamma(\beta+\alpha(nj+k))}t^{-\sigma(nj+k)}\right)(x)\\
&=\sum_{n=0}^{\infty}\frac{(\gamma)_n\,\omega^{nj+k}}{(\delta)_n\,\Gamma(\beta+\alpha(nj+k))}\left({}^cD_-^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{-(\lambda+\sigma(nj+k))}\right)(x)\\
&=\omega^k x^{\nu+\acute{\nu}-\eta-\lambda-\sigma k}\sum_{n=0}^{\infty}\frac{(\gamma)_n\,\omega^{nj}}{(\delta)_n\,\Gamma(\beta+\alpha(nj+k))}\\
&\times\frac{\Gamma\left(\acute{\mu}+\lambda+\sigma k+m+\sigma nj\right)\Gamma\left(-\nu-\acute{\nu}+\eta+\lambda+\sigma k+\sigma nj\right)}{\Gamma(\lambda+\sigma k+\sigma nj)\,\Gamma\left(-\acute{\nu}+\acute{\mu}+\lambda+\sigma k+m+\sigma nj\right)}\\
&\times\frac{\Gamma\left(-\acute{\nu}-\mu+\eta+\lambda+\sigma k+m+\sigma nj\right)}{\Gamma\left(-\nu+\acute{\nu}-\mu+\eta+\lambda+\sigma k+m+\sigma nj\right)}x^{-\sigma nj}\\
&=\omega^k x^{\nu+\acute{\nu}-\eta-\lambda-\sigma k}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\delta)_n\,\Gamma(\alpha nj+\beta+\alpha k)}\\
&\times\left[\frac{\Gamma(1+n)\Gamma\left(\acute{\mu}+\lambda+\sigma k+m+\sigma nj\right)\Gamma\left(-\nu-\acute{\nu}+\eta+\lambda+\sigma k+\sigma nj\right)}{\Gamma(\lambda+\sigma k+\sigma nj)\,\Gamma\left(-\acute{\nu}+\acute{\mu}+\lambda+\sigma k+m+\sigma nj\right)}\right.\\
&\times\left.\frac{\Gamma\left(-\acute{\nu}-\mu+\eta+\lambda+\sigma k+m+\sigma nj\right)}{\Gamma\left(-\nu+\acute{\nu}-\mu+\eta+\lambda+\sigma k+m+\sigma nj\right)}\right]\frac{(\omega^j x^{-\sigma j})^n}{n!}.
\end{aligned}
$$

By using (13.1.27) and in view of (13.1.23) and (13.1.26), we obtain the result. □

**Corollary 13.4.5.** *Let* $\nu,\mu,\rho,\lambda,\alpha,\beta,\gamma,\delta,\omega,\sigma\in\mathbb{C}$ *such that* $\Re(\lambda+\sigma k)+m>$ *max* $\{\Re(\mu)+m,\Re(-\nu-\rho)\}$, $m=[\Re(\nu)]+1$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, *and* $j\geq 1$, $k\geq 0$. *Let* ${}^cD_-^{\nu,\mu,\rho}$ *be the right-sided operator of the Caputo-type Saigo fractional derivative. Then, the following formula holds true:*

$$
\left({}^cD_-^{\nu,\mu,\rho}\left[t^{-\lambda}E^{j,k}_{\alpha,\beta,\gamma,\delta}(\omega t^{-\sigma})\right]\right)(x)=\omega^k x^{\mu-\lambda-\sigma k}E^{\gamma,\delta}_{\alpha j,\beta+\alpha k}(\omega^j x^{-\sigma j})
$$
$$
*\,{}_3\Psi_2\left[\begin{array}{c}(1,1),(-\mu+\lambda+\sigma k,\sigma j),(\nu+\rho+\lambda+\sigma k+m,\sigma j)\\(\lambda+\sigma k,\sigma j),(-\mu+\rho+\lambda+\sigma k+m,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^j\right]. \tag{13.4.13}
$$

**Corollary 13.4.6.** *Let $\rho, \nu, \lambda, \alpha, \beta, \gamma, \delta, \omega, \sigma \in \mathbb{C}$ such that $\Re(\lambda + \sigma k) + m > \max\{m, \Re(-\nu - \rho)\}$, $m = [\Re(\nu)] + 1$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, and $j \geq 1$, $k \geq 0$. Let ${}^{c}D^{-}_{\rho,\nu}$ be the right-sided operator of the Caputo-type Erdélyi–Kober fractional derivative. Then, the following formula holds true:*

$$\left({}^{c}D^{-}_{\rho,\nu}\left[t^{-\lambda}E^{j,k}_{\alpha,\beta,\gamma,\delta}(\omega t^{-\sigma})\right]\right)(x) = \omega^{k}x^{-\lambda-\sigma k}E^{\gamma,\delta}_{\alpha j,\beta+\alpha k}(\omega^{j}x^{-\sigma j})$$
$$* {}_{2}\Psi_{1}\left[\begin{array}{c}(1,1),(\rho+\nu+\lambda+\sigma k+m,\sigma j)\\(\rho+\lambda+\sigma k+m,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^{j}\right]. \tag{13.4.14}$$

**Corollary 13.4.7.** *When $\gamma = \delta = 1$, Eq. (13.4.12) reduces to the following form:*

$$\left({}^{c}D^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}_{-}\left[t^{-\lambda}E^{j,k}_{\alpha,\beta}(\omega t^{-\sigma})\right]\right)(x) = \omega^{k}x^{\nu+\acute{\nu}-\eta-\lambda-\sigma k}E_{\alpha j,\beta+\alpha k}(\omega^{j}x^{-\sigma j})$$
$$* {}_{4}\Psi_{3}\left[\begin{array}{c}(1,1),(\acute{\mu}+\lambda+\sigma k+m,\sigma j),(-\nu-\acute{\nu}+\eta+\lambda+\sigma k,\sigma j),\\(\lambda+\sigma k,\sigma j),(-\acute{\nu}+\acute{\mu}+\lambda+\sigma k+m,\sigma j),\end{array}\right.$$
$$\left.\begin{array}{c}(-\acute{\nu}-\mu+\eta+\lambda+\sigma k+m,\sigma j)\\(-\nu+\acute{\nu}-\mu+\eta+\lambda+\sigma k+m,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^{j}\right]. \tag{13.4.15}$$

**Corollary 13.4.8.** *When $j = 1$ and $k = 0$, Eq. (13.4.12) reduces to the following form:*

$$\left({}^{c}D^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}_{-}\left[t^{-\lambda}E^{\gamma,\delta}_{\alpha,\beta}(\omega t^{-\sigma})\right]\right)(x) = x^{\nu+\acute{\nu}-\eta-\lambda}E^{\gamma,\delta}_{\alpha,\beta}(\omega x^{-\sigma})$$
$$* {}_{4}\Psi_{3}\left[\begin{array}{c}(1,1),(\acute{\mu}+\lambda+m,\sigma),(-\nu-\acute{\nu}+\eta+\lambda,\sigma),(-\acute{\nu}-\mu+\eta+\lambda+m,\sigma)\\(\lambda,\sigma),(-\acute{\nu}+\acute{\mu}+\lambda+m,\sigma),(-\nu+\acute{\nu}-\mu+\eta+\lambda+m,\sigma)\end{array}\middle|\omega x^{-\sigma}\right]. \tag{13.4.16}$$

Finally, if we let $\mu = -\nu$ in Corollaries 13.4.1 and 13.4.5, respectively, we obtain the left- and right-sided Caputo derivative of $E^{j,k}_{\alpha,\beta,\gamma,\delta}(z)$.

## CRediT authorship contribution statement

All authors contributed equally to this article. All authors have read and agreed to the published version of the manuscript.

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## Data availability

Not applicable.

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CHAPTER

# 14 Fractional integration and differentiation of the generalized arbitrary order Mittag-Leffler-type function

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## 14.1 Introduction

Fractional calculus is a branch of mathematical analysis focused on the properties of functions and operators defined by arbitrary non-integer order derivatives and integrals. This field has gained increased attention across various fields of science and engineering, including physics, chemistry, biology, fluid dynamics, astrophysics, electrical engineering, image processing and others. For more details about the fractal calculus and its applications, interested readers can refer to [7–9,17]. Recently, researchers have extensively studied fractional calculus, developing new fractional integral and derivative operators that have gained significant attention due to their wide applications in diverse fields. In particular, various fractional operators such as Riemann–Liouville, Erdélyi–Kober, Caputo, Saigo, Hilfer, and Marichev–Saigo–Maeda fractional operators are present, see, for example, [10,20,21].

We recall here the generalized hypergeometric fractional integral and fractional derivative operators involving Appell's function $F_3$, introduced by Marichev [13] and later extended and studied by Saigo and Maeda [16]. These operators are known as the Marichev–Saigo–Maeda operators. The generalized fractional calculus operators with the Appell function $F_3$ in their kernel are defined as follows.

Let $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta \in \mathbb{C}$ with $\Re(\eta) > 0$, $x \in \mathbb{R}^+$, then the left and right fractional integral operators are defined as follows (see [16]):

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta} f\right)(x) = \frac{x^{-\nu}}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} t^{-\acute{\nu}} F_3\left(\nu, \acute{\nu}, \mu, \acute{\mu}; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt \tag{14.1.1}$$



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and

$$\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}f\right)(x)$$
$$=\frac{x^{-\acute{\nu}}}{\Gamma(\eta)}\int_x^{\infty}(t-x)^{\eta-1}t^{-\nu}F_3\left(\nu,\acute{\nu},\mu,\acute{\mu};\eta;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)dt, \quad (14.1.2)$$

where $F_3$ is defined as follows (see [19]):

$$F_3\left(\nu,\acute{\nu},\mu,\acute{\mu};\eta;x,t\right)=\sum_{m,n=0}^{\infty}\frac{(\nu)_m(\acute{\nu})_n(\mu)_m(\acute{\mu})_n}{(\eta)_{m+n}}\frac{x^m}{m!}\frac{t^n}{n!}, \quad (max\{|x|,|t|\}<1). \quad (14.1.3)$$

Here, $(\nu)_m$ denotes the Pochhammer symbol defined in terms of the familiar Gamma function $\Gamma$ by (see, e.g., [18]):

$$(\nu)_m=\frac{\Gamma(\nu+m)}{\Gamma(\nu)}=\begin{cases}1 & (m=0),\\ \nu(\nu+1)\ldots(\nu+m-1) & (m\in\mathbb{N}:=\{1,2,\ldots\}).\end{cases}$$

These operators are reduced to the following Saigo fractional integral operators (see [15]):

$$\left(I_{0+}^{\nu+\mu,0,-\rho,0,\nu}f\right)(x)$$
$$=\left(I_{0+}^{\nu,\mu,\rho}f\right)(x)$$
$$=\frac{x^{-\nu-\mu}}{\Gamma(\nu)}\int_0^{x}(x-t)^{\nu-1}{}_2F_1\left(\nu+\mu,-\rho;\nu;1-\frac{t}{x}\right)f(t)dt, \quad \rho\in\mathbb{C} \quad (14.1.4)$$

and

$$\left(I_{-}^{\nu+\mu,0,-\rho,0,\nu}f\right)(x)$$
$$=\left(I_{-}^{\nu,\mu,\rho}f\right)(x)$$
$$=\frac{1}{\Gamma(\nu)}\int_x^{\infty}(t-x)^{\nu-1}t^{-\nu-\mu}{}_2F_1\left(\nu+\mu,-\rho;\nu;1-\frac{x}{t}\right)f(t)dt, \quad \rho\in\mathbb{C}, \quad (14.1.5)$$

where ${}_2F_1$ is the Gauss hypergeometric series defined by (see [19]):

$${}_2F_1(\nu,\mu;\rho;x)=\sum_{m=0}^{\infty}\frac{(\nu)_m(\mu)_m}{(\rho)_m}\frac{x^m}{m!}, \quad |x|<1. \quad (14.1.6)$$

Let $\nu,\acute{\nu},\mu,\acute{\mu},\eta\in\mathbb{C}$ with $\Re(\eta)>0$, $x\in\mathbb{R}^+$, then the left and right generalized fractional differentiation operators involving the Appell function $F_3$ as a kernel are defined by (see [16]):

$$\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta} f\right)(x) = \left(I_{0+}^{-\acute{\nu},-\nu,-\acute{\mu},-\mu,-\eta} f\right)(x)$$
$$= \left(\frac{d}{dx}\right)^m \left(I_{0+}^{-\acute{\nu},-\nu,-\acute{\mu}+m,-\mu,-\eta+m} f\right)(x) \tag{14.1.7}$$

and

$$\left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta} f\right)(x) = \left(I_{-}^{-\acute{\nu},-\nu,-\acute{\mu},-\mu,-\eta} f\right)(x)$$
$$= \left(-\frac{d}{dx}\right)^m \left(I_{-}^{-\acute{\nu},-\nu,-\acute{\mu},-\mu+m,-\eta+m} f\right)(x), \tag{14.1.8}$$

where $m = [\Re(\eta)] + 1$ and $[\Re(\eta)]$ denotes the integer part of $\Re(\eta)$.

These operators are reduced to the following Saigo fractional derivative operators (see [15]):

$$\left(D_{0+}^{\nu+\mu,0,-\rho,0,\nu} f\right)(x) = \left(D_{0+}^{\nu,\mu,\rho} f\right)(x), \quad \rho \in \mathbb{C} \tag{14.1.9}$$

and

$$\left(D_{-}^{\nu+\mu,0,-\rho,0,\nu} f\right)(x) = \left(D_{-}^{\nu,\mu,\rho} f\right)(x), \quad \rho \in \mathbb{C}. \tag{14.1.10}$$

If we set $\mu = -\nu$ in (14.1.4), (14.1.5), (14.1.9), and (14.1.10), the Saigo fractional integral and derivative operators reduce to the Riemann–Liouville fractional integral and derivative operators, which are defined as follows (see [8]):

$$\left(I_{0+}^{\nu,-\nu,\rho} f\right) = \left(I_{0+}^{\nu} f\right)(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt, \tag{14.1.11}$$

$$\left(I_{-}^{\nu,-\nu,\rho} f\right) = \left(I_{-}^{\nu} f\right)(x) = \frac{1}{\Gamma(\nu)} \int_x^\infty (t-x)^{\nu-1} f(t)dt, \tag{14.1.12}$$

$$\left(D_{0+}^{\nu,-\nu,\rho} f\right) = \left(D_{0+}^{\nu} f\right)(x) = \left(\frac{d}{dx}\right)^m \left(I_{0+}^{m-\nu} f\right)(x) \tag{14.1.13}$$

and

$$\left(D_{-}^{\nu,-\nu,\rho} f\right) = \left(D_{-}^{\nu} f\right)(x) = \left(-\frac{d}{dx}\right)^m \left(I_{-}^{m-\nu} f\right)(x), \tag{14.1.14}$$

where $\nu \in \mathbb{C}$, $\Re(\nu) > 0$, $m = [\Re(\nu)] + 1$ and $x \in \mathbb{R}^+$.

When $\mu = 0$ in (14.1.4), (14.1.5), (14.1.9), and (14.1.10), the Saigo fractional integral and derivative operators reduce to the Erdélyi–Kober fractional integral and derivative operators, which are defined as follows (see [9]):

$$\left(I_{0+}^{\nu,0,\rho} f\right)(x) = \left(I_{\rho,\nu}^{+} f\right)(x) = \frac{x^{-\nu-\rho}}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} t^\rho f(t)dt, \tag{14.1.15}$$

$$\left(I_{-}^{\nu,0,\rho,}f\right)(x) = \left(K_{\rho,\nu}^{-}f\right)(x)) = \frac{x^{\rho}}{\Gamma(\nu)}\int_{x}^{\infty}(t-x)^{\nu-1}t^{-\nu-\rho}f(t)dt, \quad (14.1.16)$$

$$\left(D_{0+}^{\nu,0,\rho}f\right)(x) = \left(D_{\rho,\nu}^{+}f\right)(x) = \left(\frac{d}{dx}\right)^{m}\left(I_{0+}^{-\nu+m,-\nu,\nu+\rho-m}f\right)(x) \quad (14.1.17)$$

and

$$\left(D_{-}^{\nu,0,\rho}f\right)(x) = \left(D_{\rho,\nu}^{-}f\right)(x) = \left(-\frac{d}{dx}\right)^{m}\left(I_{-}^{-\nu+m,-\nu,\nu+\rho}f\right)(x), \quad (14.1.18)$$

where $\nu \in \mathbb{C}$, $\Re(\nu) > 0$, $m = [\Re(\nu)] + 1$ and $x \in \mathbb{R}^{+}$.

Further, the image formulas for a power function, under operators (14.1.1), (14.1.2), (14.1.7), and (14.1.8) are given by [16]:

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda-1}\right)(x) = \frac{\Gamma(\lambda)\Gamma\left(\lambda+\eta-\nu-\acute{\nu}-\mu\right)\Gamma\left(\lambda+\acute{\mu}-\acute{\nu}\right)}{\Gamma\left(\lambda+\acute{\mu}\right)\Gamma\left(\lambda+\eta-\nu-\acute{\nu}\right)\Gamma\left(\lambda+\eta-\acute{\nu}-\mu\right)}x^{\lambda-\nu-\acute{\nu}+\eta-1}, \quad (14.1.19)$$

where $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta \in \mathbb{C}$ and $\Re(\eta) > 0, \Re(\lambda) > max\left\{0, \Re(\nu+\acute{\nu}+\mu-\eta), \Re(\acute{\nu}-\acute{\mu})\right\}$,

$$\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda-1}\right)(x) = \frac{\Gamma(1-\lambda-\mu)\Gamma\left(1-\lambda-\eta+\nu+\acute{\nu}\right)\Gamma\left(1-\lambda+\nu-\acute{\mu}-\eta\right)}{\Gamma(1-\lambda)\Gamma(1-\lambda+\nu-\mu)\Gamma\left(1-\lambda+\nu+\acute{\nu}+\acute{\mu}-\eta\right)} \times x^{\lambda-\nu-\acute{\nu}+\eta-1}, \quad (14.1.20)$$

where $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta \in \mathbb{C}$ and $\Re(\eta) > 0, \Re(\lambda) < 1 + max\left\{\Re(-\mu), \Re(\nu+\acute{\nu}-\eta), \Re(\nu+\acute{\mu}-\eta)\right\}$,

$$\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda-1}\right)(x) = \frac{\Gamma(\lambda)\Gamma\left(\lambda-\eta+\nu+\acute{\nu}+\acute{\mu}\right)\Gamma(\lambda-\mu+\nu)}{\Gamma(\lambda-\mu)\Gamma\left(\lambda-\eta+\nu+\acute{\nu}\right)\Gamma\left(\lambda-\eta+\nu+\acute{\mu}\right)}x^{\lambda+\nu+\acute{\nu}-\eta-1}, \quad (14.1.21)$$

where $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta \in \mathbb{C}$ and $\Re(\eta) > 0, \Re(\lambda) > max\left\{0, \Re(\eta-\nu-\acute{\nu}-\acute{\mu}), \Re(\mu-\nu)\right\}$ and

$$\left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda-1}\right)(x) = \frac{\Gamma\left(1-\lambda+\acute{\mu}\right)\Gamma\left(1-\lambda-\acute{\nu}-\mu+\eta\right)\Gamma\left(1-\lambda-\nu-\acute{\nu}+\eta\right)}{\Gamma(1-\lambda)\Gamma\left(1-\lambda-\acute{\nu}+\acute{\mu}\right)\Gamma\left(1-\lambda-\nu-\acute{\nu}-\mu+\eta\right)} \times x^{\lambda+\nu+\acute{\nu}-\eta-1}, \quad (14.1.22)$$

where $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta \in \mathbb{C}$ and $\Re(\eta) > 0, \Re(\lambda) < 1 + max\left\{\Re(\acute{\mu}), \Re(\eta-\nu-\acute{\nu}), \Re(\eta-\acute{\nu}-\mu)\right\}$.

The Mittag-Leffler function and its generalization appear in special functions as a solution of fractional integro-differential equations having arbitrary order. The importance of such functions in applied mathematics and engineering sciences is steadily increasing. Some interesting applications of the Mittag-Leffler function are considered in the study of quantum mechanics, electric networks, random walks, Levy flights and kinetic equations, interested readers can refer to the recent works [4–6,12] and the references cited therein. In addition to fractional calculus, the Mittag-Leffler function also plays an important role in several branches of science and engineering like applied physics, statistics, quantum mechanics, mechanics, thermodynamics, telecommunications, electrical engineering, and more.

In recent years, Mittag-Leffler functions have garnered significant attention in the field of special functions, prompting numerous researchers to explore their generalizations and applications. Pathan and Bin-Saad [14] introduced and studied the function $E_{\alpha,\beta}^{j,k}(z)$, which is defined as:

$$E_{\alpha,\beta}^{j,k}(z) = \sum_{n=0}^{\infty} \frac{z^{nj+k}}{\Gamma(\beta + \alpha(nj+k))}, \quad \alpha, \beta, z \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, j \geq 1, k \geq 0. \tag{14.1.23}$$

Very recently, Bin-Saad and Younis [2] investigated a new generalization of the arbitrary order Mittag-Leffler-type function $E_{\alpha,\beta}^{j,k}(z)$, which is defined as:

$${}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n}}{\Gamma(\beta + \alpha(nj+k))} z^{nj+k}, \tag{14.1.24}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\alpha j) > \Re(\delta)$, and $j \geq 1, k \geq 0$.

Clearly, when $j = 1$ and $k = 0$ in (14.1.25) yields the extended Mittag-Leffler-type function given by Garg et al. [3]:

$${}_{\gamma,\delta}E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n}}{\Gamma(\alpha n + \beta)} z^{n}, \tag{14.1.25}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\alpha) > \Re(\delta)$.

The Fox–Wright function ${}_p\Psi_q(z)$, which is a generalization of a hypergeometric function, is defined as follows (see, e.g., [8,19]):

$${}_p\Psi_q\left[\begin{array}{c} (d_1, D_1), \ldots, (d_p, D_p) \\ (e_1, E_1), \ldots, (e_q, E_q) \end{array} \middle| z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(d_i + D_i n)}{\prod_{j=1}^{q} \Gamma\left(e_j + E_j n\right)} \frac{z^n}{n!}, \tag{14.1.26}$$

where $d_i, D_i, e_j, E_j, z \in \mathbb{C}$, $\Re(d_i) > 0$, $\Re(D_i) > 0$, $i = 1, \ldots, p$, $\Re(e_i) > 0$, $\Re(E_i) > 0$, $j = 1, \ldots, q$, and $1 + \Re\left(\sum_{j=1}^{q} E_j - \sum_{i=1}^{p} D_i\right) \geq 0$.

In this chapter, we establish the Marichev–Saigo–Maeda fractional integration and differentiation of the generalized Mittag-Leffler-type function of arbitrary order. As special cases, the corresponding assertions for the Saigo, Erdélyi–Kober, and Riemann–Liouville fractional operators are also deduced.

## 14.2 Marichev–Saigo–Maeda fractional integrals with the function ${}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}(z)$

In this section, we study Marichev–Saigo–Maeda fractional integrals of the generalized arbitrary order Mittag-Leffler-type function.

**Theorem 14.2.1.** *Let* $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta, \lambda, \alpha, \beta, \gamma, \delta, \sigma, \omega \in \mathbb{C}$ *with* $\Re(\lambda + \sigma k) > max\{0, \Re(\nu + \acute{\nu} + \mu - \eta), \Re(\acute{\nu} - \acute{\mu})\}$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, *and* $j \geq 1$, $k \geq 0$. *Let* $I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}$ *be the left-sided operator of the Marichev–Saigo–Maeda fractional integral. Then,*

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}\left(\omega t^{\sigma}\right)\right]\right)(x) = \frac{\omega^k x^{-\nu-\acute{\nu}+\eta+\lambda+\sigma k-1}}{\Gamma(\gamma)}$$
$$\times {}_5\Psi_4\left[\begin{matrix}(1,1),(\gamma,\delta),(\lambda+\sigma k,\sigma j),(-\acute{\nu}+\acute{\mu}+\lambda+\sigma k,\sigma j),\\ (\beta+\alpha k,\alpha j),(\acute{\mu}+\lambda+\sigma k,\sigma j),(-\nu-\acute{\nu}+\eta+\lambda+\sigma k,\sigma j),\end{matrix}\right.$$
$$\left.\begin{matrix}(-\nu-\acute{\nu}-\mu+\eta+\lambda+\sigma k,\sigma j)\\ (-\acute{\nu}-\mu+\eta+\lambda+\sigma k,\sigma j)\end{matrix}\middle|(\omega x^{\sigma})^j\right]. \tag{14.2.1}$$

*Proof.* Consider the left-sided Marichev–Saigo–Maeda fractional integral operator with the representation of (14.1.24), we find:

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}\left(\omega t^{\sigma}\right)\right]\right)(x)$$
$$= \omega^k \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n}\,\omega^{nj}}{\Gamma\left(\beta+\alpha(nj+k)\right)}\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda+\sigma(nj+k)-1}\right)(x).$$

Applying (14.1.19), we obtain:

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}\left(\omega t^{\sigma}\right)\right]\right)(x)$$
$$= \frac{\omega^k x^{-\nu-\acute{\nu}+\eta+\lambda+\sigma k-1}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\omega^{nj}\,\Gamma(\gamma+\delta n)}{\Gamma\left(\beta+\alpha(nj+k)\right)}$$
$$\times \frac{\Gamma(\lambda+\sigma k+\sigma nj)\,\Gamma\left(-\acute{\nu}+\acute{\mu}+\lambda+\sigma k+\sigma nj\right)\Gamma\left(-\nu-\acute{\nu}-\mu+\eta+\lambda+\sigma k+\sigma nj\right)}{\Gamma\left(\acute{\mu}+\lambda+\sigma k+\sigma nj\right)\Gamma\left(-\nu-\acute{\nu}+\eta+\lambda+\sigma k+\sigma nj\right)\Gamma\left(-\acute{\nu}-\mu+\eta+\lambda+\sigma k+\sigma nj\right)}x^{\sigma nj}$$
$$= \frac{\omega^k x^{-\nu-\acute{\nu}+\eta+\lambda+\sigma k-1}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(1+n)\,\Gamma(\gamma+\delta n)}{\Gamma(\beta+\alpha k+\alpha nj)}$$
$$\times \frac{\Gamma(\lambda+\sigma k+\sigma nj)\,\Gamma\left(-\acute{\nu}+\acute{\mu}+\lambda+\sigma k+\sigma nj\right)\Gamma\left(-\nu-\acute{\nu}-\mu+\eta+\lambda+\sigma k+\sigma nj\right)}{\Gamma\left(\acute{\mu}+\lambda+\sigma k+\sigma nj\right)\Gamma\left(-\nu-\acute{\nu}+\eta+\lambda+\sigma k+\sigma nj\right)\Gamma\left(-\acute{\nu}-\mu+\eta+\lambda+\sigma k+\sigma nj\right)}$$
$$\times \frac{(\omega^j x^{\sigma j})^n}{n!},$$

which in view of (14.1.26), yields the desired result (14.2.1). □

**Corollary 14.2.1.** *Let $\nu, \mu, \rho, \lambda, \alpha, \beta, \gamma, \delta, \omega, \sigma \in \mathbb{C}$ such that $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\nu) > 0$, $\Re(\lambda + \sigma k) >$ max $\{0, \Re(\mu - \rho)\}$, and $j \geq 1$, $k \geq 0$. Then, the following left fractional integral formula holds true:*

$$\left(I_{0+}^{\nu,\mu,\rho}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}\left(\omega t^{\sigma}\right)\right]\right)(x) = \frac{\omega^k x^{-\mu+\lambda+\sigma k-1}}{\Gamma(\gamma)} \times {}_4\Psi_3\left[\begin{array}{c} (1,1), (\gamma,\delta), (\lambda+\sigma k, \sigma j), (-\mu+\rho+\lambda+\sigma k, \sigma j) \\ (\beta+\alpha k, \alpha j), (-\mu+\lambda+\sigma k, \sigma j), (\nu+\rho+\lambda+\sigma k, \sigma j) \end{array} \middle| (\omega x^{\sigma})^j\right]. \tag{14.2.2}$$

**Corollary 14.2.2.** *Let $\nu, \rho, \lambda, \alpha, \beta, \gamma, \delta, \omega, \sigma \in \mathbb{C}$ such that $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\nu) > 0$, $\Re(\lambda + \sigma k) > -\Re(\rho)$, and $j \geq 1$, $k \geq 0$. Then, the following left fractional integral formula holds true:*

$$\left(I_{\rho,\nu}^{+}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}\left(\omega t^{\sigma}\right)\right]\right)(x) = \frac{\omega^k x^{\lambda+\sigma k-1}}{\Gamma(\gamma)} \times {}_3\Psi_2\left[\begin{array}{c} (1,1), (\gamma,\delta), (\rho+\lambda+\sigma k, \sigma j) \\ (\beta+\alpha k, \alpha j), (\rho+\nu+\lambda+\sigma k, \sigma j) \end{array} \middle| (\omega x^{\sigma})^j\right]. \tag{14.2.3}$$

**Corollary 14.2.3.** *Let $\nu, \lambda, \alpha, \beta, \gamma, \delta, \omega, \sigma \in \mathbb{C}$ such that $\Re(\nu) > 0$, $\Re(\lambda) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\sigma) > 0$, and $j \geq 1$, $k \geq 0$. Then, the following left fractional integral formula holds true:*

$$\left(I_{0+}^{\nu}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}\left(\omega t^{\sigma}\right)\right]\right)(x) = \frac{\omega^k x^{\nu+\lambda+\sigma k-1}}{\Gamma(\gamma)} \times {}_3\Psi_2\left[\begin{array}{c} (1,1), (\gamma,\delta), (\lambda+\sigma k, \sigma j) \\ (\beta+\alpha k, \alpha j), (\nu+\lambda+\sigma k, \sigma j) \end{array} \middle| (\omega x^{\sigma})^j\right]. \tag{14.2.4}$$

For $\delta = 1$ in (14.2.1), Theorem 14.2.1 yields the following corollary.

**Corollary 14.2.4.** *Let $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta, \lambda, \alpha, \beta, \gamma, \sigma, \omega \in \mathbb{C}$ such that $\Re(\lambda + \sigma k) >$ max $\left\{0, \Re(\nu + \acute{\nu} + \mu - \eta), \Re(\acute{\nu} - \acute{\mu})\right\}$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and $j \geq 1$, $k \geq 0$. Then, the following left fractional integral formula holds true:*

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}{}_{\gamma,1}E_{\alpha,\beta}^{j,k}\left(\omega t^{\sigma}\right)\right]\right)(x) = \frac{\omega^k x^{-\nu-\acute{\nu}+\eta+\lambda+\sigma k-1}}{\Gamma(\gamma)} \times {}_5\Psi_4\left[\begin{array}{c} (1,1), (\gamma,1), (\lambda+\sigma k, \sigma j), (-\acute{\nu}+\acute{\mu}+\lambda+\sigma k, \sigma j), \\ (\beta+\alpha k, \alpha j), (\acute{\mu}+\lambda+\sigma k, \sigma j), (-\nu-\acute{\nu}+\eta+\lambda+\sigma k, \sigma j), \end{array}\right.$$
$$\left.\begin{array}{c} (-\nu-\acute{\nu}-\mu+\eta+\lambda+\sigma k, \sigma j) \\ (-\acute{\nu}-\mu+\eta+\lambda+\sigma k, \sigma j) \end{array} \middle| (\omega x^{\sigma})^j\right]. \tag{14.2.5}$$

For $\delta = 0$ in (14.2.1), Theorem 14.2.1 yields the following corollary.

**Corollary 14.2.5.** *Let $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta, \lambda, \alpha, \beta, \sigma, \omega \in \mathbb{C}$ such that $\Re(\lambda + \sigma k) > max\{0, \Re(\nu + \acute{\nu} + \mu - \eta), \Re(\acute{\nu} - \acute{\mu})\}$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and $j \geq 1$, $k \geq 0$. Then, the following left fractional integral formula holds true:*

$$\left(I_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E_{\alpha,\beta}^{j,k}\left(\omega t^{\sigma}\right)\right]\right)(x) = \omega^k x^{-\nu-\acute{\nu}+\eta+\lambda+\sigma k-1}$$
$$\times\, {}_4\Psi_4\left[\begin{matrix}(1,1),(\lambda+\sigma k,\sigma j),(-\acute{\nu}+\acute{\mu}+\lambda+\sigma k,\sigma j),\\ (\beta+\alpha k,\alpha j),(\acute{\mu}+\lambda+\sigma k,\sigma j),(-\nu-\acute{\nu}+\eta+\lambda+\sigma k,\sigma j),\end{matrix}\right.$$
$$\left.\begin{matrix}(-\nu-\acute{\nu}-\mu+\eta+\lambda+\sigma k,\sigma j)\\ (-\acute{\nu}-\mu+\eta+\lambda+\sigma k,\sigma j)\end{matrix}\,\middle|\,(\omega x^{\sigma})^j\right]. \tag{14.2.6}$$

Further, if $j = 1$ and $k = 0$ in (14.2.1), then we have the well-known result involving the left-sided Marichev–Saigo–Maeda fractional integral operator (see [1]).

**Theorem 14.2.2.** *Let $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta, \lambda, \alpha, \beta, \gamma, \delta, \sigma, \omega \in \mathbb{C}$ with $\Re(\lambda - \sigma k) < 1 + min\{\Re(-\mu), \Re(\nu + \acute{\nu} - \eta), \Re(\nu + \acute{\mu} - \eta)\}$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and $j \geq 1$, $k \geq 0$. Let $I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}$ be the right-sided operator of the Marichev–Saigo–Maeda fractional integral. Then,*

$$\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}\left(\omega t^{-\sigma}\right)\right]\right)(x) = \frac{\omega^k x^{-\nu-\acute{\nu}+\eta+\lambda-\sigma k-1}}{\Gamma(\gamma)}$$
$$\times\, {}_5\Psi_4\left[\begin{matrix}(1,1),(\gamma,\delta),(-\mu-\lambda+\sigma k+1,\sigma j),(\nu+\acute{\nu}-\eta-\lambda+\sigma k+1,\sigma j),\\ (\beta+\alpha k,\alpha j),(-\lambda+\sigma k+1,\sigma j),(\nu-\mu-\lambda+\sigma k+1,\sigma j),\end{matrix}\right.$$
$$\left.\begin{matrix}(\nu+\acute{\mu}-\eta-\lambda+\sigma k+1,\sigma j)\\ (\nu+\acute{\nu}+\acute{\mu}-\eta-\lambda+\sigma k+1,\sigma j)\end{matrix}\,\middle|\,(\omega x^{-\sigma})^j\right]. \tag{14.2.7}$$

*Proof.* Considering the right-sided Marichev–Saigo–Maeda fractional integral operator with Eq. (14.1.24), we find:

$$\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}\left(\omega t^{-\sigma}\right)\right]\right)(x)$$
$$= \sum_{n=0}^{\infty}\frac{\omega^{nj+k}\,(\gamma)_{\delta n}}{\Gamma\left(\beta+\alpha(nj+k)\right)}\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda-\sigma(nj+k)-1}\right)(x).$$

Using (14.1.20), we have:

$$\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}\left(\omega t^{-\sigma}\right)\right]\right)(x)$$
$$= \omega^k x^{-\nu-\acute{\nu}+\eta+\lambda-\sigma k-1}\sum_{n=0}^{\infty}\frac{\omega^{nj}\,(\gamma)_{\delta n}}{\Gamma\left(\beta+\alpha(nj+k)\right)}\frac{\Gamma\left(-\mu-\lambda+\sigma k+\sigma nj+1\right)}{\Gamma\left(-\lambda+\sigma k+\sigma nj+1\right)}$$
$$\times\frac{\Gamma\left(\nu+\acute{\nu}-\eta-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(\nu+\acute{\mu}-\eta-\lambda+\sigma k+\sigma nj+1\right)}{\Gamma\left(\nu-\mu-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(\nu+\acute{\nu}+\acute{\mu}-\eta-\lambda+\sigma k+\sigma nj+1\right)}x^{-\sigma nj}$$

$$= \frac{\omega^k x^{-\nu-\acute{\nu}+\eta+\lambda-\sigma k-1}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(1+n)\Gamma(\gamma+\delta n)\Gamma(-\mu-\lambda+\sigma k+\sigma nj+1)}{\Gamma(\beta+\alpha k+\alpha nj)\Gamma(-\lambda+\sigma k+\sigma nj+1)}$$
$$\times \frac{\Gamma\left(\nu+\acute{\nu}-\eta-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(\nu+\acute{\mu}-\eta-\lambda+\sigma k+\sigma nj+1\right)}{\Gamma(\nu-\mu-\lambda+\sigma k+\sigma nj+1)\Gamma\left(\nu+\acute{\nu}+\acute{\mu}-\eta-\lambda+\sigma k+\sigma nj+1\right)}$$
$$\times \frac{(\omega^j x^{-\sigma j})^n}{n!},$$

which in view of (14.1.26), yields the required result (14.2.7). □

**Corollary 14.2.6.** *Let $\nu, \mu, \rho, \lambda, \alpha, \beta, \gamma, \delta, \omega, \sigma \in \mathbb{C}$ such that $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\nu) > 0$, $\Re(\lambda-\sigma k) < 1+min\{\Re(\mu), \Re(\rho)\}$, and $j \geq 1$, $k \geq 0$. Then, the following right fractional integral formula holds true:*

$$\left(I_{-}^{\nu,\mu,\rho}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}\left(\omega t^{-\sigma}\right)\right]\right)(x) = \frac{\omega^k x^{-\mu+\lambda-\sigma k-1}}{\Gamma(\gamma)}$$
$$\times {}_4\Psi_3\left[\begin{array}{c}(1,1),(\gamma,\delta),(\mu-\lambda+\sigma k+1,\sigma j),(\rho-\lambda+\sigma k+1,\sigma j)\\(\beta+\alpha k,\alpha j),(-\lambda+\sigma k+1,\sigma j),(\nu+\mu+\rho-\lambda+\sigma k+1,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^j\right]. \tag{14.2.8}$$

**Corollary 14.2.7.** *Let $\rho, \nu, \lambda, \alpha, \beta, \gamma, \delta, \omega, \sigma \in \mathbb{C}$ such that $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\nu) > 0$, $\Re(\lambda-\sigma k) < 1+\Re(\rho)$, and $j \geq 1$, $k \geq 0$. Then, the following right fractional integral formula holds true:*

$$\left(K_{\rho,\nu}^{-}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}\left(\omega t^{-\sigma}\right)\right]\right)(x) = \frac{\omega^k x^{\lambda-\sigma k-1}}{\Gamma(\gamma)}$$
$$\times {}_3\Psi_2\left[\begin{array}{c}(1,1),(\gamma,\delta),(\rho-\lambda+\sigma k+1,\sigma j)\\(\beta+\alpha k,\alpha j),(\rho+\nu-\lambda+\sigma k+1,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^j\right]. \tag{14.2.9}$$

**Corollary 14.2.8.** *Let $\nu, \lambda, \alpha, \beta, \gamma, \delta, \omega, \sigma \in \mathbb{C}$ such that $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $0 < \Re(\nu) < 1-\Re(\lambda-\sigma k)$, and $j \geq 1$, $k \geq 0$. Then, the following right fractional integral formula holds true:*

$$\left(I_{-}^{\nu}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}\left(\omega t^{-\sigma}\right)\right]\right)(x) = \frac{\omega^k x^{\nu+\lambda-\sigma k-1}}{\Gamma(\gamma)}$$
$$\times {}_3\Psi_2\left[\begin{array}{c}(1,1),(\gamma,\delta),(-\nu-\lambda+\sigma k+1,\sigma j)\\(\beta+\alpha k,\alpha j),(-\lambda+\sigma k+1,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^j\right]. \tag{14.2.10}$$

For $\delta = 1$ in (14.2.7), Theorem 14.2.2 yields the following corollary.

**Corollary 14.2.9.** *Let $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta, \lambda, \alpha, \beta, \gamma, \sigma, \omega \in \mathbb{C}$ such that $\Re(\lambda-\sigma k) < 1+min\left\{\Re(-\mu), \Re(\nu+\acute{\nu}-\eta), \Re(\nu+\acute{\mu}-\eta)\right\}$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and $j \geq 1$, $k \geq 0$. Then, the following right fractional integral formula holds true:*

$$\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}{}_{\gamma,1}E_{\alpha,\beta}^{j,k}\left(\omega t^{-\sigma}\right)\right]\right)(x) = \frac{\omega^k x^{-\nu-\acute{\nu}+\eta+\lambda-\sigma k-1}}{\Gamma(\gamma)}$$

$$\times {}_5\Psi_4\left[\begin{array}{c}(1,1),(\gamma,1),(-\mu-\lambda+\sigma k+1,\sigma j),(\nu+\acute{\nu}-\eta-\lambda+\sigma k+1,\sigma j),\\(\beta+\alpha k,\alpha j),(-\lambda+\sigma k+1,\sigma j),(\nu-\mu-\lambda+\sigma k+1,\sigma j),\end{array}\right.$$

$$\left.\begin{array}{c}(\nu+\acute{\mu}-\eta-\lambda+\sigma k+1,\sigma j)\\(\nu+\acute{\nu}+\acute{\mu}-\eta-\lambda+\sigma k+1,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^j\right]. \tag{14.2.11}$$

For $\delta=0$ in (14.2.7), Theorem 14.2.2 yields the following corollary.

**Corollary 14.2.10.** *Let $\nu,\acute{\nu},\mu,\acute{\mu},\eta,\lambda,\alpha,\beta,\sigma,\omega\in\mathbb{C}$ such that $\Re(\lambda-\sigma k)<1+min\left\{\Re(-\mu),\Re(\nu+\acute{\nu}-\eta),\Re(\nu+\acute{\mu}-\eta)\right\}$, $\Re(\eta)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, and $j\geq 1$, $k\geq 0$. Then, the following right fractional integral formula holds true:*

$$\left(I_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E_{\alpha,\beta}^{j,k}\left(\omega t^{-\sigma}\right)\right]\right)(x)=\omega^k x^{-\nu-\acute{\nu}+\eta+\lambda-\sigma k-1}$$

$$\times {}_4\Psi_4\left[\begin{array}{c}(1,1),(-\mu-\lambda+\sigma k+1,\sigma j),(\nu+\acute{\nu}-\eta-\lambda+\sigma k+1,\sigma j),\\(\beta+\alpha k,\alpha j),(-\lambda+\sigma k+1,\sigma j),(\nu-\mu-\lambda+\sigma k+1,\sigma j),\end{array}\right.$$

$$\left.\begin{array}{c}(\nu+\acute{\mu}-\eta-\lambda+\sigma k+1,\sigma j)\\(\nu+\acute{\nu}+\acute{\mu}-\eta-\lambda+\sigma k+1,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^j\right]. \tag{14.2.12}$$

Further, if $j=1$ and $k=0$ in (14.2.7), then we have the well-known result involving the right-sided Marichev–Saigo–Maeda fractional integral operator (see [1]).

## 14.3 Marichev–Saigo–Maeda fractional derivatives with the function ${}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}(z)$

In this section, we study the Marichev–Saigo–Maeda fractional derivatives of the generalized arbitrary order Mittag-Leffler-type function.

**Theorem 14.3.1.** *Let $\nu,\acute{\nu},\mu,\acute{\mu},\eta,\lambda,\alpha,\beta,\gamma,\delta,\sigma,\omega\in\mathbb{C}$ with $\Re(\lambda+\sigma k)>max\{0,\Re(-\nu-\acute{\nu}-\acute{\mu}+\eta),\Re(\mu-\nu)\}$, $\Re(\eta)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, and $j\geq 1$, $k\geq 0$. Let $D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}$ be the left-sided operator of the Marichev–Saigo–Maeda fractional derivative. Then,*

$$\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}\left(\omega t^{\sigma}\right)\right]\right)(x)=\frac{\omega^k x^{\nu+\acute{\nu}-\eta+\lambda+\sigma k-1}}{\Gamma(\gamma)}$$

$$\times {}_5\Psi_4\left[\begin{array}{c}(1,1),(\gamma,\delta),(\lambda+\sigma k,\sigma j),(\nu-\mu+\lambda+\sigma k,\sigma j),\\(\beta+\alpha k,\alpha j),(-\mu+\lambda+\sigma k,\sigma j),(\nu+\acute{\nu}-\eta+\lambda+\sigma k,\sigma j),\end{array}\right.$$

$$\left.\begin{array}{c}(\nu+\acute{\nu}+\acute{\mu}-\eta+\lambda+\sigma k,\sigma j)\\(\nu+\acute{\mu}-\eta+\lambda+\sigma k,\sigma j)\end{array}\middle|(\omega x^{\sigma})^j\right]. \tag{14.3.1}$$

*Proof.* In view of (14.1.24), (14.1.7), and the left-hand side of (14.3.1), we obtain:

$$\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}(\omega t^{\sigma})\right]\right)(x)$$
$$=\omega^{k}\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n}\,\omega^{nj}}{\Gamma\left(\beta+\alpha(nj+k)\right)}\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda+\sigma(nj+k)-1}\right)(x).$$

Now, applying (14.1.21), we have:

$$\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}(\omega t^{\sigma})\right]\right)(x)$$
$$=\frac{\omega^{k}x^{\nu+\acute{\nu}-\eta+\lambda+\sigma k-1}}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\gamma+\delta n\right)\omega^{nj}}{\Gamma\left(\beta+\alpha(nj+k)\right)}$$
$$\times\frac{\Gamma\left(\lambda+\sigma k+\sigma nj\right)\Gamma\left(\nu-\mu+\lambda+\sigma k+\sigma nj\right)\Gamma\left(\nu+\acute{\nu}+\acute{\mu}-\eta+\lambda+\sigma k+\sigma nj\right)}{\Gamma\left(-\mu+\lambda+\sigma k+\sigma nj\right)\Gamma\left(\nu+\acute{\nu}-\eta+\lambda+\sigma k+\sigma nj\right)\Gamma\left(\nu+\acute{\mu}-\eta+\lambda+\sigma k+\sigma nj\right)}x^{\sigma nj}$$
$$=\frac{\omega^{k}x^{\nu+\acute{\nu}-\eta+\lambda+\sigma k-1}}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma\left(1+n\right)\Gamma\left(\gamma+\delta n\right)}{\Gamma\left(\beta+\alpha k+\alpha nj\right)}$$
$$\times\frac{\Gamma\left(\lambda+\sigma k+\sigma nj\right)\Gamma\left(\nu-\mu+\lambda+\sigma k+\sigma nj\right)\Gamma\left(\nu+\acute{\nu}+\acute{\mu}-\eta+\lambda+\sigma k+\sigma nj\right)}{\Gamma\left(-\mu+\lambda+\sigma k+\sigma nj\right)\Gamma\left(\nu+\acute{\nu}-\eta+\lambda+\sigma k+\sigma nj\right)\Gamma\left(\nu+\acute{\mu}-\eta+\lambda+\sigma k+\sigma nj\right)}$$
$$\times\frac{(\omega^{j}x^{\sigma j})^{n}}{ni}.$$

This, in accordance with (14.1.26), completes the proof. □

**Corollary 14.3.1.** *Let* $\nu,\mu,\rho,\lambda,\alpha,\beta,\gamma,\delta,\omega,\sigma\in\mathbb{C}$ *such that* $\Re(\lambda+\sigma k)>-min\{0,\Re(\nu+\mu+\rho)\}$, $\Re(\alpha)>0$, $\Re(\beta)>0$, *and* $j\geq 1$, $k\geq 0$. *Then, the following left fractional derivative formula holds true:*

$$\left(D_{0+}^{\nu,\mu,\rho}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}(\omega t^{\sigma})\right]\right)(x)=\frac{\omega^{k}x^{\mu+\lambda+\sigma k-1}}{\Gamma(\gamma)}$$
$$\times{}_{4}\Psi_{3}\left[\begin{array}{c}(1,1),(\gamma,\delta),(\lambda+\sigma k,\sigma j),(\nu+\mu+\rho+\lambda+\sigma k,\sigma j)\\(\beta+\alpha k,\alpha j),(\mu+\lambda+\sigma k,\sigma j),(\rho+\lambda+\sigma k,\sigma j)\end{array}\middle|(\omega x^{\sigma})^{j}\right].\tag{14.3.2}$$

**Corollary 14.3.2.** *Let* $\rho,\nu,\lambda,\alpha,\beta,\gamma,\delta,\omega,\sigma\in\mathbb{C}$ *such that* $\Re(\lambda+\sigma k)>-\Re(\rho+\nu)$, $\Re(\alpha)>0$, $\Re(\beta)>0$, *and* $j\geq 1$, $k\geq 0$. *Then, the following left fractional derivative formula holds true:*

$$\left(D_{\rho,\nu}^{+}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}(\omega t^{\sigma})\right]\right)(x)=\frac{\omega^{k}x^{\lambda+\sigma k-1}}{\Gamma(\gamma)}$$
$$\times{}_{3}\Psi_{2}\left[\begin{array}{c}(1,1),(\gamma,\delta),(\rho+\nu+\lambda+\sigma k,\sigma j)\\(\beta+\alpha k,\alpha j),(\rho+\lambda+\sigma k,\sigma j)\end{array}\middle|(\omega x^{\sigma})^{j}\right].\tag{14.3.3}$$

**Corollary 14.3.3.** *Let $\nu, \lambda, \alpha, \beta, \gamma, \delta, \omega, \sigma \in \mathbb{C}$ such that $\Re(\nu) > 0$, $\Re(\lambda) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\sigma) > 0$, and $j \geq 1$, $k \geq 0$. Then, the following left fractional derivative formula holds true:*

$$\left(D_{0+}^{\nu}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}(\omega t^{\sigma})\right]\right)(x) = \frac{\omega^k x^{-\nu+\lambda+\sigma k-1}}{\Gamma(\gamma)} \times {}_3\Psi_2\left[\begin{array}{c}(1,1),(\gamma,\delta),(\lambda+\sigma k,\sigma j)\\(\beta+\alpha k,\alpha j),(-\nu+\lambda+\sigma k,\sigma j)\end{array}\middle|(\omega x^{\sigma})^j\right]. \tag{14.3.4}$$

For $\delta = 1$ in (14.3.1), Theorem 14.3.1 yields the following corollary.

**Corollary 14.3.4.** *Let $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta, \lambda, \alpha, \beta, \gamma, \sigma, \omega \in \mathbb{C}$ with $\Re(\lambda+\sigma k) > max\{0, \Re(-\nu-\acute{\nu}-\acute{\mu}+\eta), \Re(\mu-\nu)\}$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and $j \geq 1$, $k \geq 0$. Then, the following left fractional derivative formula holds true:*

$$\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}{}_{\gamma,1}E_{\alpha,\beta}^{j,k}\left(\omega t^{\sigma}\right)\right]\right)(x) = \frac{\omega^k x^{\nu+\acute{\nu}-\eta+\lambda+\sigma k-1}}{\Gamma(\gamma)} \times {}_5\Psi_4\left[\begin{array}{c}(1,1),(\gamma,1),(\lambda+\sigma k,\sigma j),(\nu-\mu+\lambda+\sigma k,\sigma j),\\(\beta+\alpha k,\alpha j),(-\mu+\lambda+\sigma k,\sigma j),(\nu+\acute{\nu}-\eta+\lambda+\sigma k,\sigma j),\end{array}\right.$$
$$\left.\begin{array}{c}(\nu+\acute{\nu}+\acute{\mu}-\eta+\lambda+\sigma k,\sigma j)\\(\nu+\acute{\mu}-\eta+\lambda+\sigma k,\sigma j)\end{array}\middle|(\omega x^{\sigma})^j\right]. \tag{14.3.5}$$

For $\delta = 0$ in (14.3.1), Theorem 14.3.1 yields the following corollary.

**Corollary 14.3.5.** *Let $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta, \lambda, \alpha, \beta, \sigma, \omega \in \mathbb{C}$ with $\Re(\lambda+\sigma k) > max\{0, \Re(-\nu-\acute{\nu}-\acute{\mu}+\eta), \Re(\mu-\nu)\}$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and $j \geq 1$, $k \geq 0$. Then, the following left fractional derivative formula holds true:*

$$\left(D_{0+}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}E_{\alpha,\beta}^{j,k}\left(\omega t^{\sigma}\right)\right]\right)(x) = \omega^k x^{\nu+\acute{\nu}-\eta+\lambda+\sigma k-1} \times {}_4\Psi_4\left[\begin{array}{c}(1,1),(\lambda+\sigma k,\sigma j),(\nu-\mu+\lambda+\sigma k,\sigma j),\\(\beta+\alpha k,\alpha j),(-\mu+\lambda+\sigma k,\sigma j),(\nu+\acute{\nu}-\eta+\lambda+\sigma k,\sigma j),\end{array}\right.$$
$$\left.\begin{array}{c}(\nu+\acute{\nu}+\acute{\mu}-\eta+\lambda+\sigma k,\sigma j)\\(\nu+\acute{\mu}-\eta+\lambda+\sigma k,\sigma j)\end{array}\middle|(\omega x^{\sigma})^j\right]. \tag{14.3.6}$$

Further, if $j = 1$ and $k = 0$ in (14.3.1), then we have the well-known result involving the left-sided Marichev–Saigo–Maeda fractional derivative operator (see [11]).

**Theorem 14.3.2.** *Let $\nu, \acute{\nu}, \mu, \acute{\mu}, \eta, \lambda, \alpha, \beta, \gamma, \delta, \omega, \sigma \in \mathbb{C}$ with $\Re(\lambda-\sigma k) < 1 + min\{\Re(\acute{\mu}), \Re(-\nu-\acute{\nu}+\eta), \Re(-\acute{\nu}-\mu+\eta)\}$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and $j \geq 1$, $k \geq 0$. Let $D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}$ be the right-sided operator of the Marichev–Saigo–Maeda fractional derivative. Then,*

$$\left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}\left(\omega t^{-\sigma}\right)\right]\right)(x) = \frac{\omega^k x^{\nu+\acute{\nu}-\eta+\lambda-\sigma k-1}}{\Gamma(\gamma)}$$

$$\times\, {}_5\Psi_4\left[\begin{matrix}(1,1),(\gamma,\delta),(\acute{\mu}-\lambda+\sigma k+1,\sigma j),(-\nu-\acute{\nu}+\eta-\lambda+\sigma k+1,\sigma j),\\ (\beta+\alpha k,\alpha j),(-\lambda+\sigma k+1,\sigma j),(-\acute{\nu}+\acute{\mu}-\lambda+\sigma k+1,\sigma j),\end{matrix}\right.$$
$$\left.\begin{matrix}(-\acute{\nu}-\mu+\eta-\lambda+\sigma k+1,\sigma j)\\ (-\nu-\acute{\nu}-\mu+\eta-\lambda+\sigma k+1,\sigma j)\end{matrix}\,\middle|\,(-\omega x^{\sigma})^{j}\right]. \tag{14.3.7}$$

*Proof.* In view of (14.1.24), (14.1.8), and the left-hand side of (14.3.7), we obtain:

$$\left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}(\omega t^{-\sigma})\right]\right)(x)$$
$$=\omega^{k}\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n}\,\omega^{nj}}{\Gamma\left(\beta+\alpha(nj+k)\right)}\left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}t^{\lambda-\sigma(nj+k)-1}\right)(x).$$

Applying (14.1.22), we find:

$$\left(D_{-}^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}(\omega t^{-\sigma})\right]\right)(x)$$
$$=\frac{\omega^{k}x^{\nu+\acute{\nu}-\eta+\lambda-\sigma k-1}}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+\delta n)\,\omega^{nj}x^{-\sigma nj}}{\Gamma\left(\beta+\alpha(nj+k)\right)}$$
$$\times\frac{\Gamma\left(\acute{\mu}-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(-\nu-\acute{\nu}+\eta-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(-\acute{\nu}-\mu+\eta-\lambda+\sigma k+\sigma nj+1\right)}{\Gamma\left(-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(-\acute{\nu}+\acute{\mu}-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(-\nu-\acute{\nu}-\mu+\eta-\lambda+\sigma k+\sigma nj+1\right)}$$
$$=\frac{\omega^{k}x^{\nu+\acute{\nu}-\eta+\lambda-\sigma k-1}}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma\left(1+n\right)\Gamma(\gamma+\delta n)}{\Gamma\left(\beta+\alpha k+\alpha nj\right)}$$
$$\times\frac{\Gamma\left(\acute{\mu}-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(-\nu-\acute{\nu}+\eta-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(-\acute{\nu}-\mu+\eta-\lambda+\sigma k+\sigma nj+1\right)}{\Gamma\left(-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(-\acute{\nu}+\acute{\mu}-\lambda+\sigma k+\sigma nj+1\right)\Gamma\left(-\nu-\acute{\nu}-\mu+\eta-\lambda+\sigma k+\sigma nj+1\right)}$$
$$\times\frac{(\omega^{j}x^{-\sigma j})^{n}}{n!}.$$

This, in accordance with (14.1.26), completes the proof. □

**Corollary 14.3.6.** *Let $\nu,\mu,\rho,\lambda,\alpha,\beta,\gamma,\delta,\omega,\sigma\in\mathbb{C}$ such that $\Re(\lambda-\sigma k)<1+min\{\Re(-\mu)-m,\Re(\nu+\rho)\}$, $m=[\Re(\nu)]+1$, $\Re(\alpha)>0$, $\Re(\beta)>0$, and $j\geq 1$, $k\geq 0$. Then, the following right fractional derivative formula holds true:*

$$\left(D_{-}^{\nu,\mu,\rho}\left[t^{\lambda-1}{}_{\gamma,\delta}E_{\alpha,\beta}^{j,k}(\omega t^{-\sigma})\right]\right)(x)=\frac{\omega^{k}x^{\mu+\lambda-\sigma k-1}}{\Gamma(\gamma)}$$
$$\times\,{}_4\Psi_3\left[\begin{matrix}(1,1),(\gamma,\delta),(-\mu-\lambda+\sigma k+1,\sigma j),(\nu+\rho-\lambda+\sigma k+1,\sigma j)\\ (\beta+\alpha k,\alpha j),(-\lambda+\sigma k+1,\sigma j),(-\mu+\rho-\lambda+\sigma k+1,\sigma j)\end{matrix}\,\middle|\,(\omega x^{-\sigma})^{j}\right]. \tag{14.3.8}$$

**Corollary 14.3.7.** *Let $\rho,\nu,\lambda,\alpha,\beta,\gamma,\delta,\omega,\sigma\in\mathbb{C}$ such that $\Re(\nu)>0$, $\Re(\lambda-\sigma k)<\Re(\rho+\nu)-[\Re(\nu)]$, $\Re(\alpha)>0$, $\Re(\beta)>0$, and $j\geq 1$, $k\geq 0$. Then, the following right* *fractional derivative formula holds true:*

$$\left(D^{-}_{\rho,\nu}\left[t^{\lambda-1}{}_{\gamma,\delta}E^{j,k}_{\alpha,\beta}(\omega t^{-\sigma})\right]\right)(x)=\frac{\omega^k x^{\lambda-\sigma k-1}}{\Gamma(\gamma)}$$
$$\times\, {}_3\Psi_2\left[\begin{matrix}(1,1),(\gamma,\delta),(\rho+\nu-\lambda+\sigma k+1,\sigma j)\\(\beta+\alpha k,\alpha j),(\rho-\lambda+\sigma k+1,\sigma j)\end{matrix}\,\middle|\,(\omega x^{-\sigma})^j\right]. \tag{14.3.9}$$

**Corollary 14.3.8.** *Let $\nu,\lambda,\alpha,\beta,\gamma,\delta,\omega,\sigma\in\mathbb{C}$ such that $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\nu)>0$, $\Re(\lambda-\sigma k)<\Re(\nu)-[\Re(\nu)]$, and $j\geq 1$, $k\geq 0$. Then, the following right fractional derivative formula holds true:*

$$\left(D^{\nu}_{-}\left[t^{\lambda-1}{}_{\gamma,\delta}E^{j,k}_{\alpha,\beta}(\omega t^{-\sigma})\right]\right)(x)=\frac{\omega^k x^{-\nu+\lambda-\sigma k-1}}{\Gamma(\gamma)}$$
$$\times\, {}_3\Psi_2\left[\begin{matrix}(1,1),(\gamma,\delta),(\nu-\lambda+\sigma k+1,\sigma j)\\(\beta+\alpha k,\alpha j),(-\lambda+\sigma k+1,\sigma j)\end{matrix}\,\middle|\,(\omega x^{-\sigma})^j\right]. \tag{14.3.10}$$

For $\delta=1$ in (14.3.7), Theorem 14.3.2 yields the following corollary.

**Corollary 14.3.9.** *Let $\nu,\acute{\nu},\mu,\acute{\mu},\eta,\lambda,\alpha,\beta,\gamma,\omega,\sigma\in\mathbb{C}$ with $\Re(\lambda-\sigma k)<1+min\left\{\Re(\acute{\mu}),\Re(-\nu-\acute{\nu}+\eta),\Re(-\acute{\nu}-\mu+\eta)\right\}$, $\Re(\eta)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, and $j\geq 1$, $k\geq 0$. Then, the following right fractional derivative formula holds true:*

$$\left(D^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}_{-}\left[t^{\lambda-1}{}_{\gamma,1}E^{j,k}_{\alpha,\beta}\left(\omega t^{-\sigma}\right)\right]\right)(x)=\frac{\omega^k x^{\nu+\acute{\nu}-\eta+\lambda-\sigma k-1}}{\Gamma(\gamma)}$$
$$\times\, {}_5\Psi_4\left[\begin{matrix}(1,1),(\gamma,1),(\acute{\mu}-\lambda+\sigma k+1,\sigma j),(-\nu-\acute{\nu}+\eta-\lambda+\sigma k+1,\sigma j),\\(\beta+\alpha k,\alpha j),(-\lambda+\sigma k+1,\sigma j),(-\acute{\nu}+\acute{\mu}-\lambda+\sigma k+1,\sigma j),\end{matrix}\right.$$
$$\left.\begin{matrix}(-\acute{\nu}-\mu+\eta-\lambda+\sigma k+1,\sigma j)\\(-\nu-\acute{\nu}-\mu+\eta-\lambda+\sigma k+1,\sigma j)\end{matrix}\,\middle|\,(-\omega x^{\sigma})^j\right]. \tag{14.3.11}$$

For $\delta=0$ in (14.3.7), Theorem 14.3.2 yields the following corollary.

**Corollary 14.3.10.** *Let $\nu,\acute{\nu},\mu,\acute{\mu},\eta,\lambda,\alpha,\beta,\omega,\sigma\in\mathbb{C}$ with $\Re(\lambda-\sigma k)<1+min\left\{\Re(\acute{\mu}),\Re(-\nu-\acute{\nu}+\eta),\Re(-\acute{\nu}-\mu+\eta)\right\}$, $\Re(\eta)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, and $j\geq 1$, $k\geq 0$. Then, the following right fractional derivative formula holds true:*

$$\left(D^{\nu,\acute{\nu},\mu,\acute{\mu},\eta}_{-}\left[t^{\lambda-1}E^{j,k}_{\alpha,\beta}\left(\omega t^{-\sigma}\right)\right]\right)(x)=\omega^k x^{\nu+\acute{\nu}-\eta+\lambda-\sigma k-1}$$
$$\times\, {}_4\Psi_4\left[\begin{matrix}(1,1),(\acute{\mu}-\lambda+\sigma k+1,\sigma j),(-\nu-\acute{\nu}+\eta-\lambda+\sigma k+1,\sigma j),\\(\beta+\alpha k,\alpha j),(-\lambda+\sigma k+1,\sigma j),(-\acute{\nu}+\acute{\mu}-\lambda+\sigma k+1,\sigma j),\end{matrix}\right.$$
$$\left.\begin{matrix}(-\acute{\nu}-\mu+\eta-\lambda+\sigma k+1,\sigma j)\\(-\nu-\acute{\nu}-\mu+\eta-\lambda+\sigma k+1,\sigma j)\end{matrix}\,\middle|\,(-\omega x^{\sigma})^j\right]. \tag{14.3.12}$$

Further, if $j=1$ and $k=0$ in (14.3.7), then we have the well-known result involving the right-sided Marichev–Saigo–Maeda fractional derivative operator (see [11]).

## CRediT authorship contribution statement

All authors contributed equally to this article. All authors have read and agreed to the published version of the manuscript.

## Declaration of competing interest

The authors declare no conflict of interest.

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## Data availability

Not applicable.

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CHAPTER

# 15 Generalized fractional order kinetic equations involving multi-index Mittag-Leffler functions of several variables

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## 15.1 Introduction

Many scholars have investigated the well-known Mittag-Leffler function (MLf) and its extensions (see [1–9]). These functions became more interesting because of their applications in the study of various fields like mathematical physics, as well as fractional order differential and integral equations (see [10–13]). The Swedish mathematician Gosta Mittag-Leffler studied the elementary Mittag-Leffler function $E(x)$ and defined [3] it as:

$$E_l(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(nl+1)}, \tag{15.1}$$

where $x \in \mathbb{C}$, $l > 0$.

Wiman [9] gave its generalization as shown below:

$$E_{l,m}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(nl+m)}, \tag{15.2}$$

where $x, l, m \in \mathbb{C}$; $\Re(l) > 0$, $\Re(m) > 0$.

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In 1971, the extended form of these above definitions of the MLf was introduced by Prabhakar in the following form [6]:

$$E_{l,m}^{\rho}(x) = \sum_{n=0}^{\infty} \frac{(\rho)_n}{\Gamma(nl+m)} \frac{x^n}{n!}, \tag{15.3}$$

where $x, l, m, \rho \in \mathbb{C}$; $\Re(l) > 0, \Re(m) > 0$, and $(\lambda)_n$ represents the Pochhammer symbol defined in [17] as:

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n=0 \\ \lambda(\lambda+1)...(\lambda+n-1), & \text{if } n=1,2,3... \end{cases} . \tag{15.4}$$

Shukla and Prajapati gave a generalization of Prabhakar's MLf in 2007 [8] as follows:

$$E_{l,m}^{\rho,\sigma}(x) = \sum_{n=0}^{\infty} \frac{(\rho)_{n\sigma}}{\Gamma(nl+m)} \frac{x^n}{n!}, \tag{15.5}$$

where $x, l, m, \rho \in \mathbb{C}$; $\Re(l) > 0, \Re(m) > 0, \Re(\rho) > 0$, and $\sigma \in (0,1) \cup \mathbb{N}$.

Further, Saxena and Nishimoto [14] also generalized the Mittag-Leffler functions as shown below:

$$E_{p,q}[(\gamma_1, \alpha_1), ..., (\gamma_j, \alpha_j); x] = \sum_{n=0}^{\infty} \frac{(p)_{qn}}{\Pi_{m=1}^{j} \Gamma(\alpha_m + \gamma_m n)} \frac{x^n}{n!}, \tag{15.6}$$

where, $\gamma_m, \alpha_m, p, q \in \mathbb{C}$; $\Re(\gamma_m) > 0$, $\Re(\alpha_m) > 0$; $\Re(q) > 0$; $\{m = 1, 2, .., j\}$, and $\Re\{\sum_{i=1}^{j} \gamma_i\} > max\{0, \Re(q) - 1\}$.

An extended form of the multivariable Mittag-Leffler function was studied by Gautam [15] and Saxena et al. [16] and defined in the following manner:

$$E_{(\gamma_1 \cdots \gamma_r);\alpha}^{(p_1 \cdots p_r)}(x_1, x_2, ..., x_r) = \sum_{n_1, n_2, ..., n_r = 0}^{\infty} \frac{\Pi_{i=1}^{r} (p_i)_{n_i}}{\Gamma(\alpha + \sum_{i=1}^{r} \gamma_i n_i)} \frac{x_1^{n_1} \cdots x_r^{n_r}}{n_1! \cdots n_r!}, \tag{15.7}$$

$$E_{(\gamma_1 \cdots \gamma_r);\alpha}^{(p_1 \cdots p_r);(q_1 \cdots q_r)}(x_1, x_2, ..., x_r) = \sum_{n_1, n_2, ..., n_r = 0}^{\infty} \frac{\Pi_{i=1}^{r} (p_i)_{q_i n_i}}{\Gamma(\alpha + \sum_{i=1}^{r} \gamma_i n_i)} \frac{x_1^{n_1} \cdots x_r^{n_r}}{n_1! \cdots n_r!}, \tag{15.8}$$

where, $\gamma_m, \alpha, p_m, q_m \in \mathbb{C}$; $\Re(\gamma_m) > 0$, $\Re(q_m) > 0$; $\alpha \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}$, and $\{m = 1, 2, .., r\}$.

Recently, Jaimini and Sharma et al. [12] have defined a unified definition of the Mittag-Leffler function of multivariables as:

$$E^{(p_r);(q_r)}_{(\gamma_1^{(r)}),\ldots,(\gamma_j^{(r)});\alpha_1,\ldots,\alpha_j}(x_1, x_2, \ldots, x_r) = \sum_{n_1,n_2,\ldots,n_r=0}^{\infty} \frac{\Pi_{i=1}^{r}(p_i)_{q_i n_i}}{\Pi_{m=1}^{j}\Gamma(\alpha_m + \sum_{i=1}^{r}\gamma_m^{(i)} n_i)} \times \frac{x_1^{n_1}\cdots x_r^{n_r}}{n_1!\cdots n_r!}, \tag{15.9}$$

where, $\gamma_m^{(i)}, \alpha_m, p_i, q_i \in \mathbb{C}$; $\Re(\gamma_m^{(i)}) > 0$, $\Re(q_m) > 0$; $\alpha_m \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$; $(\gamma_m^{(i)}) = \gamma'_m, \gamma''_m, \ldots, \gamma_m^{(i)}$; $\{m = 1, 2, .., j\}$ and $\{i = 1, 2, \ldots, r\}$.

## 15.2 Fractional differential and kinetic equations

Fractional differential equations became more important due to their wide applications in various fields such as physics, chemical engineering, environmental science, signal processing, biology, chemistry, and medicine.

Fractional kinetic equations are specifically used in chemical engineering to simulate the behavior of ions in electrochemical systems, like fuel cells, batteries, and supercapacitors, where complicated electrode structures and electrolyte interactions frequently cause anomalous diffusion in charge and mass transport. By incorporating memory effects, nonlocal behaviors, and anomalous diffusion, fractional kinetic equations provide powerful tools for understanding and modeling complex systems in chemistry and chemical engineering.

Initially, Haubold and Mathai [18] represented the rate of change of a reaction relation with the rate of destruction, and the rate of production in the following fractional differential equation:

$$\frac{dN}{dt} = -d(N_t) + p(N_t), \tag{15.10}$$

where the time dependent $N = N(t)$ is the rate of reaction, $d = d(N)$ signifies the rate of destruction, $p = p(N)$ is the production rate, and $N_t(t^*) = N(t - t^*)$, for positive parameter $t^*$.

The fractional generalization of the standard kinetic equation is provided by Haubold and Mathai [18] as:

$$N(t) - N_0 = -c^{\mu}\,{}_0D_t^{-1} N(t). \tag{15.11}$$

Here, ${}_0D_t^{-1}$ represents a special case of the Riemann–Liouville integral operator, defined as:

$${}_0D_u^{-\mu} f(u) = \frac{1}{\Gamma(\mu)} \int_0^u (u - s)^{\mu-1} f(s)ds, u > 0, \Re(\mu) > 0. \tag{15.12}$$

In the next section, we will derive the solution of fractional differential equations by Laplace transforms.

## 15.3 Solution of generalized fractional order kinetic and other differential equations

**Lemma 15.3.1.** *Let* $min\{\Re(\nu), \Re(\mu_i), \Re(s)\} > 0$ *with* $\omega_i > 0$ *and* $i = \{1, 2, ..., r\}$, *then for multi-index Mittag-Leffler function* $[E^{(p_r);(q_r)}_{(\gamma_1^{(r)}),...,(\gamma_j^{(r)});\alpha_1,...,\alpha_j}(x_1\omega_1^{\mu_1}t^{\mu_1}, x_2\omega_2^{\mu_2}t^{\mu_2}, ..., x_r\omega_r^{\mu_r}t^{\mu_r}); s]$ *the following Laplace transform in [12] holds true:*

$$\mathcal{L}\{t^{\nu-1}E^{(p_r);(q_r)}_{(\gamma_1^{(r)}),...,(\gamma_j^{(r)});\alpha_1,...,\alpha_j}(x_1\omega_1^{\mu_1}t^{\mu_1}, x_2\omega_2^{\mu_2}t^{\mu_2}, ..., x_r\omega_r^{\mu_r}t^{\mu_r}); s\}$$

$$= \sum_{n_1,n_2,...,n_r=0}^{\infty} \frac{\Pi_{i=1}^{r}(p_i)_{q_i n_i}}{\Pi_{m=1}^{j}\Gamma(\alpha_m + \sum_{i=1}^{r}\gamma_m^{(i)}n_i)} \frac{(x_1\omega_1^{\mu_1})^{n_1}\cdots(x_r\omega_r^{\mu_r})^{n_r}}{n_1!\cdots n_r!} \frac{\Gamma(\nu + \sum_{i=1}^{r}\mu_i n_i)}{s^{(\nu+\sum_{i=1}^{r}\mu_i n_i)}}$$

$$= s^{-\nu}\Pi_{m=1}^{j}\left(\frac{1}{\Gamma(\alpha_m)}\right)\Gamma(\nu)$$

$$\times F^{1:1;...;1}_{j:0;...;0}\left[\begin{array}{c} (\nu:\mu_1,...,\mu_r):(p_1,q_1);...;(p_r,q_r); \\ (\alpha_m:\gamma_m',...,\gamma_m^{(r)})_{1,j}:-;...;-; \end{array} x_1\omega_1^{\mu_1}s^{-\mu_1}, ..., x_r\omega_r^{\mu_r}s^{-\mu_r}\right]. \tag{15.13}$$

**Remark 1.** For $\nu = \alpha_1$, $\mu_i = \gamma_1^{(i)}$, the lemma becomes as shown below.

**Lemma 15.3.2.** *Let* $min\{\Re(\alpha_1), \Re(\gamma_1^{(i)}), \Re(s)\} > 0$; $\omega_i > 0$, *and* $i = \{1, 2, ..., r\}$, *then the following Laplace transform is given in [12, p. 3, Eq. 10] as:*

$$\mathcal{L}\{t^{\alpha_1-1}E^{(p_r);(q_r)}_{(\gamma_1^{(r)}),...,(\gamma_j^{(r)});\alpha_1,...,\alpha_j}(x_1\omega_1^{\gamma_1'}t^{\gamma_1'}, x_2\omega_2^{\gamma_1''}t^{\gamma_1''}, ..., x_r\omega_r^{\gamma_1^{(r)}}t^{\gamma_1^{(r)}}); s\}$$

$$= \sum_{n_1,n_2,...,n_r=0}^{\infty} \frac{\Pi_{i=1}^{r}(p_i)_{q_i n_i}}{\Pi_{m=2}^{j}\Gamma(\alpha_m + \sum_{i=1}^{r}\gamma_m^{(i)}n_i)} \frac{(x_1\omega_1^{\gamma_1'})^{n_1}\cdots(x_r\omega_r^{\gamma_1^{(r)}})^{n_r}}{n_1!\cdots n_r!} \frac{1}{s^{(\alpha_1+\sum_{i=1}^{r}\gamma_1^{(i)}n_i)}}$$

$$= s^{-\alpha_1}\Pi_{m=2}^{j}\left(\frac{1}{\Gamma(\alpha_m)}\right)$$

$$\times F^{0:1;\ldots;1}_{j-1:0;\ldots;0}\left[\begin{array}{c} -:(p_1,q_1);\ldots;(p_r,q_r); \\ (\alpha_m:\gamma'_m,\ldots,\gamma^{(r)}_m)_{2,j}:-;\ldots;-; \end{array} x_1\omega_1^{\gamma'_1}s^{-\gamma'_1},\ldots,x_r\omega_r^{\gamma^{(r)}_1}s^{-\gamma^{(r)}_1}\right]. \tag{15.14}$$

**Theorem 15.3.3.** *For* $\nu, a, \rho, \mu_i, \omega_i \in \mathbb{R}^+$; $\gamma^{(i)}_m, \alpha_m, p_i, q_i \in \mathbb{C}$; $\Re(\gamma^{(i)}_m) > 0, \Re(q_m) > 0$; $\alpha_m \notin \mathbb{Z}^-_0 = \{0, -1, -2, \ldots\}$; $(\gamma^{(i)}_m) = \gamma'_m, \gamma''_m, \ldots, \gamma^{(i)}_m$; $\{m = 1, 2, .., j\}$ *and* $\{i = 1, 2, \ldots, r\}$, *the solution to the generalized fractional kinetic equation:*

$$N(t) - N_0 t^{\nu-1} E^{(p_r);(q_r)}_{(\gamma^{(r)}_1),\ldots,(\gamma^{(r)}_j);\alpha_1,\ldots,\alpha_j}(x_1\omega_1^{\mu_1}t^{\mu_1}, x_2\omega_2^{\mu_2}t^{\mu_2}, \ldots, x_r\omega_r^{\mu_r}t^{\mu_r}) = -a^{\rho}\,{}_0D_t^{-\rho}N(t) \tag{15.15}$$

*can be determined as follows:*

$$N(t) = \frac{N_0 t^{\nu-1}\Gamma(\nu)}{\Pi_{m=1}^{j}\Gamma\alpha_m}\sum_{k=0}^{\infty}\frac{(-a^{\rho}t^{\rho})^k}{\Gamma(\nu+\rho k)} \times F^{1:1;\ldots;1}_{j+1:0;\ldots;0}\left[\begin{array}{c} (\nu:\mu_1,\ldots,\mu_r):(p_1,q_1);\ldots;(p_r,q_r); \\ (\alpha_m:\gamma'_m,\ldots,\gamma^{(r)}_m)_{1,j},(\nu+\rho k:\mu_1,\ldots,\mu_r):-;\ldots;-; \end{array} x_1\omega_1^{\mu_1}t^{\mu_1},\ldots,x_r\omega_r^{\mu_r}t^{\mu_r}\right], \tag{15.16}$$

$$N(t) = N_0 t^{\nu-1}\sum_{n_1,n_2,\ldots,n_r=0}^{\infty}\frac{\Pi_{i=1}^{r}(p_i)_{q_i n_i}}{\Pi_{m=1}^{j}\Gamma(\alpha_m+\sum_{i=1}^{r}\gamma^{(i)}_m n_i)} \times \frac{(x_1\omega_1^{\mu_1}t^{\mu_1})^{n_1}\cdots(x_r\omega_r^{\mu_r}t^{\mu_r})^{n_r}}{n_1!\cdots n_r!} \times \Gamma(\nu+\sum_{i=1}^{r}\mu_i n_i)E_{\rho,\nu+\sum_{i=1}^{r}\mu_i n_i}(-a^{\rho}t^{\rho}), \tag{15.17}$$

*where,* $(\alpha_m;\gamma'_m,\ldots,\gamma^{(r)}_m)_{1,j} = (\alpha_1;\gamma'_1,\ldots,\gamma^{(r)}_1),\ldots,(\alpha_j;\gamma'_j,\ldots,\gamma^{(r)}_j)$ *and* $F[x_1\omega_1^{\mu_1}t^{\mu_1}, \ldots, x_r\omega_r^{\mu_r}t^{\mu_r}]$ *represents the Lauricella function [21, p. 37, Eqs. (21)–(23)].*

*Proof.* Erdely et al. [20] derived the Riemann–Liouville fractional integral operator's Laplace transform as:

$$\mathcal{L}\{{}_0D_t^{-\rho}N(t);s\} = s^{-\rho}N(s), \tag{15.18}$$

where, $\mathcal{L}\{N(t);s\} = N(s)$.

Applying Laplace transform [19] in Eq. (15.15), we have:

$$N(s) - N_0 \sum_{n_1,n_2,\ldots,n_r=0}^{\infty} \frac{\Pi_{i=1}^{r}(p_i)_{q_i n_i}}{\Pi_{m=1}^{j}\Gamma(\alpha_m + \sum\limits_{i=1}^{r}\gamma_m^{(i)} n_i)} \frac{\Gamma(\nu + \sum\limits_{i=1}^{r}\mu_i n_i)}{s^{(\nu+\sum\limits_{i=1}^{r}\mu_i n_i)}} \times \frac{(x_1\omega_1^{\mu_1})^{n_1}\cdots(x_r\omega_r^{\mu_r})^{n_r}}{n_1!\cdots n_r!} = -a^{\rho}s^{-\rho}N(s), \tag{15.19}$$

$$N(s) = \frac{N_0}{(1+(\frac{a}{s})^{\rho})} \sum_{n_1,n_2,\ldots,n_r=0}^{\infty} \frac{\Pi_{i=1}^{r}(p_i)_{q_i n_i}}{\Pi_{m=1}^{j}\Gamma(\alpha_m + \sum\limits_{i=1}^{r}\gamma_m^{(i)} n_i)} \times \frac{(x_1\omega_1^{\mu_1})^{n_1}\cdots(x_r\omega_r^{\mu_r})^{n_r}}{n_1!\cdots n_r!} \frac{\Gamma(\nu + \sum\limits_{i=1}^{r}\mu_i n_i)}{s^{(\nu+\sum\limits_{i=1}^{r}\mu_i n_i)}}. \tag{15.20}$$

Using the geometric series:

$$\left(1+\left(\frac{a}{s}\right)^{\rho}\right)^{-1} = \sum_{k=0}^{\infty}\left(-\frac{a}{s}\right)^{\rho k}, \tag{15.21}$$

$$N(s) = N_0 \sum_{k=0}^{\infty}(-a^{\rho})^k \sum_{n_1,n_2,\ldots,n_r=0}^{\infty} \frac{\Pi_{i=1}^{r}(p_i)_{q_i n_i}}{\Pi_{m=1}^{j}\Gamma(\alpha_m + \sum\limits_{i=1}^{r}\gamma_m^{(i)} n_i)} \times \frac{(x_1\omega_1^{\mu_1})^{n_1}\cdots(x_r\omega_r^{\mu_r})^{n_r}}{n_1!\cdots n_r!} \frac{\Gamma(\nu + \sum\limits_{i=1}^{r}\mu_i n_i)}{s^{(\nu+\rho k+\sum\limits_{i=1}^{r}\mu_i n_i)}}. \tag{15.22}$$

Now, by taking the inverse Laplace transform in Eq. (15.22) with the following identity:

$$\mathcal{L}\{t^{\mu}; s\} = \frac{\Gamma(\mu+1)}{s^{\mu+1}} \tag{15.23}$$

$$\Leftrightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^{\mu+1}}\right\} = \frac{t^{\mu}}{\Gamma(\mu+1)}, \tag{15.24}$$

$$N(t) = N_0 t^{\nu-1} \sum_{k=0}^{\infty}(-a^{\rho}t^{\rho})^k \sum_{n_1,n_2,\ldots,n_r=0}^{\infty} \frac{\Pi_{i=1}^{r}(p_i)_{q_i n_i}}{\Pi_{m=1}^{j}\Gamma(\alpha_m + \sum\limits_{i=1}^{r}\gamma_m^{(i)} n_i)}$$

$$\times \frac{(x_1\omega_1^{\mu_1})^{n_1}\cdots(x_r\omega_r^{\mu_r})^{n_r}}{n_1!\cdots n_r!}\,\frac{\Gamma(\nu+\sum_{i=1}^{r}\mu_i n_i)t^{(\sum_{i=1}^{r}\mu_i n_i)}}{\Gamma(\nu+\rho k+\sum_{i=1}^{r}\mu_i n_i)}. \quad (15.25)$$

Using Eq. (15.4) and the definition of the generalized Lauricella function, we can arrive at our desired result (15.16).

Again, use of Eq. (15.2) in Eq. (15.25), gives the result (15.17). □

**Theorem 15.3.4.** *For $\nu, a, \rho, \mu_i, \omega_i \in \mathbb{R}^+$; $n \in \mathbb{N}$; $\gamma_m^{(i)}, \alpha_m, p_i, q_i \in \mathbb{C}$; $\Re(\gamma_m^{(i)}) > 0$, $\Re(q_m) > 0$; $\alpha_m \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}$; $(\gamma_m^{(i)}) = \gamma_m', \gamma_m'', ..., \gamma_m^{(i)}$; $\{m = 1, 2, .., j\}$, and $\{i = 1, 2, ..., r\}$, the solution of the generalized fractional differential equation:*

$$N(t) - N_0 t^{\nu-1} E^{(p_r);(q_r)}_{(\gamma_1^{(r)}),...,(\gamma_j^{(r)});\alpha_1,...,\alpha_j}(x_1\omega_1^{\mu_1}t^{\mu_1}, x_2\omega_2^{\mu_2}t^{\mu_2}, ..., x_r\omega_r^{\mu_r}t^{\mu_r})$$
$$= -\left\{\sum_{k=1}^{n}\binom{n}{k}a^{\rho k}{}_0D_t^{-\rho k}\right\}N(t) \quad (15.26)$$

*i.e.,*

$$(1 + a_0^{\rho} D_t^{-\rho})^n N(t)$$
$$= N_0 t^{\nu-1} E^{(p_r);(q_r)}_{(\gamma_1^{(r)}),...,(\gamma_j^{(r)});\alpha_1,...,\alpha_j}(x_1\omega_1^{\mu_1}t^{\mu_1}, x_2\omega_2^{\mu_2}t^{\mu_2}, ..., x_r\omega_r^{\mu_r}t^{\mu_r}) \quad (15.27)$$

*can be given as follows:*

$$N(t) = \frac{N_0 t^{\nu-1}\Gamma(\nu)}{\Pi_{m=1}^{j}\Gamma\alpha_m}\sum_{k=0}^{\infty}\frac{(n)_k}{k!}\frac{(-a^{\rho}t^{\rho})^k}{\Gamma(\nu+\rho k)}$$
$$\times F^{1:1;...;1}_{j+1:0;...;0}\left[\begin{matrix}(\nu:\mu_1,...,\mu_r):(p_1,q_1);...;(p_r,q_r);\\ (\alpha_m:\gamma_m',...,\gamma_m^{(r)})_{1,j},(\nu+\rho k:\mu_1,...,\mu_r):-;...;-;\end{matrix}\; x_1\omega_1^{\mu_1}t^{\mu_1},...,x_r\omega_r^{\mu_r}t^{\mu_r}\right], \quad (15.28)$$

$$N(t) = N_0 t^{\nu-1}\sum_{n_1,n_2,...,n_r=0}^{\infty}\frac{\Pi_{i=1}^{r}(p_i)_{q_i n_i}}{\Pi_{m=1}^{j}\Gamma(\alpha_m+\sum_{i=1}^{r}\gamma_m^{(i)}n_i)}$$
$$\times \frac{(x_1\omega_1^{\mu_1}t^{\mu_1})^{n_1}\cdots(x_r\omega_r^{\mu_r}t^{\mu_r})^{n_r}}{n_1!\cdots n_r!}$$
$$\times \Gamma(\nu+\sum_{i=1}^{r}\mu_i n_i)E^{n}_{\rho,\nu+\sum_{i=1}^{r}\mu_i n_i}(-a^{\rho}t^{\rho}). \quad (15.29)$$

*Proof.* Using Eq. (15.18), we have:

$$\mathcal{L}\{(1+a^{\rho}{}_0D_t^{-\rho})^n N(t);s\} = \sum_{k=0}^{n}\binom{n}{k} a^{\rho k}\mathcal{L}\{{}_0D_t^{-\rho k}N(t);s\}$$
$$= \sum_{k=0}^{n}\binom{n}{k} a^{\rho k}s^{-\rho k}N(s) = (1+a^{\rho}s^{-\rho})^n N(s). \tag{15.30}$$

Applying the Laplace transform in Eq. (15.27), we have:

$$(1+a^{\rho}s^{-\rho})^n N(s) = N_0 \sum_{n_1,n_2,\ldots,n_r=0}^{\infty} \frac{\Pi_{i=1}^{r}(p_i)_{q_i n_i}}{\Pi_{m=1}^{j}\Gamma(\alpha_m + \sum_{i=1}^{r}\gamma_m^{(i)}n_i)}$$
$$\times \frac{(x_1\omega_1^{\mu_1})^{n_1}\cdots(x_r\omega_r^{\mu_r})^{n_r}}{n_1!\cdots n_r!}\frac{\Gamma(\nu+\sum_{i=1}^{r}\mu_i n_i)}{s^{(\nu+\sum_{i=1}^{r}\mu_i n_i)}}, \tag{15.31}$$

$$N(s) = \frac{N_0}{(1+(\frac{a}{s})^{\rho})^n} \sum_{n_1,n_2,\ldots,n_r=0}^{\infty} \frac{\Pi_{i=1}^{r}(p_i)_{q_i n_i}}{\Pi_{m=1}^{j}\Gamma(\alpha_m + \sum_{i=1}^{r}\gamma_m^{(i)}n_i)}$$
$$\times \frac{(x_1\omega_1^{\mu_1})^{n_1}\cdots(x_r\omega_r^{\mu_r})^{n_r}}{n_1!\cdots n_r!}\frac{\Gamma(\nu+\sum_{i=1}^{r}\mu_i n_i)}{s^{(\nu+\sum_{i=1}^{r}\mu_i n_i)}}. \tag{15.32}$$

Using the geometric series:

$$\left(1+\left(\frac{a}{s}\right)^{\rho}\right)^{-n} = \sum_{k=0}^{\infty}\frac{(n)_k}{k!}\left(-\frac{a}{s}\right)^{\rho k}, \tag{15.33}$$

$$N(s) = N_0\sum_{k=0}^{\infty}(-a^{\rho})^k\frac{(n)_k}{k!} \sum_{n_1,n_2,\ldots,n_r=0}^{\infty} \frac{\Pi_{i=1}^{r}(p_i)_{q_i n_i}}{\Pi_{m=1}^{j}\Gamma(\alpha_m + \sum_{i=1}^{r}\gamma_m^{(i)}n_i)}$$
$$\times \frac{(x_1\omega_1^{\mu_1})^{n_1}\cdots(x_r\omega_r^{\mu_r})^{n_r}}{n_1!\cdots n_r!}\frac{\Gamma(\nu+\sum_{i=1}^{r}\mu_i n_i)}{s^{(\nu+\rho k+\sum_{i=1}^{r}\mu_i n_i)}}. \tag{15.34}$$

Now, by taking the inverse Laplace transform in Eq. (15.34) with the identities (15.23) and (15.24):

$$N(t) = N_0 t^{\nu-1} \sum_{k=0}^{\infty} (-a^{\rho} t^{\rho})^k \frac{(n)_k}{k!} \sum_{n_1,n_2,\ldots,n_r=0}^{\infty} \frac{\Pi_{i=1}^{r} (p_i)_{q_i n_i}}{\Pi_{m=1}^{j} \Gamma(\alpha_m + \sum_{i=1}^{r} \gamma_m^{(i)} n_i)}$$

$$\times \frac{(x_1 \omega_1^{\mu_1})^{n_1} \cdots (x_r \omega_r^{\mu_r})^{n_r}}{n_1! \cdots n_r!} \frac{\Gamma(\nu + \sum_{i=1}^{r} \mu_i n_i) t^{(\sum_{i=1}^{r} \mu_i n_i)}}{\Gamma(\nu + \rho k + \sum_{i=1}^{r} \mu_i n_i)}. \tag{15.35}$$

Using Eq. (15.4), and the definition of the generalized Lauricella function, we can arrive at our desired result (15.28).

Again, use of Eq. (15.3) in Eq. (15.35), gives the result (15.29). □

**Theorem 15.3.5.** *Let* $a, \rho, \omega_i \in \mathbb{R}^+$ *with* $\gamma_m^{(i)}, \alpha_m, p_i, q_i \in \mathbb{C}; n \in \mathbb{N}; \Re(\gamma_m^{(i)}) > 0$, $\Re(q_m) > 0; \alpha_m \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}; (\gamma_m^{(i)}) = \gamma_m', \gamma_m'', \ldots, \gamma_m^{(i)}; \{m = 1, 2, .., j\}$, *and* $\{i = 1, 2, \ldots, r\}$, *then the solution of the generalized fractional differential equation:*

$$N(t) - N_0 t^{\alpha_1 - 1} E^{(p_r);(q_r)}_{(\gamma_1^{(r)}),\ldots,(\gamma_j^{(r)});\alpha_1,\ldots,\alpha_j} (x_1 \omega_1^{\gamma_1'} t^{\gamma_1'}, x_2 \omega_2^{\gamma_1''} t^{\gamma_1''}, \ldots, x_r \omega_r^{\gamma_1^{(r)}} t^{\gamma_1^{(r)}})$$
$$= -\left\{ \sum_{k=1}^{n} \binom{n}{k} a^{\rho k} {}_0D_t^{-\rho k} \right\} N(t) \tag{15.36}$$

*i.e.,*

$$(1 + a_0^{\rho} D_t^{-\rho})^n N(t)$$
$$= N_0 t^{\alpha_1 - 1} E^{(p_r);(q_r)}_{(\gamma_1^{(r)}),\ldots,(\gamma_j^{(r)});\alpha_1,\ldots,\alpha_j} (x_1 \omega_1^{\gamma_1'} t^{\gamma_1'}, x_2 \omega_2^{\gamma_1''} t^{\gamma_1''}, \ldots, x_r \omega_r^{\gamma_1^{(r)}} t^{\gamma_1^{(r)}}) \tag{15.37}$$

*can be given as follows:*

$$N(t) = \frac{N_0 t^{\alpha_1 - 1}}{\Pi_{m=2}^{j} \Gamma(\alpha_m)} \sum_{k=0}^{\infty} \frac{(n)_k}{k!} \frac{(-a^{\rho} t^{\rho})^k}{\Gamma(\alpha_1 + \rho k)}$$
$$\times F^{0:1;\ldots;1}_{j:0;\ldots;0} \left[ \begin{matrix} - : (p_1, q_1); \ldots; (p_r, q_r); \\ (\alpha_m : \gamma_m', \ldots, \gamma_m^{(r)})_{2,j}, (\alpha_1 + \rho k : \gamma_1', \ldots, \gamma_1^{(r)}) : -; \ldots; -; \end{matrix} \; x_1 \omega_1^{\gamma_1'} t^{\gamma_1'}, \ldots, x_r \omega_r^{\gamma_1^{(r)}} t^{\gamma_1^{(r)}} \right], \tag{15.38}$$

$$N(t) = N_0 t^{\alpha_1 - 1} \sum_{n_1,n_2,\ldots,n_r=0}^{\infty} \frac{\Pi_{i=1}^{r} (p_i)_{q_i n_i}}{\Pi_{m=2}^{j} \Gamma(\alpha_m + \sum_{i=1}^{r} \gamma_m^{(i)} n_i)}$$

$$\times \frac{(x_1\omega_1^{\gamma_1'} t^{\gamma_1'})^{n_1} \cdots (x_r\omega_r^{\gamma_1^{(r)}} t^{\gamma_1^{(r)}})^{n_r}}{n_1! \cdots n_r!} E^{n}_{\rho, \alpha_1 + \sum_{i=1}^{r} \gamma_1^{(i)} n_i}(-a^{\rho} t^{\rho}). \quad (15.39)$$

*Proof.* In a similar manner, we can derive this theorem's solution. □

## 15.4 Special cases

When we set the values of the parameters to specific values, we obtain the following special cases.

If we set $r = 1$, $q_1 = 1$, the outcomes described in Theorems 15.3.3, 15.3.4, and 15.3.5 simplify into the following form.

**Corollary 15.4.1.** *For* $\nu, a, \rho, \mu_1, \omega_1 \in \mathbb{R}^+$; $\gamma_m', \alpha_m$, $p_1 \in \mathbb{C}$; $\Re(\gamma_m') > 0$, $q_1 = 1$; $\alpha_m \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}$, *and* $\{m = 1, 2, .., j\}$, *the solution to the generalized fractional kinetic equation:*

$$N(t) - N_0 t^{\nu-1} E^{p_1}_{(\gamma_1', \ldots, \gamma_j'); \alpha_1, \ldots, \alpha_j}(x_1\omega_1^{\mu_1} t^{\mu_1}) = -a^{\rho}{}_0D_t^{-\rho} N(t) \quad (15.40)$$

*is determined as follows:*

$$N(t) = \frac{N_0 t^{\nu-1}}{\Gamma p_1} \sum_{k=0}^{\infty} (-a^{\rho} t^{\rho})^k {}_2\Psi_{j+1}\left[\begin{matrix} (p_1, 1), (\nu, \mu_1) \\ (\alpha_m, \gamma_m')_{1,j}, (\nu + \rho k, \mu_1) \end{matrix} ; x_1\omega_1^{\mu_1} t^{\mu_1}\right] \quad (15.41)$$

$$= N_0 t^{\nu-1} \sum_{n_1=0}^{\infty} \frac{(p_1)_{n_1} \Gamma(\nu + \mu_1 n_1)}{\Pi_{m=1}^{j} \Gamma(\alpha_m + \gamma_m' n_1)} \frac{(x_1\omega_1^{\mu_1} t^{\mu_1})^{n_1}}{n_1!} E_{\rho, \nu + \mu_1 n_1}(-a^{\rho} t^{\rho}), \quad (15.42)$$

*where* ${}_2\Psi_{j+1}[.]$ *represents the Fox–Wright function defined in [21, p. 21].*

**Corollary 15.4.2.** *For* $\nu, a, \rho, \mu_1, \omega_1 \in \mathbb{R}^+$; $n \in \mathbb{N}$; $\gamma_m', \alpha_m, p_1, q_1 \in \mathbb{C}$; $\Re(\gamma_m') > 0$, $q_1 = 1$; $\alpha_m \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}$, *and* $\{m = 1, 2, .., j\}$, *the solution to the generalized fractional kinetic equation:*

$$N(t) - N_0 t^{\nu-1} E^{p_1}_{(\gamma_1', \ldots, \gamma_j'); \alpha_1, \ldots, \alpha_j}(x_1\omega_1^{\mu_1} t^{\mu_1}) = -\left\{\sum_{k=1}^{n} \binom{n}{k} a^{\rho k}{}_0D_t^{-\rho k}\right\} N(t) \quad (15.43)$$

*i.e.,*

$$(1 + a^{\rho}{}_0D_t^{-\rho})^n N(t) = N_0 t^{\nu-1} E^{p_1}_{(\gamma_1', \ldots, \gamma_j'); \alpha_1, \ldots, \alpha_j}(x_1\omega_1^{\mu_1} t^{\mu_1}) \quad (15.44)$$

*is determined as follows:*

$$N(t)=\frac{N_0t^{\nu-1}}{\Gamma p_1}\sum_{k=0}^{\infty}\frac{(n)_k}{k!}(-a^{\rho}t^{\rho})^k{}_2\Psi_{j+1}\left[\begin{matrix}(p_1,1),(\nu,\mu_1)\\ \\ (\alpha_m,\gamma'_m)_{1,j},(\nu+\rho k,\mu_1)\end{matrix};x_1\omega_1^{\mu_1}t^{\mu_1}\right] \tag{15.45}$$

$$=N_0t^{\nu-1}\sum_{n_1=0}^{\infty}\frac{(p_1)_{n_1}\Gamma(\nu+\mu_1n_1)}{\Pi_{m=1}^{j}\Gamma(\alpha_m+\gamma'_mn_1)}\frac{(x_1\omega_1^{\mu_1}t^{\mu_1})^{n_1}}{n_1!}E^{n}_{\rho,\nu+\mu_1n_1}(-a^{\rho}t^{\rho}). \tag{15.46}$$

**Corollary 15.4.3.** *Let $a,\rho,\omega_1\in\mathbb{R}^+$ with $\gamma'_m,\alpha_m,p_1,q_1\in\mathbb{C}; n\in\mathbb{N}; \Re(\gamma'_m)>0$, $q_1=1; \alpha_m\notin\mathbb{Z}_0^-=\{0,-1,-2,...\}$, and $\{m=1,2,..,j\}$, then the generalized fractional kinetic equation:*

$$N(t)-N_0t^{\alpha_1-1}E^{p_1}_{(\gamma'_1,...,\gamma'_j);\alpha_1,...,\alpha_j}(x_1\omega_1^{\gamma'_1}t^{\gamma'_1})=-\left\{\sum_{k=1}^{n}\binom{n}{k}a^{\rho k}{}_0D_t^{-\rho k}\right\}N(t) \tag{15.47}$$

*has a solution as follows:*

$$N(t)=\frac{N_0t^{\alpha_1-1}}{\Gamma p_1}\sum_{k=0}^{\infty}\frac{(n)_k}{k!}(-a^{\rho}t^{\rho})^k{}_1\Psi_j\left[\begin{matrix}(p_1,1)\\ \\ (\alpha_m,\gamma'_m)_{2,j},(\alpha_1+\rho k,\gamma'_1)\end{matrix};x_1\omega_1^{\gamma'_1}t^{\gamma'_1}\right] \tag{15.48}$$

$$=N_0t^{\alpha_1-1}\sum_{n_1=0}^{\infty}\frac{(p_1)_{n_1}}{\Pi_{m=2}^{j}\Gamma(\alpha_m+\gamma'_mn_1)}\frac{(x_1\omega_1^{\gamma'_1}t^{\gamma'_1})^{n_1}}{n_1!}E^{n}_{\rho,\alpha_1+\gamma'_1n_1}(-a^{\rho}t^{\rho}). \tag{15.49}$$

If we choose $q_1,...,q_r=1$ the results outlined in Theorems 15.3.3, 15.3.4, and 15.3.5 take on the following form.

**Corollary 15.4.4.** *For $\nu,a,\rho,\mu_i,\omega_i\in\mathbb{R}^+;\gamma_m^{(i)},\alpha_m,\ p_i\in\mathbb{C};\Re(\gamma_m^{(i)})>0;\alpha_m\notin\mathbb{Z}_0^-=\{0,-1,-2,...\}$; $(\gamma_m^{(i)})=\gamma'_m,\gamma''_m,...,\gamma_m^{(i)}$; $\{m=1,2,..,j\}$ and $\{i=1,2,...,r\}$, the solution to the generalized fractional kinetic equation:*

$$N(t)-N_0t^{\nu-1}E^{(p_r)}_{(\gamma_1^{(r)}),...,(\gamma_j^{(r)});\alpha_1,...,\alpha_j}(x_1\omega_1^{\mu_1}t^{\mu_1},x_2\omega_2^{\mu_2}t^{\mu_2},...,x_r\omega_r^{\mu_r}t^{\mu_r})$$
$$=-a^{\rho}{}_0D_t^{-\rho}N(t) \tag{15.50}$$

*is determined as follows:*

$$N(t)=\frac{N_0t^{\nu-1}\Gamma(\nu)}{\Pi_{m=1}^{j}\Gamma\alpha_m}\sum_{k=0}^{\infty}\frac{(-a^{\rho}t^{\rho})^k}{\Gamma(\nu+\rho k)}$$
$$\times F^{1:1;...;1}_{j+1:0;...;0}\left[\begin{matrix}(\nu:\mu_1,...,\mu_r):(p_1,1);...;(p_r,1);\\ \\ (\alpha_m:\gamma'_m,...,\gamma_m^{(r)})_{1,j},(\nu+\rho k:\mu_1,...,\mu_r):-;...;-;\end{matrix}\ x_1\omega_1^{\mu_1}t^{\mu_1},...,x_r\omega_r^{\mu_r}t^{\mu_r}\right], \tag{15.51}$$

$$N(t) = N_0 t^{\nu-1} \sum_{n_1,n_2,\ldots,n_r=0}^{\infty} \frac{\Pi_{i=1}^{r}(p_i)_{n_i}}{\Pi_{m=1}^{j}\Gamma(\alpha_m + \sum_{i=1}^{r}\gamma_m^{(i)} n_i)} \times \frac{(x_1\omega_1^{\mu_1}t^{\mu_1})^{n_1}\cdots(x_r\omega_r^{\mu_r}t^{\mu_r})^{n_r}}{n_1!\cdots n_r!} \times \Gamma(\nu + \sum_{i=1}^{r}\mu_i n_i) E_{\rho,\nu+\sum_{i=1}^{r}\mu_i n_i}(-a^\rho t^\rho). \quad (15.52)$$

**Corollary 15.4.5.** *For* $\nu, a, \rho, \mu_i, \omega_i \in \mathbb{R}^+; n \in \mathbb{N}; \gamma_m^{(i)}, \alpha_m,\ p_i \in \mathbb{C}; \Re(\gamma_m^{(i)}) > 0; \alpha_m \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}; (\gamma_m^{(i)}) = \gamma'_m, \gamma''_m, \ldots, \gamma_m^{(i)}; \{m = 1, 2, .., j\}$ *and* $\{i = 1, 2, \ldots, r\}$*, the solution to the generalized fractional kinetic equation:*

$$N(t) - N_0 t^{\nu-1} E^{(p_r)}_{(\gamma_1^{(r)}),\ldots,(\gamma_j^{(r)});\alpha_1,\ldots,\alpha_j}(x_1\omega_1^{\mu_1}t^{\mu_1}, x_2\omega_2^{\mu_2}t^{\mu_2}, \ldots, x_r\omega_r^{\mu_r}t^{\mu_r}) = -\left\{\sum_{k=1}^{n}\binom{n}{k} a^{\rho k}\,{}_0D_t^{-\rho k}\right\}N(t) \quad (15.53)$$

*i.e.,*

$$(1 + a_0^\rho D_t^{-\rho})^n N(t) = N_0 t^{\nu-1} E^{(p_r)}_{(\gamma_1^{(r)}),\ldots,(\gamma_j^{(r)});\alpha_1,\ldots,\alpha_j}(x_1\omega_1^{\mu_1}t^{\mu_1}, x_2\omega_2^{\mu_2}t^{\mu_2}, \ldots, x_r\omega_r^{\mu_r}t^{\mu_r}) \quad (15.54)$$

*is determined as follows:*

$$N(t) = \frac{N_0 t^{\nu-1}\Gamma(\nu)}{\Pi_{m=1}^{j}\Gamma\alpha_m} \sum_{k=0}^{\infty} \frac{(n)_k}{k!} \frac{(-a^\rho t^\rho)^k}{\Gamma(\nu+\rho k)} \times F^{1:1;\ldots;1}_{j+1:0;\ldots;0}\left[\begin{matrix}(\nu:\mu_1,\ldots,\mu_r):(p_1,1);\ldots;(p_r,1);\\ (\alpha_m:\gamma'_m,\ldots,\gamma_m^{(r)})_{1,j},(\nu+\rho k:\mu_1,\ldots,\mu_r):-;\ldots;-;\end{matrix}\ x_1\omega_1^{\mu_1}t^{\mu_1},\ldots,x_r\omega_r^{\mu_r}t^{\mu_r}\right], \quad (15.55)$$

$$N(t) = N_0 t^{\nu-1} \sum_{n_1,n_2,\ldots,n_r=0}^{\infty} \frac{\Pi_{i=1}^{r}(p_i)_{n_i}}{\Pi_{m=1}^{j}\Gamma(\alpha_m + \sum_{i=1}^{r}\gamma_m^{(i)} n_i)} \times \frac{(x_1\omega_1^{\mu_1}t^{\mu_1})^{n_1}\cdots(x_r\omega_r^{\mu_r}t^{\mu_r})^{n_r}}{n_1!\cdots n_r!} \times \Gamma(\nu + \sum_{i=1}^{r}\mu_i n_i) E^{n}_{\rho,\nu+\sum_{i=1}^{r}\mu_i n_i}(-a^\rho t^\rho). \quad (15.56)$$

**Corollary 15.4.6.** *Let* $a, \rho, \omega_i \in \mathbb{R}^+$ *with* $\gamma_m^{(i)}, \alpha_m, p_i \in \mathbb{C}; n \in \mathbb{N}; \Re(\gamma_m^{(i)}) > 0; \alpha_m \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}; (\gamma_m^{(i)}) = \gamma'_m, \gamma''_m, ..., \gamma_m^{(i)}; \{m = 1, 2, .., j\}$ *and* $\{i = 1, 2, ..., r\}$, *then the generalized fractional kinetic equation:*

$$N(t) - N_0 t^{\alpha_1 - 1} E^{(p_r)}_{(\gamma_1^{(r)}),...,(\gamma_j^{(r)});\alpha_1,...,\alpha_j}(x_1 \omega_1^{\gamma'_1} t^{\gamma'_1}, x_2 \omega_2^{\gamma''_1} t^{\gamma''_1}, ..., x_r \omega_r^{\gamma_1^{(r)}} t^{\gamma_1^{(r)}})$$
$$= -\left\{\sum_{k=1}^{n} \binom{n}{k} a^{\rho k} {}_0D_t^{-\rho k}\right\} N(t) \quad (15.57)$$

*has a solution as follows:*

$$N(t) = \frac{N_0 t^{\alpha_1 - 1}}{\Pi_{m=2}^{j} \Gamma(\alpha_m)} \sum_{k=0}^{\infty} \frac{(n)_k}{k!} \frac{(-a^\rho t^\rho)^k}{\Gamma(\alpha_1 + \rho k)}$$
$$\times F^{0:1;...;1}_{j:0;...;0}\left[\begin{matrix} - : (p_1, 1); ...; (p_r, 1); \\ (\alpha_m : \gamma'_m, ..., \gamma_m^{(r)})_{2,j}, (\alpha_1 + \rho k : \gamma'_1, ..., \gamma_1^{(r)}) : -; ...; -; \end{matrix} \quad x_1 \omega_1^{\gamma'_1} t^{\gamma'_1}, ..., x_r \omega_r^{\gamma_1^{(r)}} t^{\gamma_1^{(r)}}\right] \quad (15.58)$$

*and*

$$N(t) = N_0 t^{\alpha_1 - 1} \sum_{n_1, n_2, ..., n_r = 0}^{\infty} \frac{\Pi_{i=1}^{r} (p_i)_{n_i}}{\Pi_{m=2}^{j} \Gamma(\alpha_m + \sum_{i=1}^{r} \gamma_m^{(i)} n_i)}$$
$$\times \frac{(x_1 \omega_1^{\gamma'_1} t^{\gamma'_1})^{n_1} \cdots (x_r \omega_r^{\gamma_1^{(r)}} t^{\gamma_1^{(r)}})^{n_r}}{n_1! \cdots n_r!} E^{n}_{\rho, \alpha_1 + \sum_{i=1}^{r} \gamma_1^{(i)} n_i}(-a^\rho t^\rho). \quad (15.59)$$

## 15.5 Conclusion

In this chapter, we used the Laplace transform to solve the generalized fractional differential equations involving the multi-index Mittag-Leffler function in several variables. Choosing appropriate values of parameters, we can extract various known results from our main findings, some of them are shown below:

(i) If we assume $n = 1,\ j = 1, r = 1, q_1 = 1, x_1 = -1,\ p_1 = \gamma,\ \gamma'_1 = \rho = \nu, \alpha_1 = \mu$, $\omega_1 = a = c$ in Theorem 15.3.5, then we obtain the known result [22, Theorem 1, p. 658] and the correct form of the result [23, Theorem 1, p. 35].

(ii) If we consider $j = 1, r = 1, q_1 = 1, x_1 = -1, \omega_1 = a = c, \gamma'_1 = \rho = \nu, \alpha_1 = \delta$, $p_1 = \lambda$ in Theorem 15.3.5, then we can obtain the result [24, p. 230, Theorem 5] that turned into [24, p. 229, Theorem 4] at $p_1 = 1$.

(iii) For $n = r = x_1 = 1,\ \nu = \lambda,\ (\gamma'_i = \alpha_i, \alpha_i = \beta_i; i = 1, 2, ..., j),\ \mu_1 = \rho,\ \rho = \nu$, $a = c,\ \omega_1^{\mu_1} = \partial,\ p_1 = \gamma,\ q_1 = \kappa$, Theorem 15.3.4 converts into the former result of

Kumar et al. [25, Theorem 2.3, p. 461] and gives the correct form of the theorem [25, Theorem 3.4, p. 465].

(iv) In Theorem 15.3.3, if we assign $r = j = \nu = x_1 = \omega_1 = \mu_1 = 1, \alpha_1 = \lambda, \gamma_1' = \rho$, $p_1 = \gamma$, $q_1 = \kappa$, $\rho = \nu$, $a = d$, then it converts into the result of Nisar et al. [27, Corollary 2.1].

(v) If we consider $j = \nu = 1, \alpha_1 = \lambda, \rho = \nu, (\omega_i = d, \gamma_1^{(i)} = \rho_i, \mu_i = \nu, p_i = \gamma_i, q_i = l_i; i = 1, 2, ..., r)$ in Theorem 15.3.3, then we arrive at the earlier proved result [26, Theorem 1, p. 284].

Similarly, considering appropriate values of parameters in Theorems 15.3.3 and 15.3.4, we can achieve the remaining results of Chand et al. [26] and other corollaries of Nisar et al. [27].

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CHAPTER

# 16 Certain integrals involving $k$-type $R$- and $G$-functions

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## 16.1 Introduction and definitions

Many special functions in fractional calculus are defined via infinite summation or infinite products. During the last two decades, several authors extensively studied interesting and useful extensions of many of the special functions (see, for example, [1,3–5,9–12,6,14–16,21,22,26]). Our present work is largely motivated from the above mentioned work.

Recently, Diaz and Pariguan [8] introduced the $k$-Pochhammer symbol and $k$-gamma function defined as:

$$(\gamma)_{n,k} = \begin{cases} \dfrac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)} & (\gamma \in \mathbb{C}\setminus k\mathbb{Z}^-;\ k \in \mathbb{R}^+;\ n \in \mathbb{N}) \\ \gamma(\gamma+k)\ldots(\gamma+(n-1)k) & (\gamma \in \mathbb{C};\ k \in \mathbb{R};\ n \in \mathbb{N}). \end{cases} \tag{16.1.1}$$

The relation with the classical Euler's gamma function (see [7,19]) is given by:

$$\Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1}\Gamma\left(\frac{\gamma}{k}\right) \qquad (\gamma \in \mathbb{C}\setminus k\mathbb{Z}^-;\ k \in \mathbb{R}^+), \tag{16.1.2}$$

when $k=1$, Eq. (16.1.1) reduces to the classical Pochhammer symbol and Euler's gamma function, respectively, (see [20]) given as:

$$(\gamma)_n = \begin{cases} 1 & (\gamma \in \mathbb{C};\ n=0) \\ \gamma(\gamma+1)\ldots(\gamma+(n-1)) & (\gamma \in \mathbb{C};\ n \in \mathbb{N}) \end{cases}$$
$$= \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} \qquad (\gamma \in \mathbb{C}\setminus k\mathbb{Z}_0^-). \tag{16.1.3}$$

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Lorenzo and Hartley [13] presented two special functions, known as the $R$-function and the $G$-function as:

$$R_{\alpha,\nu}[a,c,x] = \sum_{n=0}^{\infty} \frac{a^n (x-c)^{(n+1)\alpha-\nu-1}}{\Gamma[(n+1)\alpha-\nu]} \tag{16.1.4}$$

$$(\alpha, \nu \in \mathbb{C};\ \Re(\alpha-\nu) > 0;\ x > c > 0;\ a > 0)$$

and

$$G_{\alpha,\nu,\mu}[a,c,x] = \sum_{n=0}^{\infty} \frac{(\mu)_n a^n (x-c)^{(n+\mu)\alpha-\nu-1}}{(n!)\Gamma[(n+\mu)\alpha-\nu]} \tag{16.1.5}$$

$$(\alpha, \nu, \mu \in \mathbb{C};\ \Re(\mu\alpha-\nu) > 0;\ x > c > 0;\ a > 0).$$

In particular, for $c = 0$, we have:

$$R_{\alpha,\nu}[a,x] = R_{\alpha,\nu}[a,0,x] = \sum_{n=0}^{\infty} \frac{a^n x^{(n+1)\alpha-\nu-1}}{\Gamma[(n+1)\alpha-\nu]} \tag{16.1.6}$$

$$(\alpha, \nu \in \mathbb{C};\ \Re(\alpha-\nu) > 0;\ x > 0;\ a > 0)$$

and

$$G_{\alpha,\nu,\mu}[a,x] = G_{\alpha,\nu,\mu}[a,0,x] = \sum_{n=0}^{\infty} \frac{(\mu)_n a^n x^{(n+\mu)\alpha-\nu-1}}{(n!)\Gamma[(n+\mu)\alpha-\nu]} \tag{16.1.7}$$

$$(\alpha, \nu, \mu \in \mathbb{C};\ \Re(\mu\alpha-\nu) > 0;\ x > 0;\ a > 0).$$

Here, we use extensions of the $R$- and $G$-functions given in Eqs. (16.1.4), (16.1.5), (16.1.6), and (16.1.7) given as follows:

$$R^q_{k,\alpha,\nu}[a,c,x] = \sum_{n=0}^{\infty} \frac{a^{nq} (x-c)^{(nq+1)\alpha-\nu-1}}{\Gamma_k[(nq+1)\alpha-\nu]} \tag{16.1.8}$$

$$(\alpha, \nu, q \in \mathbb{C};\ \Re(\alpha-\nu) > 0;\ \Re(q) > 0;\ k \in \mathbb{R};\ x > c > 0;\ a > 0)$$

and

$$G^q_{k,\alpha,\nu,\mu}[a,c,x] = \sum_{n=0}^{\infty} \frac{(\mu)_{nq} a^{nq} (x-c)^{(nq+\mu)\alpha-\nu-1}}{(n!)\Gamma_k[(nq+\mu)\alpha-\nu]} \tag{16.1.9}$$

$$(\alpha, \nu, \mu, q \in \mathbb{C};\ \Re(\mu\alpha-\nu) > 0;\ \Re(q) > 0;\ k \in \mathbb{R};\ x > c > 0;\ a > 0).$$

In particular, for $c = 0$, we have:

$$R^q_{k,\alpha,\nu}[a,x] = R^q_{k,\alpha,\nu}[a,0,x] = \sum_{n=0}^{\infty} \frac{a^{nq} x^{(nq+1)\alpha-\nu-1}}{\Gamma_k[(nq+1)\alpha-\nu]}, \tag{16.1.10}$$

$$(\alpha, \nu, q \in \mathbb{C};\ \Re(\alpha-\nu) > 0;\ \Re(q) > 0;\ k \in \mathbb{R};\ x > 0;\ a > 0)$$

and

$$G^{q}_{k,\alpha,\nu,\mu}[a,x]=G^{q}_{k,\alpha,\nu,\mu}[a,0,x]=\sum_{n=0}^{\infty}\frac{(\mu)_{nq}a^{nq}x^{(nq+\mu)\alpha-\nu-1}}{(n!)\Gamma_k[(nq+\mu)\alpha-\nu]}, \tag{16.1.11}$$

$(\alpha,\nu,\mu,q\in\mathbb{C};\ \Re(\mu\alpha-\nu)>0;\ \Re(q)>0;\ k\in\mathbb{R};\ x>0;\ a>0)$.

For the present study, we also require the Fox–Wright function ${}_p\Psi_q$, which is defined as (see [23]):

$$\begin{aligned}{}_p\Psi_q[z]&={}_p\Psi_q\left[\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p);\\(b_1,\beta_1),\ldots,(b_q,\beta_q);\end{matrix}\ z\right]={}_p\Psi_q\left[\begin{matrix}(a_i,\alpha_i)_{1,p};\\(b_j,\beta_j)_{1,q};\end{matrix}\ z\right]\\&=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)}\frac{z^n}{n!},\end{aligned} \tag{16.1.12}$$

where the coefficients $\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q\in\mathbb{R}^+$ are such that:

$$1+\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i\geq 0. \tag{16.1.13}$$

Prabhakar and Suman [17] defined the polynomials $L_n^{(\eta,\tau)}(x)$ as:

$$L_n^{(\eta,\tau)}(x)=\frac{\Gamma(\eta n+\tau+1)}{\Gamma(n+1)}\sum_{r=0}^{n}\frac{(-n)_r x^r}{r!\Gamma(\eta r+\tau+1)}, \tag{16.1.14}$$

where $\eta\in\mathbb{C}^+$, $\tau\in\mathbb{C}^+_{-1}$ and $n\in\mathbb{N}$

If $\eta=1$, then Eq. (16.1.14) reduces to:

$$L_n^{(1,\tau)}(x)=\frac{\Gamma(n+\tau+1)}{\Gamma(n+1)}\sum_{r=0}^{n}\frac{(-n)_r x^r}{r!\Gamma(r+\tau+1)}=L_n^{\tau}(x), \tag{16.1.15}$$

where $L_n^{\tau}(x)$ is a generalized Laguerre polynomial ([18]).

The Konhauser polynomials of the second kind ([24]) is defined as:

$$Z_n^{\tau}(x;r)=\frac{\Gamma(rn+\tau+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{x^{rj}}{\Gamma(rj+\tau+1)}, \tag{16.1.16}$$

where $\tau\in\mathbb{C}^+_{-1}$, $n\in\mathbb{N}$ and $r\in\mathbb{Z}$, such that

$$L_n^{(r,\tau)}(x^r)=Z_n^{\tau}(x;r) \tag{16.1.17}$$

$$L_n^{\tau}(x)=Z_n^{\tau}(x;1). \tag{16.1.18}$$

The polynomial $Z_n^{(\eta,\tau)}(x;r)$ is defined [25] as:

$$Z_n^{(\eta,\tau)}(x;r)=\sum_{j=0}^{n}\frac{\Gamma(rn+\tau+1)(-1)^j x^{rj}}{j!\Gamma(rj+\tau+1)\Gamma(\eta n-\eta j+1)}, \tag{16.1.19}$$

where $\eta\in\mathbb{C}^+$, $\tau\in\mathbb{C}^+_{-1}$, $n\in\mathbb{N}$, and $r\in\mathbb{Z}$.

From Eqs. (16.1.16) and (16.1.19), we obtain:

$$Z_n^{(1,\tau)}(x;r)=Z_n^{\tau}(x;r). \tag{16.1.20}$$

If $\eta\in\mathbb{N}$ then Eq. (16.1.19) can be written in the following form:

$$Z_n^{(\eta,\tau)}(x;r)=\frac{\Gamma(rn+\tau+1)}{\Gamma(\eta n+1)}\sum_{m=0}^{n}\frac{(-\eta n)_{\eta m}x^{rm}}{m!\Gamma(rm+\tau+1)(-1)^{(\eta-1)m}}. \tag{16.1.21}$$

The set of polynomials $L_n^{(\eta,\tau)}(\gamma;x)$ is defined [25] as:

$$L_n^{(\eta,\tau)}(\gamma;x)=\sum_{r=0}^{n}\frac{\Gamma(\eta n+\tau+1)(-1)^r x^r}{r!\Gamma(\eta r+\tau+1)\Gamma(\gamma n-\gamma r+1)}, \tag{16.1.22}$$

where $\eta,\gamma\in\mathbb{C}^+$, $\tau\in\mathbb{C}^+_{-1}$, $n\in\mathbb{N}$.

From Eqs. (16.1.14) and (16.1.16), we have:

$$L_n^{(\eta,\tau)}(1;x)=L_n^{(\eta,\tau)}(x), \tag{16.1.23}$$

such that:

$$L_n^{(r,\tau)}\left(\eta;x^r\right)=Z_n^{(\eta,\tau)}(x;r) \tag{16.1.24}$$

$$Z_n^{(1,\tau)}(x;1)=L_n^{\tau}(\eta;x) \tag{16.1.25}$$

$$Z_n^{(1,\tau)}(x;1)=Z_n^{\tau}(x;1)=L_n^{\tau}(x) \tag{16.1.26}$$

$$L_n^{(1,\tau)}(1;x)=L_n^{(1,\tau)}(x)=L_n^{\tau}(x). \tag{16.1.27}$$

We also require some facts listed below:

$$(-x)_n=(-1)^n(x-n+1)_n \tag{16.1.28}$$

$$(x+y)_n=\sum_{j=0}^{n}\binom{n}{j}(x)_j(y)_{n-j}, \tag{16.1.29}$$

$$(x)_{n+m}=(x)_n(x+n)_m \tag{16.1.30}$$

and

$$\binom{x}{n}=\frac{(-1)^n}{n!}(-x)_n. \tag{16.1.31}$$

## 16.2 Integral formulae involving $k$-type $R$- and $G$-functions

**Theorem 1.** *If* $\lambda, \delta, \alpha, \nu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$, *then*

$$\frac{1}{\Gamma(\delta)} \int_0^1 u^{\lambda-1}(1-u)^{\delta-1} R^q_{k,\alpha,\nu}[a, zu]\, du = \frac{z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} {}_2\Psi_2\left[ \begin{array}{c} (\lambda+\alpha-\nu-1, \alpha q), (1,1) \\ (\frac{\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda+\alpha-\nu+\delta-1, \alpha q) \end{array} \middle| \left(\frac{az^\alpha}{k^{\alpha/k}}\right)^q \right]. \tag{16.2.1}$$

*Proof.* For our convenience, denote the left-hand side of Eq. (16.2.1) by $\mathscr{I}$, then using the result given in Eq. (16.1.10), also by interchanging the order of summation and integration, we have:

$$\mathscr{I} = \frac{1}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{a^{nq} z^{(nq+1)\alpha-\nu-1}}{\Gamma_k[(nq+1)\alpha-\nu]} \int_0^1 u^{\lambda+(nq+1)\alpha-\nu-2}(1-u)^{\delta-1} du. \tag{16.2.2}$$

After simplification, Eq. (16.2.2) reduces to:

$$\mathscr{I} = \sum_{n=0}^{\infty} \frac{a^{nq} z^{(nq+1)\alpha-\nu-1}}{\Gamma_k[(nq+1)\alpha-\nu]} \frac{\Gamma[\lambda+(nq+1)\alpha-\nu-1]}{\Gamma[\lambda+(nq+1)\alpha-\nu+\delta-1]} \tag{16.2.3}$$

and on applying the result given in Eq. (16.1.2), Eq. (16.2.3) reduces to the following form:

$$\mathscr{I} = \frac{z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \sum_{n=0}^{\infty} \frac{\Gamma[\lambda+\alpha-\nu-1+\alpha nq]\Gamma(n+1)}{\Gamma\left[\frac{\alpha nq}{k}+\frac{\alpha}{k}-\frac{\nu}{k}\right]\Gamma[\lambda+\alpha-\nu+\delta-1+\alpha nq]} \frac{1}{n!} \left(\frac{az^\alpha}{k^{\alpha/k}}\right)^{nq}. \tag{16.2.4}$$

Interpreting the above result in view of Eq. (16.1.12), we have the required result. □

**Theorem 2.** *If* $\lambda, \delta, \alpha, \nu, \mu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\mu\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$, *then*

$$\frac{1}{\Gamma(\delta)} \int_0^1 u^{\lambda-1}(1-u)^{\delta-1} G^q_{k,\alpha,\nu,\mu}[a, zu]\, du = \frac{z^{\mu\alpha-\nu-1}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)} {}_2\Psi_2\left[ \begin{array}{c} (\lambda+\mu\alpha-\nu-1, \alpha q), (\mu, q) \\ (\frac{\mu\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda+\mu\alpha-\nu+\delta-1, \alpha q) \end{array} \middle| \left(\frac{az^\alpha}{k^{\alpha/k}}\right)^q \right]. \tag{16.2.5}$$

*Proof.* The proof is similar to that of Theorem 1. □

**Theorem 3.** *$\lambda, \delta, \alpha, \nu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$, then*

$$\frac{1}{\Gamma(\delta)} \int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1} R^q_{k,\alpha,\nu}[a, z(s-t)]\, ds = \frac{(x-t)^{\delta+\lambda+\alpha-\nu-2} z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \times {}_2\Psi_2\left[\begin{matrix} (\lambda+\alpha-\nu-1, \alpha q), (1,1) \\ (\frac{\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda+\alpha-\nu+\delta-1, \alpha q) \end{matrix} \middle| \left(\frac{a[z(x-t)]^\alpha}{k^{\alpha/k}}\right)^q\right]. \tag{16.2.6}$$

*Proof.* First, we denote the left-hand side of Eq. (16.2.6) by $\mathscr{I}$, then changing the variable $s$ to $u = \frac{s-t}{x-t}$, further using the definition given in Eq. (16.1.10), after interchanging the order of summation and integration, we have:

$$\mathscr{I} = \frac{(x-t)^{\delta+\lambda-1}}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{a^{nq}[z(x-t)]^{(nq+1)\alpha-\nu-1}}{\Gamma_k[(nq+1)\alpha-\nu]} \int_0^1 (1-u)^{\delta-1} u^{\lambda+(nq+1)\alpha-\nu-2} du. \tag{16.2.7}$$

After simplification, Eq. (16.2.7) reduces to the following form:

$$\mathscr{I} = \frac{(x-t)^{\delta+\lambda-1}}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{a^{nq}[z(x-t)]^{(nq+1)\alpha-\nu-1}}{\Gamma_k[(nq+1)\alpha-\nu]} B(\delta, \lambda+(nq+1)\alpha-\nu-1). \tag{16.2.8}$$

Further, on applying the result given in Eq. (16.1.2), the above result reduces to the following form:

$$\mathscr{I} = \frac{(x-t)^{\delta+\lambda+\alpha-\nu-2} z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \times \sum_{n=0}^{\infty} \frac{\Gamma[\lambda+\alpha-\nu-1+\alpha nq]\Gamma(1+n)}{\Gamma\left[\frac{\alpha}{k}-\frac{\nu}{k}+\frac{\alpha nq}{k}\right]\Gamma[\lambda+\alpha-\nu+\delta-1+\alpha nq]} \frac{1}{n!} \left(\frac{a[z(x-t)]^\alpha}{k^{\alpha/k}}\right)^{nq}. \tag{16.2.9}$$

Interpreting Eq. (16.2.9) in view of (16.1.12), we easily arrive at the required result (16.2.6). □

**Theorem 4.** *$\lambda, \delta, \alpha, \nu, \mu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\mu\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$, then*

$$\frac{1}{\Gamma(\delta)} \int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1} G^q_{k,\alpha,\nu,\mu}[a, z(s-t)]\, ds = \frac{(x-t)^{\delta+\lambda+\mu\alpha-\nu-2} z^{\mu\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)}$$

$$\times\, {}_2\Psi_2\left[\begin{array}{c}(\lambda+\mu\alpha-\nu-1,\alpha q),(\mu,q)\\ (\frac{\mu\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}),(\lambda+\mu\alpha-\nu+\delta-1,\alpha q)\end{array}\middle|\left(\frac{a[z(x-t)]^{\alpha}}{k^{\alpha/k}}\right)^{q}\right]. \tag{16.2.10}$$

*Proof.* The proof is similar to that of Theorem 3, therefore we omit the details. □

**Theorem 5.** *If* $\lambda,\delta,\alpha,\nu,q\in\mathbb{C}; z>0; a>0; \Re(\alpha-\nu)>0; \Re(\delta)>0; \Re(\lambda)>0; \Re(q)>0; k\in\mathbb{R}$, *then*

$$\frac{1}{\Gamma(\delta)}\int_0^x t^{\delta-1}(x-t)^{\lambda-1}R^{q}_{k,\alpha,\nu}\left[a,z(x-t)^{\rho}\right]dt=\frac{z^{\alpha-\nu-1}x^{\lambda+\delta+\rho(\alpha-\nu-1)}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}}$$
$$\times\, {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\alpha-\nu-1),\rho\alpha q),(1,1)\\ (\frac{\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}),(\lambda+\delta+\rho(\alpha-\nu-1),\rho\alpha q)\end{array}\middle|\left(\frac{a\,(zx^{\rho})^{\alpha}}{k^{\alpha/k}}\right)^{q}\right]. \tag{16.2.11}$$

*Proof.* Denote the left-hand side of Eq. (16.2.11) by $\mathscr{I}$ and using the result (16.1.10), then interchanging the order of summation and integration, we have:

$$\mathscr{I}=\frac{1}{\Gamma(\delta)}\sum_{n=0}^{\infty}\frac{a^{nq}z^{(nq+1)\alpha-\nu-1}x^{\lambda+\delta+\rho((nq+1)\alpha-\nu-1)-1}}{\Gamma_k\left[(nq+1)\alpha-\nu\right]}$$
$$\times\int_0^1 u^{\delta-1}(1-u)^{\lambda+\rho((nq+1)\alpha-\nu-1)-1}du. \tag{16.2.12}$$

After simplification, Eq. (16.2.12) reduces to:

$$\mathscr{I}=\sum_{n=0}^{\infty}\frac{a^{nq}z^{(nq+1)\alpha-\nu-1}x^{\lambda+\delta+\rho((nq+1)\alpha-\nu-1)-1}}{\Gamma_k\left[(nq+1)\alpha-\nu\right]}\frac{\Gamma\left[\lambda+\rho((nq+1)\alpha-\nu-1)\right]}{\Gamma\left[\lambda+\delta+\rho((nq+1)\alpha-\nu-1)\right]}. \tag{16.2.13}$$

Further, on applying the result given in Eq. (16.1.2), Eq. (16.2.13) reduces to the following form:

$$\mathscr{I}=\frac{z^{\alpha-\nu-1}x^{\lambda+\delta+\rho(\alpha-\nu-1)}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}}\sum_{n=0}^{\infty}\frac{\Gamma\left[\lambda+\rho(\alpha-\nu-1)+\rho\alpha nq\right]\Gamma(1+n)}{\Gamma\left[\frac{\alpha nq}{k}+\frac{\alpha}{k}-\frac{\nu}{k}\right]\Gamma\left[\lambda+\delta+\rho(\alpha-\nu-1)+\rho\alpha nq\right]}$$
$$\times\frac{1}{n!}\left(\frac{a\,(zx^{\rho})^{\alpha}}{k^{\alpha/k}}\right)^{nq}. \tag{16.2.14}$$

Interpreting the above equation with the help of (16.1.12), we have the required result (16.2.11). □

**Theorem 6.** *If* $\lambda, \delta, \alpha, \nu, \mu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\mu\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(\lambda) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$, *then*

$$\frac{1}{\Gamma(\delta)} \int_0^x t^{\delta-1}(x-t)^{\lambda-1} G^q_{k,\alpha,\nu,\mu}\left[a, z(x-t)^\rho\right] dt = \frac{z^{\mu\alpha-\nu-1} x^{\lambda+\delta+\rho(\mu\alpha-\nu-1)}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)}$$
$$\times\, {}_2\Psi_2\left[\begin{array}{c} (\lambda+\rho(\mu\alpha-\nu-1), \rho\alpha q), (\mu, q) \\ (\frac{\mu\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda+\delta+\rho(\mu\alpha-\nu-1), \rho\alpha q) \end{array} \middle| \left(\frac{a\,(zx^\rho)^\alpha}{k^{\alpha/k}}\right)^q\right]. \tag{16.2.15}$$

*Proof.* The proof of Theorem 6 is similar to that of Theorem 5, therefore we omit the details. □

In our investigation, the following lemma is also required [2].

**Lemma 1.** *If* $\vartheta, \eta, \zeta, \xi \in \mathbb{C}^+$, $\omega, \tau \in \mathbb{C}^+_{-1}$, $m, n \in \mathbb{N}$, *then*

$$\begin{aligned} &L_m^{(\vartheta,\omega)}(\xi; x) L_n^{(\eta,\tau)}(\zeta; x) \\ &= \sum_{h=0}^{m+n}\sum_{j=0}^{h} \frac{\Gamma(\vartheta m+\omega+1)\Gamma(\eta n+\tau+1)}{\Gamma(h-j+1)\Gamma(\xi(m-j)+1)\Gamma(\zeta(n-h+j)+1)} \\ &\quad\times \frac{(-x)^h}{\Gamma(\vartheta j+\omega+1)\Gamma(\eta(h-j)+\tau+1)\Gamma(j+1)}. \end{aligned} \tag{16.2.16}$$

**Note 1.** To investigate Theorems 7–12, ${}_1\Delta^{m,n,\xi,\zeta}_{\vartheta,\omega,\eta,\tau}$ is defined as:

$$\begin{aligned} {}_1\Delta^{m,n,\xi,\zeta}_{\vartheta,\omega,\eta,\tau} &= \sum_{j=0}^{h} \binom{h}{j} \frac{\Gamma(\vartheta m+\omega+1)\Gamma(\eta n+\tau+1)}{\Gamma(\xi(m-j)+1)\Gamma(h+1)} \\ &\quad\times \frac{(-1)^h}{\Gamma(\vartheta j+\omega+1)\Gamma(\eta(h-j)+\tau+1)\Gamma(\zeta(n-h+j)+1)}. \end{aligned} \tag{16.2.17}$$

**Theorem 7.** *If* $\lambda, \delta, \alpha, \nu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$; $\vartheta, \eta, \xi, \zeta, \sigma \in \mathbb{C}^+$; $\omega, \tau \in \mathbb{C}^+_{-1}$; $m, n \in \mathbb{N}$, *then*

$$\begin{aligned} &\frac{1}{\Gamma(\delta)} \int_0^1 u^{\lambda-1}(1-u)^{\delta-1} L_m^{(\vartheta,\omega)}[\xi; \sigma(1-u)] L_n^{(\eta,\tau)}[\zeta; \sigma(1-u)] R^q_{k,\alpha,\nu}\left[a, zu^\rho\right] du \\ &= \frac{z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \\ &\quad\times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h\, {}_1\Delta^{m,n,\xi,\zeta}_{\vartheta,\omega,\eta,\tau}\, {}_2\Psi_2 \\ &\quad\times \left[\begin{array}{c} (\lambda+\rho(\alpha-\nu-1), \rho\alpha q), (1,1) \\ (\frac{\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda+\delta+h+\rho(\alpha-\nu-1), \rho\alpha q) \end{array} \middle| \left(\frac{az^\alpha}{k^{\alpha/k}}\right)^q\right]. \end{aligned} \tag{16.2.18}$$

*Proof.* For convenience, denote the left-hand side of Eq. (16.2.18) by $\mathscr{I}$, then using Eqs. (16.1.10) and (16.2.16), further interchanging the order of summation and integration, we have:

$$\mathscr{I} = \sum_{h=0}^{m+n} \sum_{j=0}^{h} \frac{\Gamma(\vartheta m+\omega+1)\Gamma(\eta n+\tau+1)(\sigma)^h}{\Gamma(h-j+1)\Gamma(\xi(m-j)+1)\Gamma(j+1)}$$
$$\times \frac{(-1)^h}{\Gamma(\vartheta j+\omega+1)\Gamma(\eta(h-j)+\tau+1)\Gamma(\zeta(n-h+j)+1)} \sum_{n=0}^{\infty} \frac{a^{nq} z^{(nq+1)\alpha-\nu-1}}{\Gamma_k\left[(nq+1)\alpha-\nu\right]}$$
$$\times \frac{1}{\Gamma(\delta)} \int_0^1 u^{\lambda+\rho((nq+1)\alpha-\nu-1)-1}(1-u)^{\delta+h-1} du. \tag{16.2.19}$$

Using the integral formula in Eq. (16.2.19), after simplification, we have:

$$\mathscr{I} = \sum_{h=0}^{m+n} \sum_{j=0}^{h} \binom{h}{j} \frac{\Gamma(\vartheta m+\omega+1)\Gamma(\eta n+\tau+1)}{\Gamma(\xi(m-j)+1)\Gamma(h+1)}$$
$$\times \frac{(-1)^h}{\Gamma(\vartheta j+\omega+1)\Gamma(\eta(h-j)+\tau+1)\Gamma(\zeta(n-h+j)+1)} \frac{z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} (\delta)_h (\sigma)^h$$
$$\times \sum_{n=0}^{\infty} \frac{\Gamma\left[\lambda+\rho(\alpha-\nu-1)+\rho\alpha nq\right]\Gamma(1+n)}{\Gamma\left[\frac{\alpha nq}{k}+\frac{\alpha}{k}-\frac{\nu}{k}\right]\Gamma\left[\lambda+\rho(\alpha-\nu-1)+\delta+h+\rho\alpha nq\right]} \frac{1}{n!} \left(\frac{a z^{\alpha}}{k^{\alpha/k}}\right)^{nq}. \tag{16.2.20}$$

Further, Eq. (16.2.20) can be written as:

$$\mathscr{I} = \frac{z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \sum_{h=0}^{m+n} {}_1\Delta_{\vartheta,\omega,\eta,\tau}^{m,n,\xi,\zeta} (\delta)_h (\sigma)^h$$
$$\times \sum_{n=0}^{\infty} \frac{\Gamma\left[\lambda+\rho(\alpha-\nu-1)+\rho\alpha nq\right]\Gamma(1+n)}{\Gamma\left[\frac{\alpha nq}{k}+\frac{\alpha}{k}-\frac{\nu}{k}\right]\Gamma\left[\lambda+\rho(\alpha-\nu-1)+\delta+h+\rho\alpha nq\right]} \frac{1}{n!} \left(\frac{a z^{\alpha}}{k^{\alpha/k}}\right)^{nq}, \tag{16.2.21}$$

where ${}_1\Delta_{\vartheta,\omega,\eta,\tau}^{m,n,\xi,\zeta}(\delta)_h(\sigma)^h$ is defined in Eq. (16.2.17).

Now, interpreting Eq. (16.2.21) with the help of (16.1.12), we have the required result. □

**Theorem 8.** *If* $\lambda, \delta, \alpha, \nu, \mu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\mu\alpha-\nu) > 0$; $\Re(\delta) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$; $\vartheta, \eta, \xi, \zeta, \sigma \in \mathbb{C}^+$; $\omega, \tau \in \mathbb{C}_{-1}^+$; $m, n \in \mathbb{N}$, *then*

$$\frac{1}{\Gamma(\delta)} \int_0^1 u^{\lambda-1}(1-u)^{\delta-1} L_m^{(\vartheta,\omega)}[\xi;\sigma(1-u)] L_n^{(\eta,\tau)}[\zeta;\sigma(1-u)] G_{k,\alpha,\nu,\mu}^{q}\left[a, z u^{\rho}\right] du$$

$$= \frac{z^{\mu\alpha-\nu-1}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)} \times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h {}_1\Delta_{\vartheta,\omega,\eta,\tau}^{m,n,\xi,\zeta} {}_2\Psi_2 \times \left[ \begin{array}{c} (\lambda+\rho(\mu\alpha-\nu-1),\rho\alpha q),(\mu,q) \\ (\frac{\mu\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}),(\lambda+\delta+h+\rho(\mu\alpha-\nu-1),\rho\alpha q) \end{array} \Bigg| \left(\frac{az^\alpha}{k^{\alpha/k}}\right)^q \right]. \tag{16.2.22}$$

**Theorem 9.** *If* $\lambda,\delta,\alpha,\nu,q \in \mathbb{C}$; $z>0$; $a>0$; $\Re(\alpha-\nu)>0$; $\Re(\delta)>0$; $\Re(q)>0$; $k\in\mathbb{R}$; $\vartheta,\eta,\xi,\zeta,\sigma\in\mathbb{C}^+$; $\omega,\tau\in\mathbb{C}_{-1}^+$; $m,n\in\mathbb{N}$, *then*

$$\frac{1}{\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1} L_m^{(\vartheta,\omega)}[\xi;\sigma(x-s)] \times L_n^{(\eta,\tau)}[\zeta;\sigma(x-s)] R_{k,\alpha,\nu}^q\left[a,z(s-t)^\rho\right]du$$
$$= \frac{z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h {}_1\Delta_{\vartheta,\omega,\eta,\tau}^{m,n,\xi,\zeta} {}_2\Psi_2 \times \left[ \begin{array}{c} (\lambda+\rho(\alpha-\nu-1),\rho\alpha q),(1,1) \\ (\frac{\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}),(\lambda+\delta+h+\rho(\alpha-\nu-1),\rho\alpha q) \end{array} \Bigg| \left(\frac{az^\alpha}{k^{\alpha/k}}\right)^q \right]. \tag{16.2.23}$$

**Theorem 10.** *If* $\lambda,\delta,\alpha,\nu,\mu,q \in \mathbb{C}$; $z>0$; $a>0$; $\Re(\mu\alpha-\nu)>0$; $\Re(\delta)>0$; $\Re(q)>0$; $k\in\mathbb{R}$; $\vartheta,\eta,\xi,\zeta,\sigma\in\mathbb{C}^+$; $\omega,\tau\in\mathbb{C}_{-1}^+$; $m,n\in\mathbb{N}$, *then*

$$\frac{1}{\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1} L_m^{(\vartheta,\omega)}[\xi;\sigma(x-s)] \times L_n^{(\eta,\tau)}[\zeta;\sigma(x-s)] G_{k,\alpha,\nu,\mu}^q\left[a,z(s-t)^\rho\right]du$$
$$= \frac{z^{\mu\alpha-\nu-1}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)} \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h {}_1\Delta_{\vartheta,\omega,\eta,\tau}^{m,n,\xi,\zeta} \times {}_2\Psi_2 \left[ \begin{array}{c} (\lambda+\rho(\mu\alpha-\nu-1),\rho\alpha q),(\mu,q) \\ (\frac{\mu\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}),(\lambda+\delta+h+\rho(\mu\alpha-\nu-1),\rho\alpha q) \end{array} \Bigg| \left(\frac{az^\alpha}{k^{\alpha/k}}\right)^q \right]. \tag{16.2.24}$$

**Theorem 11.** *If* $\lambda,\delta,\alpha,\nu,q \in \mathbb{C}$; $z>0$; $a>0$; $\Re(\alpha-\nu)>0$; $\Re(\delta)>0$; $\Re(\lambda)>0$; $\Re(q)>0$; $k\in\mathbb{R}$; $\vartheta,\eta,\xi,\zeta,\sigma\in\mathbb{C}^+$; $\omega,\tau\in\mathbb{C}_{-1}^+$; $m,n\in\mathbb{N}$, *then*

$$\frac{1}{\Gamma(\delta)}\int_0^x t^{\delta-1}(x-t)^{\lambda-1} L_m^{(\vartheta,\omega)}[\xi;\sigma t] L_n^{(\eta,\tau)}[\zeta;\sigma t] R_{k,\alpha,\nu}^q\left[a,z(x-t)^\rho\right]dt$$
$$= \frac{z^{\alpha-\nu-1}x^{\lambda+\delta+h+\rho(\alpha-\nu-1)}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}}$$

$$\times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h {}_1\Delta_{\vartheta,\omega,\eta,\tau}^{m,n,\xi,\zeta} {}_2\Psi_2$$
$$\times \left[ \begin{array}{c} (\lambda + \rho(\alpha - \nu - 1), \rho\alpha q), (1, 1) \\ (\frac{\alpha}{k} - \frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda + \delta + h + \rho(\alpha - \nu - 1), \rho\alpha q) \end{array} \middle| \left( \frac{a\,(zx^{\rho})^{\alpha}}{k^{\alpha/k}} \right)^{q} \right]. \tag{16.2.25}$$

**Theorem 12.** *If* $\lambda, \delta, \alpha, \nu, \mu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\mu\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(\lambda) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$; $\vartheta, \eta, \xi, \zeta, \sigma \in \mathbb{C}^{+}$; $\omega, \tau \in \mathbb{C}^{+}_{-1}$; $m, n \in \mathbb{N}$, *then*

$$\frac{1}{\Gamma(\delta)} \int_0^x t^{\delta-1}(x-t)^{\lambda-1} L_m^{(\vartheta,\omega)}[\xi;\sigma t] L_n^{(\eta,\tau)}[\zeta;\sigma t] G^{q}_{k,\alpha,\nu,\mu}\left[a, z(x-t)^{\rho}\right] dt$$
$$= \frac{z^{\mu\alpha-\nu-1} x^{\lambda+\delta+h+\rho(\mu\alpha-\nu-1)}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)}$$
$$\times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h {}_1\Delta_{\vartheta,\omega,\eta,\tau}^{m,n,\xi,\zeta} {}_2\Psi_2$$
$$\times \left[ \begin{array}{c} (\lambda + \rho(\mu\alpha - \nu - 1), \rho\alpha q), (\mu, q) \\ (\frac{\mu\alpha}{k} - \frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda + \delta + h + \rho(\mu\alpha - \nu - 1), \rho\alpha q) \end{array} \middle| \left( \frac{a\,(zx^{\rho})^{\alpha}}{k^{\alpha/k}} \right)^{q} \right]. \tag{16.2.26}$$

*Proof.* Proofs of Theorems 8–12 would run parallel to that of Theorem 7, therefore we omit the details. □

**Note 2.** To investigate Theorems 13–18, ${}_2\Delta_{\vartheta,\omega,\eta,\tau}^{m,n,\xi,\zeta}$ is defined as:

$${}_2\Delta_{\vartheta,\omega,\eta,\tau}^{m,n,\xi,\zeta} = \frac{\Gamma(\vartheta m + \omega + 1)\,\Gamma(\eta n + \tau + 1)}{\Gamma(\zeta m + 1)\,\Gamma(\xi n + 1)} \times \sum_{j=0}^{h} \left[ \binom{h}{j} \frac{(-1)^{h-\zeta(h-j)-\xi j}(-\zeta m)_{\zeta(h-j)}(-\xi n)_{\xi j}}{\Gamma(\vartheta j + \omega + 1)\Gamma(\eta(h-j) + \tau + 1)} \right]. \tag{16.2.27}$$

**Theorem 13.** *If* $\lambda, \delta, \alpha, \nu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$; $\vartheta, \eta, \sigma \in \mathbb{C}^{+}$; $m, n, \xi, \zeta \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}^{+}_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)} \int_0^1 u^{\lambda-1}(1-u)^{\delta-1} L_m^{(\vartheta,\omega)}[\xi;\sigma(1-u)] L_n^{(\eta,\tau)}[\zeta;\sigma(1-u)] R^{q}_{k,\alpha,\nu}\left[a, zu^{\rho}\right] du$$
$$= \frac{z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}}$$
$$\times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h {}_2\Delta_{\vartheta,\omega,\eta,\tau}^{m,n,\xi,\zeta} {}_2\Psi_2$$
$$\times \left[ \begin{array}{c} (\lambda + \rho(\alpha - \nu - 1), \rho\alpha q), (1, 1) \\ (\frac{\alpha}{k} - \frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda + \delta + h + \rho(\alpha - \nu - 1), \rho\alpha q) \end{array} \middle| \left( \frac{az^{\alpha}}{k^{\alpha/k}} \right)^{q} \right]. \tag{16.2.28}$$

*Proof.* For $\xi, \zeta \in \mathbb{N}$, employing the results given in Eqs. (16.1.28) and (16.1.3), Eq. (16.2.20) reduces to:

$$\mathscr{I} = \frac{z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \frac{\Gamma(\vartheta m+\omega+1)\Gamma(\eta n+\tau+1)}{\Gamma(\zeta m+1)\Gamma(\xi n+1)} \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h$$
$$\times \sum_{j=0}^{h} \left[ \binom{h}{j} \frac{(-1)^{h-\zeta(h-j)-\xi j}(-\zeta m)_{\zeta(h-j)}(-\xi n)_{\xi j}}{\Gamma(\vartheta j+\omega+1)\Gamma(\eta(h-j)+\tau+1)} \right]$$
$$\times \sum_{n=0}^{\infty} \frac{\Gamma[\lambda+\rho(\alpha-\nu-1)+\rho\alpha nq]\Gamma(1+n)}{\Gamma\left[\frac{\alpha nq}{k}+\frac{\alpha}{k}-\frac{\nu}{k}\right]\Gamma[\lambda+\rho(\alpha-\nu-1)+\delta+h+\rho\alpha nq]} \frac{1}{n!} \left(\frac{az^{\alpha}}{k^{\alpha/k}}\right)^{nq}. \tag{16.2.29}$$

Interpreting Eq. (16.2.29) in view of (16.1.12), we have the required result (16.2.28). □

**Theorem 14.** *If* $\lambda, \delta, \alpha, \nu, \mu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\mu\alpha-\nu) > 0$; $\Re(\delta) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$; $\vartheta, \eta, \sigma \in \mathbb{C}^+$; $m, n, \xi, \zeta \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}^+_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)} \int_0^1 u^{\lambda-1}(1-u)^{\delta-1} L_m^{(\vartheta,\omega)}[\xi;\sigma(1-u)] L_n^{(\eta,\tau)}[\zeta;\sigma(1-u)] G^q_{k,\alpha,\nu,\mu}\left[a, zu^{\rho}\right] du$$
$$= \frac{z^{\mu\alpha-\nu-1}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)}$$
$$\times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h {}_2\Delta^{m,n,\xi,\zeta}_{\vartheta,\omega,\eta,\tau} {}_2\Psi_2$$
$$\times \left[ \begin{matrix} (\lambda+\rho(\mu\alpha-\nu-1), \rho\alpha q), (\mu, q) \\ (\frac{\mu\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda+\delta+h+\rho(\mu\alpha-\nu-1), \rho\alpha q) \end{matrix} \middle| \left(\frac{az^{\alpha}}{k^{\alpha/k}}\right)^q \right]. \tag{16.2.30}$$

**Theorem 15.** *If* $\lambda, \delta, \alpha, \nu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\alpha-\nu) > 0$; $\Re(\delta) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$; $\vartheta, \eta, \sigma \in \mathbb{C}^+$; $m, n, \xi, \zeta \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}^+_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)} \int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1} L_m^{(\vartheta,\omega)}[\xi;\sigma(x-s)] L_n^{(\eta,\tau)}[\zeta;\sigma(x-s)]$$
$$\times R^q_{k,\alpha,\nu}\left[a, z(s-t)^{\rho}\right] du = \frac{z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}}$$
$$\times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h {}_2\Delta^{m,n,\xi,\zeta}_{\vartheta,\omega,\eta,\tau} {}_2\Psi_2$$
$$\times \left[ \begin{matrix} (\lambda+\rho(\alpha-\nu-1), \rho\alpha q), (1, 1) \\ (\frac{\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda+\delta+h+\rho(\alpha-\nu-1), \rho\alpha q) \end{matrix} \middle| \left(\frac{az^{\alpha}}{k^{\alpha/k}}\right)^q \right]. \tag{16.2.31}$$

**Theorem 16.** *If* $\lambda, \delta, \alpha, \nu, \mu, q \in \mathbb{C};\ z > 0; a > 0; \Re(\mu\alpha - \nu) > 0;\ \Re(\delta) > 0;\ \Re(q) > 0;\ k \in \mathbb{R};\ \vartheta, \eta, \sigma \in \mathbb{C}^+;\ m, n, \xi, \zeta \in \mathbb{N};\ \omega, \tau \in \mathbb{C}^+_{-1}$, *then*

$$\begin{aligned}
&\frac{1}{\Gamma(\delta)} \int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1} L_m^{(\vartheta,\omega)}[\xi; \sigma(x-s)] \\
&\quad \times L_n^{(\eta,\tau)}[\zeta; \sigma(x-s)] G^q_{k,\alpha,\nu,\mu}\left[a, z(s-t)^\rho\right] du \\
&= \frac{z^{\mu\alpha-\nu-1}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)} \\
&\quad \times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h {}_2\Delta^{m,n,\xi,\zeta}_{\vartheta,\omega,\eta,\tau} {}_2\Psi_2 \\
&\quad \times \left[ \begin{array}{c} (\lambda + \rho(\mu\alpha - \nu - 1), \rho\alpha q), (\mu, q) \\ (\frac{\mu\alpha}{k} - \frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda + \delta + h + \rho(\mu\alpha - \nu - 1), \rho\alpha q) \end{array} \middle| \left(\frac{a z^\alpha}{k^{\alpha/k}}\right)^q \right].
\end{aligned} \tag{16.2.32}$$

**Theorem 17.** *If* $\lambda, \delta, \alpha, \nu, q \in \mathbb{C}; z > 0; a > 0; \Re(\alpha - \nu) > 0; \Re(\delta) > 0; \Re(\lambda) > 0;\ \Re(q) > 0;\ k \in \mathbb{R};\ \vartheta, \eta, \sigma \in \mathbb{C}^+;\ m, n, \xi, \zeta \in \mathbb{N};\ \omega, \tau \in \mathbb{C}^+_{-1}$, *then*

$$\begin{aligned}
&\frac{1}{\Gamma(\delta)} \int_0^x t^{\delta-1}(x-t)^{\lambda-1} L_m^{(\vartheta,\omega)}[\xi; \sigma t] L_n^{(\eta,\tau)}[\zeta; \sigma t] R^q_{k,\alpha,\nu}\left[a, z(x-t)^\rho\right] dt \\
&= \frac{z^{\alpha-\nu-1} x^{\lambda+\delta+h+\rho(\alpha-\nu-1)}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \\
&\quad \times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h {}_2\Delta^{m,n,\xi,\zeta}_{\vartheta,\omega,\eta,\tau} {}_2\Psi_2 \\
&\quad \times \left[ \begin{array}{c} (\lambda + \rho(\alpha - \nu - 1), \rho\alpha q), (1, 1) \\ (\frac{\alpha}{k} - \frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda + \delta + h + \rho(\alpha - \nu - 1), \rho\alpha q) \end{array} \middle| \left(\frac{a\,(z x^\rho)^\alpha}{k^{\alpha/k}}\right)^q \right].
\end{aligned} \tag{16.2.33}$$

**Theorem 18.** *If* $\lambda, \delta, \alpha, \nu, \mu, q \in \mathbb{C}; z > 0; a > 0; \Re(\mu\alpha - \nu) > 0; \Re(\delta) > 0;\ \Re(\lambda) > 0;\ \Re(q) > 0;\ k \in \mathbb{R};\ \vartheta, \eta, \sigma \in \mathbb{C}^+;\ m, n, \xi, \zeta \in \mathbb{N};\ \omega, \tau \in \mathbb{C}^+_{-1}$, *then*

$$\begin{aligned}
&\frac{1}{\Gamma(\delta)} \int_0^x t^{\delta-1}(x-t)^{\lambda-1} L_m^{(\vartheta,\omega)}[\xi; \sigma t] L_n^{(\eta,\tau)}[\zeta; \sigma t] G^q_{k,\alpha,\nu,\mu}\left[a, z(x-t)^\rho\right] dt \\
&= \frac{z^{\mu\alpha-\nu-1} x^{\lambda+\delta+h+\rho(\mu\alpha-\nu-1)}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)} \\
&\quad \times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h {}_2\Delta^{m,n,\xi,\zeta}_{\vartheta,\omega,\eta,\tau} {}_2\Psi_2 \\
&\quad \times \left[ \begin{array}{c} (\lambda + \rho(\mu\alpha - \nu - 1), \rho\alpha q), (\mu, q) \\ (\frac{\mu\alpha}{k} - \frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda + \delta + h + \rho(\mu\alpha - \nu - 1), \rho\alpha q) \end{array} \middle| \left(\frac{a\,(z x^\rho)^\alpha}{k^{\alpha/k}}\right)^q \right].
\end{aligned} \tag{16.2.34}$$

*Proof.* Proofs of Theorems 14–18 are similar to that of Theorem 13. □

## 16.3 Special cases

On setting $\xi = \zeta = 1$, the results in Theorems 13, 14, 15, 16, 17, and 18 reduce to the following form:

**Corollary 1.** *If* $\lambda, \delta, \alpha, \nu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$; $\vartheta, \eta, \sigma \in \mathbb{C}^+$; $m, n \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}^+_{-1}$, *then*

$$\begin{aligned}
&\frac{1}{\Gamma(\delta)} \int_0^1 u^{\lambda-1}(1-u)^{\delta-1} L_m^{(\vartheta,\omega)}[\sigma(1-u)] L_n^{(\eta,\tau)}[\sigma(1-u)] R^q_{k,\alpha,\nu}\left[a, zu^{\rho}\right] du \\
&= \frac{z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \\
&\quad \times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h \frac{\Gamma(\vartheta m+\omega+1)\,\Gamma(\eta n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)} \\
&\quad \times \sum_{j=0}^{h} \left[ \binom{h}{j} \frac{(-m)_{(h-j)}(-n)_j}{\Gamma(\vartheta j+\omega+1)\Gamma(\eta(h-j)+\tau+1)} \right] \\
&\quad \times {}_2\Psi_2\left[ \begin{matrix} (\lambda+\rho(\alpha-\nu-1), \rho\alpha q), (1,1) \\ (\frac{\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda+\delta+h+\rho(\alpha-\nu-1), \rho\alpha q) \end{matrix} \,\middle|\, \left(\frac{az^{\alpha}}{k^{\alpha/k}}\right)^q \right].
\end{aligned} \tag{16.3.1}$$

**Corollary 2.** *If* $\lambda, \delta, \alpha, \nu, \mu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\mu\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$; $\vartheta, \eta, \sigma \in \mathbb{C}^+$; $m, n \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}^+_{-1}$, *then*

$$\begin{aligned}
&\frac{1}{\Gamma(\delta)} \int_0^1 u^{\lambda-1}(1-u)^{\delta-1} L_m^{(\vartheta,\omega)}[\sigma(1-u)] L_n^{(\eta,\tau)}[\sigma(1-u)] G^q_{k,\alpha,\nu,\mu}\left[a, zu^{\rho}\right] du \\
&= \frac{z^{\mu\alpha-\nu-1}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)} \\
&\quad \times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h \frac{\Gamma(\vartheta m+\omega+1)\,\Gamma(\eta n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)} \\
&\quad \times \sum_{j=0}^{h} \left[ \binom{h}{j} \frac{(-m)_{(h-j)}(-n)_j}{\Gamma(\vartheta j+\omega+1)\Gamma(\eta(h-j)+\tau+1)} \right] \\
&\quad \times {}_2\Psi_2\left[ \begin{matrix} (\lambda+\rho(\mu\alpha-\nu-1), \rho\alpha q), (\mu, q) \\ (\frac{\mu\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda+\delta+h+\rho(\mu\alpha-\nu-1), \rho\alpha q) \end{matrix} \,\middle|\, \left(\frac{az^{\alpha}}{k^{\alpha/k}}\right)^q \right].
\end{aligned} \tag{16.3.2}$$

**Corollary 3.** *If* $\lambda, \delta, \alpha, \nu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$; $\vartheta, \eta, \sigma \in \mathbb{C}^+$; $m, n \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}^+_{-1}$, *then*

$$\begin{aligned}
&\frac{1}{\Gamma(\delta)} \int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1} L_m^{(\vartheta,\omega)}[\sigma(x-s)] L_n^{(\eta,\tau)}[\sigma(x-s)] \\
&\quad \times R^q_{k,\alpha,\nu}\left[a, z(s-t)^\rho\right] du = \frac{z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \\
&\quad \times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h \frac{\Gamma(\vartheta m+\omega+1)\,\Gamma(\eta n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)} \\
&\quad \times \sum_{j=0}^{h} \left[ \binom{h}{j} \frac{(-m)_{(h-j)}(-n)_j}{\Gamma(\vartheta j+\omega+1)\Gamma(\eta(h-j)+\tau+1)} \right] \\
&\quad \times {}_2\Psi_2\left[ \begin{array}{c} (\lambda+\rho(\alpha-\nu-1), \rho\alpha q), (1,1) \\ (\frac{\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda+\delta+h+\rho(\alpha-\nu-1), \rho\alpha q) \end{array} \middle| \left(\frac{az^\alpha}{k^{\alpha/k}}\right)^q \right].
\end{aligned} \tag{16.3.3}$$

**Corollary 4.** *If* $\lambda, \delta, \alpha, \nu, \mu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\mu\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$; $\vartheta, \eta, \sigma \in \mathbb{C}^+$; $m, n \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}^+_{-1}$, *then*

$$\begin{aligned}
&\frac{1}{\Gamma(\delta)} \int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1} L_m^{(\vartheta,\omega)}[\sigma(x-s)] L_n^{(\eta,\tau)}[\sigma(x-s)] \\
&\quad \times G^q_{k,\alpha,\nu,\mu}\left[a, z(s-t)^\rho\right] du = \frac{z^{\mu\alpha-\nu-1}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)} \\
&\quad \times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h \frac{\Gamma(\vartheta m+\omega+1)\,\Gamma(\eta n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)} \\
&\quad \times \sum_{j=0}^{h} \left[ \binom{h}{j} \frac{(-m)_{(h-j)}(-n)_j}{\Gamma(\vartheta j+\omega+1)\Gamma(\eta(h-j)+\tau+1)} \right] \\
&\quad \times {}_2\Psi_2\left[ \begin{array}{c} (\lambda+\rho(\mu\alpha-\nu-1), \rho\alpha q), (\mu, q) \\ (\frac{\mu\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda+\delta+h+\rho(\mu\alpha-\nu-1), \rho\alpha q) \end{array} \middle| \left(\frac{az^\alpha}{k^{\alpha/k}}\right)^q \right].
\end{aligned} \tag{16.3.4}$$

**Corollary 5.** *If* $\lambda, \delta, \alpha, \nu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(\lambda) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$; $\vartheta, \eta, \sigma \in \mathbb{C}^+$; $m, n \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}^+_{-1}$, *then*

$$\begin{aligned}
&\frac{1}{\Gamma(\delta)} \int_0^x t^{\delta-1}(x-t)^{\lambda-1} L_m^{(\vartheta,\omega)}[\sigma t] L_n^{(\eta,\tau)}[\sigma t] R^q_{k,\alpha,\nu}\left[a, z(x-t)^\rho\right] dt \\
&= \frac{z^{\alpha-\nu-1} x^{\lambda+\delta+h+\rho(\alpha-\nu-1)}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}}
\end{aligned}$$

$$\times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h \frac{\Gamma(\vartheta m+\omega+1)\Gamma(\eta n+\tau+1)}{\Gamma(m+1)\Gamma(n+1)}$$
$$\times \sum_{j=0}^{h} \left[ \binom{h}{j} \frac{(-m)_{(h-j)}(-n)_j}{\Gamma(\vartheta j+\omega+1)\Gamma(\eta(h-j)+\tau+1)} \right]$$
$$\times {}_2\Psi_2 \left[ \begin{matrix} (\lambda+\rho(\alpha-\nu-1),\rho\alpha q),(1,1) \\ (\frac{\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}),(\lambda+\delta+h+\rho(\alpha-\nu-1),\rho\alpha q) \end{matrix} \middle| \left( \frac{a\,(zx^{\rho})^{\alpha}}{k^{\alpha/k}} \right)^{q} \right]. \tag{16.3.5}$$

**Corollary 6.** *If* $\lambda,\delta,\alpha,\nu,\mu,q \in \mathbb{C}$; $z>0$; $a>0$; $\Re(\mu\alpha-\nu)>0$; $\Re(\delta)>0$; $\Re(\lambda)>0$; $\Re(q)>0$; $k\in\mathbb{R}$; $\vartheta,\eta,\sigma \in \mathbb{C}^+$; $m,n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^+_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)} \int_0^x t^{\delta-1}(x-t)^{\lambda-1} L_m^{(\vartheta,\omega)}[\sigma t] L_n^{(\eta,\tau)}[\sigma t] G^q_{k,\alpha,\nu,\mu}\left[a, z(x-t)^{\rho}\right] dt$$
$$= \frac{z^{\mu\alpha-\nu-1} x^{\lambda+\delta+h+\rho(\mu\alpha-\nu-1)}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)}$$
$$\times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h \frac{\Gamma(\vartheta m+\omega+1)\Gamma(\eta n+\tau+1)}{\Gamma(m+1)\Gamma(n+1)}$$
$$\times \sum_{j=0}^{h} \left[ \binom{h}{j} \frac{(-m)_{(h-j)}(-n)_j}{\Gamma(\vartheta j+\omega+1)\Gamma(\eta(h-j)+\tau+1)} \right]$$
$$\times {}_2\Psi_2 \left[ \begin{matrix} (\lambda+\rho(\mu\alpha-\nu-1),\rho\alpha q),(\mu,q) \\ (\frac{\mu\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}),(\lambda+\delta+h+\rho(\mu\alpha-\nu-1),\rho\alpha q) \end{matrix} \middle| \left( \frac{a\,(zx^{\rho})^{\alpha}}{k^{\alpha/k}} \right)^{q} \right]. \tag{16.3.6}$$

On setting $\vartheta=\eta=\xi=\zeta=1$ and using Eq. (16.1.18), we have $L_n^{1,\omega}(1;x) = Z_n^{(1,\omega)}(x;1)$; the results in Theorems 13, 14, 15, 16, 17, and 18 reduce to the following form.

**Corollary 7.** *If* $\lambda,\delta,\alpha,\nu,q \in \mathbb{C}$; $z>0$; $a>0$; $\Re(\alpha-\nu)>0$; $\Re(\delta)>0$; $\Re(q)>0$; $k\in\mathbb{R}$; $\sigma\in\mathbb{C}^+$; $m,n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^+_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)} \int_0^1 u^{\lambda-1}(1-u)^{\delta-1} L_m^{(1,\omega)}[\sigma(1-u)] L_n^{(1,\tau)}[\sigma(1-u)] R^q_{k,\alpha,\nu}\left[a, zu^{\rho}\right] du$$
$$= \frac{z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}}$$
$$\times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h \frac{\Gamma(m+\omega+1)\Gamma(n+\tau+1)}{\Gamma(m+1)\Gamma(n+1)}$$
$$\times \sum_{j=0}^{h} \left[ \binom{h}{j} \frac{(-m)_{(h-j)}(-n)_j}{\Gamma(j+\omega+1)\Gamma(h-j+\tau+1)} \right]$$

$$\times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\alpha-\nu-1),\rho\alpha q),(1,1)\\(\frac{\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}),(\lambda+\delta+h+\rho(\alpha-\nu-1),\rho\alpha q)\end{array}\left|\left(\frac{az^{\alpha}}{k^{\alpha/k}}\right)^{q}\right.\right]. \tag{16.3.7}$$

**Corollary 8.** *If* $\lambda,\delta,\alpha,\nu,\mu,q\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\mu\alpha-\nu)>0$; $\Re(\delta)>0$; $\Re(q)>0$; $k\in\mathbb{R}$; $\sigma\in\mathbb{C}^{+}$; $m,n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^{+}_{-1}$, *then*

$$\begin{aligned}&\frac{1}{\Gamma(\delta)}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}L_m^{(1,\omega)}[\sigma(1-u)]L_n^{(1,\tau)}[\sigma(1-u)]G^{q}_{k,\alpha,\nu,\mu}\left[a,zu^{\rho}\right]du\\&=\frac{z^{\mu\alpha-\nu-1}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)}\\&\quad\times\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h\frac{\Gamma(m+\omega+1)\,\Gamma(n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)}\\&\quad\times\sum_{j=0}^{h}\left[\binom{h}{j}\frac{(-m)_{(h-j)}(-n)_j}{\Gamma(j+\omega+1)\Gamma(h-j+\tau+1)}\right]\\&\quad\times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\mu\alpha-\nu-1),\rho\alpha q),(\mu,q)\\(\frac{\mu\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}),(\lambda+\delta+h+\rho(\mu\alpha-\nu-1),\rho\alpha q)\end{array}\left|\left(\frac{az^{\alpha}}{k^{\alpha/k}}\right)^{q}\right.\right].\end{aligned} \tag{16.3.8}$$

**Corollary 9.** *If* $\lambda,\delta,\alpha,\nu,q\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\alpha-\nu)>0$; $\Re(\delta)>0$; $\Re(q)>0$; $k\in\mathbb{R}$; $\sigma\in\mathbb{C}^{+}$; $m,n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^{+}_{-1}$, *then*

$$\begin{aligned}&\frac{1}{\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1}L_m^{(1,\omega)}[\sigma(x-s)]L_n^{(1,\tau)}[\sigma(x-s)]\\&\quad\times R^{q}_{k,\alpha,\nu}\left[a,z(s-t)^{\rho}\right]du=\frac{z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}}\\&\quad\times\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h\frac{\Gamma(m+\omega+1)\,\Gamma(n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)}\\&\quad\times\sum_{j=0}^{h}\left[\binom{h}{j}\frac{(-m)_{(h-j)}(-n)_j}{\Gamma(j+\omega+1)\Gamma(h-j+\tau+1)}\right]\\&\quad\times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\alpha-\nu-1),\rho\alpha q),(1,1)\\(\frac{\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}),(\lambda+\delta+h+\rho(\alpha-\nu-1),\rho\alpha q)\end{array}\left|\left(\frac{az^{\alpha}}{k^{\alpha/k}}\right)^{q}\right.\right].\end{aligned} \tag{16.3.9}$$

**Corollary 10.** *If* $\lambda,\delta,\alpha,\nu,\mu,q\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\mu\alpha-\nu)>0$; $\Re(\delta)>0$; $\Re(q)>0$; $k\in\mathbb{R}$; $\sigma\in\mathbb{C}^{+}$; $m,n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^{+}_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1}L_m^{(1,\omega)}[\sigma(x-s)]$$

$$\times L_n^{(1,\tau)}[\sigma(x-s)]G_{k,\alpha,\nu,\mu}^{q}\left[a, z(s-t)^{\rho}\right]du$$
$$= \frac{z^{\mu\alpha-\nu-1}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)}$$
$$\times \sum_{h=0}^{m+n}(\delta)_h(\sigma)^h \frac{\Gamma(m+\omega+1)\,\Gamma(n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)}$$
$$\times \sum_{j=0}^{h}\left[\binom{h}{j}\frac{(-m)_{(h-j)}(-n)_j}{\Gamma(j+\omega+1)\Gamma(h-j+\tau+1)}\right]$$
$$\times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\mu\alpha-\nu-1),\rho\alpha q),(\mu,q)\\ (\frac{\mu\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}),(\lambda+\delta+h+\rho(\mu\alpha-\nu-1),\rho\alpha q)\end{array}\middle|\left(\frac{az^{\alpha}}{k^{\alpha/k}}\right)^{q}\right]. \tag{16.3.10}$$

**Corollary 11.** *If* $\lambda,\delta,\alpha,\nu,q\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\alpha-\nu)>0$; $\Re(\delta)>0$; $\Re(\lambda)>0$; $\Re(q)>0$; $k\in\mathbb{R}$; $\sigma\in\mathbb{C}^{+}$; $m,n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}_{-1}^{+}$, *then*

$$\frac{1}{\Gamma(\delta)}\int_0^x t^{\delta-1}(x-t)^{\lambda-1}L_m^{(1,\omega)}[\sigma t]L_n^{(1,\tau)}[\sigma t]R_{k,\alpha,\nu}^{q}\left[a, z(x-t)^{\rho}\right]dt$$
$$= \frac{z^{\alpha-\nu-1}x^{\lambda+\delta+h+\rho(\alpha-\nu-1)}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}}$$
$$\times \sum_{h=0}^{m+n}(\delta)_h(\sigma)^h \frac{\Gamma(m+\omega+1)\,\Gamma(n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)}$$
$$\times \sum_{j=0}^{h}\left[\binom{h}{j}\frac{(-m)_{(h-j)}(-n)_j}{\Gamma(j+\omega+1)\Gamma(h-j+\tau+1)}\right]$$
$$\times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\alpha-\nu-1),\rho\alpha q),(1,1)\\ (\frac{\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}),(\lambda+\delta+h+\rho(\alpha-\nu-1),\rho\alpha q)\end{array}\middle|\left(\frac{a\,(zx^{\rho})^{\alpha}}{k^{\alpha/k}}\right)^{q}\right]. \tag{16.3.11}$$

**Corollary 12.** *If* $\lambda,\delta,\alpha,\nu,\mu,q\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\mu\alpha-\nu)>0$; $\Re(\delta)>0$; $\Re(\lambda)>0$; $\Re(q)>0$; $k\in\mathbb{R}$; $\sigma\in\mathbb{C}^{+}$; $m,n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}_{-1}^{+}$, *then*

$$\frac{1}{\Gamma(\delta)}\int_0^x t^{\delta-1}(x-t)^{\lambda-1}L_m^{(1,\omega)}[\sigma t]L_n^{(1,\tau)}[\sigma t]G_{k,\alpha,\nu,\mu}^{q}\left[a, z(x-t)^{\rho}\right]dt$$
$$= \frac{z^{\mu\alpha-\nu-1}x^{\lambda+\delta+h+\rho(\mu\alpha-\nu-1)}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)}$$
$$\times \sum_{h=0}^{m+n}(\delta)_h(\sigma)^h \frac{\Gamma(m+\omega+1)\,\Gamma(n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)}$$

$$\times \sum_{j=0}^{h}\left[\binom{h}{j}\frac{(-m)_{(h-j)}(-n)_j}{\Gamma(j+\omega+1)\Gamma(h-j+\tau+1)}\right]$$
$$\times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\mu\alpha-\nu-1),\rho\alpha q),(\mu,q)\\ (\frac{\mu\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}),(\lambda+\delta+h+\rho(\mu\alpha-\nu-1),\rho\alpha q)\end{array}\Bigg|\left(\frac{a\,(zx^{\rho})^{\alpha}}{k^{\alpha/k}}\right)^{q}\right]. \tag{16.3.12}$$

On setting $\vartheta=\eta=0$, $\xi=\zeta=1$, the results in Theorems 13, 14, 15, 16, 17, and 18 reduce to the following form.

**Corollary 13.** *If $\lambda,\delta,\alpha,\nu,q\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\alpha-\nu)>0$; $\Re(\delta)>0$; $\Re(q)>0$; $k\in\mathbb{R}$; $\sigma\in\mathbb{C}^{+}$; $n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^{+}_{-1}$, then*

$$\frac{1}{\Gamma(\delta)}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}[1-\sigma(1-u)]^n R^{q}_{k,\alpha,\nu}\left[a,zu^{\rho}\right]du=\frac{z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}}$$
$$\times\sum_{h=0}^{n}(\delta)_h(\sigma)^h(-n)_h$$
$$\times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\alpha-\nu-1),\rho\alpha q),(1,1)\\ (\frac{\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}),(\lambda+\delta+h+\rho(\alpha-\nu-1),\rho\alpha q)\end{array}\Bigg|\left(\frac{az^{\alpha}}{k^{\alpha/k}}\right)^{q}\right]. \tag{16.3.13}$$

*Proof.* On setting $\vartheta=\eta=0$, $\xi=\zeta=1$, Eq. (16.2.28) reduces to the following form:

$$\frac{1}{\Gamma(\delta)}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}L_m^{(0,\omega)}[1;\sigma(1-u)]L_n^{(0,\tau)}[1;\sigma(1-u)]R^{q}_{k,\alpha,\nu}\left[a,zu^{\rho}\right]du$$
$$=\frac{z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}}\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h\frac{1}{\Gamma(m+1)\Gamma(n+1)}\sum_{j=0}^{h}\left[\binom{h}{j}(-m)_{(h-j)}(-n)_j\right]$$
$$\times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\alpha-\nu-1),\rho\alpha q),(1,1)\\ (\frac{\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}),(\lambda+\delta+h+\rho(\alpha-\nu-1),\rho\alpha q)\end{array}\Bigg|\left(\frac{az^{\alpha}}{k^{\alpha/k}}\right)^{q}\right]. \tag{16.3.14}$$

Using the result (16.1.29), Eq. (16.3.14) reduces to:

$$\frac{1}{\Gamma(\delta)}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}L_m^{(0,\omega)}[1;\sigma(1-u)]L_n^{(0,\tau)}[1;\sigma(1-u)]R^{q}_{k,\alpha,\nu}\left[a,zu^{\rho}\right]du$$
$$=\frac{z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}}\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h\frac{(-m-n)_h}{\Gamma(m+1)\Gamma(n+1)}$$
$$\times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\alpha-\nu-1),\rho\alpha q),(1,1)\\ (\frac{\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}),(\lambda+\delta+h+\rho(\alpha-\nu-1),\rho\alpha q)\end{array}\Bigg|\left(\frac{az^{\alpha}}{k^{\alpha/k}}\right)^{q}\right]. \tag{16.3.15}$$

Also, applying the case (16.1.14) and using the results (16.1.28)–(16.1.31), we have:

$$L_n^{(0,\tau)}(x) = \frac{1}{\Gamma(n+1)} \sum_{r=0}^{h} \binom{h}{r} (-x)^r = \frac{1}{\Gamma(n+1)} (1-x)^n. \tag{16.3.16}$$

Using the above result (16.3.16), Eq. (16.3.15) reduces to:

$$\begin{aligned} &\frac{1}{\Gamma(\delta)} \int_0^1 u^{\lambda-1} (1-u)^{\delta-1} \frac{[1-\sigma(1-u)]^{m+n}}{\Gamma(m+1)\Gamma(n+1)} R^q_{k,\alpha,\nu}\left[a, zu^{\rho}\right] du \\ &= \frac{z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h \frac{(-m-n)_h}{\Gamma(m+1)\Gamma(n+1)} \\ &\quad \times {}_2\Psi_2\left[ \begin{matrix} (\lambda+\rho(\alpha-\nu-1), \rho\alpha q), (1,1) \\ (\frac{\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda+\delta+h+\rho(\alpha-\nu-1), \rho\alpha q) \end{matrix} \,\middle|\, \left(\frac{a z^{\alpha}}{k^{\alpha/k}}\right)^q \right]. \end{aligned} \tag{16.3.17}$$

Replacing $m+n$ by $n$ we have the required result (16.3.13). □

**Corollary 14.** *If* $\lambda, \delta, \alpha, \nu, \mu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\mu\alpha-\nu) > 0$; $\Re(\delta) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$; $\sigma \in \mathbb{C}^+$; $n \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}^+_{-1}$, *then*

$$\begin{aligned} &\frac{1}{\Gamma(\delta)} \int_0^1 u^{\lambda-1} (1-u)^{\delta-1} [1-\sigma(1-u)]^n G^q_{k,\alpha,\nu,\mu}\left[a, zu^{\rho}\right] du = \frac{z^{\mu\alpha-\nu-1}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)} \\ &\quad \times \sum_{h=0}^{n} (\delta)_h (\sigma)^h (-n)_h \\ &\quad \times {}_2\Psi_2\left[ \begin{matrix} (\lambda+\rho(\mu\alpha-\nu-1), \rho\alpha q), (\mu, q) \\ (\frac{\mu\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda+\delta+h+\rho(\mu\alpha-\nu-1), \rho\alpha q) \end{matrix} \,\middle|\, \left(\frac{a z^{\alpha}}{k^{\alpha/k}}\right)^q \right]. \end{aligned} \tag{16.3.18}$$

**Corollary 15.** *If* $\lambda, \delta, \alpha, \nu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\alpha-\nu) > 0$; $\Re(\delta) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$; $\sigma \in \mathbb{C}^+$; $n \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}^+_{-1}$, *then*

$$\begin{aligned} &\frac{1}{\Gamma(\delta)} \int_t^x (x-s)^{\delta-1} (s-t)^{\lambda-1} [1-\sigma(x-s)]^n R^q_{k,\alpha,\nu}\left[a, z(s-t)^{\rho}\right] du = \frac{z^{\alpha-\nu-1}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \\ &\quad \times \sum_{h=0}^{n} (\delta)_h (\sigma)^h (-n)_h \\ &\quad \times {}_2\Psi_2\left[ \begin{matrix} (\lambda+\rho(\alpha-\nu-1), \rho\alpha q), (1,1) \\ (\frac{\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda+\delta+h+\rho(\alpha-\nu-1), \rho\alpha q) \end{matrix} \,\middle|\, \left(\frac{a z^{\alpha}}{k^{\alpha/k}}\right)^q \right]. \end{aligned} \tag{16.3.19}$$

**Corollary 16.** *If* $\lambda, \delta, \alpha, \nu, \mu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\mu\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$; $\sigma \in \mathbb{C}^{+}$; $n \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}^{+}_{-1}$, *then*

$$\begin{aligned}
&\frac{1}{\Gamma(\delta)} \int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1}[1-\sigma(x-s)]^n G^q_{k,\alpha,\nu,\mu}\left[a, z(s-t)^\rho\right] du \\
&\quad = \frac{z^{\mu\alpha-\nu-1}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)} \\
&\qquad \times \sum_{h=0}^{n} (\delta)_h (\sigma)^h (-n)_h \\
&\qquad \times {}_2\Psi_2\left[\begin{matrix} (\lambda+\rho(\mu\alpha-\nu-1), \rho\alpha q), (\mu, q) \\ (\frac{\mu\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda+\delta+h+\rho(\mu\alpha-\nu-1), \rho\alpha q) \end{matrix} \middle| \left(\frac{a z^\alpha}{k^{\alpha/k}}\right)^q\right].
\end{aligned} \tag{16.3.20}$$

**Corollary 17.** *If* $\lambda, \delta, \alpha, \nu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(\lambda) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$; $\sigma \in \mathbb{C}^{+}$; $n \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}^{+}_{-1}$, *then*

$$\begin{aligned}
&\frac{1}{\Gamma(\delta)} \int_0^x t^{\delta-1}(x-t)^{\lambda-1}[1-\sigma t]^n R^q_{k,\alpha,\nu}\left[a, z(x-t)^\rho\right] dt \\
&\quad = \frac{z^{\alpha-\nu-1} x^{\lambda+\delta+h+\rho(\alpha-\nu-1)}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \\
&\qquad \times \sum_{h=0}^{n} (\delta)_h (\sigma)^h (-n)_h \\
&\qquad \times {}_2\Psi_2\left[\begin{matrix} (\lambda+\rho(\alpha-\nu-1), \rho\alpha q), (1, 1) \\ (\frac{\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda+\delta+h+\rho(\alpha-\nu-1), \rho\alpha q) \end{matrix} \middle| \left(\frac{a\,(z x^\rho)^\alpha}{k^{\alpha/k}}\right)^q\right].
\end{aligned} \tag{16.3.21}$$

**Corollary 18.** *If* $\lambda, \delta, \alpha, \nu, \mu, q \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\mu\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(\lambda) > 0$; $\Re(q) > 0$; $k \in \mathbb{R}$; $\sigma \in \mathbb{C}^{+}$; $n \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}^{+}_{-1}$, *then*

$$\begin{aligned}
&\frac{1}{\Gamma(\delta)} \int_0^x t^{\delta-1}(x-t)^{\lambda-1}[1-\sigma t]^n G^q_{k,\alpha,\nu,\mu}\left[a, z(x-t)^\rho\right] dt \\
&\quad = \frac{z^{\mu\alpha-\nu-1} x^{\lambda+\delta+h+\rho(\mu\alpha-\nu-1)}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)} \\
&\qquad \times \sum_{h=0}^{n} (\delta)_h (\sigma)^h (-n)_h \\
&\qquad \times {}_2\Psi_2\left[\begin{matrix} (\lambda+\rho(\mu\alpha-\nu-1), \rho\alpha q), (\mu, q) \\ (\frac{\mu\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\lambda+\delta+h+\rho(\mu\alpha-\nu-1), \rho\alpha q) \end{matrix} \middle| \left(\frac{a\,(z x^\rho)^\alpha}{k^{\alpha/k}}\right)^q\right].
\end{aligned} \tag{16.3.22}$$

**Remark 1.** If we take $k = q = 1$, the results in Theorems 1–18 reduce to the results involving the R- and G-functions, which are listed in Appendix 16.A.

**Remark 2.** If we choose $k = q = 1$, the results in Corollaries 1–18 reduce to the results involving the R- and G-functions, which are listed in Appendix 16.B.

## Appendix 16.A Results obtained from Section 16.2

**Corollary 16.A.1.** *If* $\lambda, \delta, \alpha, \nu \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\alpha - \nu) > 0$; $\Re(\delta) > 0$, *then*

$$\begin{aligned}&\frac{1}{\Gamma(\delta)} \int_0^1 u^{\lambda-1}(1-u)^{\delta-1} R_{\alpha,\nu}[a, zu]\, du \\ &= z^{\alpha-\nu-1} {}_2\Psi_2\left[\begin{array}{c}(\lambda+\alpha-\nu-1, \alpha), (1, 1)\\ (\alpha-\nu, \alpha), (\lambda+\alpha-\nu+\delta-1, \alpha)\end{array}\middle| az^{\alpha}\right].\end{aligned} \tag{16.A.1}$$

**Corollary 16.A.2.** *If* $\lambda, \delta, \alpha, \nu, \mu \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\mu\alpha - \nu) > 0$; $\Re(\delta) > 0$, *then*

$$\begin{aligned}&\frac{1}{\Gamma(\delta)} \int_0^1 u^{\lambda-1}(1-u)^{\delta-1} G_{\alpha,\nu,\mu}[a, zu]\, du \\ &= \frac{z^{\mu\alpha-\nu-1}}{\Gamma(\mu)} {}_2\Psi_2\left[\begin{array}{c}(\lambda+\mu\alpha-\nu-1, \alpha), (\mu, 1)\\ (\mu\alpha-\nu, \alpha), (\lambda+\mu\alpha-\nu+\delta-1, \alpha)\end{array}\middle| az^{\alpha}\right].\end{aligned} \tag{16.A.2}$$

**Corollary 16.A.3.** $\lambda, \delta, \alpha, \nu \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\alpha - \nu) > 0$; $\Re(\delta) > 0$, *then*

$$\begin{aligned}&\frac{1}{\Gamma(\delta)} \int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1} R_{\alpha,\nu}[a, z(s-t)]\, ds = (x-t)^{\delta+\lambda+\alpha-\nu-2} z^{\alpha-\nu-1} \\ &\times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\alpha-\nu-1, \alpha), (1, 1)\\ (\alpha-\nu, \alpha), (\lambda+\alpha-\nu+\delta-1, \alpha)\end{array}\middle| a[z(x-t)]^{\alpha}\right].\end{aligned} \tag{16.A.3}$$

**Corollary 16.A.4.** $\lambda, \delta, \alpha, \nu, \mu \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\mu\alpha - \nu) > 0$; $\Re(\delta) > 0$, *then*

$$\begin{aligned}&\frac{1}{\Gamma(\delta)} \int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1} G_{\alpha,\nu,\mu}[a, z(s-t)]\, ds = \frac{(x-t)^{\delta+\lambda+\mu\alpha-\nu-2} z^{\mu\alpha-\nu-1}}{\Gamma(\mu)} \\ &\times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\mu\alpha-\nu-1, \alpha), (\mu, 1)\\ (\mu\alpha-\nu, \alpha), (\lambda+\mu\alpha-\nu+\delta-1, \alpha)\end{array}\middle| a[z(x-t)]^{\alpha}\right].\end{aligned} \tag{16.A.4}$$

**Corollary 16.A.5.** *If* $\lambda, \delta, \alpha, \nu \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(\lambda) > 0$, *then*

$$\begin{aligned}&\frac{1}{\Gamma(\delta)} \int_0^x t^{\delta-1}(x-t)^{\lambda-1} R_{\alpha,\nu}\left[a, z(x-t)^{\rho}\right] dt = z^{\alpha-\nu-1} x^{\lambda+\delta+\rho(\alpha-\nu-1)} \\ &\times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\alpha-\nu-1), \rho\alpha), (1, 1)\\ (\alpha-\nu, \alpha), (\lambda+\delta+\rho(\alpha-\nu-1), \rho\alpha)\end{array}\middle| a\left(zx^{\rho}\right)^{\alpha}\right].\end{aligned} \tag{16.A.5}$$

**Corollary 16.A.6.** *If* $\lambda, \delta, \alpha, \nu, \mu \in \mathbb{C}; z > 0; a > 0; \Re(\mu\alpha - \nu) > 0; \Re(\delta) > 0; \Re(\lambda) > 0$*, then*

$$\frac{1}{\Gamma(\delta)}\int_0^x t^{\delta-1}(x-t)^{\lambda-1}G_{\alpha,\nu,\mu}\left[a, z(x-t)^{\rho}\right]dt = \frac{z^{\mu\alpha-\nu-1}x^{\lambda+\delta+\rho(\mu\alpha-\nu-1)}}{\Gamma(\mu)} \times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\mu\alpha-\nu-1),\rho\alpha),(\mu,1)\\(\mu\alpha-\nu,\alpha),(\lambda+\delta+\rho(\mu\alpha-\nu-1),\rho\alpha)\end{array}\middle| a\left(zx^{\rho}\right)^{\alpha}\right]. \tag{16.A.6}$$

**Corollary 16.A.7.** *If* $\lambda, \delta, \alpha, \nu \in \mathbb{C};\ z > 0; a > 0; \Re(\alpha-\nu) > 0;\ \Re(\delta) > 0;\ \vartheta, \eta, \xi, \zeta, \sigma \in \mathbb{C}^{+};\ m, n \in \mathbb{N};\ \omega, \tau \in \mathbb{C}^{+}_{-1}$*, then*

$$\frac{1}{\Gamma(\delta)}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}L_m^{(\vartheta,\omega)}[\xi;\sigma(1-u)]L_n^{(\eta,\tau)}[\zeta;\sigma(1-u)]R_{\alpha,\nu}\left[a, zu^{\rho}\right]du$$
$$= z^{\alpha-\nu-1}\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h\,{}_1\Delta^{m,n,\xi,\zeta}_{\vartheta,\omega,\eta,\tau} \times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\alpha-\nu-1),\rho\alpha),(1,1)\\(\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\alpha-\nu-1),\rho\alpha)\end{array}\middle| az^{\alpha}\right]. \tag{16.A.7}$$

**Corollary 16.A.8.** *If* $\lambda, \delta, \alpha, \nu, \mu \in \mathbb{C}; z > 0; a > 0; \Re(\mu\alpha - \nu) > 0; \Re(\delta) > 0;\ \vartheta, \eta, \xi, \zeta, \sigma \in \mathbb{C}^{+};\ m, n \in \mathbb{N};\ \omega, \tau \in \mathbb{C}^{+}_{-1}$*, then*

$$\frac{1}{\Gamma(\delta)}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}L_m^{(\vartheta,\omega)}[\xi;\sigma(1-u)]L_n^{(\eta,\tau)}[\zeta;\sigma(1-u)]G_{\alpha,\nu,\mu}\left[a, zu^{\rho}\right]du$$
$$= \frac{z^{\mu\alpha-\nu-1}}{\Gamma(\mu)}\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h\,{}_1\Delta^{m,n,\xi,\zeta}_{\vartheta,\omega,\eta,\tau} \times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\mu\alpha-\nu-1),\rho\alpha),(\mu,1)\\(\mu\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\mu\alpha-\nu-1),\rho\alpha)\end{array}\middle| az^{\alpha}\right]. \tag{16.A.8}$$

**Corollary 16.A.9.** *If* $\lambda, \delta, \alpha, \nu \in \mathbb{C};\ z > 0; a > 0; \Re(\alpha-\nu) > 0;\ \Re(\delta) > 0;\ \vartheta, \eta, \xi, \zeta, \sigma \in \mathbb{C}^{+};\ m, n \in \mathbb{N};\ \omega, \tau \in \mathbb{C}^{+}_{-1}$*, then*

$$\frac{1}{\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1}L_m^{(\vartheta,\omega)}[\xi;\sigma(x-s)]L_n^{(\eta,\tau)}[\zeta;\sigma(x-s)] \times R_{\alpha,\nu}\left[a, z(s-t)^{\rho}\right]du$$
$$= z^{\alpha-\nu-1}\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h\,{}_1\Delta^{m,n,\xi,\zeta}_{\vartheta,\omega,\eta,\tau} \times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\alpha-\nu-1),\rho\alpha),(1,1)\\(\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\alpha-\nu-1),\rho\alpha)\end{array}\middle| az^{\alpha}\right]. \tag{16.A.9}$$

**Corollary 16.A.10.** *If* $\lambda,\delta,\alpha,\nu,\mu\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\mu\alpha-\nu)>0$; $\Re(\delta)>0$; $\vartheta,\eta,\xi,\zeta,\sigma\in\mathbb{C}^+$; $m,n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^+_{-1}$, *then*

$$\begin{aligned}&\frac{1}{\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1}L_m^{(\vartheta,\omega)}[\xi;\sigma(x-s)]L_n^{(\eta,\tau)}[\zeta;\sigma(x-s)]\\&\quad\times G_{\alpha,\nu,\mu}\left[a,z(s-t)^{\rho}\right]du\\&=\frac{z^{\mu\alpha-\nu-1}}{\Gamma(\mu)}\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h{}_1\Delta^{m,n,\xi,\zeta}_{\vartheta,\omega,\eta,\tau}\\&\quad\times{}_2\Psi_2\left[\begin{matrix}(\lambda+\rho(\mu\alpha-\nu-1),\rho\alpha),(\mu,1)\\(\mu\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\mu\alpha-\nu-1),\rho\alpha)\end{matrix}\middle|az^{\alpha}\right].\end{aligned}\tag{16.A.10}$$

**Corollary 16.A.11.** *If* $\lambda,\delta,\alpha,\nu\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\alpha-\nu)>0$; $\Re(\delta)>0$; $\Re(\lambda)>0$; $\vartheta,\eta,\xi,\zeta,\sigma\in\mathbb{C}^+$; $m,n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^+_{-1}$, *then*

$$\begin{aligned}&\frac{1}{\Gamma(\delta)}\int_0^x t^{\delta-1}(x-t)^{\lambda-1}L_m^{(\vartheta,\omega)}[\xi;\sigma t]L_n^{(\eta,\tau)}[\zeta;\sigma t]R_{\alpha,\nu}\left[a,z(x-t)^{\rho}\right]dt\\&=z^{\alpha-\nu-1}x^{\lambda+\delta+h+\rho(\alpha-\nu-1)}\\&\quad\times\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h{}_1\Delta^{m,n,\xi,\zeta}_{\vartheta,\omega,\eta,\tau}\\&\quad\times{}_2\Psi_2\left[\begin{matrix}(\lambda+\rho(\alpha-\nu-1),\rho\alpha),(1,1)\\(\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\alpha-\nu-1),\rho\alpha)\end{matrix}\middle|a\left(zx^{\rho}\right)^{\alpha}\right].\end{aligned}\tag{16.A.11}$$

**Corollary 16.A.12.** *If* $\lambda,\delta,\alpha,\nu,\mu\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\mu\alpha-\nu)>0$; $\Re(\delta)>0$; $\Re(\lambda)>0$; $\vartheta,\eta,\xi,\zeta,\sigma\in\mathbb{C}^+$; $m,n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^+_{-1}$, *then*

$$\begin{aligned}&\frac{1}{\Gamma(\delta)}\int_0^x t^{\delta-1}(x-t)^{\lambda-1}L_m^{(\vartheta,\omega)}[\xi;\sigma t]L_n^{(\eta,\tau)}[\zeta;\sigma t]G_{\alpha,\nu,\mu}\left[a,z(x-t)^{\rho}\right]dt\\&=\frac{z^{\mu\alpha-\nu-1}x^{\lambda+\delta+h+\rho(\mu\alpha-\nu-1)}}{\Gamma(\mu)}\\&\quad\times\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h{}_1\Delta^{m,n,\xi,\zeta}_{\vartheta,\omega,\eta,\tau}\\&\quad\times{}_2\Psi_2\left[\begin{matrix}(\lambda+\rho(\mu\alpha-\nu-1),\rho\alpha),(\mu,1)\\(\mu\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\mu\alpha-\nu-1),\rho\alpha)\end{matrix}\middle|a\left(zx^{\rho}\right)^{\alpha}\right].\end{aligned}\tag{16.A.12}$$

**Corollary 16.A.13.** *If* $\lambda,\delta,\alpha,\nu\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\alpha-\nu)>0$; $\Re(\delta)>0$; $\vartheta,\eta,\sigma\in\mathbb{C}^+$; $m,n,\xi,\zeta\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^+_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}L_m^{(\vartheta,\omega)}[\xi;\sigma(1-u)]L_n^{(\eta,\tau)}[\zeta;\sigma(1-u)]R_{\alpha,\nu}\left[a,zu^{\rho}\right]du$$

$$= z^{\alpha-\nu-1}\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h{}_2\Delta_{\vartheta,\omega,\eta,\tau}^{m,n,\xi,\zeta}$$
$$\times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\alpha-\nu-1),\rho\alpha),(1,1)\\(\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\alpha-\nu-1),\rho\alpha)\end{array}\middle| az^{\alpha}\right]. \quad (16.A.13)$$

**Corollary 16.A.14.** *If* $\lambda,\delta,\alpha,\nu,\mu\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\mu\alpha-\nu)>0$; $\Re(\delta)>0$; $\vartheta,\eta,\sigma\in\mathbb{C}^+$; $m,n,\xi,\zeta\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^+_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}L_m^{(\vartheta,\omega)}[\xi;\sigma(1-u)]L_n^{(\eta,\tau)}[\zeta;\sigma(1-u)]G_{\alpha,\nu,\mu}\left[a,zu^{\rho}\right]du$$
$$=\frac{z^{\mu\alpha-\nu-1}}{\Gamma(\mu)}\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h{}_2\Delta_{\vartheta,\omega,\eta,\tau}^{m,n,\xi,\zeta}$$
$$\times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\mu\alpha-\nu-1),\rho\alpha),(\mu,1)\\(\mu\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\mu\alpha-\nu-1),\rho\alpha)\end{array}\middle| az^{\alpha}\right]. \quad (16.A.14)$$

**Corollary 16.A.15.** *If* $\lambda,\delta,\alpha,\nu\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\alpha-\nu)>0$; $\Re(\delta)>0$; $\vartheta,\eta,\sigma\in\mathbb{C}^+$; $m,n,\xi,\zeta\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^+_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1}L_m^{(\vartheta,\omega)}[\xi;\sigma(x-s)]L_n^{(\eta,\tau)}[\zeta;\sigma(x-s)]$$
$$\times R_{\alpha,\nu}\left[a,z(s-t)^{\rho}\right]du$$
$$= z^{\alpha-\nu-1}\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h{}_2\Delta_{\vartheta,\omega,\eta,\tau}^{m,n,\xi,\zeta}$$
$$\times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\alpha-\nu-1),\rho\alpha),(1,1)\\(\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\alpha-\nu-1),\rho\alpha)\end{array}\middle| az^{\alpha}\right]. \quad (16.A.15)$$

**Corollary 16.A.16.** *If* $\lambda,\delta,\alpha,\nu,\mu\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\mu\alpha-\nu)>0$; $\Re(\delta)>0$; $\vartheta,\eta,\sigma\in\mathbb{C}^+$; $m,n,\xi,\zeta\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^+_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1}L_m^{(\vartheta,\omega)}[\xi;\sigma(x-s)]L_n^{(\eta,\tau)}[\zeta;\sigma(x-s)]$$
$$\times G_{\alpha,\nu,\mu}\left[a,z(s-t)^{\rho}\right]du$$
$$=\frac{z^{\mu\alpha-\nu-1}}{\Gamma(\mu)}\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h{}_2\Delta_{\vartheta,\omega,\eta,\tau}^{m,n,\xi,\zeta}$$
$$\times {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\mu\alpha-\nu-1),\rho\alpha),(\mu,1)\\(\mu\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\mu\alpha-\nu-1),\rho\alpha)\end{array}\middle| az^{\alpha}\right]. \quad (16.A.16)$$

**Corollary 16.A.17.** *If* $\lambda, \delta, \alpha, \nu \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(\lambda) > 0$; $\vartheta, \eta, \sigma \in \mathbb{C}^+$; $m, n, \xi, \zeta \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}^+_{-1}$, *then*

$$\begin{aligned}
&\frac{1}{\Gamma(\delta)} \int_0^x t^{\delta-1}(x-t)^{\lambda-1} L_m^{(\vartheta,\omega)}[\xi;\sigma t] L_n^{(\eta,\tau)}[\zeta;\sigma t] R_{\alpha,\nu}\left[a, z(x-t)^\rho\right] dt \\
&\quad = z^{\alpha-\nu-1} x^{\lambda+\delta+h+\rho(\alpha-\nu-1)} \\
&\quad\quad \times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h {}_2\Delta_{\vartheta,\omega,\eta,\tau}^{m,n,\xi,\zeta} \\
&\quad\quad \times {}_2\Psi_2\left[\begin{array}{c} (\lambda+\rho(\alpha-\nu-1), \rho\alpha), (1,1) \\ (\alpha-\nu, \alpha), (\lambda+\delta+h+\rho(\alpha-\nu-1), \rho\alpha) \end{array} \middle| a\left(zx^\rho\right)^\alpha \right].
\end{aligned} \tag{16.A.17}$$

**Corollary 16.A.18.** *If* $\lambda, \delta, \alpha, \nu, \mu \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\mu\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(\lambda) > 0$; $\vartheta, \eta, \sigma \in \mathbb{C}^+$; $m, n, \xi, \zeta \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}^+_{-1}$, *then*

$$\begin{aligned}
&\frac{1}{\Gamma(\delta)} \int_0^x t^{\delta-1}(x-t)^{\lambda-1} L_m^{(\vartheta,\omega)}[\xi;\sigma t] L_n^{(\eta,\tau)}[\zeta;\sigma t] G_{\alpha,\nu,\mu}\left[a, z(x-t)^\rho\right] dt \\
&\quad = \frac{z^{\mu\alpha-\nu-1} x^{\lambda+\delta+h+\rho(\mu\alpha-\nu-1)}}{\Gamma(\mu)} \\
&\quad\quad \times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h {}_2\Delta_{\vartheta,\omega,\eta,\tau}^{m,n,\xi,\zeta} \\
&\quad\quad \times {}_2\Psi_2\left[\begin{array}{c} (\lambda+\rho(\mu\alpha-\nu-1), \rho\alpha), (\mu,1) \\ (\mu\alpha-\nu, \alpha), (\lambda+\delta+h+\rho(\mu\alpha-\nu-1), \rho\alpha) \end{array} \middle| a\left(zx^\rho\right)^\alpha \right].
\end{aligned} \tag{16.A.18}$$

## Appendix 16.B Further special cases of Section 16.3

**Corollary 16.B.1.** *If* $\lambda, \delta, \alpha, \nu \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\vartheta, \eta, \sigma \in \mathbb{C}^+$; $m, n \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}^+_{-1}$, *then*

$$\begin{aligned}
&\frac{1}{\Gamma(\delta)} \int_0^1 u^{\lambda-1}(1-u)^{\delta-1} L_m^{(\vartheta,\omega)}[\sigma(1-u)] L_n^{(\eta,\tau)}[\sigma(1-u)] R_{\alpha,\nu}\left[a, zu^\rho\right] du \\
&\quad = z^{\alpha-\nu-1} \\
&\quad\quad \times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h \frac{\Gamma(\vartheta m+\omega+1)\,\Gamma(\eta n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)} \\
&\quad\quad \times \sum_{j=0}^{h} \left[ \binom{h}{j} \frac{(-m)_{(h-j)}(-n)_j}{\Gamma(\vartheta j+\omega+1)\Gamma(\eta(h-j)+\tau+1)} \right]
\end{aligned}$$

$$\times\, {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\alpha-\nu-1),\rho\alpha),(1,1)\\(\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\alpha-\nu-1),\rho\alpha)\end{array}\middle|az^{\alpha}\right]. \quad (16.B.1)$$

**Corollary 16.B.2.** *If* $\lambda,\delta,\alpha,\nu,\mu\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\mu\alpha-\nu)>0$; $\Re(\delta)>0$; $\vartheta,\eta,\sigma\in\mathbb{C}^+$; $m,n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^+_{-1}$, *then*

$$\begin{aligned}&\frac{1}{\Gamma(\delta)}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}L_m^{(\vartheta,\omega)}[\sigma(1-u)]L_n^{(\eta,\tau)}[\sigma(1-u)]G_{\alpha,\nu,\mu}\left[a,zu^{\rho}\right]du\\&=\frac{z^{\mu\alpha-\nu-1}}{\Gamma(\mu)}\\&\times\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h\frac{\Gamma(\vartheta m+\omega+1)\,\Gamma(\eta n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)}\\&\times\sum_{j=0}^{h}\left[\begin{pmatrix}h\\j\end{pmatrix}\frac{(-m)_{(h-j)}(-n)_j}{\Gamma(\vartheta j+\omega+1)\Gamma(\eta(h-j)+\tau+1)}\right]\\&\times\, {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\mu\alpha-\nu-1),\rho\alpha),(\mu,1)\\(\mu\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\mu\alpha-\nu-1),\rho\alpha)\end{array}\middle|az^{\alpha}\right].\end{aligned} \quad (16.B.2)$$

**Corollary 16.B.3.** *If* $\lambda,\delta,\alpha,\nu\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\alpha-\nu)>0$; $\Re(\delta)>0$; $\vartheta,\eta,\sigma\in\mathbb{C}^+$; $m,n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^+_{-1}$, *then*

$$\begin{aligned}&\frac{1}{\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1}L_m^{(\vartheta,\omega)}[\sigma(x-s)]L_n^{(\eta,\tau)}[\sigma(x-s)]\\&\quad\times R_{\alpha,\nu}\left[a,z(s-t)^{\rho}\right]du=z^{\alpha-\nu-1}\\&\quad\times\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h\frac{\Gamma(\vartheta m+\omega+1)\,\Gamma(\eta n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)}\\&\quad\times\sum_{j=0}^{h}\left[\begin{pmatrix}h\\j\end{pmatrix}\frac{(-m)_{(h-j)}(-n)_j}{\Gamma(\vartheta j+\omega+1)\Gamma(\eta(h-j)+\tau+1)}\right]\\&\quad\times\, {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\alpha-\nu-1),\rho\alpha),(1,1)\\(\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\alpha-\nu-1),\rho\alpha)\end{array}\middle|az^{\alpha}\right].\end{aligned} \quad (16.B.3)$$

**Corollary 16.B.4.** *If* $\lambda,\delta,\alpha,\nu,\mu\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\mu\alpha-\nu)>0$; $\Re(\delta)>0$; $\vartheta,\eta,\sigma\in\mathbb{C}^+$; $m,n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^+_{-1}$, *then*

$$\begin{aligned}&\frac{1}{\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1}L_m^{(\vartheta,\omega)}[\sigma(x-s)]L_n^{(\eta,\tau)}[\sigma(x-s)]\\&\quad\times G_{\alpha,\nu,\mu}\left[a,z(s-t)^{\rho}\right]du=\frac{z^{\mu\alpha-\nu-1}}{\Gamma(\mu)}\\&\quad\times\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h\frac{\Gamma(\vartheta m+\omega+1)\,\Gamma(\eta n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)}\end{aligned}$$

$$\times \sum_{j=0}^{h}\left[\binom{h}{j}\frac{(-m)_{(h-j)}(-n)_j}{\Gamma(\vartheta j+\omega+1)\Gamma(\eta(h-j)+\tau+1)}\right]$$

$$\times {}_2\Psi_2\left[\begin{matrix}(\lambda+\rho(\mu\alpha-\nu-1),\rho\alpha),(\mu,1)\\(\mu\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\mu\alpha-\nu-1),\rho\alpha)\end{matrix}\middle| az^{\alpha}\right]. \quad (16.B.4)$$

**Corollary 16.B.5.** *If* $\lambda,\delta,\alpha,\nu\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\alpha-\nu)>0$; $\Re(\delta)>0$; $\Re(\lambda)>0$; $\vartheta,\eta,\sigma\in\mathbb{C}^{+}$; $m,n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^{+}_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)}\int_0^x t^{\delta-1}(x-t)^{\lambda-1}L_m^{(\vartheta,\omega)}[\sigma t]L_n^{(\eta,\tau)}[\sigma t]R_{\alpha,\nu}\left[a,z(x-t)^{\rho}\right]dt$$
$$=z^{\alpha-\nu-1}x^{\lambda+\delta+h+\rho(\alpha-\nu-1)}$$
$$\times\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h\frac{\Gamma(\vartheta m+\omega+1)\,\Gamma(\eta n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)}$$
$$\times \sum_{j=0}^{h}\left[\binom{h}{j}\frac{(-m)_{(h-j)}(-n)_j}{\Gamma(\vartheta j+\omega+1)\Gamma(\eta(h-j)+\tau+1)}\right]$$
$$\times {}_2\Psi_2\left[\begin{matrix}(\lambda+\rho(\alpha-\nu-1),\rho\alpha),(1,1)\\(\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\alpha-\nu-1),\rho\alpha)\end{matrix}\middle| a\left(zx^{\rho}\right)^{\alpha}\right]. \quad (16.B.5)$$

**Corollary 16.B.6.** *If* $\lambda,\delta,\alpha,\nu,\mu\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\mu\alpha-\nu)>0$; $\Re(\delta)>0$; $\Re(\lambda)>0$; $\vartheta,\eta,\sigma\in\mathbb{C}^{+}$; $m,n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^{+}_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)}\int_0^x t^{\delta-1}(x-t)^{\lambda-1}L_m^{(\vartheta,\omega)}[\sigma t]L_n^{(\eta,\tau)}[\sigma t]G_{\alpha,\nu,\mu}\left[a,z(x-t)^{\rho}\right]dt$$
$$=\frac{z^{\mu\alpha-\nu-1}x^{\lambda+\delta+h+\rho(\mu\alpha-\nu-1)}}{\Gamma(\mu)}$$
$$\times\sum_{h=0}^{m+n}(\delta)_h(\sigma)^h\frac{\Gamma(\vartheta m+\omega+1)\,\Gamma(\eta n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)}$$
$$\times \sum_{j=0}^{h}\left[\binom{h}{j}\frac{(-m)_{(h-j)}(-n)_j}{\Gamma(\vartheta j+\omega+1)\Gamma(\eta(h-j)+\tau+1)}\right]$$
$$\times {}_2\Psi_2\left[\begin{matrix}(\lambda+\rho(\mu\alpha-\nu-1),\rho\alpha),(\mu,1)\\(\mu\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\mu\alpha-\nu-1),\rho\alpha)\end{matrix}\middle| a\left(zx^{\rho}\right)^{\alpha}\right]. \quad (16.B.6)$$

**Corollary 16.B.7.** *If* $\lambda,\delta,\alpha,\nu\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\alpha-\nu)>0$; $\Re(\delta)>0$; $\sigma\in\mathbb{C}^{+}$; $m,n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^{+}_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}L_m^{(1,\omega)}[\sigma(1-u)]L_n^{(1,\tau)}[\sigma(1-u)]R_{\alpha,\nu}\left[a,zu^{\rho}\right]du$$

$$= z^{\alpha-\nu-1}$$

$$\times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h \frac{\Gamma(m+\omega+1)\,\Gamma(n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)}$$

$$\times \sum_{j=0}^{h} \left[ \binom{h}{j} \frac{(-m)_{(h-j)}(-n)_j}{\Gamma(j+\omega+1)\Gamma(h-j+\tau+1)} \right]$$

$$\times {}_2\Psi_2 \left[ \begin{array}{c} (\lambda+\rho(\alpha-\nu-1), \rho\alpha), (1,1) \\ (\alpha-\nu, \alpha), (\lambda+\delta+h+\rho(\alpha-\nu-1), \rho\alpha) \end{array} \middle| az^{\alpha} \right]. \qquad (16.B.7)$$

**Corollary 16.B.8.** *If* $\lambda, \delta, \alpha, \nu, \mu \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\mu\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\sigma \in \mathbb{C}^+$; $m, n \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}^+_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)} \int_0^1 u^{\lambda-1}(1-u)^{\delta-1} L_m^{(1,\omega)}[\sigma(1-u)] L_n^{(1,\tau)}[\sigma(1-u)] G_{\alpha,\nu,\mu}\left[a, zu^{\rho}\right] du$$

$$= \frac{z^{\mu\alpha-\nu-1}}{\Gamma(\mu)}$$

$$\times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h \frac{\Gamma(m+\omega+1)\,\Gamma(n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)}$$

$$\times \sum_{j=0}^{h} \left[ \binom{h}{j} \frac{(-m)_{(h-j)}(-n)_j}{\Gamma(j+\omega+1)\Gamma(h-j+\tau+1)} \right]$$

$$\times {}_2\Psi_2 \left[ \begin{array}{c} (\lambda+\rho(\mu\alpha-\nu-1), \rho\alpha), (\mu,1) \\ (\mu\alpha-\nu, \alpha), (\lambda+\delta+h+\rho(\mu\alpha-\nu-1), \rho\alpha) \end{array} \middle| az^{\alpha} \right]. \qquad (16.B.8)$$

**Corollary 16.B.9.** *If* $\lambda, \delta, \alpha, \nu \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\sigma \in \mathbb{C}^+$; $m, n \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}^+_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)} \int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1} L_m^{(1,\omega)}[\sigma(x-s)] L_n^{(1,\tau)}[\sigma(x-s)]$$

$$\times R_{\alpha,\nu}\left[a, z(s-t)^{\rho}\right] du = z^{\alpha-\nu-1}$$

$$\times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h \frac{\Gamma(m+\omega+1)\,\Gamma(n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)}$$

$$\times \sum_{j=0}^{h} \left[ \binom{h}{j} \frac{(-m)_{(h-j)}(-n)_j}{\Gamma(j+\omega+1)\Gamma(h-j+\tau+1)} \right]$$

$$\times {}_2\Psi_2 \left[ \begin{array}{c} (\lambda+\rho(\alpha-\nu-1), \rho\alpha), (1,1) \\ (\alpha-\nu, \alpha), (\lambda+\delta+h+\rho(\alpha-\nu-1), \rho\alpha) \end{array} \middle| az^{\alpha} \right]. \qquad (16.B.9)$$

**Corollary 16.B.10.** *If* $\lambda, \delta, \alpha, \nu, \mu \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\mu\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\sigma \in \mathbb{C}^{+}$; $m, n \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}_{-1}^{+}$, *then*

$$\frac{1}{\Gamma(\delta)} \int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1} L_m^{(1,\omega)}[\sigma(x-s)] L_n^{(1,\tau)}[\sigma(x-s)] \times G_{\alpha,\nu,\mu}\left[a, z(s-t)^{\rho}\right] du = \frac{z^{\mu\alpha-\nu-1}}{\Gamma(\mu)} \times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h \frac{\Gamma(m+\omega+1)\,\Gamma(n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)} \times \sum_{j=0}^{h} \left[ \binom{h}{j} \frac{(-m)_{(h-j)}(-n)_j}{\Gamma(j+\omega+1)\Gamma(h-j+\tau+1)} \right] \times {}_2\Psi_2 \left[ \begin{matrix} (\lambda+\rho(\mu\alpha-\nu-1), \rho\alpha), (\mu, 1) \\ (\mu\alpha-\nu, \alpha), (\lambda+\delta+h+\rho(\mu\alpha-\nu-1), \rho\alpha) \end{matrix} \middle| az^{\alpha} \right]. \quad (16.B.10)$$

**Corollary 16.B.11.** *If* $\lambda, \delta, \alpha, \nu \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(\lambda) > 0$; $\sigma \in \mathbb{C}^{+}$; $m, n \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}_{-1}^{+}$, *then*

$$\frac{1}{\Gamma(\delta)} \int_0^x t^{\delta-1}(x-t)^{\lambda-1} L_m^{(1,\omega)}[\sigma t] L_n^{(1,\tau)}[\sigma t] R_{\alpha,\nu}\left[a, z(x-t)^{\rho}\right] dt = z^{\alpha-\nu-1} x^{\lambda+\delta+h+\rho(\alpha-\nu-1)} \times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h \frac{\Gamma(m+\omega+1)\,\Gamma(n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)} \times \sum_{j=0}^{h} \left[ \binom{h}{j} \frac{(-m)_{(h-j)}(-n)_j}{\Gamma(j+\omega+1)\Gamma(h-j+\tau+1)} \right] \times {}_2\Psi_2 \left[ \begin{matrix} (\lambda+\rho(\alpha-\nu-1), \rho\alpha), (1, 1) \\ (\alpha-\nu, \alpha), (\lambda+\delta+h+\rho(\alpha-\nu-1), \rho\alpha) \end{matrix} \middle| a\left(zx^{\rho}\right)^{\alpha} \right]. \quad (16.B.11)$$

**Corollary 16.B.12.** *If* $\lambda, \delta, \alpha, \nu, \mu \in \mathbb{C}$; $z > 0$; $a > 0$; $\Re(\mu\alpha - \nu) > 0$; $\Re(\delta) > 0$; $\Re(\lambda) > 0$; $\sigma \in \mathbb{C}^{+}$; $m, n \in \mathbb{N}$; $\omega, \tau \in \mathbb{C}_{-1}^{+}$, *then*

$$\frac{1}{\Gamma(\delta)} \int_0^x t^{\delta-1}(x-t)^{\lambda-1} L_m^{(1,\omega)}[\sigma t] L_n^{(1,\tau)}[\sigma t] G_{\alpha,\nu,\mu}\left[a, z(x-t)^{\rho}\right] dt = \frac{z^{\mu\alpha-\nu-1} x^{\lambda+\delta+h+\rho(\mu\alpha-\nu-1)}}{\Gamma(\mu)} \times \sum_{h=0}^{m+n} (\delta)_h (\sigma)^h \frac{\Gamma(m+\omega+1)\,\Gamma(n+\tau+1)}{\Gamma(m+1)\,\Gamma(n+1)}$$

$$\times \sum_{j=0}^{h}\left[\binom{h}{j}\frac{(-m)_{(h-j)}(-n)_j}{\Gamma(j+\omega+1)\Gamma(h-j+\tau+1)}\right]$$
$$\times\, {}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\mu\alpha-\nu-1),\rho\alpha),(\mu,1)\\(\mu\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\mu\alpha-\nu-1),\rho\alpha)\end{array}\middle| a\left(zx^{\rho}\right)^{\alpha}\right]. \tag{16.B.12}$$

**Corollary 16.B.13.** *If* $\lambda,\delta,\alpha,\nu\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\alpha-\nu)>0$; $\Re(\delta)>0$; $\sigma\in\mathbb{C}^{+}$; $n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^{+}_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}[1-\sigma(1-u)]^n R_{\alpha,\nu}\left[a,zu^{\rho}\right]du = z^{\alpha-\nu-1}$$
$$\times\sum_{h=0}^{n}(\delta)_h(\sigma)^h(-n)_h\,{}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\alpha-\nu-1),\rho\alpha),(1,1)\\(\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\alpha-\nu-1),\rho\alpha)\end{array}\middle| az^{\alpha}\right]. \tag{16.B.13}$$

**Corollary 16.B.14.** *If* $\lambda,\delta,\alpha,\nu,\mu\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\mu\alpha-\nu)>0$; $\Re(\delta)>0$; $\sigma\in\mathbb{C}^{+}$; $n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^{+}_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}[1-\sigma(1-u)]^n G_{\alpha,\nu,\mu}\left[a,zu^{\rho}\right]du = \frac{z^{\mu\alpha-\nu-1}}{\Gamma(\mu)}$$
$$\times\sum_{h=0}^{n}(\delta)_h(\sigma)^h(-n)_h\,{}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\mu\alpha-\nu-1),\rho\alpha),(\mu,1)\\(\mu\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\mu\alpha-\nu-1),\rho\alpha)\end{array}\middle| az^{\alpha}\right]. \tag{16.B.14}$$

**Corollary 16.B.15.** *If* $\lambda,\delta,\alpha,\nu\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\alpha-\nu)>0$; $\Re(\delta)>0$; $\sigma\in\mathbb{C}^{+}$; $n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^{+}_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1}[1-\sigma(x-s)]^n R_{\alpha,\nu}\left[a,z(s-t)^{\rho}\right]du = z^{\alpha-\nu-1}$$
$$\times\sum_{h=0}^{n}(\delta)_h(\sigma)^h(-n)_h\,{}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\alpha-\nu-1),\rho\alpha),(1,1)\\(\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\alpha-\nu-1),\rho\alpha)\end{array}\middle| az^{\alpha}\right]. \tag{16.B.15}$$

**Corollary 16.B.16.** *If* $\lambda,\delta,\alpha,\nu,\mu\in\mathbb{C}$; $z>0$; $a>0$; $\Re(\mu\alpha-\nu)>0$; $\Re(\delta)>0$; $\sigma\in\mathbb{C}^{+}$; $n\in\mathbb{N}$; $\omega,\tau\in\mathbb{C}^{+}_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{\lambda-1}[1-\sigma(x-s)]^n G_{\alpha,\nu,\mu}\left[a,z(s-t)^{\rho}\right]du = \frac{z^{\mu\alpha-\nu-1}}{\Gamma(\mu)}$$
$$\times\sum_{h=0}^{n}(\delta)_h(\sigma)^h(-n)_h\,{}_2\Psi_2\left[\begin{array}{c}(\lambda+\rho(\mu\alpha-\nu-1),\rho\alpha),(\mu,1)\\(\mu\alpha-\nu,\alpha),(\lambda+\delta+h+\rho(\mu\alpha-\nu-1),\rho\alpha)\end{array}\middle| az^{\alpha}\right]. \tag{16.B.16}$$

**Corollary 16.B.17.** *If* $\lambda, \delta, \alpha, \nu \in \mathbb{C}; z > 0; a > 0; \Re(\alpha - \nu) > 0; \Re(\delta) > 0; \Re(\lambda) > 0; \sigma \in \mathbb{C}^{+}; n \in \mathbb{N}; \omega, \tau \in \mathbb{C}^{+}_{-1}$, *then*

$$\frac{1}{\Gamma(\delta)} \int_0^x t^{\delta-1}(x-t)^{\lambda-1}[1-\sigma t]^n R_{\alpha,\nu}\left[a, z(x-t)^{\rho}\right] dt = z^{\alpha-\nu-1} x^{\lambda+\delta+h+\rho(\alpha-\nu-1)} \times \sum_{h=0}^{n} (\delta)_h (\sigma)^h (-n)_h {}_2\Psi_2 \left[ \begin{array}{c} (\lambda+\rho(\alpha-\nu-1), \rho\alpha), (1,1) \\ (\alpha-\nu, \alpha), (\lambda+\delta+h+\rho(\alpha-\nu-1), \rho\alpha) \end{array} \middle| a\left(zx^{\rho}\right)^{\alpha} \right]. \tag{16.B.17}$$

**Corollary 16.B.18.** *If* $\lambda, \delta, \alpha, \nu, \mu \in \mathbb{C}; z > 0; a > 0; \Re(\mu\alpha - \nu) > 0; \Re(\delta) > 0; \Re(\lambda) > 0; \sigma \in \mathbb{C}^{+}; n \in \mathbb{N}; \omega, \tau \in \mathbb{C}^{+}_{-1}$, *then*

$$\begin{aligned} &\frac{1}{\Gamma(\delta)} \int_0^x t^{\delta-1}(x-t)^{\lambda-1}[1-\sigma t]^n G_{\alpha,\nu,\mu}\left[a, z(x-t)^{\rho}\right] dt \\ &= \frac{z^{\mu\alpha-\nu-1} x^{\lambda+\delta+h+\rho(\mu\alpha-\nu-1)}}{\Gamma(\mu)} \\ &\quad \times \sum_{h=0}^{n} (\delta)_h (\sigma)^h (-n)_h \\ &\quad \times {}_2\Psi_2 \left[ \begin{array}{c} (\lambda+\rho(\mu\alpha-\nu-1), \rho\alpha), (\mu,1) \\ (\mu\alpha-\nu, \alpha), (\lambda+\delta+h+\rho(\mu\alpha-\nu-1), \rho\alpha) \end{array} \middle| a\left(zx^{\rho}\right)^{\alpha} \right]. \end{aligned} \tag{16.B.18}$$

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CHAPTER

# 17 Fractional integrals and solutions of fractional kinetic equations involving $R$- and $G$-functions

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## 17.1 Introduction and preliminaries

Recently, Diaz and Pariguan [16] introduced the $k$-Pochhammer symbol and $k$-gamma function defined as follows:

$$(\gamma)_{n,k} := \begin{cases} \dfrac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)} & (k \in \mathbb{R};\ \gamma \in \mathbb{C} \setminus \{0\}) \\ \gamma(\gamma+k)\ldots(\gamma+(n-1)k) & (n \in \mathbb{N};\ \gamma \in \mathbb{C}). \end{cases} \tag{17.1.1}$$

They gave the relation with the classical Euler's gamma function (see [13,17]) as:

$$\Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1}\Gamma\left(\frac{\gamma}{k}\right), \tag{17.1.2}$$

where $\gamma \in \mathbb{C}$, $k \in \mathbb{R}$ and $n \in \mathbb{N}$.

When $k = 1$, (17.1.1) reduces to the classical Pochhammer symbol and Euler's gamma function, respectively, (see [18]).

Also, let $\gamma \in \mathbb{C}$, $k, s \in \mathbb{R}$, then the following identity holds:

$$\Gamma_s(\gamma) = \left(\frac{s}{k}\right)^{\frac{\gamma}{s}-1}\Gamma_k\left(\frac{k\gamma}{s}\right) \tag{17.1.3}$$

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and in particular

$$\Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1}\Gamma\left(\frac{\gamma}{k}\right). \tag{17.1.4}$$

Further, let $\gamma \in \mathbb{C}$, $k, s \in \mathbb{R}$ and $\gamma \in \mathbb{C}$, then the following identity holds:

$$(\gamma)_{nq,s} = \left(\frac{s}{k}\right)^{nq}\left(\frac{k\gamma}{s}\right)_{nq} \tag{17.1.5}$$

and in particular

$$(\gamma)_{nq,k} = (k)^{nq}\left(\frac{\gamma}{k}\right)_{nq}. \tag{17.1.6}$$

For more details of the $k$-Pochhammer symbol, the $k$-special function, and the fractional Fourier transform one can refer to the papers by Romero et al. [17,18].

Lorenzo and Hartley [15] presented certain special functions, known as the $R$-function and $G$-function, by means of the following series representations:

$$R_{\alpha,\nu}[a,c,x] = \sum_{n=0}^{\infty}\frac{a^n(x-c)^{(n+1)\alpha-\nu-1}}{\Gamma[(n+1)\alpha-\nu]}, \quad \alpha,\nu \in \mathbb{C}; \Re(\alpha-\nu) > 0; x > c > 0; a > 0 \tag{17.1.7}$$

and

$$G_{\alpha,\nu,\mu}[a,c,x] = \sum_{n=0}^{\infty}\frac{(\mu)_n a^n (x-c)^{(n+\mu)\alpha-\nu-1}}{(n!)\Gamma[(n+\mu)\alpha-\nu]}, \quad \alpha,\nu,\mu \in \mathbb{C}; \Re(\alpha\mu-\nu) > 0; x > c > 0; a > 0. \tag{17.1.8}$$

In particular, for $c = 0$, we have:

$$R_{\alpha,\nu}[a,x] = R_{\alpha,\nu}[a,0,x] = \sum_{n=0}^{\infty}\frac{a^n x^{(n+1)\alpha-\nu-1}}{\Gamma[(n+1)\alpha-\nu]}, \quad \alpha,\nu \in \mathbb{C}; \Re(\alpha-\nu) > 0; x > 0, a > 0 \tag{17.1.9}$$

and

$$G_{\alpha,\nu,\mu}[a,x] = G_{\alpha,\nu,\mu}[a,0,x] = \sum_{n=0}^{\infty}\frac{(\mu)_n a^n x^{(n+\mu)\alpha-\nu-1}}{(n!)\Gamma[(n+\mu)\alpha-\nu]}, \quad \alpha,\nu,\mu \in \mathbb{C}; \Re(\alpha\mu-\nu) > 0; x > 0, a > 0. \tag{17.1.10}$$

Here, we extend the $R$- and $G$-functions given in Eqs. (17.1.7), (17.1.8), (17.1.9), and (17.1.10) as follows:

$$R^q_{k,\alpha,\nu}[a,c,x] = \sum_{n=0}^{\infty} \frac{a^{nq}(x-c)^{(nq+1)\alpha-\nu-1}}{\Gamma_k[(nq+1)\alpha-\nu]}, \quad (17.1.11)$$
$$\alpha,\nu,q\in\mathbb{C};\ \Re(\alpha-\nu)>0, \Re(q)>0;\ k\in\mathbb{R};\ x>c>0;\ a>0$$

and

$$G^q_{k,\alpha,\nu,\mu}[a,c,x] = \sum_{n=0}^{\infty} \frac{(\mu)_{nq}a^{nq}(x-c)^{(nq+\mu)\alpha-\nu-1}}{(n!)\Gamma_k[(nq+\mu)\alpha-\nu]}, \quad (17.1.12)$$
$$\alpha,\nu,\mu,q\in\mathbb{C};\ \Re(\alpha\mu-\nu)>0, \Re(q)>0;\ k\in\mathbb{R};\ x>c>0;\ a>0.$$

In particular, for $c=0$, we have:

$$R^q_{k,\alpha,\nu}[a,x] = R^q_{k,\alpha,\nu}[a,0,x] = \sum_{n=0}^{\infty} \frac{a^{nq}x^{(nq+1)\alpha-\nu-1}}{\Gamma_k[(nq+1)\alpha-\nu]}, \quad (17.1.13)$$
$$\alpha,\nu,q\in\mathbb{C};\ \Re(\alpha-\nu)>0, \Re(q)>0;\ k\in\mathbb{R};\ x>0;\ a>0$$

and

$$G^q_{k,\alpha,\nu,\mu}[a,x] = G^q_{k,\alpha,\nu,\mu}[a,0,x] = \sum_{n=0}^{\infty} \frac{(\mu)_{nq}a^{nq}x^{(nq+\mu)\alpha-\nu-1}}{(n!)\Gamma_k[(nq+\mu)\alpha-\nu]}, \quad (17.1.14)$$
$$\alpha,\nu,\mu,q\in\mathbb{C};\ \Re(\alpha\mu-\nu)>0, \Re(q)>0;\ k\in\mathbb{R};\ x>0;\ a>0.$$

The Fox–Wright function ${}_p\Psi_q$ is defined as (see, for details, Srivastava and Karisson [19]):

$$\begin{aligned} {}_p\Psi_q[z] &= {}_p\Psi_q\left[\begin{array}{l}(a_1,\alpha_1),\ldots,(a_p,\alpha_p);\\(b_1,\beta_1),\ldots,(b_q,\beta_q);\end{array} z\right] \\ &= {}_p\Psi_q\left[\begin{array}{l}(a_i,\alpha_i)_{1,p};\\(b_j,\beta_j)_{1,q};\end{array} z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)}\frac{z^n}{n!}, \end{aligned} \quad (17.1.15)$$

where the coefficients $\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q\in\mathbb{R}^+$ are such that:

$$1+\sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i \geq 0. \quad (17.1.16)$$

## 17.2 Fractional integration

In this section, we will establish some fractional integral formulas for the generalized $R$- and $G$-functions. To do this, we need to recall the following pair of Saigo hypergeometric fractional integral operators.

For $x > 0, \lambda, \sigma, \vartheta \in \mathbb{C}$ and $\Re(\lambda) > 0$, we have:

$$\left(I_{0,x}^{\lambda,\sigma,\vartheta} f(t)\right)(x) = \frac{x^{-\lambda-\sigma}}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} {}_2F_1\left(\lambda+\sigma, -\vartheta; \lambda; 1-\frac{t}{x}\right) f(t)\, dt \tag{17.2.1}$$

and

$$\left(J_{x,\infty}^{\lambda,\sigma,\vartheta} f(t)\right)(x) = \frac{1}{\Gamma(\lambda)} \int_x^\infty (t-x)^{\lambda-1} t^{-\lambda-\sigma} {}_2F_1\left(\lambda+\sigma, -\vartheta; \lambda; 1-\frac{x}{t}\right) f(t)\, dt, \tag{17.2.2}$$

where the ${}_2F_1(.)$, a special case of the generalized hypergeometric function, is the Gauss hypergeometric function.

The operator $I_{0,x}^{\lambda,\sigma,\vartheta}(.)$ contains the Riemann–Liouville $R_{0,x}^{\lambda}(.)$ fractional integral operators by means of the following relationships:

$$\left(R_{0,x}^{\lambda} f(t)\right)(x) = \left(I_{0,x}^{\lambda,-\lambda,\vartheta} f(t)\right)(x) = \frac{1}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} f(t)\, dt \tag{17.2.3}$$

$$\left(W_{x,\infty}^{\lambda} f(t)\right)(x) = \left(J_{x,\infty}^{\lambda,-\lambda,\vartheta} f(t)\right)(x) = \frac{1}{\Gamma(\lambda)} \int_x^\infty (t-x)^{\lambda-1} f(t)\, dt. \tag{17.2.4}$$

It is noted that the operator (17.2.2) unifies the Erdêlyi–Kober fractional integral operators as follows:

$$\left(E_{0,x}^{\lambda,\vartheta} f(t)\right)(x) = \left(I_{0,x}^{\lambda,0,\vartheta} f(t)\right)(x) = \frac{x^{-\lambda-\vartheta}}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} t^{\eta} f(t)\, dt \tag{17.2.5}$$

$$\left(K_{x,\infty}^{\lambda,\vartheta} f(t)\right)(x) = \left(J_{x,\infty}^{\lambda,0,\vartheta} f(t)\right)(x) = \frac{x^{\vartheta}}{\Gamma(\lambda)} \int_x^\infty (t-x)^{\lambda-1} t^{-\lambda-\vartheta} f(t)\, dt. \tag{17.2.6}$$

The following lemmas, proved in Kilbas and Sebastian [20], are useful to prove our main results.

**Lemma 1** (Kilbas and Sebastian 2008, [20]). *Let $\lambda, \sigma, \vartheta \in \mathbb{C}$ be such that $\Re(\lambda) > 0$, $\Re(\rho) > max[0, \Re(\sigma-\vartheta)]$, then*

$$\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\vartheta-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\vartheta)} x^{\rho-\sigma-1}. \tag{17.2.7}$$

**Lemma 2** (Kilbas and Sebastian 2008, [20]). *Let* $\lambda, \sigma, \vartheta \in \mathbb{C}$ *be such that* $\Re(\lambda) > 0$, $\Re(\rho) < 1 + min[\Re(\sigma), \Re(\vartheta)]$, *then*

$$\left(J_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1}\right)(x) = \frac{\Gamma(\sigma-\rho+1)\Gamma(\vartheta-\rho+1)}{\Gamma(1-\rho)\Gamma(\lambda+\sigma+\vartheta-\rho+1)} x^{\rho-\sigma-1}. \tag{17.2.8}$$

The main results are given in the following theorem.

**Theorem 1.** *Let* $\lambda, \sigma, \vartheta, \alpha, \nu, q \in \mathbb{C}$; $k \in \mathbb{R}$, $\Re(q) > 0$, $\Re(\alpha - \nu) > 0$, $\Re(\lambda) > 0$, $a > 0$ *be such that* $\Re((nq+1)\alpha + \rho - \nu - 1) > max[0, \Re(\sigma - \vartheta)]$ *and* $t > 0$, *then*

$$\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} R_{k,\alpha,\nu}^{q}(a,t)\right)(x) = \frac{x^{\alpha+\rho-\nu-\sigma-2}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \times {}_3\Psi_3\left[\begin{matrix} (\alpha+\rho-\nu-1,\alpha q), (\alpha+\rho+\vartheta-\nu-\sigma-1,\alpha q), (1,1) \\ (\frac{\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\alpha+\rho-\nu-\sigma-1,\alpha q), (\alpha+\rho+\lambda+\vartheta-\nu-1,\alpha q) \end{matrix} \middle| \left(\frac{ax^{\alpha}}{k^{\alpha/k}}\right)^{q}\right]. \tag{17.2.9}$$

*Proof.* For convenience, we denote the left-hand side of the result (17.2.9) by $\mathscr{I}$. Using (17.1.13), and then changing the order of integration and summation, which is valid under the conditions of Theorem 1, we have:

$$\mathscr{I} = \sum_{n=0}^{\infty} \frac{a^{nq}}{\Gamma_k[(nq+1)\alpha - \nu]} \left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{(nq+1)\alpha+\rho-\nu-2}\right)(x). \tag{17.2.10}$$

Applying the result (17.2.7), Eq. (17.2.10) reduces to:

$$\mathscr{I} = x^{\alpha+\rho-\nu-\sigma-2} \sum_{n=0}^{\infty} \frac{a^{nq}}{\Gamma_k[(nq+1)\alpha-\nu]} \times \frac{\Gamma[(nq+1)\alpha+\rho-\nu-1]\Gamma[(nq+1)\alpha+\rho-\nu-1+\vartheta-\sigma]}{\Gamma[(nq+1)\alpha+\rho-\nu-1-\sigma]\Gamma[(nq+1)\alpha+\rho-\nu-1+\lambda+\vartheta]} x^{nq\alpha}, \tag{17.2.11}$$

which can be written as

$$\mathscr{I} = \frac{x^{\alpha+\rho-\nu-\sigma-2}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \sum_{n=0}^{\infty} \frac{\Gamma[(nq+1)\alpha+\rho-\nu-1]}{\Gamma\left[\frac{(nq+1)\alpha}{k} - \frac{\nu}{k}\right]\Gamma[(nq+1)\alpha+\rho-\nu-\sigma-1]} \times \frac{\Gamma[(nq+1)\alpha+\rho+\vartheta-\nu-\sigma-1]\Gamma(n+1)}{\Gamma[(nq+1)\alpha+\rho+\lambda+\vartheta-\nu-1]} \frac{1}{n!} \left(\frac{ax^{\alpha}}{k^{\alpha/k}}\right)^{nq}. \tag{17.2.12}$$

Interpreting the above equation with the help of (17.1.15), we have the required result. □

**Theorem 2.** *Let* $\lambda, \sigma, \vartheta, \alpha, \nu, q \in \mathbb{C}$; $k \in \mathbb{R}$, $\Re(q) > 0$, $\Re(\alpha-\nu) > 0$, $\Re(\lambda) > 0$, $a > 0$ *be such that* $\Re((nq+1)\alpha+\rho-\nu) < 2 + min[\Re(\sigma), \Re(\vartheta)]$ *and* $t > 0$, *then*

$$\left(J_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} R_{k,\alpha,\nu}^{q}(a, 1/t)\right)(x) = \frac{x^{\rho-\alpha+\nu-\sigma}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \times {}_3\Psi_3\left[\begin{array}{c}(\sigma-\rho+\alpha-\nu, \alpha q), (\vartheta-\rho+\alpha-\nu, \alpha q), (1,1) \\ (\frac{\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\alpha-\rho-\nu, \alpha q), (\lambda+\sigma+\vartheta-\rho+\alpha-\nu, \alpha q)\end{array} \middle| \left(\frac{a}{k^{\alpha/k}x^{\alpha}}\right)^{q}\right]. \tag{17.2.13}$$

*Proof.* For convenience, we denote the left-hand side of the result (17.2.13) by $\mathscr{J}$. Using (17.1.13), and then changing the order of integration and summation, which is valid under the conditions of Theorem 2, we have:

$$\mathscr{J} = \sum_{n=0}^{\infty} \frac{a^{nq}}{\Gamma_k[(nq+1)\alpha-\nu]} \left(J_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-(nq+1)\alpha+\nu+1-1}\right)(x). \tag{17.2.14}$$

Applying the result (17.2.8), Eq. (17.2.14) reduces to:

$$\mathscr{J} = \sum_{n=0}^{\infty} \frac{a^{nq}\Gamma[\sigma-\rho+nq\alpha+\alpha-\nu]}{\Gamma_k[(nq+1)\alpha-\nu]\,\Gamma[nq\alpha+\alpha-\rho-\nu]} \times \frac{\Gamma[\vartheta-\rho+nq\alpha+\alpha-\nu]}{\Gamma[\lambda+\sigma+\vartheta-\rho+nq\alpha+\alpha-\nu]} x^{\rho-(nq+1)\alpha+\nu-\sigma}, \tag{17.2.15}$$

which can be written as:

$$\mathscr{J} = \frac{x^{\rho-\alpha+\nu-\sigma}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \sum_{n=0}^{\infty} \frac{\Gamma[\sigma-\rho+\alpha-\nu+nq\alpha]}{\Gamma\left[\frac{\alpha}{k}-\frac{\nu}{k}+\frac{nq\alpha}{k}\right]\Gamma[\alpha-\rho-\nu+nq\alpha]} \times \frac{\Gamma[\vartheta-\rho+\alpha-\nu+nq\alpha]\Gamma(1+n)}{\Gamma[\lambda+\sigma+\vartheta-\rho+\alpha-\nu+nq\alpha]} \frac{1}{n!} \left(\frac{a}{k^{\alpha/k}x^{\alpha}}\right)^{nq}. \tag{17.2.16}$$

In view of the definition of the Fox–Wright function (17.1.15) we obtain the desired result. □

**Theorem 3.** *Let* $\lambda, \sigma, \vartheta, \alpha, \nu, \mu, q \in \mathbb{C}$; $k \in \mathbb{R}$, $\Re(q) > 0$, $\Re(\alpha\mu-\nu) > 0$, $\Re(\lambda) > 0$, $a > 0$ *be such that* $\Re((nq+\mu)\alpha+\rho-\nu-1) > max[0, \Re(\sigma-\vartheta)]$ *and* $t > 0$, *then*

$$\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} G_{k,\alpha,\nu,\mu}^{q}(a, t)\right)(x) = \frac{x^{\mu\alpha+\rho-\nu-\sigma-2}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)} \times {}_3\Psi_3\left[\begin{array}{c}(\mu\alpha+\rho-\nu-1, \alpha q), (\mu\alpha+\rho+\vartheta-\nu-\sigma-1, \alpha q), (\mu, q) \\ (\frac{\mu\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\mu\alpha+\rho-\nu-\sigma-1, \alpha q), (\mu\alpha+\rho+\lambda+\vartheta-\nu-1, \alpha q)\end{array} \middle| \left(\frac{ax^{\alpha}}{k^{\alpha/k}}\right)^{q}\right]. \tag{17.2.17}$$

*Proof.* The proof of this theorem is the same as that of Theorem 1. □

**Theorem 4.** *Let* $\lambda, \sigma, \vartheta, \alpha, \nu, \mu, q \in \mathbb{C}$; $k \in \mathbb{R}$, $\Re(q) > 0$, $\Re(\alpha\mu - \nu) > 0$, $\Re(\lambda) > 0$, $a > 0$ *be such that* $\Re((nq+\mu)\alpha + \rho - \nu) < 2 + min[\Re(\sigma), \Re(\vartheta)]$ *and* $t > 0$, *then*

$$\left(J_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} G_{k,\alpha,\nu,\mu}^{q}(a, 1/t)\right)(x) = \frac{x^{\rho-\mu\alpha+\nu-\sigma}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)}$$
$$\times {}_3\Psi_3\left[\begin{array}{c} (\sigma-\rho+\mu\alpha-\nu, \alpha q), (\vartheta-\rho+\mu\alpha-\nu, \alpha q), (\mu, q) \\ (\frac{\mu\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\mu\alpha-\rho-\nu, \alpha q), (\lambda+\sigma+\vartheta-\rho+\mu\alpha-\nu, \alpha q) \end{array} \middle| \left(\frac{a}{k^{\alpha/k}x^{\alpha}}\right)^{q}\right]. \tag{17.2.18}$$

*Proof.* The proof of this theorem is the same as that of Theorem 2. □

**Remark 1.** We can easily obtain the Riemann–Liouville and Erdêlyi–Kober fractional integrals for the R- and G-functions by setting the values of the parameters and using the relations given in Eqs. (17.2.3), (17.2.4), (17.2.5), and (17.2.6).

## 17.3 Image formulas associated with integral transforms

In this section, we establish certain theorems involving the results obtained in the previous section associated with the integral transforms like the Beta transform, Laplace transform, and Whittaker transform.

### 17.3.1 Beta transform

The Beta transform of $f(z)$ is defined as [14]:

$$B\{f(z) : a, b\} = \int_0^1 z^{a-1}(1-z)^{b-1} f(z)dz. \tag{17.3.1}$$

**Theorem 5.** *Let* $\lambda, \sigma, \vartheta, \alpha, \nu, q \in \mathbb{C}$; $k \in \mathbb{R}$, $\Re(q) > 0$, $\Re(\alpha - \nu) > 0$, $\Re(\lambda) > 0$, $a > 0$ *be such that* $\Re((nq+1)\alpha + \rho - \nu - 1) > max[0, \Re(\sigma - \vartheta)]$ *and* $t > 0$, *then*

$$B\left\{\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} R_{k,\alpha,\nu}^{q}(a, tz)\right)(x) : l, m\right\} = \Gamma(m)\frac{x^{\alpha+\rho-\nu-\sigma-2}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}}$$
$$\times {}_3\Psi_3\left[\begin{array}{c} (\alpha+\rho-\nu-1, \alpha q), (\alpha+\rho+\vartheta-\nu-\sigma-1, \alpha q), \\ (\frac{\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\alpha+\rho-\nu-\sigma-1, \alpha q), \end{array}\right. \tag{17.3.2}$$
$$\left.\begin{array}{c} (l+\alpha-\nu, q\alpha), (1,1) \\ (\alpha+\rho+\lambda+\vartheta-\nu-1, \alpha q), (l+m+\alpha-\nu, q\alpha) \end{array} \middle| \left(\frac{ax^{\alpha}}{k^{\alpha/k}}\right)^{q}\right].$$

*Proof.* For convenience, we denote the left-hand side of the result (17.3.2) by $\mathscr{B}$. Using the definition of the beta transform, the left-hand side of (17.3.2) becomes:

$$\mathscr{B} = \int_0^1 z^{l-1}(1-z)^{m-1}\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} R_{k,\alpha,\nu}^{q}(a, tz)\right)(x)dz. \tag{17.3.3}$$

Further, using (17.1.13) and then changing the order of integration and summation, which is valid under the conditions of Theorem 1, we have:

$$\mathscr{B} = \sum_{n=0}^{\infty} \frac{a^{nq}}{\Gamma_k[(nq+1)\alpha-\nu]} \left(I_{0+}^{\lambda,\sigma,\vartheta} t^{(nq+1)\alpha+\rho-\nu-2}\right)(x) \times \int_0^1 z^{l+(nq+1)\alpha+\rho-\nu-2}(1-z)^{m-1}dz. \tag{17.3.4}$$

Applying the result (17.2.7), after simplification Eq. (17.3.4) reduces to:

$$\mathscr{B} = \frac{x^{\alpha+\rho-\nu-\sigma-2}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \sum_{n=0}^{\infty} \frac{\Gamma[(nq+1)\alpha+\rho-\nu-1]}{\Gamma\left[\frac{(nq+1)\alpha}{k}-\frac{\nu}{k}\right]\Gamma[(nq+1)\alpha+\rho-\nu-\sigma-1]} \times \frac{\Gamma[(nq+1)\alpha+\rho+\vartheta-\nu-\sigma-1]\Gamma(n+1)}{\Gamma[(nq+1)\alpha+\rho+\lambda+\vartheta-\nu-1]} \frac{1}{n!}\left(\frac{ax^{\alpha}}{k^{\alpha/k}}\right)^{nq} \times \int_0^1 z^{l+(nq+1)\alpha-\nu-1}(1-z)^{m-1}dz. \tag{17.3.5}$$

Applying the definition of the beta transform, Eq. (17.3.5) reduces to:

$$\mathscr{B} = \Gamma(m)\frac{x^{\alpha+\rho-\nu-\sigma-2}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \sum_{n=0}^{\infty} \frac{\Gamma[(nq+1)\alpha+\rho-\nu-1]}{\Gamma\left[\frac{(nq+1)\alpha}{k}-\frac{\nu}{k}\right]\Gamma[(nq+1)\alpha+\rho-\nu-\sigma-1]} \times \frac{\Gamma[(nq+1)\alpha+\rho+\vartheta-\nu-\sigma-1]\Gamma(n+1)}{\Gamma[(nq+1)\alpha+\rho+\lambda+\vartheta-\nu-1]} \times \frac{\Gamma(l+(nq+1)\alpha-\nu)}{\Gamma(l+m+(nq+1)\alpha-\nu)} \frac{1}{n!}\left(\frac{ax^{\alpha}}{k^{\alpha/k}}\right)^{nq}. \tag{17.3.6}$$

Interpreting the above equation with the help of (17.1.15), we have the required result. □

**Theorem 6.** *Let* $\lambda, \sigma, \vartheta, \alpha, \nu, q \in \mathbb{C}$; $k \in \mathbb{R}$, $\Re(q) > 0$, $\Re(\alpha-\nu) > 0$, $\Re(\lambda) > 0$, $a > 0$ *be such that* $\Re((nq+1)\alpha+\rho-\nu) < 2+min[\Re(\sigma), \Re(\vartheta)]$ *and* $t > 0$, *then*

$$B\left\{\left(J_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} R_{k,\alpha,\nu}^{q}(a, z/t)\right)(x) : l, m\right\} = \Gamma(m)\frac{x^{\rho-\alpha+\nu-\sigma}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \times {}_3\Psi_3\left[\begin{matrix} (\sigma-\rho+\alpha-\nu, \alpha q), (\vartheta-\rho+\alpha-\nu, \alpha q), (l+\alpha-\nu, q\alpha), (1,1) \\ (\frac{\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\alpha-\rho-\nu, \alpha q), (\lambda+\sigma+\vartheta-\rho+\alpha-\nu, \alpha q), (l+m+\alpha-\nu, q\alpha) \end{matrix} \middle| \left(\frac{a}{k^{\alpha/k}x^{\alpha}}\right)^{q}\right]. \tag{17.3.7}$$

*Proof.* The proof of this theorem is the same as that of Theorem 3. □

**Theorem 7.** *Let* $\lambda, \sigma, \vartheta, \alpha, \nu, \mu, q \in \mathbb{C}$; $k \in \mathbb{R}$, $\Re(q) > 0$, $\Re(\alpha\mu - \nu) > 0$, $\Re(\lambda) > 0$, $a > 0$ *be such that* $\Re((nq+\mu)\alpha + \rho - \nu - 1) > max[0, \Re(\sigma - \vartheta)]$ *and* $t > 0$, *then*

$$B\left\{\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} G_{k,\alpha,\nu,\mu}^{q}(a, zt)\right)(x) : l, m\right\} = \Gamma(m) \frac{x^{\mu\alpha+\rho-\nu-\sigma-2}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)}$$
$$\times {}_3\Psi_3\left[\begin{array}{c} (\mu\alpha+\rho-\nu-1, \alpha q), (\mu\alpha+\rho+\vartheta-\nu-\sigma-1, \alpha q), \\ (\frac{\mu\alpha}{k} - \frac{\nu}{k}, \frac{\alpha q}{k}), (\mu\alpha+\rho-\nu-\sigma-1, \alpha q), \end{array}\right. \tag{17.3.8}$$
$$\left.\begin{array}{c} (\mu, q), (l+\alpha\mu-\nu, q\alpha) \\ (\mu\alpha+\rho+\lambda+\vartheta-\nu-1, \alpha q), (l+m+\alpha\mu-\nu, q\alpha) \end{array}\right| \left(\frac{ax^{\alpha}}{k^{\alpha/k}}\right)^{q}\Bigg].$$

*Proof.* The proof of this theorem is the same as that of Theorem 1. □

**Theorem 8.** *Let* $\lambda, \sigma, \vartheta, \alpha, \nu, \mu, q \in \mathbb{C}$; $k \in \mathbb{R}$, $\Re(q) > 0$, $\Re(\alpha\mu - \nu) > 0$, $\Re(\lambda) > 0$, $a > 0$ *be such that* $\Re((nq+\mu)\alpha + \rho - \nu) < 2 + min[\Re(\sigma), \Re(\vartheta)]$ *and* $t > 0$, *then*

$$B\left\{\left(J_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} G_{k,\alpha,\nu,\mu}^{q}(a, z/t)\right)(x) : l, m\right\} = \Gamma(m) \frac{x^{\rho-\mu\alpha+\nu-\sigma}}{k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)}$$
$$\times {}_3\Psi_3\left[\begin{array}{c} (\sigma-\rho+\mu\alpha-\nu, \alpha q), (\vartheta-\rho+\mu\alpha-\nu, \alpha q), \\ (\frac{\mu\alpha}{k} - \frac{\nu}{k}, \frac{\alpha q}{k}), (\mu\alpha-\rho-\nu, \alpha q), \end{array}\right. \tag{17.3.9}$$
$$\left.\begin{array}{c} (\mu, q), (l+\alpha\mu-\nu, q\alpha) \\ (\lambda+\sigma+\vartheta-\rho+\mu\alpha-\nu, \alpha q), (l+m+\alpha\mu-\nu, q\alpha) \end{array}\right| \left(\frac{a}{k^{\alpha/k}x^{\alpha}}\right)^{q}\Bigg].$$

*Proof.* The proof of this theorem is the same as that of Theorem 2. □

## 17.3.2 Laplace transform

The Laplace transform of $f(z)$ is defined as [14]:

$$L\{f(z)\} = \int_0^{\infty} e^{-sz} f(z) dz. \tag{17.3.10}$$

**Theorem 9.** *Let* $\lambda, \sigma, \vartheta, \alpha, \nu, q \in \mathbb{C}$; $k \in \mathbb{R}$, $\Re(q) > 0$, $\Re(\alpha - \nu) > 0$, $\Re(\lambda) > 0$, $a > 0$ *be such that* $\Re((nq+1)\alpha + \rho - \nu - 1) > max[0, \Re(\sigma - \vartheta)]$ *and* $t > 0$, *then*

$$L\left\{z^{l-1}\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} R_{k,\alpha,\nu}^{q}(a, tz)\right)(x)\right\} = \frac{x^{\alpha+\rho-\nu-\sigma-2}}{s^{l+\alpha-\nu-1}k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}}$$
$$\times {}_4\Psi_3\left[\begin{array}{c} (\alpha+\rho-\nu-1, \alpha q), (\alpha+\rho+\vartheta-\nu-\sigma-1, \alpha q), \\ (\frac{\alpha}{k} - \frac{\nu}{k}, \frac{\alpha q}{k}), (\alpha+\rho-\nu-\sigma-1, \alpha q) \end{array}\right. \tag{17.3.11}$$
$$\left.\begin{array}{c} (l+\alpha-\nu-1, \alpha q), (1, 1), \\ (\alpha+\rho+\lambda+\vartheta-\nu-1, \alpha q) \end{array}\right| \left(\frac{ax^{\alpha}}{s^{\alpha}k^{\alpha/k}}\right)^{q}\Bigg].$$

*Proof.* For convenience, we denote the left-hand side of the result (17.3.11) by $\mathscr{L}$. Then, applying the Laplace transform, we have:

$$\mathscr{L} = \int_0^\infty e^{-sz} z^{l-1} \left( I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} R_{k,\alpha,\nu}^{q}(a, tz) \right)(x) dz. \tag{17.3.12}$$

Further, using (17.1.13) and then changing the order of integration and summation, which is valid under the conditions of Theorem 1, we have:

$$\begin{aligned}\mathscr{L} &= \frac{x^{\alpha+\rho-\nu-\sigma-2}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \sum_{n=0}^{\infty} \frac{\Gamma[(nq+1)\alpha+\rho-\nu-1]}{\Gamma\left[\frac{(nq+1)\alpha}{k}-\frac{\nu}{k}\right]\Gamma[(nq+1)\alpha+\rho-\nu-\sigma-1]} \\ &\times \frac{\Gamma[(nq+1)\alpha+\rho+\vartheta-\nu-\sigma-1]\Gamma(n+1)}{\Gamma[(nq+1)\alpha+\rho+\lambda+\vartheta-\nu-1]} \frac{1}{n!} \left(\frac{ax^\alpha}{k^{\alpha/k}}\right)^{nq} \\ &\times \int_0^\infty e^{-sz} z^{l+(nq+1)\alpha-\nu-2} dz\end{aligned} \tag{17.3.13}$$

$$\begin{aligned}\mathscr{L} &= \frac{x^{\alpha+\rho-\nu-\sigma-2}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \sum_{n=0}^{\infty} \frac{\Gamma[(nq+1)\alpha+\rho-\nu-1]}{\Gamma\left[\frac{(nq+1)\alpha}{k}-\frac{\nu}{k}\right]\Gamma[(nq+1)\alpha+\rho-\nu-\sigma-1]} \\ &\times \frac{\Gamma[(nq+1)\alpha+\rho+\vartheta-\nu-\sigma-1]\Gamma(n+1)}{\Gamma[(nq+1)\alpha+\rho+\lambda+\vartheta-\nu-1]} \frac{1}{n!} \left(\frac{ax^\alpha}{k^{\alpha/k}}\right)^{nq} \\ &\times \frac{\Gamma[l+(nq+1)\alpha-\nu-1]}{s^{l+(nq+1)\alpha-\nu-1}}\end{aligned} \tag{17.3.14}$$

$$\begin{aligned}\mathscr{L} &= \frac{x^{\alpha+\rho-\nu-\sigma-2}}{s^{l+\alpha-\nu-1} k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \sum_{n=0}^{\infty} \frac{\Gamma[(nq+1)\alpha+\rho-\nu-1]}{\Gamma\left[\frac{(nq+1)\alpha}{k}-\frac{\nu}{k}\right]\Gamma[(nq+1)\alpha+\rho-\nu-\sigma-1]} \\ &\times \frac{\Gamma[(nq+1)\alpha+\rho+\vartheta-\nu-\sigma-1]\Gamma(n+1)}{\Gamma[(nq+1)\alpha+\rho+\lambda+\vartheta-\nu-1]} \frac{\Gamma[l+(nq+1)\alpha-\nu-1]}{n!} \\ &\times \left(\frac{ax^\alpha}{sk^{\alpha/k}}\right)^{nq}.\end{aligned} \tag{17.3.15}$$

Interpreting the above equation with the help of (17.1.15), we have the required result. □

**Theorem 10.** *Let* $\lambda, \sigma, \vartheta, \alpha, \nu, q \in \mathbb{C}; k \in \mathbb{R}$, $\Re(q) > 0$, $\Re(\alpha-\nu) > 0$, $\Re(\lambda) > 0$, $a > 0$ *be such that* $\Re((nq+1)\alpha+\rho-\nu) < 2 + min[\Re(\sigma), \Re(\vartheta)]$ *and* $t > 0$, *then*

$$\begin{aligned}L\left\{z^{l-1}\left(J_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} R_{k,\alpha,\nu}^{q}(a, z/t)\right)(x)\right\} &= \frac{x^{\rho-\alpha+\nu-\sigma}}{s^{l+\alpha-\nu-1} k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \\ \times {}_4\Psi_3 \Bigg[ &\begin{matrix} (\sigma-\rho+\alpha-\nu, \alpha q), (\vartheta-\rho+\alpha-\nu, \alpha q), \\ (\frac{\alpha}{k}-\frac{\nu}{k}, \frac{\alpha q}{k}), (\alpha-\rho-\nu, \alpha q), \end{matrix}\end{aligned} \tag{17.3.16}$$

$$\left. \begin{matrix} (l+\alpha-\nu-1,\alpha q),(1,1) \\ (\lambda+\sigma+\vartheta-\rho+\alpha-\nu,\alpha q) \end{matrix} \right| \left( \frac{a}{s^{\alpha}k^{\alpha/k}x^{\alpha}} \right)^{q} \Bigg].$$

*Proof.* The proof of this theorem runs parallel to that of Theorem 5. □

**Theorem 11.** *Let* $\lambda, \sigma, \vartheta, \alpha, \nu, \mu, q \in \mathbb{C}$; $k \in \mathbb{R}$, $\Re(q) > 0$, $\Re(\alpha\mu - \nu) > 0$, $\Re(\lambda) > 0$, $a > 0$ *be such that* $\Re((nq+\mu)\alpha + \rho - \nu - 1) > max[0, \Re(\sigma - \vartheta)]$ *and* $t > 0$, *then*

$$L\left\{z^{l-1}\left(I_{0,x}^{\lambda,\sigma,\vartheta}t^{\rho-1}G_{k,\alpha,\nu,\mu}^{q}(a,tz)\right)(x)\right\} = \frac{x^{\mu\alpha+\rho-\nu-\sigma-2}}{s^{l+\mu\alpha-\nu-1}k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)}$$
$$\times {}_4\Psi_3\Bigg[ \begin{matrix} (\mu\alpha+\rho-\nu-1,\alpha q),(\mu\alpha+\rho+\vartheta-\nu-\sigma-1,\alpha q), \\ (\frac{\mu\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}),(\mu\alpha+\rho-\nu-\sigma-1,\alpha q) \end{matrix} \tag{17.3.17}$$
$$\left. \begin{matrix} (\mu,q),(l+\mu\alpha-\nu-1,\alpha q), \\ (\mu\alpha+\rho+\lambda+\vartheta-\nu-1,\alpha q) \end{matrix} \right| \left( \frac{ax^{\alpha}}{s^{\alpha}k^{\alpha/k}} \right)^{q} \Bigg].$$

**Theorem 12.** *Let* $\lambda, \sigma, \vartheta, \alpha, \nu, \mu, q \in \mathbb{C}$; $k \in \mathbb{R}$, $\Re(q) > 0$, $\Re(\alpha\mu - \nu) > 0$, $\Re(\lambda) > 0$, $a > 0$ *be such that* $\Re((nq+\mu)\alpha + \rho - \nu) < 2 + min[\Re(\sigma), \Re(\vartheta)]$ *and* $t > 0$, *then*

$$L\left\{z^{l-1}\left(J_{x,\infty}^{\lambda,\sigma,\vartheta}t^{\rho-1}G_{k,\alpha,\nu,\mu}^{q}(a,z/t)\right)(x)\right\} = \frac{x^{\rho-\mu\alpha+\nu-\sigma}}{s^{l+\mu\alpha-\nu-1}k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)}$$
$$\times {}_4\Psi_3\Bigg[ \begin{matrix} (\sigma-\rho+\mu\alpha-\nu,\alpha q),(\vartheta-\rho+\mu\alpha-\nu,\alpha q), \\ (\frac{\mu\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}),(\mu\alpha-\rho-\nu,\alpha q), \end{matrix} \tag{17.3.18}$$
$$\left. \begin{matrix} (\mu,q),(l+\mu\alpha-\nu-1,\alpha q), \\ (\lambda+\sigma+\vartheta-\rho+\mu\alpha-\nu,\alpha q) \end{matrix} \right| \left( \frac{a}{s^{\alpha}k^{\alpha/k}x^{\alpha}} \right)^{q} \Bigg].$$

### 17.3.3 Whittaker transform

**Theorem 13.** *Let* $\lambda, \sigma, \vartheta, \alpha, \nu, q \in \mathbb{C}$; $k \in \mathbb{R}$, $\Re(q) > 0$, $\Re(\alpha - \nu) > 0$, $\Re(\lambda) > 0$, $a > 0$, $\Re(\xi \pm \omega) > \dfrac{-1}{2}$ *be such that* $\Re((nq+1)\alpha + \rho - \nu - 1) > max[0, \Re(\sigma - \vartheta)]$ *and* $t > 0$, *then*

$$\int_0^{\infty} z^{\xi-1}e^{-\eta z/2}W_{\tau,\omega}(\eta z)\left\{\left(I_{0,x}^{\lambda,\sigma,\vartheta}t^{\rho-1}R_{k,\alpha,\nu}^{q}(a,tz)\right)(x)\right\}dz$$
$$= \frac{x^{\alpha+\rho-\nu-\sigma-2}}{\eta^{\alpha+\xi-\nu-1}k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}}{}_5\Psi_4\Bigg[ \begin{matrix} (\alpha+\rho-\nu-1,\alpha q),(\alpha+\rho+\vartheta-\nu-\sigma-1,\alpha q) \\ \left(\frac{\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}\right),(\alpha+\rho-\nu-\sigma-1,\alpha q), \end{matrix}$$
$$\left. \begin{matrix} (1/2+\omega+\alpha+\xi-\nu-1,\alpha q),(1/2-\omega+\alpha+\xi-\nu-1,\alpha q),(1,1) \\ (\alpha+\rho+\lambda+\vartheta-\nu-1,\alpha q),(1/2-\tau+\alpha+\xi-\nu-1,\alpha q) \end{matrix} \right| \left( \frac{ax^{\alpha}}{\eta^{\alpha}k^{\alpha/k}} \right)^{q} \Bigg]. \tag{17.3.19}$$

*Proof.* For convenience, we denote the left-hand side of the result (17.3.19) by $\mathscr{W}$. Then, using the result from (17.2.12), after changing the order of integration and summation, we obtain:

$$\mathscr{W} = \frac{x^{\alpha+\rho-\nu-\sigma-2}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \sum_{n=0}^{\infty} \frac{\Gamma[(nq+1)\alpha+\rho-\nu-1]}{\Gamma\left[\frac{(nq+1)\alpha}{k}-\frac{\nu}{k}\right]\Gamma[(nq+1)\alpha+\rho-\nu-\sigma-1]}$$
$$\times \frac{\Gamma[(nq+1)\alpha+\rho+\vartheta-\nu-\sigma-1]\Gamma(n+1)}{\Gamma[(nq+1)\alpha+\rho+\lambda+\vartheta-\nu-1]} \frac{1}{n!}\left(\frac{ax^{\alpha}}{k^{\alpha/k}}\right)^{nq} \tag{17.3.20}$$
$$\times \int_0^{\infty} z^{(nq+1)\alpha+\xi-\nu-2} e^{-\eta z/2} W_{\tau,\omega}(\eta z)dz.$$

On substituting $\eta z = \varsigma$, (17.3.20) becomes:

$$\mathscr{W} = \frac{x^{\alpha+\rho-\nu-\sigma-2}}{k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \sum_{n=0}^{\infty} \frac{\Gamma[(nq+1)\alpha+\rho-\nu-1]}{\Gamma\left[\frac{(nq+1)\alpha}{k}-\frac{\nu}{k}\right]\Gamma[(nq+1)\alpha+\rho-\nu-\sigma-1]}$$
$$\times \frac{\Gamma[(nq+1)\alpha+\rho+\vartheta-\nu-\sigma-1]\Gamma(n+1)}{\Gamma[(nq+1)\alpha+\rho+\lambda+\vartheta-\nu-1]} \frac{1}{n!}\left(\frac{ax^{\alpha}}{k^{\alpha/k}}\right)^{nq} \tag{17.3.21}$$
$$\times \frac{1}{\eta^{(nq+1)\alpha+\xi-\nu-1}} \int_0^{\infty} \varsigma^{(nq+1)\alpha+\xi-\nu-2} e^{-\varsigma/2} W_{\tau,\omega}(\varsigma)d\varsigma.$$

Now, we use the following integral formula involving the Whittaker function:

$$\int_0^{\infty} t^{\nu-1} e^{-t/2} W_{\tau,\omega}(t)dt = \frac{\Gamma(1/2+\omega+\nu)\Gamma(1/2-\omega+\nu)}{\Gamma(1/2-\tau+\nu)}, \quad \left(\Re(\nu\pm\omega) > \frac{-1}{2}\right), \tag{17.3.22}$$

$$\mathscr{W} = \frac{x^{\alpha+\rho-\nu-\sigma-2}}{\eta^{\alpha+\xi-\nu-1} k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}} \sum_{n=0}^{\infty} \frac{\Gamma[(nq+1)\alpha+\rho-\nu-1]}{\Gamma\left[\frac{(nq+1)\alpha}{k}-\frac{\nu}{k}\right]\Gamma[(nq+1)\alpha+\rho-\nu-\sigma-1]}$$
$$\times \frac{\Gamma[(nq+1)\alpha+\rho+\vartheta-\nu-\sigma-1]\Gamma(n+1)}{\Gamma[(nq+1)\alpha+\rho+\lambda+\vartheta-\nu-1]}$$
$$\times \frac{\Gamma(1/2+\omega+(nq+1)\alpha+\xi-\nu-1)\Gamma(1/2-\omega+(nq+1)\alpha+\xi-\nu-1)}{\Gamma(1/2-\tau+(nq+1)\alpha+\xi-\nu-1)}$$
$$\times \frac{1}{n!}\left(\frac{ax^{\alpha}}{\eta^{\alpha} k^{\alpha/k}}\right)^{nq}. \tag{17.3.23}$$

Interpreting the above equation with the help of (17.1.15), we have the required result. □

**Theorem 14.** *Let* $\lambda, \sigma, \vartheta, \alpha, \nu, q \in \mathbb{C}$; $k \in \mathbb{R}$, $\Re(q) > 0$, $\Re(\alpha-\nu) > 0$, $\Re(\lambda) > 0$, $a > 0$, $\Re(\xi \pm \omega) > \frac{-1}{2}$ *be such that* $\Re((nq+1)\alpha+\rho-\nu) < 2+min[\Re(\sigma), \Re(\vartheta)]$ *and* $t > 0$, *then*

$$\int_0^\infty z^{\xi-1}e^{-\eta z/2}W_{\tau,\omega}(\eta z)\left\{\left(J_{x,\infty}^{\lambda,\sigma,\vartheta}t^{\rho-1}R_{k,\alpha,\nu}^{q}(a,z/t)\right)(x)\right\}dz$$
$$=\frac{x^{\alpha+\rho-\nu-\sigma-2}}{\eta^{\alpha+\xi-\nu-1}k^{\frac{\alpha}{k}-\frac{\nu}{k}-1}}{}_5\Psi_4\left[\begin{array}{c}(\sigma-\rho+\alpha-\nu,\alpha q),(\vartheta-\rho+\alpha-\nu,\alpha q)\\ \left(\frac{\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}\right),(\alpha-\rho-\nu,\alpha q),\end{array}\right.$$
$$\left.\begin{array}{c}(1/2+\omega+\alpha+\xi-\nu-1,\alpha q),(1/2-\omega+\alpha+\xi-\nu-1,\alpha q),(1,1)\\(\lambda+\sigma+\vartheta-\rho+\alpha-\nu,\alpha q),(1/2-\tau+(nq+1)\alpha+\xi-\nu-1)\end{array}\right|\left(\frac{a}{\eta^{\alpha}k^{\alpha/k}x^{\alpha}}\right)^{q}\right]. \tag{17.3.24}$$

**Theorem 15.** *Let* $\lambda,\sigma,\vartheta,\alpha,\nu,\mu,q\in\mathbb{C}$; $k\in\mathbb{R}$, $\Re(q)>0$, $\Re(\alpha\mu-\nu)>0$, $\Re(\lambda)>0$, $a>0$, $\Re(\xi\pm\omega)>\dfrac{-1}{2}$ *be such that* $\Re((nq+\mu)\alpha+\rho-\nu-1)>max[0,\Re(\sigma-\vartheta)]$ *and* $t>0$, *then*

$$\int_0^\infty z^{\xi-1}e^{-\eta z/2}W_{\tau,\omega}(\eta z)\left\{\left(I_{0,x}^{\lambda,\sigma,\vartheta}t^{\rho-1}G_{k,\alpha,\nu,\mu}^{q}(a,tz)\right)(x)\right\}dz$$
$$=\frac{x^{\mu\alpha+\rho-\nu-\sigma-2}}{\eta^{\mu\alpha+\xi-\nu-1}k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)}{}_5\Psi_4\left[\begin{array}{c}(\mu\alpha+\rho-\nu-1,\alpha q),(\mu\alpha+\rho+\vartheta-\nu-\sigma-1,\alpha q)\\ \left(\frac{\mu\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}\right),(\mu\alpha+\rho-\nu-\sigma-1,\alpha q),\end{array}\right.$$
$$\left.\begin{array}{c}(1/2+\omega+\mu\alpha+\xi-\nu-1,\alpha q),(1/2-\omega+\mu\alpha+\xi-\nu-1,\alpha q),(\mu,q)\\(\mu\alpha+\rho+\lambda+\vartheta-\nu-1,\alpha q),(1/2-\tau+\mu\alpha+\xi-\nu-1,\alpha q)\end{array}\right|\left(\frac{ax^{\alpha}}{\eta^{\alpha}k^{\alpha/k}}\right)^{q}\right]. \tag{17.3.25}$$

**Theorem 16.** *Let* $\lambda,\sigma,\vartheta,\alpha,\nu,\mu,q\in\mathbb{C}$; $k\in\mathbb{R}$, $\Re(q)>0$, $\Re(\alpha\mu-\nu)>0$, $\Re(\lambda)>0$, $a>0$, $\Re(\xi\pm\omega)>\dfrac{-1}{2}$ *be such that* $\Re((nq+\mu)\alpha+\rho-\nu)<2+min[\Re(\sigma),\Re(\vartheta)]$ *and* $t>0$, *then*

$$\int_0^\infty z^{\xi-1}e^{-\eta z/2}W_{\tau,\omega}(\eta z)\left\{\left(J_{x,\infty}^{\lambda,\sigma,\vartheta}t^{\rho-1}G_{k,\alpha,\nu,\mu}^{q}(a,z/t)\right)(x)\right\}dz$$
$$=\frac{x^{\mu\alpha+\rho-\nu-\sigma-2}}{\eta^{\mu\alpha+\xi-\nu-1}k^{\frac{\mu\alpha}{k}-\frac{\nu}{k}-1}\Gamma(\mu)}{}_5\Psi_4\left[\begin{array}{c}(\sigma-\rho+\mu\alpha-\nu,\alpha q),(\vartheta-\rho+\mu\alpha-\nu,\alpha q)\\ \left(\frac{\mu\alpha}{k}-\frac{\nu}{k},\frac{\alpha q}{k}\right),(\mu\alpha-\rho-\nu,\alpha q),\end{array}\right.$$
$$\left.\begin{array}{c}(1/2+\omega+\mu\alpha+\xi-\nu-1,\alpha q),(1/2-\omega+\mu\alpha+\xi-\nu-1,\alpha q),(\mu,q)\\(\lambda+\sigma+\vartheta-\rho+\mu\alpha-\nu,\alpha q),(1/2-\tau+\mu\alpha+\xi-\nu-1,\alpha q)\end{array}\right|\left(\frac{a}{\eta^{\alpha}k^{\alpha/k}x^{\alpha}}\right)^{q}\right]. \tag{17.3.26}$$

## 17.4 Applications

The importance of fractional differential equations in the field of applied science has gained more attention not only in mathematics but also in physics, dynamical systems, control systems, and engineering, to create the mathematical model of many physical phenomena. In particular, the kinetic equations describe the continuity of motion of substances. The extension and generalization of fractional kinetic equations involving many fractional operators can be found in [1–13,21,26].

In view of the effectiveness and great importance of the kinetic equation in certain astrophysical problems the authors develop a further generalized form of the fractional kinetic equation involving extended $R$- and $G$-functions.

The fractional differential equation between the rate of change of the reaction, the destruction rate, and the production rate was established by Haubold and Mathai [2], and is given as follows:

$$\frac{dN}{dt} = -d(N_t) + p(N_t), \tag{17.4.1}$$

where $N = N(t)$ is the rate of reaction, $d = d(N)$ is the rate of destruction, $p = p(N)$ is the rate of production, and $N_t$ denotes the function defined by $N_t(t^*) = N(t - t^*)$, $t^* > 0$.

In the special case of (17.4.1) for spatial fluctuations and inhomogeneities in $N(t)$ the quantities are neglected, that is, the equation:

$$\frac{dN}{dt} = -c_i N_i(t), \tag{17.4.2}$$

with the initial condition that $N_i(t = 0) = N_0$ is the number density of the species $i$ at time $t = 0$ and $c_i > 0$. If we remove the index $i$ and integrate the standard kinetic Eq. (17.4.2), we have:

$$N(t) - N_0 = -c_0 D_t^{-1} N(t), \tag{17.4.3}$$

where ${}_0D_t^{-1}$ is the special case of the Riemann–Liouville integral operator ${}_0D_t^{-\gamma}$ defined as:

$${}_0D_t^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s)ds, \qquad (t > 0,\ R(\gamma) > 0). \tag{17.4.4}$$

The fractional generalization of the standard kinetic Eq. (17.4.3) is given by Haubold and Mathai [2] as follows:

$$N(t) - N_0 = -c^{\gamma} {}_0D_t^{-1} N(t) \tag{17.4.5}$$

and they obtained the solution of (17.4.5) as follows:

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\gamma k + 1)} (ct)^{\gamma k}. \tag{17.4.6}$$

Further, [6] considered the following fractional kinetic equation:

$$N(t) - N_0 f(t) = -c^{\gamma} {}_0D_t^{-\gamma} N(t), \qquad (\Re(\gamma) > 0), \tag{17.4.7}$$

where $N(t)$ denotes the number density of a given species at time $t$, $N_0 = N(0)$ is the number density of that species at time $t = 0$, $c$ is a constant, and $f \in \mathcal{L}(0, \infty)$.

By applying the Laplace transform to (17.4.7) (see [12]):

$$L\{N(t); p\} = N_0 \frac{F(p)}{1 + c^\gamma p^{-\gamma}} = N_0 \left( \sum_{n=0}^{\infty} (-c^\gamma)^n p^{-\gamma n} \right) F(p), \\ \left( n \in N_0, \left| \frac{c}{p} \right| < 1 \right), \tag{17.4.8}$$

where the Laplace transform [22] is given by:

$$F(p) = L\{N(t); p\} = \int_0^\infty e^{-pt} f(t) dt, \qquad (\mathcal{R}(p) > 0). \tag{17.4.9}$$

## 17.5 Solution of generalized fractional kinetic equations

In this section, we will investigate the solution of the generalized fractional kinetic equations by considering extended $R$- and $G$-functions. The results are as follows.

**Remark 2.** The solutions of the fractional kinetic equations in this section are obtained in terms of the generalized Mittag-Leffler function $E_{\alpha,\beta}(x)$ (Mittag-Leffler [23]), which is defined as:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \qquad \Re(\alpha) > 0, \Re(\beta) > 0. \tag{17.5.1}$$

**Theorem 17.** *If $c > 0, d > 0, \gamma > 0; \alpha, \nu, q \in \mathbb{C}; k \in \mathbb{R}, \Re(q) > 0, \Re(\alpha - \nu) > 0, \Re(\lambda) > 0, a > 0,$ and $t > 0$, then the solution of the equation:*

$$N(t) = N_0 R^q_{k,\alpha,\nu}(a, d^\gamma t^\gamma) - c^\gamma {}_0D_t^{-\gamma} N(t) \tag{17.5.2}$$

*is given by the following formula:*

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{a^{nq} \Gamma[\gamma((nq+1)\alpha - \nu - 1) + 1]}{\Gamma_k[(nq+1)\alpha - \nu]} (d^\gamma t^\gamma)^{(nq+1)\alpha - \nu - 1} \\ \times E_{\gamma, \gamma((nq+1)\alpha - \nu - 1)+1}(-c^\gamma t^\gamma). \tag{17.5.3}$$

*Proof.* The Laplace transform of the Riemann–Liouville fractional integral operator is given by [24], [25]:

$$L\left\{ {}_0D_t^{-\gamma} f(t); p \right\} = p^{-\gamma} F(p), \tag{17.5.4}$$

where $F(p)$ is defined in (17.4.9). Now, applying the Laplace transform to both sides of (17.5.2) gives:

$$L\{N(t); p\} = N_0 L\left\{ R^q_{k,\alpha,\nu}(a, d^\gamma t^\gamma); p \right\} - c^\gamma L\left\{ {}_0D_t^{-\gamma} N(t); p \right\} \tag{17.5.5}$$

$$N(p) = N_0 \left( \int_0^\infty e^{-pt} \sum_{n=0}^{\infty} \frac{a^{nq}}{\Gamma_k\left[(nq+1)\alpha - \nu\right]} (d^\gamma t^\gamma)^{(nq+1)\alpha-\nu-1} dt \right) - c^\gamma p^{-\gamma} N(p) \tag{17.5.6}$$

$$N(p) + c^\gamma p^{-\gamma} N(p) = N_0 \sum_{n=0}^{\infty} \frac{a^{nq}}{\Gamma_k\left[(nq+1)\alpha - \nu\right]} (d^\gamma)^{(nq+1)\alpha-\nu-1} \times \int_0^\infty e^{-pt} t^{\gamma((nq+1)\alpha-\nu-1)} dt \tag{17.5.7}$$

$$= N_0 \sum_{n=0}^{\infty} \frac{a^{nq}}{\Gamma_k\left[(nq+1)\alpha - \nu\right]} (d^\gamma)^{(nq+1)\alpha-\nu-1} \frac{\Gamma[\gamma((nq+1)\alpha-\nu-1)+1]}{p^{\gamma((nq+1)\alpha-\nu-1)+1}} \tag{17.5.8}$$

$$N(p) = N_0 \sum_{n=0}^{\infty} \frac{a^{nq}\Gamma[\gamma((nq+1)\alpha-\nu-1)+1]}{\Gamma_k\left[(nq+1)\alpha - \nu\right]} (d^\gamma)^{(nq+1)\alpha-\nu-1} \times \left\{ p^{-(\gamma((nq+1)\alpha-\nu-1)+1)} \sum_{r=0}^{\infty} \left[ -\left(\frac{p}{c}\right)^{-\gamma} \right]^r \right\}. \tag{17.5.9}$$

Taking the Laplace inverse of (17.5.9), and by using:

$$L^{-1}\left\{p^{-\gamma}; t\right\} = \frac{t^{\gamma-1}}{\Gamma(\gamma)}, (R(\gamma) > 0), \tag{17.5.10}$$

we have:

$$L^{-1}\left\{N(p)\right\} = N_0 \sum_{n=0}^{\infty} \frac{a^{nq}\Gamma[\gamma((nq+1)\alpha-\nu-1)+1]}{\Gamma_k\left[(nq+1)\alpha - \nu\right]} (d^\gamma)^{(nq+1)\alpha-\nu-1} \times L^{-1}\left\{ \sum_{r=0}^{\infty} (-1)^r c^{\gamma r} p^{-(\gamma((nq+1)\alpha+r-\nu-1)+1)} \right\} \tag{17.5.11}$$

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{a^{nq}\Gamma[\gamma((nq+1)\alpha-\nu-1)+1]}{\Gamma_k\left[(nq+1)\alpha - \nu\right]} (d^\gamma)^{(nq+1)\alpha-\nu-1} \times \left\{ \sum_{r=0}^{\infty} (-1)^r c^{\gamma r} \frac{t^{\gamma((nq+1)\alpha+r-\nu-1)}}{\Gamma[\gamma((nq+1)\alpha+r-\nu-1)+1]} \right\} \tag{17.5.12}$$

$$= N_0 \sum_{n=0}^{\infty} \frac{a^{nq}\Gamma[\gamma((nq+1)\alpha-\nu-1)+1]}{\Gamma_k\left[(nq+1)\alpha - \nu\right]} (d^\gamma t^\gamma)^{(nq+1)\alpha-\nu-1} \times \left\{ \sum_{r=0}^{\infty} (-1)^r c^{\gamma r} \frac{t^{\gamma r}}{\Gamma[\gamma((nq+1)\alpha+r-\nu-1)+1]} \right\} \tag{17.5.13}$$

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{a^{nq}\Gamma[\gamma((nq+1)\alpha - \nu - 1) + 1]}{\Gamma_k[(nq+1)\alpha - \nu]} (d^\gamma t^\gamma)^{(nq+1)\alpha-\nu-1} \times E_{\gamma,\gamma((nq+1)\alpha-\nu-1)+1}(-c^\gamma t^\gamma). \tag{17.5.14}$$

□

**Theorem 18.** *If $c > 0, d > 0, \gamma > 0; \alpha, \nu, \mu, q \in \mathbb{C}; k \in \mathbb{R}, \Re(q) > 0, \Re(\alpha\mu - \nu) > 0, \Re(\lambda) > 0, a > 0$ and $t > 0$, then the solution of the equation:*

$$N(t) = N_0 G^q_{k,\alpha,\nu,\mu}(a, d^\gamma t^\gamma) - c^\gamma {}_0D_t^{-\gamma} N(t) \tag{17.5.15}$$

*is given by the following formula*

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(\mu)_{nq} a^{nq}\Gamma[\gamma((nq+\mu)\alpha - \nu - 1) + 1]}{n!\Gamma_k[(nq+\mu)\alpha - \nu]} (d^\gamma t^\gamma)^{(nq+\mu)\alpha-\nu-1} \times E_{\gamma,\gamma((nq+\mu)\alpha-\nu-1)+1}(-c^\gamma t^\gamma). \tag{17.5.16}$$

## 17.6 Special cases

Setting $c = d$, the results in Eqs. (17.5.2) and (17.5.15) reduce to the following form.

**Corollary 1.** *If $d > 0, \gamma > 0; \alpha, \nu, q \in \mathbb{C}; k \in \mathbb{R}, \Re(q) > 0, \Re(\alpha - \nu) > 0, \Re(\lambda) > 0, a > 0$, and $t > 0$, then the solution of the equation:*

$$N(t) = N_0 R^q_{k,\alpha,\nu}(a, d^\gamma t^\gamma) - d^\gamma {}_0D_t^{-\gamma} N(t) \tag{17.6.1}$$

*is given by the following formula*

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{a^{nq}\Gamma[\gamma((nq+1)\alpha - \nu - 1) + 1]}{\Gamma_k[(nq+1)\alpha - \nu]} (d^\gamma t^\gamma)^{(nq+1)\alpha-\nu-1} \times E_{\gamma,\gamma((nq+1)\alpha-\nu-1)+1}(-d^\gamma t^\gamma). \tag{17.6.2}$$

**Corollary 2.** *If $d > 0, \gamma > 0; \alpha, \nu, \mu, q \in \mathbb{C}; k \in \mathbb{R}, \Re(q) > 0, \Re(\alpha\mu - \nu) > 0, \Re(\lambda) > 0, a > 0$, and $t > 0$, then the solution of the equation:*

$$N(t) = N_0 G^q_{k,\alpha,\nu,\mu}(a, d^\gamma t^\gamma) - d^\gamma {}_0D_t^{-\gamma} N(t) \tag{17.6.3}$$

*is given by the following formula*

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(\mu)_{nq} a^{nq}\Gamma[\gamma((nq+\mu)\alpha - \nu - 1) + 1]}{n!\Gamma_k[(nq+\mu)\alpha - \nu]} (d^\gamma t^\gamma)^{(nq+\mu)\alpha-\nu-1} \times E_{\gamma,\gamma((nq+\mu)\alpha-\nu-1)+1}(-d^\gamma t^\gamma). \tag{17.6.4}$$

Setting $d = 1$ and replace $c$ by $d$, then the results in Eqs. (17.5.2) and (17.5.15) reduce to the following form.

**Corollary 3.** *If* $d > 0, \gamma > 0; \alpha, \nu, q \in \mathbb{C}; k \in \mathbb{R}, \Re(q) > 0, \Re(\alpha - \nu) > 0, \Re(\lambda) > 0,$ $a > 0$, *and* $t > 0$, *then the solution of the equation:*

$$N(t) = N_0 R^q_{k,\alpha,\nu}(a,t) - d^\gamma {}_0D_t^{-\gamma} N(t) \tag{17.6.5}$$

*is given by the following formula*

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{a^{nq}\Gamma[(nq+1)\alpha - \nu]}{\Gamma_k[(nq+1)\alpha - \nu]} (t)^{(nq+1)\alpha-\nu-1} \times E_{\gamma,(nq+1)\alpha-\nu}(-d^\gamma t^\gamma). \tag{17.6.6}$$

**Corollary 4.** *If* $d > 0, \gamma > 0; \alpha, \nu, \mu, q \in \mathbb{C}; k \in \mathbb{R}, \Re(q) > 0, \Re(\alpha\mu - \nu) > 0,$ $\Re(\lambda) > 0$, $a > 0$, *and* $t > 0$, *then the solution of the equation:*

$$N(t) = N_0 G^q_{k,\alpha,\nu,\mu}(a,t) - d^\gamma {}_0D_t^{-\gamma} N(t) \tag{17.6.7}$$

*is given by the following formula*

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(\mu)_{nq} a^{nq}\Gamma[(nq+\mu)\alpha - \nu]}{n!\Gamma_k[(nq+\mu)\alpha - \nu]} (t)^{(nq+\mu)\alpha-\nu-1} \times E_{\gamma,(nq+\mu)\alpha-\nu}(-d^\gamma t^\gamma). \tag{17.6.8}$$

If we choose $q = 1, k = 1$, then the results in Eqs. (17.5.2) and (17.5.15) reduce to the following form.

**Corollary 5.** *If* $c > 0, d > 0, \gamma > 0; \alpha, \nu, q \in \mathbb{C}; k \in \mathbb{R}, \Re(q) > 0, \Re(\alpha - \nu) > 0,$ $\Re(\lambda) > 0$, $a > 0$, *and* $t > 0$, *then the solution of the equation:*

$$N(t) = N_0 R_{\alpha,\nu}(a, d^\gamma t^\gamma) - c^\gamma {}_0D_t^{-\gamma} N(t) \tag{17.6.9}$$

*is given by the following formula*

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{a^n\Gamma[\gamma((n+1)\alpha - \nu - 1) + 1]}{\Gamma[(n+1)\alpha - \nu]} (d^\gamma t^\gamma)^{(n+1)\alpha-\nu-1} \times E_{\gamma,\gamma((n+1)\alpha-\nu-1)+1}(-c^\gamma t^\gamma). \tag{17.6.10}$$

**Corollary 6.** $c > 0, d > 0, \gamma > 0; \alpha, \nu, \mu, q \in \mathbb{C}; k \in \mathbb{R}, \Re(q) > 0, \Re(\alpha\mu - \nu) > 0,$ $\Re(\lambda) > 0$, $a > 0$, *and* $t > 0$, *then the solution of the equation:*

$$N(t) = N_0 G_{\alpha,\nu,\mu}(a, d^\gamma t^\gamma) - c^\gamma {}_0D_t^{-\gamma} N(t) \tag{17.6.11}$$

*is given by the following formula*

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(\mu)_n a^n \Gamma[\gamma((n+\mu)\alpha - \nu - 1) + 1]}{n!\Gamma[(n+\mu)\alpha - \nu]} (d^\gamma t^\gamma)^{(n+\mu)\alpha-\nu-1} \times E_{\gamma,\gamma((n+\mu)\alpha-\nu-1)+1}(-c^\gamma t^\gamma). \quad (17.6.12)$$

## 17.7 Numerical solutions of fractional kinetic equations

In this section, we establish a database for numerical solutions of the kinetic Eqs. (17.5.2) and (17.5.15) by employing Eqs. (17.5.3) and (17.5.16) for particular values of the parameters, which are given in Tables 17.1 and 17.2; their graphs are plotted in Figs. 17.1, 17.3, and 17.5 and Mesh-plots are also established in Figs. 17.2 and 17.4. For this purpose, we denote the solution of Eqs. (17.5.2):

$$N(t) = N(N_0, \alpha, \nu, q, k, a, c, d, \gamma, t)$$

and those of Eq. (17.5.15):

$$N(t) = N(N_0, \alpha, \mu, \nu, q, k, a, c, d, \gamma, t)$$

**Table 17.1** Numerical solutions of KE involving the $R$-function.

| $t$ | $\gamma = 1.0$ | $\gamma = 1.2$ | $\gamma = 1.4$ | $\gamma = 1.6$ | $\gamma = 1.8$ |
|---|---|---|---|---|---|
| **0** | **Inf** | **Inf** | **Inf** | **NaN** | **NaN** |
| 0.005 | 18.09491863 | 34.44031441 | 65.57553427 | 124.8945623 | 237.9224466 |
| 0.01 | 11.13456464 | 19.23820776 | 33.24483687 | 57.46298209 | 99.34299405 |
| 0.015 | 8.38005791 | 13.68379509 | 22.3431623 | 36.48913194 | 59.60211744 |
| 0.02 | 6.849007185 | 10.74518093 | 16.85361534 | 26.43803572 | 41.47994146 |
| 0.025 | 5.856368385 | 8.90764177 | 13.54279902 | 20.59143085 | 31.31342743 |
| 0.03 | 5.152753265 | 7.64191162 | 11.32653072 | 16.78802642 | 24.88634772 |
| 0.035 | 4.623964418 | 6.712998918 | 9.738099017 | 14.12586149 | 20.49306821 |
| 0.04 | 4.20976707 | 6.000007047 | 8.543313337 | 12.16352022 | 17.31951632 |
| 0.045 | 3.875157014 | 5.43413919 | 7.611676314 | 10.6601523 | 14.93076512 |
| 0.05 | 3.598297945 | 4.973251237 | 6.864685846 | 9.473481793 | 13.07452641 |
| 0.055 | 3.364808545 | 4.590028588 | 6.252271455 | 8.514204721 | 11.59496351 |
| 0.06 | 3.164801965 | 4.265960604 | 5.740984298 | 7.72351243 | 10.39091678 |
| 0.065 | 2.991240251 | 3.98804064 | 5.307624444 | 7.061141754 | 9.39402143 |
| 0.07 | 2.838966747 | 3.746851386 | 4.935592366 | 6.498624201 | 8.556509723 |
| 0.075 | 2.704110247 | 3.535400021 | 4.612696034 | 6.015277407 | 7.844055542 |
| 0.08 | 2.583703591 | 3.348380864 | 4.329778168 | 5.595723588 | 7.231393829 |
| 0.085 | 2.475431368 | 3.181692621 | 4.079826679 | 5.228297949 | 6.699539802 |
| 0.09 | 2.377458234 | 3.032113005 | 3.857380817 | 4.903996284 | 6.233967759 |
| 0.095 | 2.288309169 | 2.897073826 | 3.658124333 | 4.615760418 | 5.823385543 |
| 0.1 | 2.206784134 | 2.774502032 | 3.47860035 | 4.357981598 | 5.458889935 |

**Table 17.2** Numerical solutions of KE involving the $G$-function.

| $t$ | $\gamma = 1.0$ | $\gamma = 1.1$ | $\gamma = 1.2$ | $\gamma = 1.3$ | $\gamma = 1.4$ |
|---|---|---|---|---|---|
| **0** | **Inf** | **Inf** | **Inf** | **Inf** | **Inf** |
| 0.01 | 8.423051 | 12.69125 | 19.33196 | 29.74669 | 46.20517 |
| 0.02 | 7.330293 | 10.89463 | 16.36814 | 24.84035 | 38.05367 |
| 0.03 | 6.757111 | 9.963317 | 14.84926 | 22.35419 | 33.96909 |
| 0.04 | 6.377219 | 9.350749 | 13.85749 | 20.74235 | 31.33958 |
| 0.05 | 6.096877 | 8.901299 | 13.13378 | 19.57236 | 29.44079 |
| 0.06 | 5.876643 | 8.549864 | 12.57037 | 18.66538 | 27.97494 |
| 0.07 | 5.696369 | 8.263336 | 12.11271 | 17.93125 | 26.79254 |
| 0.08 | 5.544444 | 8.022701 | 11.72959 | 17.31855 | 25.80865 |
| 0.09 | 5.413596 | 7.816089 | 11.40156 | 16.79538 | 24.97073 |
| 0.1 | 5.298985 | 7.635623 | 11.11578 | 16.34068 | 24.24416 |
| 0.11 | 5.197237 | 7.475822 | 10.86332 | 15.93987 | 23.60505 |
| 0.12 | 5.105912 | 7.332728 | 10.63773 | 15.58245 | 23.03622 |
| 0.13 | 5.023188 | 7.203398 | 10.43425 | 15.26065 | 22.52498 |
| 0.14 | 4.947674 | 7.085584 | 10.24923 | 14.96855 | 22.06171 |
| 0.15 | 4.878284 | 6.977536 | 10.07985 | 14.70157 | 21.63891 |
| 0.16 | 4.814154 | 6.877863 | 9.923849 | 14.45606 | 21.25068 |
| 0.17 | 4.754589 | 6.785447 | 9.779434 | 14.22911 | 20.89228 |
| 0.18 | 4.699017 | 6.699372 | 9.64513 | 14.01833 | 20.55985 |
| 0.19 | 4.646968 | 6.618882 | 9.519718 | 13.82176 | 20.25022 |
| 0.2 | 4.598046 | 6.543345 | 9.402183 | 13.63777 | 19.96073 |

and then we develop the program in Matlab®. Employing the program, we establish a database, graphs, and mesh-plots of the solutions of the kinetic equations involving the $R$- and $G$-functions.

In our investigations, particular values of the parameters involved in the solution of the fractional kinetic equation are selected as $N_0 = 0.005$, $\alpha = 1$, $\nu = 0.2$, $\mu = 2$, $q = 2$, $k = 1$, $a = 2$, $c = 0.3$, $d = 2$. Solutions of the kinetic equations involve the Mittag-Leffler function $E_{\alpha,\beta}$, which contains an infinite number of terms, further solution of the fractional kinetic equation also contains the summation of infinite terms, which produces complexity for numerical solutions. For critical analysis of the numerical solutions, we investigated by taking the first 1000 terms of the Mittag-Leffler function and 50 terms of the summation of the Eqs. (17.5.2) and (17.5.15).

### 17.7.1 Interpretation of numerical solutions of the fractional kinetic equation involving the R-function

The Database is obtained by setting $\gamma = 1.0, 1.2, 1.4, 1.6, 1.8$ for $0 \le t \le 0.1$ in Table 17.1 and in Fig. 17.1. From this we can easily observe the behavior of the solution of the fractional kinetic equations. When $t = 0$ and $\gamma = 1.0, 1.2, 1.4$; the value of $N(t) = \infty$ and when $\gamma = 1.6, 1.8$; $N(t)$ is not defined. As $t \to \infty$, $N(t)$ approaches 0. However, $N(t) > 0$ for $t \in [0, \infty)$ for these parameters.

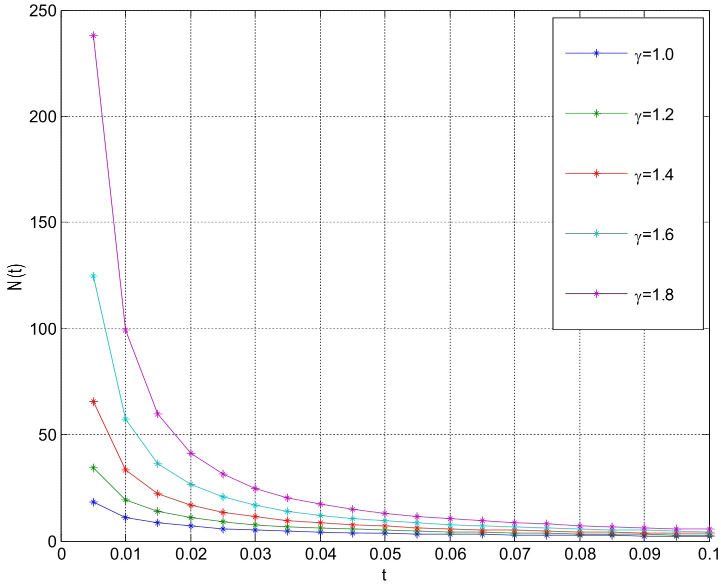


**FIGURE 17.1**

Graphical solutions of KE involving the $R$-function.

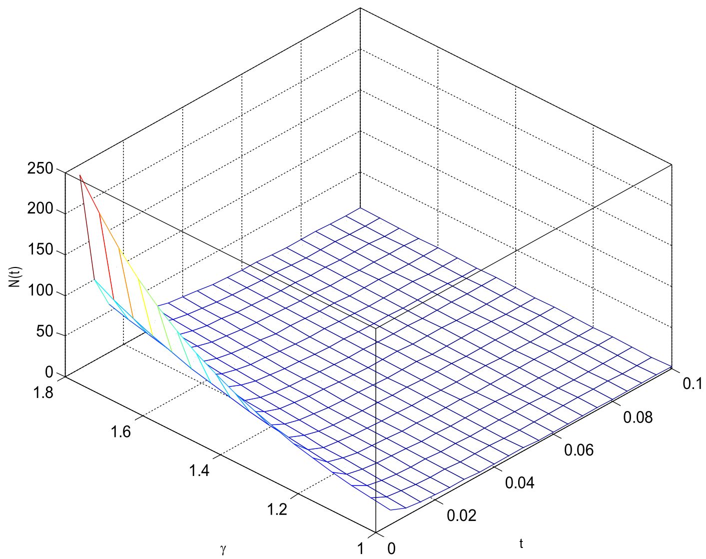


**FIGURE 17.2**

Mesh-plot of KE involving the $R$-function.

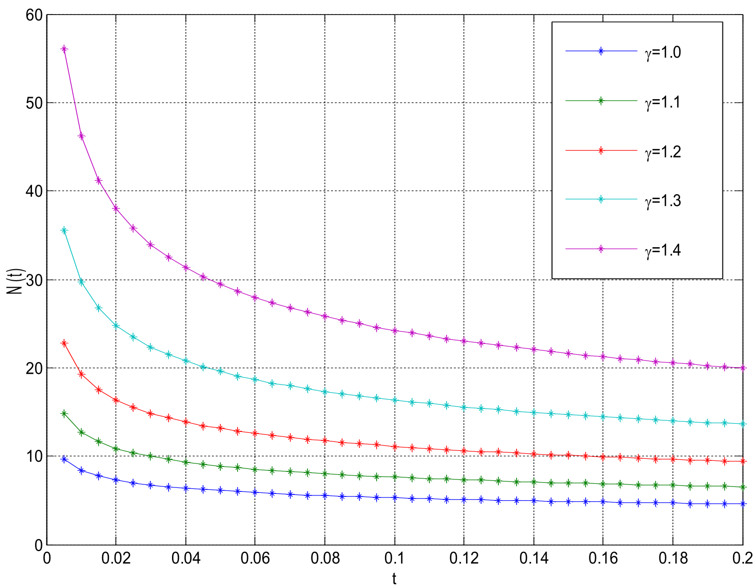


**FIGURE 17.3**

Graphical solutions of KE involving the $G$-function.

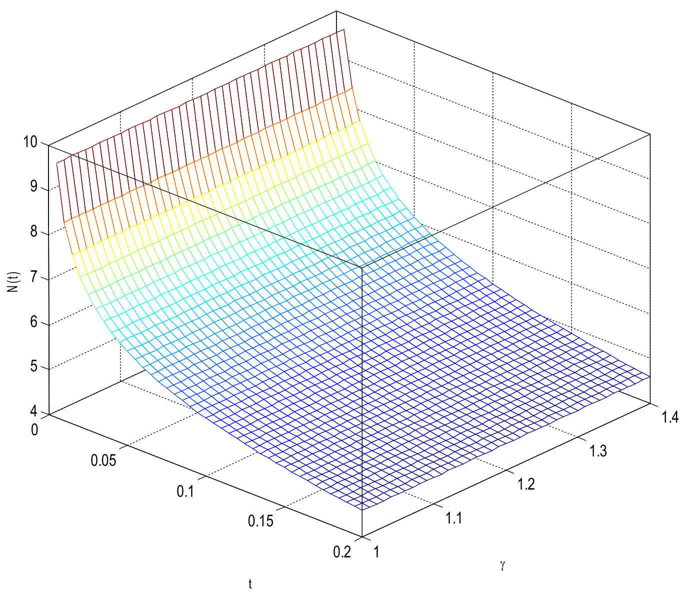


**FIGURE 17.4**

Mesh-plot of KE involving the $G$-function.

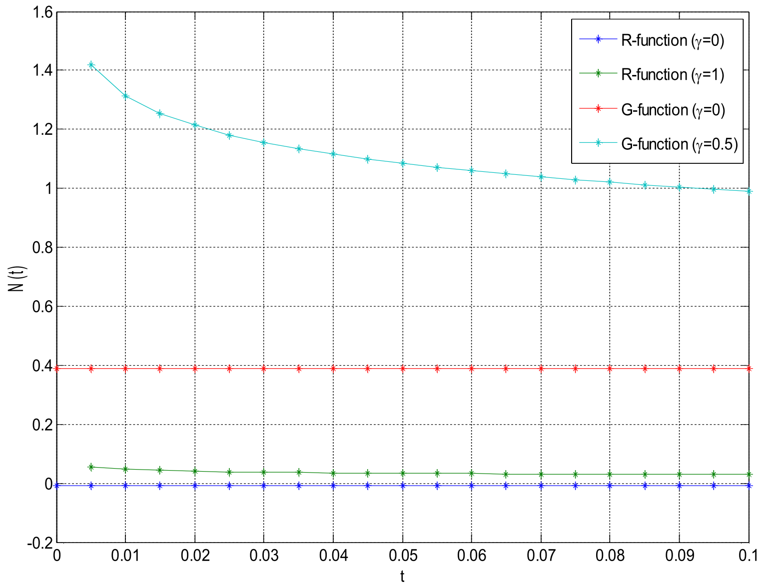


**FIGURE 17.5**

Graphical solutions of KE involving the $R$- and $G$-functions.

### 17.7.2 Interpretation of numerical solutions of the fractional kinetic equation involving the G-function

The Database is obtained by setting $\gamma = 1.0, 1.1, 1.2, 1.3, 1.4$ for $0 \leq t \leq 0.2$ in Table 17.2 and in Fig. 17.3. From this we can easily observe the behavior of the solution of fractional kinetic equations. When $t = 0$ and $\gamma = 1.0, 1.1, 1.2, 1.3, 1.4$ the value of $N(t) = \infty$. $N(t) > 0$ for $t \in [0, \infty)$ for these parameters.

### 17.7.3 Interpretation of numerical solutions of the fractional kinetic equations involving the R- and G-functions

In this case, we plot the combined graphs of solutions of the fractional kinetic equations involving $R$- and $G$-functions. For solutions of fractional kinetic equations involving the $R$-function, $\gamma = 0, 1$ and those for solutions of fractional kinetic equations involving the $G$-function $\gamma = 0, 0.5$ is chosen to plot the graph as shown in Fig. 17.5. The graph depicts that at $\gamma = 0$, the solution of the fractional kinetic equations involving the $R$- and $G$-functions remains constant for any interval of $t$, when $\gamma \neq 0$; $N(t)$ is decreasing.

The Mesh-plots in Figs. 17.2 and 17.4 depict the solutions of the fractional kinetic equations for the same parameters as we have chosen above. During our investigation, we find that the solutions of the fractional kinetic equations are not always positive, they may also be negative for some different values of the parameters. When we fix

$\gamma = 0$, the solutions of the fractional kinetic equations remain the same for fixed values of the parameters for $t \in [0, \infty)$.

## 17.8 Concluding remarks

We may also emphasize that the results derived in this chapter are of general character and can specialize to give further interesting and potentially useful formulas involving the integral transform and fractional calculus. Also, we give a new fractional generalization of the standard kinetic equation and derived solution for the same. From the close relationship of the $R$- and $G$-functions with many special functions, we can easily construct various known and new fractional kinetic equations. Also, from the numerical solutions established in Tables 17.1 and 17.2 and their graphical interpretation in Figs. 17.1, 17.2, 17.3, 17.4, and 17.5 for the $R$- and $G$-functions, we came to the conclusion that the solutions of the fractional kinetic equations are positive, negative or both and sometimes they are constant when $\gamma = 0$ for the $R$- and $G$-functions. In our investigation, we choose $N_0 = 0.005$, $\alpha = 1$, $\nu = 0.2$, $\mu = 2$, $q = 2$, $k = 1$, $a = 2$, $c = 0.3$, $d = 2$. The reader can choose any value of the parameters for further analysis of the solutions of fractional kinetic equations.

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# Extended Hypergeometric Functions and Orthogonal Polynomials

*Edited by*
**Praveen Agarwal and Clemente Cesarano**

***Extended Hypergeometric Functions and Orthogonal Polynomials*** presents a comprehensive and accessible resource for researchers and graduate students interested in exploring the rich connections between extended hypergeometric functions, orthogonal polynomials, and multivariable polynomials. Integrating all three fields and their applications in Maple, Mathematica, and MATLAB, this book fosters interdisciplinary understanding and inspires new avenues of research in mathematics, engineering, physics, and computer science. It also provides a glimpse into future research directions in these areas, including potential applications in emerging fields of applied mathematics and interdisciplinary collaborations. Each chapter begins with an introduction, includes sections on theory, followed by sections on applications, and ends with exercises, problems, references, and suggested readings.

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- Provides an integrated and up-to-date resource on extended hypergeometric functions, orthogonal polynomials, and multivariable polynomials for researchers, students, and practicing engineers
- Presents emerging trends, new developments, techniques, and applications in the fields of special functions and orthogonal polynomials
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