# Multiplicity and Concentration of Positive Solutions for Weighted Schrӧdinger Equations with Critical Nonlinearities

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**Abstract**

This work investigates the nonlinear equation of Schr ӧdinger

$$-\Delta_b w + \beta a(x) w = \eta w + w^{2^*-1}, \quad w \in \mathbb{R}^N,$$

where $2^* = \frac{2N}{N-2}$ denotes the critical Sobolev exponent with $N \geq 4$. Here, $a(x) \geq 0$ is a given potential function. Under the assumption that the parameter $\eta > 0$ is sufficiently small and $\beta > 0$ is large, we establish the existence and multiplicity of positive solutions that exhibit concentration phenomena around the potential well.

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## 1. Introduction

The nonlinear Schrӧdinger equation has garnered significant interest in recent years due to its broad range of applications in quantum mechanics, nonlinear optics, and mathematical physics. The study of solutions involving critical Sobolev exponents is particularly challenging due to the loss of compactness and the delicate variational structure.

Several researchers have explored the existence and multiplicity of positive solutions to nonlinear Schrӧdinger-type equations. For instance, in [28], the authors examined normalized solutions under critical growth conditions and discussed the impact of the potential term. Similarly, [29] addressed magnetic Schr ӧdinger equations and provided new insights into concentration phenomena under Sobolev critical exponents. Recent contributions, such as

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[30], extend these ideas by considering steep potential wells and proving the existence of multi-peak solutions in the presence of critical nonlinearity.

Motivated by these results, in this paper we investigate the following problem:

$$\begin{cases} -\Delta_b w + \beta a(x) w = \eta w + w^{2^*-1} & \text{in } \mathbb{R}^N, \\ w > 0, \quad w \in \mathscr{H}_b^1(\mathbb{R}^N), \end{cases} \tag{$PS^b_{\beta,\eta}$}$$

where $N \geq 4$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, $\beta > 0$, and $\eta \in \mathbb{R}$. The potential function $a(x)$ is assumed to satisfy the following assumptions:

$(B_1)$ $a \in C(\mathbb{R}^N, \mathbb{R})$, $a(x) \geq 0$, and $\Omega := \operatorname{int}(a^{-1}(0))$ is a nonempty bounded set with smooth boundary such that $\bar{\Omega} = a^{-1}(0)$.

$(B_2)$ There exists $M_0 > 0$ such that

$$\mathscr{L}\left\{x \in \mathbb{R}^N : a(x) \leq M_0\right\} < \infty,$$

where $\mathscr{L}$ denotes the Lebesgue measure in $\mathbb{R}^N$.

Under these hypotheses, we aim to establish the existence and multiplicity of positive solutions, particularly in the regime of small $\eta$ and large $\beta$, and investigate their concentration behavior around the potential well.

**Definition 1.1.** *Let $b \in L^p(\Omega)$ for some $1 < p < \infty$. The* weighted Sobolev space $W^{1,p}(\Omega, b)$ *is defined as the set of all real-valued, Lebesgue measurable functions $w$ defined almost everywhere in $\Omega$ such that*

$$\|w\|_{1,p,b} := \left( \int_\Omega |w(x)|^p \, dx + \int_\Omega b(x) |\nabla w(x)|^p \, dx \right)^{1/p} < +\infty.$$

*In the special case $p = 2$, we denote $W^{1,2}(\Omega, b)$ by $\mathscr{H}_b^1(\Omega)$ and define the corresponding norm as*

$$\|w\|_{\mathscr{H}_b^1} := \left( \int_\Omega |w(x)|^2 \, dx + \int_\Omega b(x) |\nabla w(x)|^2 \, dx \right)^{1/2}. \tag{1}$$

*Furthermore, the inner product in $\mathscr{H}_b^1$ is defined by*

$$(w, v) := \int_\Omega w(x) v(x) \, dx + \int_\Omega b(x) \nabla w(x) \cdot \nabla v(x) \, dx, \quad \forall w, v \in \mathscr{H}_b^1. \tag{2}$$

A solution $w_\beta$ of $(PS^b_{\beta,\eta})$ is said to be a *least energy solution* if the corresponding energy functional

$$J^b_{\beta,\eta}(w) := \int_{\mathbb{R}^N} \left( \frac{1}{2} \left( b(x) |\nabla w|^2 + (\beta a(x) - \eta) w^2 \right) - \frac{1}{2^*} |w|^{2^*} \right) dx$$

attains its minimum at $w = w_\beta$ among all nontrivial solutions of $(PS^b_{\beta,\eta})$.

A sequence of solutions $(w_n)$ to $(PS^b_{\beta_n,\eta})$ is said to *concentrate* at a solution $w$ of the following limit problem:

$$\begin{cases} -\Delta_b w = \eta w + w^{2^*-1}, & \text{in } \Omega, \\ w > 0, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

if a subsequence converges strongly to $w$ in the space $\mathscr{H}^1_b(\mathbb{R}^N)$ as $\beta_n \to \infty$.

Let

$$S := \inf_{w \in \mathscr{H}^1_b(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} b(x)|\nabla w|^2\,dx}{\left(\int_{\mathbb{R}^N} |w|^{2^*}\,dx\right)^{2/2^*}}$$

denote the best constant for the Sobolev embedding in the weighted space. We aim to prove the following results.

**Theorem 1.1.** *Suppose that conditions $(B_1)$ and $(B_2)$ are satisfied, and let $N \geq 4$. Then, for every $0 < \eta < \eta(\Omega)$, there exists a constant $\beta(\eta) > 0$ such that problem $(PS^b_{\beta,\eta})$ admits a least-energy solution $w_\beta$ for all $\beta \geq \beta(\eta)$.*

**Theorem 1.2.** *Assume that conditions $(B_1)$ and $(B_2)$ hold, and that $N \geq 4$. Then there exists a constant $0 < \eta^* < \eta_1(\Omega)$ such that, for each $0 < \eta \leq \eta^*$, there exist two constants $\Lambda(\eta) > 0$ and $0 < c(\eta) < \frac{1}{N}S^{N/2}$ with the following property: for all $\beta \geq \Lambda(\eta)$, the problem $(PS^b_{\beta,\eta})$ admits at least $\operatorname{cat}(\Omega)$ positive solutions satisfying*

$$J^b_{\beta,\eta}(w) \leq c(\eta).$$

**Theorem 1.3.** *Let $(w_n)$ be a sequence of solutions to $(PS^b_{\beta_n,\eta})$, where $\eta \in (0, \eta_1(\Omega))$ and $\beta_n \to \infty$. Suppose that the corresponding energy levels satisfy*

$$J^b_{\beta_n,\eta}(w_n) \to c < \frac{1}{N}S^{N/2} \quad \text{as } n \to \infty.$$

*Then, up to a subsequence, $(w_n)$ concentrates at a solution of problem $(D^b_\eta)$.*

## 2. Variational Compactness and Functional Setting

In this study, we consistently assume that conditions $(B_1)$ and $(B_2)$ hold, with $N \geq 4$. Let $\eta_1(\Omega)$ denote the smallest eigenvalue of the operator $-\Delta$ on the domain $\Omega$ subject to the Dirichlet boundary condition $w = 0$. Additionally, we use $|\cdot|_q$ to denote the $L^q$-norm for $q \in [1, \infty]$.

Define the Hilbert space $E_b$ as

$$E_b = \left\{ w \in \mathscr{H}^1_b(\mathbb{R}^N) : \int_{\mathbb{R}^N} a(x) w^2\,dx < \infty \right\}$$

equipped with the norm

$$\|w\|_b = \left( \|w\|^2_{\mathscr{H}^1_b} + \int_{\mathbb{R}^N} a(x) w^2 \, dx \right)^{1/2}$$

This norm is clearly equivalent to the following family of norms:

$$\|w\|_{b,\beta} = \left( \|w\|^2_{\mathscr{H}^1_b} + \beta \int_{\mathbb{R}^N} a(x) w^2 \, dx \right)^{1/2}, \quad \text{for } \beta > 0.$$

**Lemma 2.1.** *Let $\beta_n \geq 1$ and $w_n \in E_b$ be a sequence such that $\beta_n \to \infty$ and $\|w_n\|^2_{b,\beta_n} < K$ for some constant $K > 0$. Then there exists a function $w \in \mathscr{H}^1_{0,b}(\Omega)$ such that, up to a subsequence, $w_n \rightharpoonup w$ weakly in $E_b$ and $w_n \to w$ strongly in $L^2(\mathbb{R}^N)$.*

Proof. Since $\|w_n\|^2_b \leq \|w_n\|^2_{b,\beta_n} < K$, the sequence $(w_n)$ is bounded in $E_b$. Hence, by reflexivity and standard compactness arguments, there exists $w \in E_b$ such that (up to a subsequence) $w_n \rightharpoonup w$ weakly in $E_b$ and $w_n \to w$ in $L^2_{\text{loc}}(\mathbb{R}^N)$.

Define the sets

$$D_m := \left\{ x \in \mathbb{R}^N : |x| \leq m, \; a(x) \geq \frac{1}{m} \right\}, \quad m \in \mathbb{N}.$$

Then, for each $m$,

$$\int_{D_m} |w_n|^2 \, dx \leq m \int_{D_m} a(x) w_n^2 \, dx \leq \frac{mK}{\beta_n} \to 0 \quad \text{as } n \to \infty.$$

This implies that $w(x) = 0$ almost everywhere in $\mathbb{R}^N \setminus \Omega$, so $w \in \mathscr{H}^1_{0,b}(\Omega)$ due to the smoothness of $\partial\Omega$.

To show strong convergence in $L^2(\mathbb{R}^N)$, let $F := \{x \in \mathbb{R}^N : a(x) \leq M_0\}$, where $M_0$ is as in assumption $(B_2)$, and denote $F^c := \mathbb{R}^N \setminus F$. Then

$$\int_{F^c} w_n^2 \, dx \leq \frac{1}{\beta_n M_0} \int_{F^c} \beta_n a(x) w_n^2 \, dx \leq \frac{K}{\beta_n M_0} \to 0 \quad \text{as } n \to \infty.$$

Now, consider $B_R := \{x \in \mathbb{R}^N : |x| \leq R\}$ and $B_R^c := \mathbb{R}^N \setminus B_R$. Take any $r \in (1, N/(N-2))$ with Hölder conjugate $r' := r/(r-1)$. Then

$$\int_{B_R^c \cap F} |w_n - w|^2 \, dx \leq \|w_n - w\|^2_{L^{2r}} \mathscr{L}(B_R^c \cap F)^{1/r'} \leq C \|w_n - w\|^2_{E_b} \mathscr{L}(B_R^c \cap F)^{1/r'} \to 0$$

as $R \to \infty$ due to assumption $(B_2)$. Since $w_n \to w$ in $L^2_{\text{loc}}$, we also have

$$\int_{B_R} |w_n - w|^2 \, dx \to 0 \quad \text{as } n \to \infty.$$

Combining these results shows $w_n \to w$ in $L^2(\mathbb{R}^N)$.

We define the self-adjoint operator $\mathscr{B}^b_\beta := -\Delta_b + \beta a$ on the Hilbert space $L^2(\mathbb{R}^N)$, where the associated form domain is $E_b$. The inner product in $L^2(\mathbb{R}^N)$ is denoted by $(\cdot,\cdot)$. The bilinear form corresponding to $\mathscr{B}^b_\beta$ is given by

$$(\mathscr{B}^b_\beta w, v) := \int_{\mathbb{R}^N} (b(x)\nabla w \cdot \nabla v + \beta a(x) w v)\, dx, \quad \text{for all } w, v \in E_b.$$

Let $\kappa^b_\beta := \inf \sigma(\mathscr{B}^b_\beta)$ denote the lowest point in the spectrum of $\mathscr{B}^b_\beta$. It is straightforward to verify that

$$\kappa^b_\beta = \inf\left\{(\mathscr{B}^b_\beta w, w) : w \in E_b,\ \|w\|_{L^2} = 1\right\} \geq 0.$$

Moreover, the mapping $\beta \mapsto \kappa^b_\beta$ is non-decreasing due to the monotonicity of the potential term $\beta a(x)$ in $\beta$.

**Lemma 2.2.** *Let $\eta \in (0, \eta_1(\Omega))$. Then there exists a constant $\beta(\eta) > 0$ such that for all $\beta \geq \beta(\eta)$, the spectral bound satisfies*

$$\kappa^b_\beta \geq \frac{\eta + \eta_1(\Omega)}{2}.$$

*As a consequence, for all $w \in E_b$ and $\beta \geq \beta(\eta)$, we have*

$$\alpha_\eta \|w\|^2_{b,\beta} \leq \left((\mathscr{B}^b_\beta - \eta) w, w\right),$$

*where the constant $\alpha_\eta$ is defined as*

$$\alpha_\eta := \frac{\eta + \eta_1(\Omega)}{\eta_1(\Omega) + 2 + 3\eta}.$$

Proof. Suppose by contradiction that there exists a sequence $\beta_n \to \infty$ such that $\kappa^b_{\beta_n} < \frac{\eta+\eta_1(\Omega)}{2}$ for all $n$, and assume $\kappa^b_{\beta_n} \to \tau \leq \frac{\eta+\eta_1(\Omega)}{2}$.

Let $w_n \in E_b$ be a sequence satisfying $\|w_n\|_{L^2} = 1$ and

$$\left((\mathscr{B}^b_{\beta_n} - \kappa^b_{\beta_n}) w_n, w_n\right) \to 0.$$

Then, we estimate:

$$\begin{aligned}
\|w_n\|^2_{b,\beta_n} &= \int_{\mathbb{R}^N} \left(b(x)|\nabla w_n|^2 + (1 + \beta_n a(x)) w_n^2\right) \\
&= \left((\mathscr{B}^b_{\beta_n} - \kappa^b_{\beta_n}) w_n, w_n\right) + (1 + \kappa^b_{\beta_n}) \|w_n\|^2_{L^2} \\
&\leq 2(1 + \eta_1(\Omega)),
\end{aligned}$$

for all sufficiently large $n$. Hence, by Lemma 2.1, we may assume (up to subsequence) that $w_n \rightharpoonup w$ weakly in $E_b$ and strongly in $L^2(\mathbb{R}^N)$, with $\|w\|_{L^2} = 1$.

Due to the support properties induced by $a(x)$, it follows that $w \in \mathscr{H}^1_{0,b}(\Omega)$. Moreover, using weak lower semicontinuity and convergence, we obtain

$$\int_\Omega b(x)|\nabla w|^2 - \tau |w|^2 \leq \liminf_{n\to\infty} \left((\mathscr{B}^b_{\beta_n} - \kappa^b_{\beta_n}) w_n, w_n\right) = 0.$$

Thus,

$$\int_{\Omega} b(x)|\nabla w|^2 \leq \tau < \eta_1(\Omega),$$

which contradicts the variational characterization of $\eta_1(\Omega)$ as the lowest eigenvalue of $-\Delta_b$ on $\mathscr{H}^1_{0,b}(\Omega)$ with $\|w\|_{L^2} = 1$. This contradiction completes the proof.

Let us consider the energy functional associated with problem $(PS^b_{\beta,\eta})$, defined by

$$\begin{aligned} J^b_{\beta,\eta}(w) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(b(x)|\nabla w|^2 + \beta a(x)w^2 - \eta w^2\right) dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |w|^{2^*} dx \\ &= \frac{1}{2}\left((\mathscr{B}^b_\beta - \eta)w, w\right) - \frac{1}{2^*}\|w\|^{2^*}_{L^{2^*}}. \end{aligned}$$

It is straightforward to verify that $J^b_{\beta,\eta} \in C^1(E_b, \mathbb{R})$, and that its critical points correspond to weak solutions of the equation

$$-\Delta_b w + \beta a(x) w = \eta w + |w|^{2^*-2} w, \quad w \in \mathscr{H}^1_b(\mathbb{R}^N).$$

A sequence $\{w_n\} \subset E_b$ is said to be a *Palais-Smale sequence at level* $c$ (briefly, a $(PS)_c$ sequence) for $J^b_{\beta,\eta}$ if

$$J^b_{\beta,\eta}(w_n) \to c \quad \text{and} \quad \|(J^b_{\beta,\eta})'(w_n)\|_{E^*_b} \to 0 \quad \text{as } n \to \infty.$$

We say that the functional $J^b_{\beta,\eta}$ satisfies the $(PS)_c$ condition if every $(PS)_c$ sequence admits a strongly convergent subsequence in $E_b$.

**Lemma 2.3.** *Let $\eta \in (0, \eta_1(\Omega))$ and $\beta \geq \beta(\eta)$. Then every $(PS)_c$ sequence $\{w_n\} \subset E_b$ associated with the functional $J^b_{\beta,\eta}$ is bounded in $E_b$, and moreover satisfies:*

$$\lim_{n\to\infty} \left((\mathscr{B}^b_\beta - \eta)(w_n, w_n)\right) = \lim_{n\to\infty} \|w_n\|^{2^*}_{L^{2^*}} = Nc. \tag{4}$$

Proof. By assumption, $\{w_n\}$ is a $(PS)_c$ sequence, i.e.,

$$J^b_{\beta,\eta}(w_n) \to c, \quad \text{and} \quad \|(J^b_{\beta,\eta})'(w_n)\|_{E^*_b} \to 0.$$

Using the standard Pohozaev-type identities associated with critical exponents, we compute:

$$J^b_{\beta,\eta}(w_n) - \frac{1}{2^*}\langle (J^b_{\beta,\eta})'(w_n), w_n\rangle = \frac{1}{N}(\mathscr{B}^b_\beta - \eta)(w_n, w_n), \tag{5}$$

$$J^b_{\beta,\eta}(w_n) - \frac{1}{2}\langle (J^b_{\beta,\eta})'(w_n), w_n\rangle = \frac{1}{2}\|w_n\|^{2^*}_{L^{2^*}}. \tag{6}$$

Since $(J^b_{\beta,\eta})'(w_n) \to 0$ in $E^*_b$, it follows from (5) that $(\mathscr{B}^b_\beta - \eta)(w_n, w_n)$ remains bounded. Lemma 2.2 then guarantees that $\{w_n\}$ is bounded in $E_b$.

Passing to the limit in (5) and (6) yields the desired identity (4).

Let

$$S = \inf_{w \in \mathscr{H}^1_{0,b}(\Omega)} \frac{\int b(x)|\nabla w|^2\, dx}{\|w\|^2_{L^{2^*}(\Omega)}} \tag{7}$$

denote the best Sobolev constant corresponding to the weighted embedding $\mathscr{H}^1_{0,b}(\Omega) \hookrightarrow L^{2^*}(\Omega)$.

In what follows, and without loss of generality, we assume that the threshold $\beta(\eta)$ is chosen large enough so that

$$\beta(\eta) \geq \frac{\eta}{M_0},$$

ensuring that $\beta M_0 - \eta \geq 0$ holds for all $\beta \geq \beta(\eta)$.

**Proposition 2.1.** *Let $\eta \in (0, \eta_1(\Omega))$ and $\beta \geq \beta(\eta)$. Then the functional $J^b_{\beta,\eta}$ satisfies the Palais-Smale condition at any level $c < \frac{1}{N} S^{N/2}$. In other words, every sequence $\{w_n\} \subset E_b$ such that*

$$J^b_{\beta,\eta}(w_n) \to c \quad \text{and} \quad \|J^{b\prime}_{\beta,\eta}(w_n)\|_{E_b} \to 0 \quad \text{as } n \to \infty,$$

*admits a strongly convergent subsequence in $E_b$.*

Proof. By Lemma 2.3, the sequence $\{w_n\}$ is bounded in $E_b$. Thus, up to a subsequence, we may assume that $w_n \rightharpoonup w$ weakly in $E_b$, $w_n \to w$ in $L^2_{\text{loc}}(\mathbb{R}^N)$, and $w_n(x) \to w(x)$ almost everywhere in $\mathbb{R}^N$.

Standard variational arguments imply that the weak limit $w$ satisfies the limiting equation

$$-\Delta_b w + \beta a(x) w = \eta w + |w|^{2^*-2} w \quad \text{in } \mathbb{R}^N.$$

To analyze the convergence behavior of the sequence, we define the remainder sequence by

$$z_n := w_n - w.$$

By the Brezis-Lieb lemma [7; 27], we have

$$|w_n|^{2^*}_{2^*} = |w|^{2^*}_{2^*} + |z_n|^{2^*}_{2^*} + o(1). \tag{8}$$

Moreover, since $J^{b\prime}_{\beta,\eta}(w_n)(w_n) \to 0$, it follows that

$$(\mathscr{B}^b_\beta - \eta)(z_n, z_n) - |z_n|^{2^*}_{2^*} \to 0. \tag{9}$$

Using Lemma 2.3 and equations (8)–(9), we conclude

$$(\mathscr{B}^b_\beta - \eta)(z_n, z_n) \to b \quad \text{and} \quad |z_n|^{2^*}_{2^*} \to b \leq NC < S^{N/2}.$$

As in the proof of Lemma 2.1, one can show that

$$\int_F |z_n|^2 \to 0 \quad \text{as } n \to \infty,$$

where $F := \{x \in \mathbb{R}^N : a(x) \le M_0\}$, and $F^c = \mathbb{R}^N \setminus F$.

Now, by the definition of the best Sobolev constant $S$ in (7), we estimate:

$$\begin{aligned} S|z_n|_{2^*}^2 &\le \int_{\mathbb{R}^N} b(x)|\nabla z_n|^2 \\ &\le \int_{\mathbb{R}^N} b(x)|\nabla z_n|^2 + \int_{F^c} (\beta a(x) - \eta) z_n^2 \\ &\le (\mathscr{B}_\beta^b - \eta)(z_n, z_n) + \eta \int_F z_n^2 \\ &= (\mathscr{B}_\beta^b - \eta)(z_n, z_n) + o(1). \end{aligned}$$

Taking limits yields:

$$S b^{2/2^*} \le b.$$

Since $b < S^{N/2}$, it follows that $b = 0$, and therefore $z_n \to 0$ strongly in $E_b$. Hence, $w_n \to w$ in $E_b$, completing the proof.

## 3. Existence via the Nehari Manifold: Proof of Theorems 1.1 and 1.3

We consider the Nehari manifold associated with the functional $J_{\beta,\eta}^b$, defined by

$$\begin{aligned} \mathscr{M}_{\beta,\eta}^b &= \left\{ w \in E_b \setminus \{0\} : J_{\beta,\eta}^{b\prime}(w)(w) = 0 \right\} \\ &= \left\{ w \in E_b \setminus \{0\} : (\mathscr{B}_\beta^b - \eta)(w, w) = \|w\|_{2^*}^{2^*} \right\}. \end{aligned} \tag{10}$$

The Nehari manifold $\mathscr{M}_{\beta,\eta}^b$ is radially diffeomorphic to the unit sphere in $E_b$ with respect to the $L^{2^*}$-norm, namely,

$$\mathscr{V} := \{v \in E_b : \|v\|_{2^*} = 1\},$$

via the mapping

$$\mathscr{V} \to \mathscr{M}_{\beta,\eta}^b, \qquad v \mapsto \left[(\mathscr{B}_\beta^b - \eta)(v, v)\right]^{\frac{N-2}{4}} v.$$

For each $w \in \mathscr{M}_{\beta,\eta}^b$, the energy functional reduces to

$$J_{\beta,\eta}^b(w) = \frac{1}{N} \left(\mathscr{B}_\beta^b - \eta\right)(w, w).$$

Therefore, the least energy level of $J_{\beta,\eta}^b$ over the Nehari manifold is given by

$$c_{\beta,\eta}^b := \inf_{w \in \mathscr{M}_{\beta,\eta}^b} J_{\beta,\eta}^b(w) = \frac{1}{N} \inf_{v \in \mathscr{V}} \left(\left(\mathscr{B}_\beta^b - \eta\right)(v, v)\right)^{N/2}.$$

**Lemma 3.1.** *Let $\mathscr{V}_b := \{v \in E_b : \|v\|_{2^*} = 1\}$ and define the Nehari manifold*

$$\mathscr{M}_{\beta,\eta}^b := \left\{ w \in E_b \setminus \{0\} : (\mathscr{B}_\beta^b - \eta)(w, w) = \|w\|_{2^*}^{2^*} \right\}.$$

*Then, there exists a homeomorphism $\Phi : \mathscr{V}_b \to \mathscr{M}_{\beta,\eta}^b$, and thus $\mathscr{V}_b \simeq \mathscr{M}_{\beta,\eta}^b$.*

Proof. Define the mapping $\Phi : \mathscr{V}_b \to \mathscr{M}^b_{\beta,\eta}$ by

$$\Phi(v) := \left[(\mathscr{B}^b_\beta - \eta)(v,v)\right]^{\frac{N-2}{4}} v.$$

Set $c_0 := \left[(\mathscr{B}^b_\beta - \eta)(v,v)\right]^{\frac{N-2}{4}}$ so that $w = \Phi(v) = c_0 v$. Then:

$$\begin{aligned}(\mathscr{B}^b_\beta - \eta)(w,w) &= (\mathscr{B}^b_\beta - \eta)(c_0 v, c_0 v) \\ &= c_0^2(\mathscr{B}^b_\beta - \eta)(v,v) \\ &= \left[(\mathscr{B}^b_\beta - \eta)(v,v)\right]^{\frac{N}{2}} = \|w\|^{2^*}_{2^*},\end{aligned}$$

since $\|w\|_{2^*} = c_0\|v\|_{2^*} = c_0$.

Hence, $w \in \mathscr{M}^b_{\beta,\eta}$. The mapping $\Phi$ is continuous and invertible, with inverse $\Phi^{-1}(w) = w/\|w\|_{2^*} \in \mathscr{V}_b$. Thus, $\Phi$ is a homeomorphism.

**Lemma 3.2.** *Show that $\mathscr{M}^b_{\beta,\eta} \neq \emptyset$.*

Proof. Let $w \in \mathscr{H}^1_{0,b}$ be a nonzero function. Then, there exists some $t > 0$ such that $tw \in \mathscr{M}^b_{\beta,\eta}$, where $\mathscr{M}^b_{\beta,\eta}$ denotes the Nehari manifold associated with the problem.

From the definitions, we compute:

$$(\mathscr{B}^b_\beta - \eta)(tw,tw) = |tw|^{2^*}_{2^*} = t^{2^*}\|w\|^{2^*}_{2^*} = t^{2^*}L_1, \tag{11}$$

$$(\mathscr{B}^b_\beta - \eta)(tw,tw) = t^2\left(\int b(x)|\nabla w|^2 + \beta\int a(x)w^2 - \eta\int w^2\right) = t^2 L_2. \tag{12}$$

Equating (11) and (12), we get:

$$t^2 L_2 = t^{2^*}L_1 \quad \Rightarrow \quad t = \left(\frac{L_2}{L_1}\right)^{\frac{1}{2^*-2}}.$$

Thus, for $t = \left(\frac{L_2}{L_1}\right)^{\frac{1}{2^*-2}}$, we have $tw \in \mathscr{M}^b_{\beta,\eta}$. Consequently, $\mathscr{M}^b_{\beta,\eta} \neq \emptyset$.

If $\mathscr{M}^b_{\beta,\eta} \neq \emptyset$ then $J^b_{\beta,\eta} \neq +\infty$

**Proposition 3.1.** *If $w \in \mathscr{M}^b_{\beta,\eta}$ is a critical point of $J^b_{\beta,\eta}$ and satisfies $J^b_{\beta,\eta}(w) < 2c^b_{\beta,\eta}$, then $w$ does not change sign. Consequently, $|w|$ is a solution of the problem $(PS^b_{\beta,\eta})$.*

Proof. Since $w$ is a critical point of $J^b_{\beta,\eta}$, we have the relation

$$(\mathscr{B}^b_\beta - \eta)(w,v) = \int_{\mathbb{R}^N} |w|^{2^*-2} wv \, dx$$

for every $v \in E_b$. In particular, this holds for $v = w^\pm$, where $w^\pm = \max\{\pm w, 0\}$.

If both $w^+$ and $w^-$ are nonzero, then $w^\pm \in \mathscr{M}^b_{\beta,\eta}$, and it follows that

$$J^b_{\beta,\eta}(w) = J^b_{\beta,\eta}(w^+) + J^b_{\beta,\eta}(w^-) \geq 2c^b_{\beta,\eta},$$

which leads to a contradiction.

Similarly, for every domain $D \in \mathbb{R}^N$, we define the functional

$$J^b_{\eta,D}(w) = \frac{1}{2}\int_D (b(x)|\nabla w|^2 - \eta w^2) - \frac{1}{2^*}\int_D |w|^{2^*}$$
$$= \frac{1}{2}(\mathscr{B}^b_0 - \eta)(w)(w) - \frac{1}{2^*}|w|^{2^*}_{2^*} \quad \text{on} \quad \mathscr{H}^1_{0,b}(D),$$

and is associated with the problem $(D_\eta)$. The corresponding Nehari manifold is defined as

$$\mathscr{M}^b_{\eta,D} = \{w \in \mathscr{H}^1_{0,b}(D) \setminus \{0\} : (\mathscr{B}^b_0 - \eta)(w)(w) = |w|^{2^*}_{2^*}\}.$$

This manifold is radially diffeomorphic to $\mathscr{V}_{b,D} = \{v \in \mathscr{H}^1_{0,b}(D), |v|_{2^*} = 1\}$. Set

$$c^b(\eta, D) := \inf_{w \in \mathscr{M}^b_{\eta,D}} J^b_{\eta,D}(w) = \frac{1}{N}\inf_{v \in \mathscr{V}_D}((\mathscr{B}^b_0 - \eta)(v)(v))^{N/2}.$$

**Lemma 3.3.** *Let $\eta \in (0, \eta_1(\Omega))$ and $\beta \geq \beta(\eta)$. Then*

$$\frac{1}{N}(\alpha_\eta S)^{N/2} \leq c^b_{\beta,\eta} < c^b(\eta,\Omega) < \frac{1}{N}S^{N/2}.$$

Proof. By Lemma 2.2, we have the inequality

$$\alpha_\eta \|v\|^2_b \leq \alpha_\eta \|v\|^2_{b,\beta} \leq (\mathscr{B}^b_\beta - \eta)(v, v).$$

Taking the infimum over $v \in \mathscr{V}_b$ yields the first inequality. Since $\mathscr{V}_{b,\Omega} \subset \mathscr{V}_b$ and $(\mathscr{B}^b_\beta)(v, v) = (A^b_0)(v, v)$ for all $v \in \mathscr{V}_{b,\Omega}$, it follows that

$$c^b_{\beta,\eta} \leq c^b(\eta, \Omega).$$

Moreover, Brézis and Nirenberg [8] showed that for every $\eta \in (0, \eta_1(\Omega))$, we have $c^b(\eta,\Omega) < \frac{1}{N}S^{N/2}$, and that $c^b(\eta,\Omega)$ is attained at some function $\tilde{w} > 0$.

If $c^b_{\beta,\eta} = c^b(\eta,\Omega)$, then this minimum would be achieved at $\tilde{w}$. However, since $\tilde{w}$ vanishes outside $\Omega$, this contradicts the strong maximum principle. Therefore, it must hold that $c^b_{\beta,\eta} < c^b(\eta,\Omega)$.

We are now prepared to demonstrate the validity of Theorems 1.1 and 1.3.

Proof (**Proof of Theorem 1.1**). Let $\{w^\beta_n\} \subset \mathscr{M}^b_{\beta,\eta}$ be a minimizing sequence for $J^b_{\beta,\eta}$, i.e.,

$$J^b_{\beta,\eta}(w^\beta_n) \to c^b_{\beta,\eta} \quad \text{as } n \to \infty.$$

By Ekeland's variational principle [15; 27], we may assume that $\{w^\beta_n\}$ is a Palais-Smale sequence at level $c^b_{\beta,\eta}$.

Then, by Proposition 2.1, the $(PS)_c$ condition holds for all $c < \frac{1}{N}S^{N/2}$. Moreover, by Lemma 3.3, we have $c^b_{\beta,\eta} < \frac{1}{N}S^{N/2}$, ensuring compactness.

Therefore, up to a subsequence, $w^\beta_n \to w_\beta$ strongly in $E_b$, and $w_\beta$ is a least-energy critical point of $J^b_{\beta,\eta}$. Thus, $w_\beta$ is a least-energy solution to the problem $(PS^b_{\beta,\eta})$.

Proof (**Proof of Theorem 1.3**). Let $\{w_n\}$ be a sequence of solutions to $(PS^b_{\beta_n,\eta})$ such that $\eta \in (0, \eta_1(\Omega))$, $\beta_n \to \infty$, and

$$J^b_{\beta_n,\eta}(w_n) = \frac{1}{N}(\mathscr{B}^b_{\beta_n} - \eta)(w_n, w_n) \to c.$$

Assume further that the energy satisfies

$$NJ^b_{\beta_n,\eta}(w_n) - (\mathscr{B}^b_{\beta_n} - \eta)(w_n, w_n) \to Nc < S^{N/2}.$$

Then, by Lemmas 2.1 and 2.2, there exists $w \in \mathscr{H}^1_{0,b}(\Omega)$ such that, up to a subsequence, $w_n \rightharpoonup w$ weakly in $E_b$ and $w_n \to w$ strongly in $L^2(\mathbb{R}^N)$. Since each $w_n$ satisfies the Euler-Lagrange equation,

$$\int_{\mathbb{R}^N} (b(x)\nabla w_n \cdot \nabla v + \beta_n a w_n v - \eta w_n v) = \int_{\mathbb{R}^N} |w_n|^{2^*-2} w_n v, \quad \forall v \in E_b,$$

taking the limit with test functions $v \in \mathscr{H}^1_{0,b}(\Omega)$ yields

$$\int_{\mathbb{R}^N} (b(x)\nabla w \cdot \nabla v + \eta w v) = \int_{\mathbb{R}^N} |w|^{2^*-2} w v.$$

Hence, $w$ solves $(D^b_\eta)$.

Now let $t_n := w_n - w$. Using Brezis-Lieb lemma [7], and orthogonality of cross-terms, we obtain

$$(\mathscr{B}^b_{\beta_n} - \eta)(w_n, w_n) = (\mathscr{B}^b_0 - \eta)(w, w) + (\mathscr{B}^b_{\beta_n} - \eta)(t_n, t_n) + o(1),$$

and

$$|w_n|^{2^*}_{2^*} = |w|^{2^*}_{2^*} + |t_n|^{2^*}_{2^*} + o(1).$$

From the Nehari condition and energy identity, we conclude:

$$(\mathscr{B}^b_{\beta_n} - \eta)(t_n, t_n) - |t_n|^{2^*}_{2^*} \to 0.$$

Assume by contradiction that $|t_n|^{2^*}_{2^*} \to b > 0$. Then Sobolev inequality implies

$$S|t_n|^2_{2^*} \leq \|\nabla t_n\|^2_2 \leq (\mathscr{B}^b_{\beta_n} - \eta)(t_n, t_n) + o(1) = |t_n|^{2^*}_{2^*} + o(1),$$

which leads to $S^{N/2} \leq c < S^{N/2}$, a contradiction. Therefore, $|t_n|_{2^*} \to 0$ and

$$(\mathscr{B}^b_{\beta_n} - \eta)(t_n, t_n) \to 0.$$

Consequently,

$$(\mathscr{B}^b_0 - \eta)(w, w) = \lim_{n\to\infty} (\mathscr{B}^b_{\beta_n} - \eta)(w_n, w_n),$$

and since $a(x) = 0$ in $\Omega$, it follows that $\int a w_n^2 \to 0$. Thus, $w_n \to w$ in $E_b$, and the sequence concentrates at a solution of $(D^b_\eta)$.

As a consequence of theorems 1.1 and 1.3, we obtain:

**Corollary 3.1.** *For each $\eta \in (0, \eta_1(\Omega))$, we have*

$$\lim_{\beta\to\infty} c^b_{\beta,\eta} = c^b(\eta, \Omega).$$

## 4. Proof of Theorem 1.2

To prove Theorem 1.2, we adopt the topological method introduced by Benci and Cerami [6]. Since $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain, there exists a small radius $r > 0$ such that the following inclusions hold:

$$\Omega^+_{2r} := \left\{x \in \mathbb{R}^N : \operatorname{dist}(x, \Omega) < 2r\right\},$$
$$\Omega^-_r := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r\},$$

and both sets $\Omega^+_{2r}$ and $\Omega^-_r$ are homotopically equivalent to $\Omega$. Moreover, we can assume that the open ball $B_r := \{x \in \mathbb{R}^N : |x| < r\}$ is contained in $\Omega$.

Using the argument developed in the proof of Lemma 3.3, it follows that

$$c^b(\eta, \Omega) < c^b(\eta, B_r) < \frac{1}{N} S^{N/2},$$

for every $\eta \in (0, \eta_1(\Omega))$.

For any nonzero function $w \in L^{2^*}(\Omega)$, we define its *center of mass* by

$$\beta(w) := \frac{\int_\Omega |w|^{2^*} x\, dx}{\int_\Omega |w|^{2^*}\, dx}.$$

Recalling a result of Lazzo [19], we state the following lemma:

**Lemma 4.1.** *There exists a constant $\eta^\sharp = \eta^\sharp(r) \in (0, \eta_1(\Omega))$ such that for all $\eta \in (0, \eta^\sharp]$, the following statements hold:*

*(i)* $c^b(\eta, B_r) < 2c^b(\eta, \Omega)$,

*(ii) For every $w \in \mathscr{M}^b_{\eta,\Omega}$ satisfying $J^b_{\eta,\Omega}(w) \le c^b(\eta, B_r)$, the center of mass $\beta(w)$ lies in $\Omega^+_r$.*

As in [4], we choose $R > 0$ with $\bar{\Omega} \subset B_R$ and set

$$\xi(t) = \begin{cases} 1, & 0 \le t \le R, \\ R/t, & R \le t. \end{cases}$$

Define

$$\beta_0(w) := \frac{\int_{\mathbb{R}^N} \xi(|w|) x dx}{\int_{\mathbb{R}^N} |w|^{2^*} dx} \quad \text{for} \quad w \in L^{2^*}(\mathbb{R}^N \setminus \{0\})$$

**Lemma 4.2.** *There exists a constant $\eta^* = \eta^*(r) \in (0, \eta_1(\Omega))$ and, for each $\eta \in (0, \eta^*]$, a number $\Lambda(\eta) \ge \beta(\eta)$ such that:*

*(i)* $c^b(\eta, B_r) < 2c^b_{\beta,\eta}$ *for all* $\beta \ge \Lambda(\eta)$,

*(ii) For all $\beta \ge \Lambda(\eta)$ and for every $w \in \mathscr{M}^b_{\beta,\eta}$ with $J^b_{\beta,\eta}(w) \le c^b(\eta, B_r)$, we have $\beta_0(w) \in \Omega^+_{2r}$.*

Proof. Assertion (i) follows directly from Lemma 4.1 and Corollary 3.1, which imply:

$$c^b(\eta, B_r) < 2c^b(\eta, \Omega) = 2 \lim_{\beta\to\infty} c^b_{\beta,\eta} \leq 2c^b_{\beta,\eta},$$

for all $\beta \geq \Lambda(\eta)$.

We now prove (ii) by contradiction. Suppose that, for arbitrarily small $\eta$, there exists a sequence $\{w_n\}$ with $w_n \in \mathscr{M}^b_{\beta_n,\eta}$, $\beta_n \to \infty$, $J^b_{\beta_n,\eta}(w_n) \leq c^b(\eta, B_r)$, but $\beta_0(w_n) \notin \Omega^+_{2r}$.

By Lemma 2.1, up to a subsequence, $w_n \rightharpoonup w_\eta$ weakly in $E_b$, and $w_n \to w_\eta$ in $L^2(\mathbb{R}^N)$.

We consider two cases:

**Case 1:** $|w_\eta|^{2^*}_{2^*} \leq (\mathscr{B}^b_0 - \eta)(w_\eta, w_\eta)$.

Define $t_n := w_n - w_\eta$. Since $a(x) = 0$ in $\Omega$, we have:

$$(\mathscr{B}^b_{\beta_n} - \eta)(w_n, w_n) = (\mathscr{B}^b_0 - \eta)(w_\eta, w_\eta) + (\mathscr{B}^b_{\beta_n} - \eta)(t_n, t_n) + o(1),$$

and by the Brézis–Lieb Lemma [7]:

$$|w_n|^{2^*}_{2^*} = |w_\eta|^{2^*}_{2^*} + |t_n|^{2^*}_{2^*} + o(1).$$

Since $w_n \in \mathscr{M}^b_{\beta_n,\eta}$, we obtain:

$$(\mathscr{B}^b_{\beta_n} - \eta)(t_n, t_n) \leq |t_n|^{2^*}_{2^*} + o(1).$$

Assume by contradiction that $|t_n|^{2^*}_{2^*} \to b > 0$. Then:

$$S|t_n|^2_{2^*} \leq \int b(x)|\nabla t_n|^2 \leq (\mathscr{B}^b_{\beta_n} - \eta)(t_n, t_n) + o(1) \leq |t_n|^{2^*}_{2^*} + o(1),$$

which implies $S^{N/2} \leq \lim_{n\to\infty} |w_n|^{2^*}_{2^*} < S^{N/2}$, a contradiction.

Therefore, $|t_n|_{2^*} \to 0$, and hence $w_n \to w_\eta$ in $L^{2^*}(\mathbb{R}^N)$. It follows that $\beta_0(w_n) \to \beta(w_\eta)$. Since $J^b_{\eta,\Omega}(w_\eta) \leq \lim J^b_{\beta_n,\eta}(w_n) \leq c^b(\eta, B_r)$, Lemma 4.1 implies $\beta(w_\eta) \in \Omega^+_r$. This contradicts the assumption $\beta_0(w_n) \notin \Omega^+_{2r}$.

**Case 2:** $|w_\eta|^{2^*}_{2^*} > (\mathscr{B}^b_0 - \eta)(w_\eta, w_\eta)$.

Then there exists $t \in (0, 1)$ such that $tw_\eta \in \mathscr{M}^b_{\eta,\Omega}$, and hence

$$\begin{aligned} c^b(\eta, \Omega) \leq J^b_{\eta,\Omega}(tw_\eta) &= \frac{t^2}{N}(\mathscr{B}^b_0 - \eta)(w_\eta, w_\eta) \\ &< \lim_{n\to\infty} J^b_{\beta_n,\eta}(tw_n) \leq c^b(\eta, B_r). \end{aligned}$$

Thus, for $n$ sufficiently large,

$$\left||w_n|^{2^*}_{2^*} - |tw_\eta|^{2^*}_{2^*}\right| < N(c^b(\eta, B_r) - c^b(\eta, \Omega)),$$

and since $c^b(\eta, B_r) - c^b(\eta, \Omega) \to 0$ as $\eta \to 0$, it follows that:

$$|\beta_0(w_n) - \beta(tw_\eta)| < r.$$

However, by Lemma 4.1, $\beta(tw_\eta) \in \Omega^+_r$, contradicting $\beta_0(w_n) \notin \Omega^+_{2r}$. This contradiction completes the proof of (ii).

For a given function $I : M \to \mathbb{R}$, the set $I^{\leq t}$ is defined as:

$$I^{\leq t} = \{z \in M : I(z) \leq t\}.$$

which represents the level set of all points $z \in M$ where $I(z)$ does not exceed $t$.

The following is an easy consequence of Lusternik-Schnirelmann theory:

**Proposition 4.1.** *Let $J : M \to \mathbb{R}$ be an even $C^1$-functional defined on a complete, symmetric, $C^{1,1}$ submanifold $M \subset V \setminus \{0\}$ of a Banach space $V$. Assume that $J$ is bounded below and satisfies the Palais-Smale condition $(PS)_c$ for all $c \leq t$. Further, suppose there exist continuous maps*

$$X \xrightarrow{L} J^{\leq t} \xrightarrow{\beta} Y$$

*such that the composition $\beta \circ L$ is a homotopy equivalence and $\beta(z) = \beta(-z)$ for all $z \in M \cap J^{\leq t}$. Then $J$ admits at least $Cat(X)$ pairs of distinct critical points $\{z, -z\}$ with $J(z) = J(-z) \leq t$.*

Proof (**Proof of Theorem 1.2**). Let $0 < \eta < \eta^*$ and $\beta \geq \Lambda(\eta)$. We define the maps

$$\Omega_r^- \xrightarrow{L} \mathscr{M}^b_{\beta,\eta} \cap J^{\leq c^b(\eta,B_r)}_{\beta,\eta} \xrightarrow{\beta_0} \Omega^+_{2r},$$

where $\beta_0$ denotes the barycenter map introduced earlier. Lemma 4.2 ensures that $\beta_0$ is well defined on this subset of the Nehari manifold.

Let $w_r \in \mathscr{H}^1_{0,b}(B_r) \subset E_b$ be a positive minimizer of $J^b_{\eta,B_r}$ over $\mathscr{M}^b_{\eta,B_r}$. For each $x \in \Omega_r^-$, define the translated function $l(x) := w_r(\cdot - x)$. Since $B_r(x) \subset \Omega$, we have $l(x) \in \mathscr{M}^b_{\beta,\eta}$ and

$$J^b_{\beta,\eta}(l(x)) = J^b_{\eta,B_r}(w_r) = c^b(\eta, B_r).$$

Moreover, the radial symmetry of $w_r$ yields $\beta_0(l(x)) = x$ for all $x \in \Omega_r^-$. Thus, the composition $\beta_0 \circ L$ is the identity map on $\Omega_r^-$ and therefore a homotopy equivalence.

Additionally, since $J^b_{\beta,\eta}(w) = J^b_{\beta,\eta}(-w)$ and $\beta_0(w) = \beta_0(-w)$ for all $w \in E_b \setminus \{0\}$, the symmetry conditions of Proposition 4.1 are satisfied.

By the inequality $c^b(\eta, B_r) < \frac{1}{N} S^{N/2}$ from [8], Proposition 2.1 guarantees that $J^b_{\beta,\eta}$ satisfies the $(PS)_c$ condition for all $c \leq c^b(\eta, B_r)$. Consequently, Proposition 4.1 applies. Invoking also Proposition 3.1 (positivity of critical points) and Lemma 4.2, we deduce that the problem $(NS^b_{\beta,\eta})$ possesses at least $Cat(\Omega)$ positive solutions.

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