

On Some Quasi-Curves in Galilean Three-Space

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Abstract: In this paper, the quasi-frame and quasi-formulas are introduced in Galilean three-space. In addition, the quasi-Bertrand and the quasi-Mannheim curves are studied. It is proven that the angle between the tangents of two quasi-Bertrand or quasi-Mannheim curves is not constant. Furthermore, the quasi-involute is studied. Moreover, we prove that there is no quasi-evolute curve in Galilean three-space. Also, we introduce quasi-Smarandache curves in Galilean three-space. Finally, we demonstrate an illustrated example to present our findings.

Keywords: Galilean space; quasi-frame; quasi-formulas; quasi-curvatures; quasi-Bertrand curves; quasi-Mannheim curves; quasi-involute; quasi-Smarandache

MSC: 53A35, 53C50.



Citation: Elsharkawy, A.; Tashkandy, Y.; Emam, W.; Cesarano, C.; Elsharkawy, N. On Some Quasi-Curves in Galilean Three-Space. *Axioms* **2023**, *12*, 823. <https://doi.org/10.3390/axioms12090823>

Academic Editor: Juan De Dios Pérez

Received: 18 July 2023

Revised: 22 August 2023

Accepted: 25 August 2023

Published: 27 August 2023



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1. Introduction

The geometry of curves in Galilean space has been studied for many years. Galilean geometry is a Cayley–Klein geometry with Galilean transformations of classical kinematics. Modern and traditional physics use the Galilean transformation group [1]. If the curve has any points of zero curvature, the Frenet frame cannot be established, and this is especially true for normal and binormal vectors. As a result, a number of mathematicians came up with frames that can deal with points in Euclidean, Minkowski, and Galilean geometry when the curvature is zero. These frames include the Bishop frame, the modified frame, the equiform frame, and the Darboux frame [2–5].

A new adapted frame that follows a space curve was provided in [6] by Dede et al. as an alternative to the Frenet frame. This design was referred to as the quasi-frame. This frame is both simpler and more precise than the Frenet frame. The fact that the quasi-frame may be seen as a generalization of the Frenet frame is one of the advantages of its use. It is defined by a vector that does not change and an angle that is formed between the quasi-normal and the principal normal to the Frenet frame. If the point has zero curvature, then the frame rotates by the angle, and the quasi-normal is defined such that it is perpendicular to both the tangent and the fixed vector. The vector that is perpendicular to both the tangent vector and the quasi-normal vector is referred to as the quasi-binormal vector. Much research on the quasi-frame has been conducted in a variety of Euclidean and Minkowski spaces and may be found in [7–10].

In 1850, the observation of J. Bertrand led to the discovery of Bertrand curves, which have become a significant subject of interest and intrigue in classical special curve theory. Bertrand curves are a particular class of curves defined by the property of having their principal normal coincide with that of another curve. Such curves are characterized by a linear relationship between the curvature and torsion of the curve. In the Galilean space

and the Lorentzian space, there are a lot of works that are associated with Bertrand curves, such as [11–13].

The identification of involutes, which are mathematical curves derived from the rolling of one curve onto another, is credited to C. Huygens, who was engaged in the effort to create a more precise timepiece. [14]. Later on, in [15], the relations between the Frenet apparatus and the involute–evolute curve pair in the space E^3 were introduced. The involute–evolute curve pair was studied by A. Turgut in R^n in [16], and the characteristics of the curves in the Galilean space were investigated in [6].

Investigating the properties of Mannheim curves in both Euclidean and Minkowski three-space, Liu and Wang made a significant discovery in 2007. They derived necessary conditions that relate to a curve's curvature and torsion, which must be satisfied for the curve to be recognized as one of the partner curves of Mannheim. The comprehensive analysis of Mannheim curves is expounded in [17], where detailed research findings can be found.

Smarandache curves can be defined as a type of regular curve characterized by the decomposition of their position vector along the Frenet frame vectors of another regular curve. Several mathematicians have studied Smarandache curves, such as [18].

The present study is organized in the following manner. In Section 3, we investigate the quasi-frame, including its relation with the Frenet frame, and quasi-formulas in Galilean three-space are investigated. In Section 4, we study quasi-Bertrand curves in the Galilean three-space and our study demonstrates that there exists a constant measure of distance between corresponding points on two quasi-Bertrand curves that operate within Galilean 3-space, but the angle between tangent lines is not constant. In Section 5, we study quasi-Mannheim curves in the Galilean three-space, and we prove that the distance between corresponding points on two quasi-Mannheim curves in Galilean three-space is constant, but the angle between tangents is not constant. In Section 6, we investigate quasi-involute curves in the Galilean three-space. Also, we prove that there is no quasi-evolute curve in Galilean three-space. In Section 7, we prove that there are no quasi-evolute curves in the Galilean three-space. In Section 8, we focused on exploring the properties and characteristics of quasi-Smarandache curves in the Galilean three-space.

2. Preliminaries

The three-dimensional Galilean space, denoted as \mathbb{G}_3 , is a genuine Cayley-Klein space equipped with the projective metric of a particular signature $(0, 0, +, +)$. The absolute of the Galilean three-space consists of the set $\{r, k, J\}$ in which r is the absolute plane in \mathbb{G}_3 , k is the absolute line in k , and J is the fixed elliptic involution of points of f . A vector $\mathbf{m} = (m_1, m_2, m_3)$ within the Galilean three-space \mathbb{G}_3 is considered non-isotropic if its initial component is not zero. Otherwise, it is called an isotropic vector. Further, a vector with a first component of magnitude 1 is classified as a non-isotropic unit vector. In \mathbb{G}_3 , the Galilean metric g is defined by

$$g(\mathbf{m}, \mathbf{n}) = \begin{cases} m_1 n_1, & \text{if } m_1 \neq 0 \text{ or } n_1 \neq 0, \\ m_2 n_2 + m_3 n_3, & \text{if } m_1 = 0 \text{ and } n_1 = 0, \end{cases}$$

where $\mathbf{m} = (m_1, m_2, m_3)$ and $\mathbf{n} = (n_1, n_2, n_3)$. Also, the Galilean norm of the vector \mathbf{q} is defined as

$$\|\mathbf{m}\| = \begin{cases} |m_1|, & \text{if } m_1 \neq 0, \\ \sqrt{m_2^2 + m_3^2}, & \text{if } m_1 = 0. \end{cases}$$

Further, the Galilean vector product of m and n is defined as

$$m \times n = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}, & \text{if } m_1 \neq 0 \text{ or } n_1 \neq 0, \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}, & \text{if } m_1 = 0 \text{ and } n_1 = 0, \end{cases}$$

where (e_1, e_2, e_3) is the usual basis of \mathbb{R}^3 [3,6,13,19].

In \mathbb{G}_3 , a curve is a mapping from an open interval I of \mathbb{R} to \mathbb{G}_3 as

$$\begin{aligned} \Gamma : I &\longrightarrow \mathbb{G}_3, \\ t &\longrightarrow \gamma(t) = (x_1(t), x_2(t), x_3(t)). \end{aligned}$$

If the curve has no inflection points ($\dot{\Gamma}(t) \times \ddot{\Gamma}(t) \neq 0$) and no isotropic tangents ($x_1(t) \neq 0$) for each $t \in I$, then it is called an admissible curve. Let an admissible differentiable curve parameterized by the Galilean invariant arc length s be given by

$$\gamma(s) = (s, x_2(s), x_3(s)).$$

Then, the curvature $\kappa(s)$ and the torsion $\tau(s)$ of the curve $\gamma(s)$ are given by

$$\begin{aligned} \kappa(s) &= \|\Gamma''(s)\| = \sqrt{x_2''^2(s) + x_3''^2(s)}, \\ \tau(s) &= \frac{\det(\Gamma'(s), \Gamma''(s), \Gamma'''(s))}{\kappa^2(s)} \end{aligned} \tag{1}$$

and the moving Frenet frame $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ of the curve $\gamma(s)$ is defined by

$$\begin{aligned} \mathbf{T}(s) &= \gamma'(s) = (1, x_2'(s), x_3'(s)), \\ \mathbf{N}(s) &= \frac{1}{\kappa(s)}\Gamma''(s) = \frac{1}{\kappa(s)}(0, x_2''(s), x_3''(s)), \\ \mathbf{B}(s) &= \mathbf{T}(s) \times \mathbf{N}(s) = \frac{1}{\kappa(s)}(0, -x_3''(s), x_2''(s)), \end{aligned} \tag{2}$$

where the vectors $\mathbf{T}(s)$, $\mathbf{N}(s)$, and $\mathbf{B}(s)$ are the tangent, principal normal, and binormal unit vector fields of the curve $\Gamma(s)$ [4,13]. On the another hand, the Frenet derivative formulas are given by

$$\frac{d}{ds} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ 0 & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}. \tag{3}$$

3. Quasi-Frame and Quasi-Equations in \mathbb{G}_3

The present section examines the quasi-frame and its correlation with the Frenet frame. Additionally, quasi-formulas are scrutinized within the context of Galilean three-space \mathbb{G}_3 . Furthermore, the quasi-curvatures are introduced as part of this investigation. Consider a curve $\alpha(s)$ in \mathbb{G}_3 . The quasi-frame composed of three orthonormal vectors, namely the unit tangent $T(s)$, the unit quasi-normal vector $N_q(s)$, and the unit quasi-binormal vector $B_q(s)$, can be defined. This quasi-frame, characterized by $\{T(s), N_q(s), B_q(s)\}$, is derived from the curve’s Frenet-Serret frame and is fundamental in a range of geometric computations as follows:

$$T = \frac{\alpha'}{\|\alpha'\|}, \quad N_q = \frac{T \times z}{\|T \times z\|}, \quad B_q = T \times N_q, \tag{4}$$

where z is the projection vector given by either $(1, 0, 0)$, $(0, 1, 0)$, or $(0, 0, 1)$. The choice of the projection vector z depends on the parallelism with unit tangent vector T . We choose here $z = (1, 0, 0)$.

Consider the Frenet frame, denoted by $\{T, N, B\}$ and $\theta(s)$ be an angle between N and N_q ; then, we can write N_q and B_q in terms of N and B as

$$N_q = \cos\theta N + \sin\theta B, \tag{5}$$

$$B_q = -\sin\theta N + \cos\theta B, \tag{6}$$

and we can write

$$N = \cos\theta N_q - \sin\theta B_q, \tag{7}$$

$$B = \sin\theta N_q + \cos\theta B_q. \tag{8}$$

Now, from Equation (3) and Equation (7), we get

$$T' = \kappa N = \kappa \cos\theta N_q - \kappa \sin\theta B_q.$$

By taking the substitution $K_1 = \kappa \cos\theta$ and $K_2 = \kappa \sin\theta$, we have

$$T' = K_1 N_q - K_2 B_q. \tag{9}$$

Similarly, by using Equations (5) and (6), N'_q and B'_q are given, respectively, by

$$N'_q = K_3 B_q, \quad B'_q = -K_3 N_q, \tag{10}$$

where $\theta' + \tau = K_3$. Therefore, the quasi-formulas in the matrix notation are given by:

$$\begin{pmatrix} T' \\ N'_q \\ B'_q \end{pmatrix} = \begin{pmatrix} 0 & K_1 & -K_2 \\ 0 & 0 & K_3 \\ 0 & -K_3 & 0 \end{pmatrix} \begin{pmatrix} T \\ N_q \\ B_q \end{pmatrix}, \tag{11}$$

The quasi-curvatures K_1, K_2 , and K_3 are represented using the Frenet curvature and torsion as

$$K_1 = \kappa \cos\theta, \quad K_2 = \kappa \sin\theta, \quad K_3 = \theta' + \tau.$$

Corollary 1. *If $\alpha(s)$ is a curve in \mathbb{G}_3 , then the quasi-curvatures K_1, K_2 , and K_3 are given, respectively, by*

$$K_1 = g(T', N_q), \quad K_2 = -g(T', B_q), \quad K_3 = g(N'_q, B_q) = -g(B'_q, N_q).$$

Corollary 2. *In the context of \mathbb{G}_3 , the quasi-frame represents a generalization of the Frenet frame. Specifically, in the event that K_2 equals zero, the quasi-frame and the Frenet frame become equivalent.*

4. Quasi-Bertrand Curves in \mathbb{G}_3

This section investigates the Bertrand curves within the framework of the quasi-frame in \mathbb{G}_3 . Our objective is to demonstrate that the distance between corresponding points on two Bertrand curves, as determined by the quasi-frame in \mathbb{G}_3 , remains invariant. However, it should be noted that the angle between corresponding tangent lines on the two Bertrand curves, when analyzed using the quasi-frame in \mathbb{G}_3 , exhibits variation and is not constant.

Definition 1. *Two curves $\alpha(s)$ and α^* in \mathbb{G}_3 are said to be quasi-Bertrand curves according to the quasi-frame if the quasi-normal line to α is the same as the quasi-normal vector to the curve α^* at the corresponding points, in other words, if N_q of the curve α coincides with N_q^* at the corresponding points.*

Theorem 1. *If $\alpha(s)$ and $\alpha^*(s)$ are quasi-Bertrand curves in \mathbb{G}_3 , then*

$$\alpha^*(s) = \alpha(s) + cN_q,$$

where c is constant.

Proof. Let $\alpha(s)$ and $\alpha^*(s)$ be quasi-Bertrand curves in \mathbb{G}_3 ; then, we can write

$$\alpha^*(s) = \alpha(s) + \lambda(s)N_q, \tag{12}$$

for some differentiable function $\lambda(s)$.

By differentiating Equation (12) with respect to s , we have

$$\frac{d\alpha^*}{ds} = T + \lambda'(s)N_q + \lambda(s)K_3B_q. \tag{13}$$

Therefore, we can deduce that $\lambda(s)$ is constant.

Hence,

$$\alpha^*(s) = \alpha(s) + cN_q,$$

where c is constant. \square

Corollary 3. *The distance between two quasi-Bertrand curves is constant.*

Theorem 2. *The angle between the tangents of two quasi-Bertrand curves in \mathbb{G}_3 is not constant.*

Proof. Let $\alpha(s)$ and $\alpha^*(s)$ be quasi-Bertrand curves in \mathbb{G}_3 and let $\theta(s)$ be the angle between the tangents of α and α^* ; then, we can write

$$\cos \theta = \frac{g(T, T^*)}{\|T\| \|T^*\|}. \tag{14}$$

By differentiating Equation (14) with respect to s , we have

$$\frac{d}{ds} \cos \theta = -K_2 g(T^*, B_q) - K_2^* g(T, B_q^*) \frac{ds^*}{ds}, \tag{15}$$

which is not zero. Therefore, θ is not constant. \square

Corollary 4. *The angle between the tangents of two Frenet Bertrand curves according to the Frenet frame in \mathbb{G}_3 is constant.*

Proof. If we put K_2 and K_2^* into Equation (15), we have

$$\frac{d}{ds} \cos \theta = 0.$$

This implies that the angle between the tangents is constant. \square

Theorem 3. *If $\alpha(s)$ and $\alpha^*(s)$ are quasi-Bertrand curves in \mathbb{G}_3 , then the quasi-frame of $\alpha^*(s)$ is given by*

$$\begin{aligned} T^* &= \frac{1}{\sqrt{1 + C^2K_3^2}} [T + CK_3B_q], \\ N_q^* &= N_q, \\ B_q^* &= \frac{1}{\sqrt{1 + C^2K_3^2}} [-CK_3e_1 + B_q]. \end{aligned}$$

Proof. By differentiating Equation (13) with respect to s , we have

$$\frac{d\alpha^*}{ds^*} \frac{ds^*}{ds} = T + CK_3B_q; \tag{16}$$

thus, we have

$$T^* = \frac{ds}{ds^*} [T + CK_3B_q],$$

and we can obtain

$$N_q^* = N_q.$$

Furthermore, we have

$$B_q^* = T^* \times N_q^* = \frac{ds}{ds^*} [-CK_3e_1 + B_q],$$

such that $\frac{ds^*}{ds} = \sqrt{1 + C^2K_3^2}$. \square

5. Quasi-Mannheim Curves in \mathbb{G}_3

This section focuses on the analysis of quasi-Mannheim curves within the framework of the quasi-frame in \mathbb{G}_3 . Our objective is to establish that the distance between corresponding points on two quasi-Mannheim curves, as determined by the quasi-frame in \mathbb{G}_3 , remains unchanged. However, it is important to note that the angle between the tangents of the two quasi-Mannheim curves in \mathbb{G}_3 is not constant.

Definition 2. Two curves $\gamma(s)$ and $\gamma^*(s)$ in \mathbb{G}_3 are said to be quasi-Mannheim curves according to the quasi-frame if the quasi-normal line to α is the same as the quasi-binormal vector to the curve α^* at the corresponding points. In this case, the curve α is called a quasi-Mannheim curve of α^* and α^* is called a quasi-Mannheim partner curve.

Theorem 4. If $\gamma(s)$ and γ^* are quasi-Mannheim curves in \mathbb{G}_3 , then

$$\gamma^* = \gamma(s) + c_1N_q,$$

where c_1 is constant.

Proof. Let $\gamma(s)$ and $\gamma^*(s)$ be quasi-Mannheim curves in \mathbb{G}_3 . Then, $\gamma^*(s)$ can be written as

$$\gamma^* = \gamma(s) + \nu(s)N_q, \tag{17}$$

for some differentiable function $\nu(s)$.

By differentiating Equation (17) with respect to s , we have

$$\frac{d\gamma^*}{ds} = T + \nu'(s)N_q + \nu(s)K_3B_q.$$

Therefore, we can deduce that $\nu(s)$ is constant.

Hence,

$$\gamma^* = \gamma(s) + c_1N_q,$$

where c_1 is constant. \square

Corollary 5. The distance between corresponding points on two Mannheim curves remains consistent.

Theorem 5. The angle between the tangents of two quasi-Mannheim curves in \mathbb{G}_3 is not constant.

Proof. Let $\gamma(s)$ and $\gamma^*(s)$ be quasi-Mannheim curves in \mathbb{G}_3 and let $\beta(s)$ be the angle between the tangents of γ and γ^* ; then,

$$\cos \beta = \frac{g(T, T^*)}{\|T\| \|T^*\|}. \tag{18}$$

By differentiating Equation (18) with respect to s , we have

$$\frac{d}{ds} \cos \beta = -K_2 g(B_q, T^*) + K_1^* \frac{ds^*}{ds} g(T, N_q^*), \tag{19}$$

which is not zero. Therefore, β is not constant. \square

Corollary 6. *The angle between the tangents of two Frenet Mannheim curves according to the Frenet frame in \mathbb{G}_3 is constant.*

Proof. If we put K_2 and K_2^* in Equation (19), we have

$$\frac{d}{ds} \cos \beta = 0.$$

This implies that the angle between the tangents is constant. \square

6. Quasi-Involute Curves in \mathbb{G}_3

In this section, we explore the properties of quasi-involute curves in \mathbb{G}_3 as described by the quasi-frame. By utilizing the quasi-frame of the original curve, we derive the corresponding quasi-frame of the involute curve and also determine the quasi-curvatures of the quasi-involute.

Definition 3. *In the context of Galilean space \mathbb{G}_3 , curves β and β^* be given. The Curve β^* is known as the involute of the curve β if the tangent vector at $\beta^*(s)$ is intersected by the tangent vector at $\beta(s)$, whenever $g(T, T^*) = 0$. Here, the quasi-frames for β and β^* are represented by T, N_q, B_q and T^*, N_q^*, B_q^* , respectively.*

In simpler terms, $\beta^(s)$ can be expressed as*

$$\beta^*(s) = \beta(s) + X(s)T(s).$$

Theorem 6. *If β and β^* are two curves in \mathbb{G}_3 and β^* is an involute of β , then*

$$\beta^*(s) = \beta(s) + (a - s)T,$$

where a is constant.

Proof. Let β^* be an involute of β ; then, we can write

$$\beta^*(s) = \beta(s) + X(s)T. \tag{20}$$

By differentiating Equation (20), we have

$$\frac{d\beta^*}{ds} = (1 + X'(s))T + XK_1N_q - XK_2B_q.$$

Since $\frac{d\beta^*}{ds}$ is orthogonal to T , we obtain

$$1 + X'(s) = 0.$$

Therefore,

$$X = a - s,$$

where a is constant. Hence,

$$\beta^*(s) = \beta(s) + (a - s)T. \tag{21}$$

□

Theorem 7. *If β and β^* are two curves in \mathbb{G}_3 and β^* is a quasi-involute of β , then the quasi-frame of the curve β^* is given as*

$$\begin{aligned} T^* &= \frac{K_1}{\sqrt{K_1^2 + K_2^2}}N_q - \frac{K_2}{\sqrt{K_1^2 + K_2^2}}B_q, \\ N_q^* &= \frac{-K_2}{\sqrt{K_1^2 + K_2^2}}N_q - \frac{K_1}{\sqrt{K_1^2 + K_2^2}}B_q, \\ B_q^* &= -e_1. \end{aligned}$$

Proof. By differentiating Equation (21), we obtain

$$\frac{d\beta^*}{ds} = (a - s)K_1N_q - (a - s)K_2B_q. \tag{22}$$

The norm of $\frac{d\beta^*}{ds}$ is $\|\frac{d\beta^*}{ds}\| = (a - s)\sqrt{K_1^2 + K_2^2}$. Then, we obtain

$$T^* = \frac{K_1}{\sqrt{K_1^2 + K_2^2}}N_q - \frac{K_2}{\sqrt{K_1^2 + K_2^2}}B_q. \tag{23}$$

Since $N_q^* = \frac{T^* \times z}{\|T^* \times z\|}$, we obtain

$$N_q^* = \frac{-K_2}{\sqrt{K_1^2 + K_2^2}}N_q - \frac{K_1}{\sqrt{K_1^2 + K_2^2}}B_q. \tag{24}$$

Finally,

$$B_q^* = T^* \times N_q^* = -e_1. \tag{25}$$

□

Corollary 7. *Let β and β^* be two curves in \mathbb{G}_3 . If β^* is a Frenet involute of β , then the Frenet frame of the curve β^* is given as*

$$\begin{aligned} T^* &= N_q, \\ N_q^* &= -B_q, \\ B_q^* &= -T. \end{aligned}$$

Theorem 8. *Let β and β^* be two curves in \mathbb{G}_3 . If β^* is a quasi-involute of β , then the quasi-curvatures of the curve β^* are given, respectively, by*

$$\begin{aligned} K_1^* &= \left| \frac{1}{(a - s)\sqrt{K_1^2 + K_2^2}} \left[\frac{K_1K_2' - K_2K_1'}{K_1^2 + K_2^2} - K_3 \right] \right|, \\ K_2^* &= 0, \\ K_3^* &= 0. \end{aligned}$$

Proof. By differentiating Equations (23)–(25) with respect to s , we obtain

$$\frac{ds^*}{ds} T^{*'} = \left[\left(\frac{K_1}{\sqrt{K_1^2 + K_2^2}} \right)' + \frac{K_2 K_3}{\sqrt{K_1^2 + K_2^2}} \right] N_q + \left[\frac{K_1 K_3}{\sqrt{K_1^2 + K_2^2}} - \left(\frac{K_2}{\sqrt{K_1^2 + K_2^2}} \right)' \right] B_q, \tag{26}$$

$$\frac{ds^*}{ds} N_q^{*'} = \left[\left(\frac{-K_2}{\sqrt{K_1^2 + K_2^2}} \right)' + \frac{K_1 K_3}{\sqrt{K_1^2 + K_2^2}} \right] N_q - \left[\frac{K_2 K_3}{\sqrt{K_1^2 + K_2^2}} + \left(\frac{K_1}{\sqrt{K_1^2 + K_2^2}} \right)' \right] B_q, \tag{27}$$

$$\frac{ds^*}{ds} B_q^{*'} = -K_1 N_q + K_2 B_q. \tag{28}$$

Since $\frac{ds^*}{ds} = \left\| \frac{d\beta^*}{ds} \right\|$, then

$$\frac{ds^*}{ds} = (c - s) \sqrt{K_1^2 + K_2^2}. \tag{29}$$

Since $K_1^* = g(T^{*'}, N_q^*)$, we obtain

$$K_1^* = \frac{1}{(c - s) \sqrt{K_1^2 + K_2^2}} \left[\frac{K_1 K_2' - K_2 K_1'}{K_1^2 + K_2^2} - K_3 \right]. \tag{30}$$

Since $K_2^* = -g(T'^*, B_q^*)$, we obtain

$$K_2^* = 0. \tag{31}$$

Since $K_3^* = -g(N_q'^*, B_q^*) = g(B_q'^*, B_q^*)$, we obtain

$$K_3^* = 0. \tag{32}$$

□

Corollary 8. Let β and β^* be two curves in the Galilean space \mathbb{G}_3 . If β^* is a Frenet involute of β , then the Frenet curvatures of the curve β^* are given as

$$\begin{aligned} \kappa^* &= \frac{-\tau}{(c - s)\kappa'} \\ \tau &= 0. \end{aligned}$$

7. Quasi-Evolute Curves in \mathbb{G}_3

In this particular section, we establish the nonexistence of a quasi-evolute curve when adopting the quasi-frame within Galilean three-space \mathbb{G}_3 . Since the Euclidean and Minkowski three-space evolute curves are well-defined, we provide a definition of the evolute curve within the Galilean three-space.

Definition 4. Assume that ζ and ζ^* are two curves present in the Galilean space \mathbb{G}_3 . We define the curve ζ^* as a quasi-evolute of the curve ζ if, and only if, the tangent vector of ζ at the point ζ intersects with the tangent vector of ζ^* at the point $\zeta^*(s)$, satisfying the zero dot product condition given by

$$g(T^*, T) = 0,$$

where T, N_q, B_q and T^*, N_q^*, B_q^* represent the quasi-frames of ζ and ζ^* , respectively. In simpler terms, $\zeta^*(s)$ can be expressed as

$$\zeta^*(s) = \zeta(s) + Y(s)N_q + Z(s)B_q.$$

Theorem 9. Let ζ and ζ^* be two curves in \mathbb{G}_3 . Then, there is no quasi-evolute curve in \mathbb{G}_3 .

Proof. Let ζ^* is an evolute of ζ ; then, we can write

$$\zeta^*(s) = \zeta(s) + Y(s)N_q + Z(s)B_q. \tag{33}$$

By differentiating Equation (33), we have

$$\frac{d\zeta^*}{ds} = T_q + (Y' - K_3Z)N_q + (YK_3 + Z')B_q.$$

Then, $\zeta^* - \zeta = YN_q + ZB_q$, which implies that $\zeta^* - \zeta$ is not parallel to $\frac{d\zeta^*}{ds}$, which is a contradiction to ζ^* being an evolute of ζ . Therefore, ζ^* is not an evolute of ζ . \square

8. Quasi-Smarandache Curves in \mathbb{G}_3

In this section, we study the quasi-Smarandache curves in Galilean three-space of three different types. In all cases, we deduce the quasi-frame of the quasi-Smarandache curve in terms of the quasi-frame of the original curve. Furthermore, the quasi-curvatures of the quasi-Smarandache curve are obtained in terms of the quasi-curvatures of the original curve. Moreover, in all cases, the Frene–Smarandache is obtained and studied in \mathbb{G}_3 .

Definition 5. If $\eta(s)$ is composed of quasi-frame vectors on another curve, then $\eta(s)$ is said to be a quasi-Smarandache curve in \mathbb{G}_3 . In the other words, if $\eta(s)$ is admissible curve in \mathbb{G}_3 and T, N_q, B_q is a quasi-frame of another curve $\epsilon(s)$, then quasi-Smarandache TN_q, TB_q , and TNB_q curves are, respectively, defined by

$$\eta_{(TN_q)} = \frac{T + N_q}{\|T + N_q\|}, \tag{34}$$

$$\eta_{(TB_q)} = \frac{T + B_q}{\|T + B_q\|}, \tag{35}$$

$$\eta_{(TN_qB_q)} = \frac{T + N_q + B_q}{\|T + N_q + B_q\|}. \tag{36}$$

8.1. TN_q -Smarandache Curve in \mathbb{G}_3

If $\eta(s)$ is an admissible curve in \mathbb{G}_3 and quasi-Smarandache TN_q curve is defined by $\eta(s) = \frac{T+N_q}{\|T+N_q\|}$, since $T + N_q$ is a unit vector, then the quasi-Smarandache TN_q curve is

$$\eta(s) = T + N_q. \tag{37}$$

Theorem 10. If $\eta(s)$ is a quasi-Smarandache TN_q curve in \mathbb{G}_3 , then the quasi-frame of $\eta(s)$ is given by

$$\begin{aligned} (T)_\eta &= A[K_1N_q + (K_3 - K_2)B_q], \\ (N_q)_\eta &= A(K_3 - K_2)N_q - AK_1B_q, \\ (B_q)_\eta &= -e_1 = (-1, 0, 0), \end{aligned}$$

where $A = \frac{1}{\sqrt{K_1^2 + (K_3 - K_2)^2}}$.

Proof. Let $\eta(s)$ be a quasi-Smarandache TN_q curve in \mathbb{G}_3 . By differentiating Equation (37) with respect to s , we have

$$T_\eta = A[K_1N_q + (K_3 - K_2)B_q], \tag{38}$$

where $A = \frac{1}{\sqrt{K_1^2 + (K_3 - K_2)^2}}$.

Since $(N_q)_\eta = \frac{T_\eta \times z}{\|T_\eta \times z\|}$, we obtain

$$(N_q)_\eta = A(K_3 - K_2)N_q - AK_1B_q. \tag{39}$$

Since $(B_q)_\eta = (T)_\eta \times (N_q)_\eta$, then

$$(B_q)_\eta = -e_1 = (-1, 0, 0). \tag{40}$$

□

Corollary 9. *If $\eta(s)$ is a Frenet–Smarandache curve in \mathbb{G}_3 , the Frenet frame of the curve $\eta(s)$ is given as*

$$(T)_\eta = \frac{1}{\sqrt{\kappa^2 + \tau^2}}[kN + \tau B], \quad (N)_\eta = \frac{1}{\sqrt{\kappa^2 + \tau^2}}[\tau N - \kappa B], \quad (B)_\eta = -e_1.$$

Theorem 11. *If $\eta(s)$ is a quasi-Smarandache curve in \mathbb{G}_3 , then the quasi-curvatures are given, respectively, by*

$$\begin{aligned} (K_1)_\eta &= A^2A'K_1K_3 + A^3K_1'K_3 - A^3K_3^3 + A^3K_2K_3^2 - A^2A'K_1K_2 - A^3K_1'K_2 + A^3K_3^2K_2 \\ &\quad - A^3K_2^2K_3 - A^2A'K_1K_3 + A^2A'K_1K_2 - A^3K_1^2K_3 - A^3K_1K_3' + A^3K_1K_2', \\ (K_2)_\eta &= 0, \\ (K_3)_\eta &= 0. \end{aligned}$$

Proof. Let $\eta(s)$ be a quasi-Smarandache curve in \mathbb{G}_3 . By differentiating Equation (38), we have

$$(T')_\eta = [AA'K_1 + A^2K_1' - A^2(K_3^2 - K_3K_2)]N_q + [AA'(K_3 - K_2) + A^2K_1K_3 + A^2K_3' - A^2K_2']B_q.$$

By differentiating Equation (39), we obtain

$$\begin{aligned} (N_q)'_\eta &= [AA'(K_3 - K_2) + A^2(K_3' - K_2') + A^2K_1K_3]N_q \\ &\quad + [-AA'K_1 + A^2K_3^2 - A^2K_2K_3 - A^2K_1']B_q, \end{aligned}$$

since

$$(B_q)'_\eta = -e_1' = 0.$$

Since $(K_1)_\eta = g(T'_\eta, (N_q)_\eta)$, we have

$$\begin{aligned} (K_1)_\eta &= A^2A'K_1K_3 + A^3K_1'K_3 - A^3K_3^3 + A^3K_2K_3^2 - A^2A'K_1K_2 - A^3K_1'K_2 + A^3K_3^2K_2 \\ &\quad - A^3K_2^2K_3 - A^2A'K_1K_3 + A^2A'K_1K_2 - A^3K_1^2K_3 - A^3K_1K_3' + A^3K_1K_2'. \end{aligned}$$

Since $(K_2)_\eta = -g(T'_\eta, (B_q)_\eta)$, we then have

$$(K_2)_\eta = 0.$$

Finally, since $(K_3)_\eta = g((N_q)'_1, (B_q)_1) = -g((B_q)'_1, N_q)$, we obtain

$$(K_3)_\eta = 0.$$

□

Corollary 10. *If $\eta(s)$ is a Frenet–Smarandache curve in \mathbb{G}_3 , then the curvature and torsion are given, respectively, by*

$$(\kappa)_\eta = A^3(\kappa'\tau - \tau'\kappa) - A^3\tau(\tau^2 - \kappa^2),$$

$$(\tau)_\eta = 0.$$

Corollary 11. *If $\eta(s)$ is a Frenet–Smarandache curve in \mathbb{G}_3 , then the Smarandache curve in \mathbb{G}_3 is a plane curve.*

8.2. TB_q -Smarandache Curve in \mathbb{G}_3

If $\rho(s)$ is an admissible curve in \mathbb{G}_3 , then a quasi-Smarandache TB_q curve is defined by

$$\rho(s) = T + B_q. \tag{41}$$

Theorem 12. *If $\rho(s)$ is quasi-Smarandache TB_q curve in \mathbb{G}_3 , then the quasi-frame of $\rho(s)$ is given by*

$$(T)_\rho = D[(K_1 - K_3)N_q - K_2B_q],$$

$$(N_q)_\rho = D[-K_2N_q - (K_1 - K_3)B_q]$$

$$(B_q)_\rho = -e_1,$$

where $D = \frac{1}{\sqrt{K_2^2 + (K_1 - K_3)^2}}$.

Corollary 12. *If $\rho(s)$ is a Frenet–Smarandache curve in \mathbb{G}_3 , then the Frenet frame of the curve $\eta(s)$ is*

$$(T)_\rho = N, \quad (N)_\rho = B, \quad (B)_\rho = -e_1.$$

Theorem 13. *If $\rho(s)$ is a quasi-Smarandache curve TB in \mathbb{G}_3 , then the quasi-curvatures are given, respectively, by*

$$(K_1)_\rho = -2D^2D'K_1K_2 + 2D^2D'K_2K_3 - D^3K_1'K_2 + D^3K_2K_3' - D^3K_2^2K_3$$

$$- D^3K_1^2K_3 + 2D^3K_1K_3^2 + D^3K_1K_2' - D^3K_3^3 - D^3K_2K_2',$$

$$(K_2)_\rho = 0,$$

$$(K_3)_\rho = 0,$$

where $D = \frac{1}{\sqrt{K_2^2 + (K_1 - K_3)^2}}$.

Corollary 13. *If $\rho(s)$ is a Frenet–Smarandache curve TB in \mathbb{G}_3 , then the curvature and torsion are given, respectively, by*

$$(\kappa)_\rho = \frac{\tau}{|(\kappa - \tau)|^{\frac{3}{2}}}[-\kappa^2 + \kappa\tau - \tau^2],$$

$$(\tau)_\rho = 0.$$

Corollary 14. *If $\eta(s)$ is a Frenet–Smarandache curve TB in \mathbb{G}_3 , then the Smarandache curve TB is always a plane curve.*

8.3. TN_qB_q -Smarandache Curve in \mathbb{G}_3

If $\epsilon(s)$ is an admissible curve in \mathbb{G}_3 , a quasi-Smarandache TN_qB_q curve is defined by

$$\epsilon(s) = T + N_q + B_q. \tag{42}$$

Theorem 14. If $\epsilon(s)$ is a quasi-Smarandache TN_qB_q curve in \mathbb{G}_3 , then the quasi-frame of $\epsilon(s)$ is given by

$$\begin{aligned} (T)_\epsilon &= E[(K_1 - K_3)N_q + (K_3 - K_2)B_q], \\ (N_q)_\epsilon &= E[(K_3 - K_2)N_q - (K_1 - K_3)B_q] \\ (B_q)_\epsilon &= -e_1, \end{aligned}$$

where $E = \frac{1}{\sqrt{(K_1 - K_3)^2 + (K_3 - K_2)^2}}$.

Corollary 15. If $\epsilon(s)$ is a Frenet–TNB–Smarandache curve in \mathbb{G}_3 , then the Frenet frame of the curve $\epsilon(s)$ is

$$(T)_\epsilon = G[(\kappa - \tau)N + \tau B], \quad N_\epsilon = G[\tau N - (\kappa - \tau)B], \quad B_\epsilon = -e_1,$$

where $G = \frac{1}{\sqrt{(\kappa - \tau)^2 + \tau^2}}$.

Theorem 15. If $\epsilon(s)$ is a quasi-Smarandache curve in \mathbb{G}_3 , then the quasi-curvatures are given, respectively, by

$$\begin{aligned} (K_1)_\epsilon &= E^3 K'_1 K_3 - 2E^3 K_3^3 + 2E^3 K_3^2 K_2 - E^3 K'_1 K_2 - E^3 K_3 K_2^2 \\ &\quad - E^3 K_1^2 K_3 + 2E^3 K_1 K_3^2 - E^3 K'_3 K_1 + E^3 K'_2 K_1, \\ (K_2)_\epsilon &= 0, \\ (K_3)_\epsilon &= 0, \end{aligned}$$

where $E = \frac{1}{\sqrt{(K_1 - K_3)^2 + (K_3 - K_2)^2}}$.

Corollary 16. If $\epsilon(s)$ is a Frenet–Smarandache curve TNB in \mathbb{G}_3 , then the curvature and torsion are given, respectively, by

$$\begin{aligned} \kappa_\epsilon &= G^3 \kappa' \tau - 2G^3 \tau^3 - G^3 \kappa^2 \tau + 2G^3 \kappa \tau^2 - G^3 \tau' \kappa, \\ \tau_\epsilon &= 0, \end{aligned}$$

where $G = \frac{1}{\sqrt{(\kappa - \tau)^2 + \tau^2}}$.

Corollary 17. If $\epsilon(s)$ is a Frenet–Smarandache curve TNB in \mathbb{G}_3 , then the Smarandache curve TNB is always a plane curve.

Example 1. Let $\alpha : I \rightarrow \mathbb{G}_3$ be a curve defined as

$$\alpha(s) = (s, \cos s, \sin s). \tag{43}$$

The quasi-frame of α is

$$\begin{aligned} T_\alpha &= (1, -\sin s, \cos s), \\ (N_q)_\alpha &= (0, \cos s, \sin s), \\ (B_q)_\alpha &= (0, -\sin s, \cos s). \end{aligned}$$

The quasi-curvatures are given as

$$(\kappa_1)_q = -1, \quad (\kappa_2)_q = 0, \quad (\kappa_3)_q = 1.$$

- The TN_q Smarandache curve is defined as

$$\eta(s) = T + N_q = (1, \cos s - \sin s, \sin s + \cos s).$$

The quasi-frame of η is

$$(T_q)_\eta = \frac{1}{\sqrt{2}}(0, -\sin s - \cos s, \cos s - \sin s),$$

$$(N_q)_\eta = \frac{1}{\sqrt{2}}(0, -\cos s - \sin s, \sin s + \cos s),$$

$$(B_q)_\eta = -e_1.$$

The quasi-curvatures of η are

$$(\kappa_1)_\eta = \frac{-1}{\sqrt{2}},$$

$$(\kappa_2)_\eta = 0,$$

$$(\kappa_3)_\eta = 0;$$

- The TB_q -Smarandache curve is defined as

$$\beta(s) = T + B_q = (1, -2 \sin s, 2 \cos s).$$

The quasi-frame of β is

$$(T_q)_\beta = \frac{1}{\sqrt{2}}(0, -2 \cos s, -2 \sin s),$$

$$(N_q)_\beta = (0, -\sin s, \cos s),$$

$$(B_q)_\beta = -e_1.$$

The quasi-curvatures of β are

$$(\kappa_1)_\beta = -1,$$

$$(\kappa_2)_\beta = 0,$$

$$(\kappa_3)_\beta = 0;$$

- The TN_qB_q -Smarandache curve is defined as

$$\epsilon(s) = T + N_q + B_q = (1, \cos s - 2 \sin s, \sin s + 2 \cos s).$$

The quasi-frame of ϵ is

$$T_\epsilon = \frac{1}{\sqrt{5}}(0, -\sin s - 2 \cos s, \cos s - 2 \sin s),$$

$$(N_q)_\epsilon = \frac{1}{5}(0, \cos s - 2 \sin s, \sin s + 2 \cos s),$$

$$(B_q)_\epsilon = -e_1.$$

The quasi-curvatures of ϵ are

$$(\kappa_1)_\beta = \frac{-1}{\sqrt{5}},$$

$$(\kappa_2)_\beta = 0,$$

$$(\kappa_3)_\beta = 0.$$

9. Conclusions

In this paper, firstly, the quasi-frame and its relation with the Frenet frame were investigated in G_3 . Moreover, we studied quasi-Bertrand curves in G_3 and we proved that the distance between corresponding points on two quasi-Bertrand curves in G_3 is constant, but the angle between tangent lines is not constant. Furthermore, we studied quasi-Mannheim curves in G_3 and we proved that the distance between corresponding points on two quasi-Mannheim curves in G_3 is constant, but the angle between tangents is not constant. Also, quasi-involute curves were investigated in G_3 and we proved that there are no quasi-evolute curves in G_3 . Finally, we studied the quasi-Smarandache curves in G_3 of three different types.

Author Contributions: Conceptualization, N.E. and A.E.; methodology, A.E.; software, N.E.; validation, C.C., W.E., and Y.T.; formal analysis, N.E.; investigation, A.E.; resources, C.C.; data curation, Y.T.; writing original draft preparation, A.E.; writing review and editing, N.E.; visualization, N.E.; supervision, W.E.; project administration, Y.T.; funding acquisition, W.E. All authors have read and agreed to the published version of the manuscript.

Funding: The study was funded by Researchers Supporting Project number (RSPD2023R749), King Saud University, Riyadh, Saudi Arabia.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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