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Degenerate Versions of Hypergeometric Bernoulli–Euler Polynomials

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Abstract—In this paper, we introduce degenerate versions of the hypergeometric Bernoulli and Euler polynomials. We demonstrate that they form Δ_λ–Appell sets and provide some of their algebraic properties, including inversion formulas, as well as the associated matrix formulation. Additionally, we focus our attention on the monomiality principle associated with them and determine the corresponding derivative and multiplicative operators.

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Dedicated to the memory of Professor S.N. Mergelyan
on the occasion of his 95th birthday anniversary

1. INTRODUCTION

As Choi has recently pointed out in his editorial note for the Special Issue *Recent Advances in Special Functions and Their Applications* [4]: "Due to their remarkable properties, a plethora of special functions have been crafted and harnessed across a diverse spectrum of fields spanning centuries. These functions have found their place in a variety of disciplines, including mathematics, physics (quantum mechanics, electrodynamics, thermodynamics, fluid dynamics, and solid-state physics), engineering, statistics, astronomy and astrophysics, computer science, economics and finance, chemistry, biology, geophysics, medicine, materials science, and environmental science. These are just a few examples, and special functions can find applications in various other scientific and engineering disciplines whenever complex mathematical relationships need to be described or solved".

Among this broad class special functions are the degenerate Bernoulli–Euler and the generalized degenerate Bernoulli–Euler polynomials of order $\alpha \in \mathbb{C}$, respectively. These polynomials were introduced by Leonard Carlitz in the seminal papers [2, 3] by means of the generating functions and series expansions

$$\frac{te_\lambda^x(t)}{e_\lambda(t)-1} = \sum_{n=0}^{\infty} \mathcal{B}_{n,\lambda}(x)\frac{t^n}{n!}, \tag{1}$$

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$$\frac{2e_\lambda^x(t)}{e_\lambda(t)+1}=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}, \tag{2}$$

$$\left(\frac{t}{e_\lambda(t)-1}\right)^{\alpha}e_\lambda^x(t)=\sum_{n=0}^{\infty}\mathcal{B}_{n,\lambda}^{(\alpha)}(x)\frac{t^n}{n!}, \tag{3}$$

$$\left(\frac{2}{e_\lambda(t)+1}\right)^{\alpha}e_\lambda^x(t)=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(\alpha)}(x)\frac{t^n}{n!}, \tag{4}$$

where for $\lambda, x, t \in \mathbb{R}$, the degenerate exponentials are defined as follows (cf. [20])

$$e_\lambda^x(t)=\begin{cases}(1+\lambda t)^{\frac{x}{\lambda}}, & \text{if} \quad \lambda\in\mathbb{R}\setminus\{0\}\\ e^{xt}, & \text{if} \quad \lambda=0,\end{cases}=\begin{cases}\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \quad |\lambda t|<1, & \text{if} \quad \lambda\in\mathbb{R}\setminus\{0\}\\ \sum_{n=0}^{\infty}x^n\frac{t^n}{n!}, & \text{if} \quad \lambda=0.\end{cases}$$

As usual, for $x=1$ we use the notation $e_\lambda(t)=e_\lambda^1(t)$, and the generalized falling factorials $(x)_{n,\lambda}$, are given by (cf. [18–20])

$$(x)_{n,\lambda}=\begin{cases}1, & \text{if} \quad n=0\\ \prod_{i=1}^{n}(x-(i-1)\lambda), & \text{if} \quad n\geq 1\\ 0, & \text{if} \quad n<0,\end{cases}$$

where $x, \lambda \in \mathbb{R}$ and $n \in \mathbb{Z}$.

It is clear that $e_0^x(t)=e^{xt}$, $(x)_{n,0}=x^n$, and the series expansions (1)–(4) are valid in a suitable neighborhood of $t=0$, providing degenerate versions of the classical Bernoulli and Euler polynomials, respectively.

Since their introduction, these families of special polynomials have been intensively studied. Motivated by a number of interesting and recent contributions [12, 18–20, 23–26] in which the authors provide inversion type formulas for the degenerate Bernoulli/Euler polynomials, identities of symmetry for degenerate Euler polynomials, several properties of degenerate Bernstein polynomials, degenerate differential/difference operators, matrix-inversion formulas for mixed-type hypergeometric Bernoulli–Gegenbauer polynomials, and different properties and applications for hypergeometric Bernoulli/Euler polynomials, in the present paper we consider separately degenerate versions of hypergeometric Bernoulli/Euler polynomials, and study some of their properties. More precisely, we demonstrate that they form Δ_λ-Appell sets and provide their corresponding inversion formulas, as well as the associated matrix formulation. Additionally, we focus our attention on the monomiality principle associated with these two families of special polynomials and determine their derivative and multiplicative operators.

The paper is organized as follows. Section 2 serves as a preliminary section containing the necessary definitions, notation, and terminology, including a brief summary of Δ_λ-Appell polynomials and some relevant properties of hypergeometric Bernoulli/Euler polynomials. In Section 3, we demonstrate that the sequences of hypergeometric Bernoulli/Euler polynomials are Δ_λ-Appell sequences (Theorem 1) and provide the corresponding inversion formulas (Theorem 2), as well as the associated matrix formulation (Corollary 1). Section 4 is devoted to the study of the monomiality principle associated with such special polynomials (Theorem3).

2. NOTATION AND BACKGROUND

Throughout this paper, let $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ denote, respectively, the set of all natural numbers, the set of all non-negative integers, the set of all integers, the set of all real numbers, and the set of all complex numbers. Furthermore, the convention $0^0=1$ will be adopted. Also, we denote by $\mathbb{P}_n$ the linear space of polynomials with real coefficients and degree less than or equal to n.

For $w \in \mathbb{C}$ and $k \in \mathbb{Z}$, we use the notations $w^{(k)}$ and $(w)_k$ for the rising and falling factorials, respectively, i.e.,

$$w^{(k)} = \begin{cases} 1, & \text{if} \quad k = 0 \\ \prod_{i=1}^{k}(w+i-1), & \text{if} \quad k \geq 1 \\ 0, & \text{if} \quad n < 0 \end{cases} \qquad \text{and} \qquad (w)_k = \begin{cases} 1, & \text{if} \quad k = 0 \\ \prod_{i=1}^{k}(w-i+1), & \text{if} \quad k \geq 1 \\ 0, & \text{if} \quad n < 0. \end{cases}$$

Remark. It is not difficult to see that for $n \in \mathbb{N}_0$ the set $\{1, x, x(x-\lambda), \ldots, (x)_{n,\lambda}\}$ is a Newton basis for $\mathbb{P}_n$.

Following the ideas from [6, 14], we consider the notion of Δ_λ-Appell sequences, as follows: let $I \subset \mathbb{R}$ be any interval, $f : I \to \mathbb{R}$ be a function and $\lambda \in \mathbb{R}^+$. We denote Δ_λ the finite difference operator given by $\Delta_\lambda[f](x) = f(x+\lambda) - f(x)$.

Let us define the finite difference operator of order k, with $k \in \mathbb{N}$, as

$$\Delta_\lambda^k[f](x) = \Delta_\lambda\left[\Delta_\lambda^{k-1}[f]\right](x) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f(x+j\lambda),$$

where the conventions $\Delta_\lambda^0 = I$ and $\Delta_\lambda^1 = \Delta_\lambda$ are adopted, being I the identity operator.

A sequence of polynomials $\{p_n^\lambda(x)\}_{n\geq 0} \equiv \{p_n(x)\}_{n\geq 0}$ is called Δ_λ-Appell sequence if it satisfies the following conditions

(1) $\deg(p_n(x)) = n$, for $n \geq 0$,

(2) $\Delta_\lambda\left[p_n\right](x) = n\lambda p_{n-1}(x)$, for all $n \geq 1$.

For instance, for each $\lambda \in \mathbb{R}$ the generalized $\{(x)_{n,\lambda}\}_{n\geq 0}$ falling factorials are Δ_λ-Appell sequences and they satisfy the following relation [6]

$$\Delta_\lambda^k[(x)_{n,\lambda}] = \frac{n!}{(n-k)!}(x)_{n-k,\lambda}, \quad 0 \leq k \leq n.$$

Furthermore, if we consider the degenerate exponential $e_\lambda^x(t)$ as a function of the variable x, that is $f_{t,\lambda}(x) = e_\lambda^x(t)$, then we have

$$\Delta_\lambda[e_\lambda^x](t) = \lambda t e_\lambda^x(t). \tag{5}$$

Δ_λ-Appell polynomials was considered and studied from an algebraic point of view by Costabile and Longo in [6]. More precisely, given the power series

$$a(t) = \alpha_0 + \frac{t}{1!}\alpha_1 + \frac{t^2}{2!}\alpha_2 + \cdots + \frac{t^n}{n!}\alpha_n + \cdots, \quad \alpha_0 \neq 0 \tag{6}$$

with $\alpha_i \in \mathbb{R}$, $i \in \mathbb{N}_0$, and $\{p_n(x)\}_{n\in\mathbb{N}_0}$ the sequence of Δ_λ-Appell polynomials determined by the series expansion of the product functions $a(t)(1+\lambda t)^{\frac{x}{\lambda}}$, i.e.,

$$a(t)(1+\lambda t)^{\frac{x}{\lambda}} = p_0(x) + \frac{t}{1!}p_1(x) + \frac{t^2}{2!}p_2(x) + \cdots + \frac{t^n}{n!}p_n(x) + \cdots.$$

Costabile and Longo [6] obtained that if $\{\beta_j\}_{j\in\mathbb{N}_0}$ ($\beta_0 \neq 0$) is the sequence of Taylor coefficients for the series expansion of $\frac{1}{a(t)}$, then

$$\begin{cases} p_0(x) = \frac{1}{\beta_0}, \\ p_n(x) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & (x)_1 & \cdots & (x)_{n-1} & (x)_n \\ \beta_0 & \beta_1 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} & & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \quad n = 1, 2, \ldots \end{cases} \tag{7}$$

Consequently, the polynomial characterization (7) allows to prove some known general properties, by using elementary linear algebra tools as in the setting of Appell polynomials (see, [5] for more details).

2.1. Hypergeometric Bernoulli/Euler Polynomials

For a fixed $m \in \mathbb{N}$, the hypergeometric Bernoulli polynomials are defined by means of the following generating function [11, 13, 22–24]

$$\frac{t^m e^{xt}}{e^t - \sum_{l=0}^{m-1} \frac{t^l}{l!}} = \sum_{n=0}^{\infty} B_n^{[m-1]}(x) \frac{t^n}{n!}, \quad |t| < 2\pi, \tag{8}$$

and the hypergeometric Bernoulli numbers are defined by $B_n^{[m-1]} := B_n^{[m-1]}(0)$ for all $n \geq 0$. The hypergeometric Bernoulli polynomials also are called generalized Bernoulli polynomials of level m [24, 25]. It is clear that if $m = 1$ in (8), then we recover the definition of the classical Bernoulli polynomials $B_n(x)$ and classical Bernoulli numbers, respectively, i.e., $B_n(x) = B_n^{[0]}(x)$ and $B_n = B_n^{[0]}$, respectively, for all $n \geq 0$.

The first four hypergeometric Bernoulli polynomials are as follows

$$B_0^{[m-1]}(x) = m!, \quad B_1^{[m-1]}(x) = m!\left(x - \frac{1}{m+1}\right),$$

$$B_2^{[m-1]}(x) = m!\left(x^2 - \frac{2}{m+1}x + \frac{2}{(m+1)^2(m+2)}\right),$$

$$B_3^{[m-1]}(x) = m!\left(x^3 - \frac{3}{m+1}x^2 + \frac{6}{(m+1)^2(m+2)}x + \frac{6(m-1)}{(m+1)^3(m+2)(m+3)}\right).$$

Similarly, for a fixed $m \in \mathbb{N}$, the hypergeometric Euler polynomials are defined as follows (see for instance, [26])

$$\frac{2^m e^{xt}}{e^t + \sum_{l=0}^{m-1} \frac{t^l}{\pi}} = \sum_{n=0}^{\infty} E_n^{[m-1]}(x) \frac{t^n}{n!}, \quad |t| < \pi, \tag{9}$$

and the hypergeometric Euler numbers are defined by $E_n^{[m-1]} := E_n^{[m-1]}(0)$, for all $n \geq 0$. The hypergeometric Euler polynomials also are called generalized Euler polynomials of level m [25]. Again, if $m = 1$ in (9), then we recover the definition of the classical Euler polynomials $E_n(x)$, and classical Euler numbers, respectively, i.e., $E_n(x) = E_n^{[0]}(x)$, and $\mathcal{E}_n = 2^n E_n^{[0]}\left(\frac{1}{2}\right) = 2^n E_n\left(\frac{1}{2}\right)$, respectively, for all $n \geq 0$.

For instance, if $m = 3$, then the first six hypergeometric Euler polynomials are

$$E_0^{[2]}(x) = 4, \quad E_1^{[2]}(x) = 4(x-1), \quad E_2^{[2]}(x) = 4(x-1)^2, \quad E_3^{[2]}(x) = 4x^3 - 12x^2 + 12x - 2,$$

$$E_4^{[2]}(x) = 4x^4 - 16x^3 + 24x^2 - 8x - 10, \quad E_5^{[2]}(x) = 4x^5 - 20x^4 + 40x^3 - 20x^2 - 50x + 58.$$

The following results collect some properties satisfied by these two families of polynomials (cf. [22, 26]).

Proposition 1. *For a fixed $m \in \mathbb{N}$, let $\left\{B_n^{[m-1]}(x)\right\}_{n\geq 0}$ be the hypergeometric Bernoulli polynomials. Then, the following statements hold*

a) Summation formula. For every $n \geq 0$, $B_n^{[m-1]}(x) = \sum_{k=0}^{n} \binom{n}{k} B_k^{[m-1]} x^{n-k}$.

b) Differential relations (Appell polynomial sequences). For $n, j \geq 0$ with $0 \leq j \leq n$, we have

$$\left[B_n^{[m-1]}(x)\right]^{(j)} = \frac{n!}{(n-j)!} B_{n-j}^{[m-1]}(x).$$

c) *Inversion formula [22, Equation (2.6)]. For every* $n \geq 0$,

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(m+k)!} B_{n-k}^{[m-1]}(x).$$

Consequently, the set $\left\{B_0^{[m-1]}(x), B_1^{[m-1]}(x), \ldots, B_n^{[m-1]}(x)\right\}$ *is a basis for* $\mathbb{P}_n$ (cf. [24]).

d) *Recurrence relation [22, Lemma 3.2]. For every* $n \geq 1$,

$$B_n^{[m-1]}(x) = \left(x - \frac{1}{m+1}\right) B_{n-1}^{[m-1]}(x) - \frac{1}{n(m-1)!} \sum_{k=0}^{n-2} \binom{n}{k} B_{n-k}^{[m-1]} B_k^{[m-1]}(x).$$

e) *Integral formulas*

$$\int_{x_0}^{x_1} B_n^{[m-1]}(x)dx = \frac{1}{n+1}\left[B_{n+1}^{[m-1]}(x_1) - B_{n+1}^{[m-1]}(x_0)\right]$$
$$= \sum_{k=0}^{n} \frac{1}{n-k+1} \binom{n}{k} B_k^{[m-1]}((x_1)^{n-k+1} - (x_0)^{n-k+1}).$$

$$B_n^{[m-1]}(x) = n \int_0^x B_{n-1}^{[m-1]}(t)dt + B_n^{[m-1]}.$$

f) *[22, Theorem 3.1] Differential equation. For every* $n \geq 1$*, the polynomial* $B_n^{[m-1]}(x)$ *satisfies the following differential equation*

$$\frac{B_n^{[m-1]}}{n!} y^{(n)} + \frac{B_{n-1}^{[m-1]}}{(n-1)!} y^{(n-1)} + \cdots + \frac{B_2^{[m-1]}}{2!} y''$$
$$+ (m-1)! \left(\frac{1}{m+1} - x\right) y' + n(m-1)! y = 0.$$

Proposition 2. *For a fixed* $m \in \mathbb{N}$*, let* $\left\{E_n^{[m-1]}(x)\right\}_{n\geq 0}$ *be the hypergeometric Euler polynomials. Then, the following statements hold*

a) *Summation formulas. For every* $n \geq 0$,

$$E_n^{[m-1]}(x+y) = \sum_{k=0}^{n} \binom{n}{k} y^k E_{n-k}^{[m-1]}(x) = \sum_{k=0}^{n} \binom{n}{k} E_k^{[m-1]}(y) x^{n-k}.$$

In particular,

$$E_n^{[m-1]}(x) = \sum_{k=0}^{n} \binom{n}{k} E_k^{[m-1]} x^{n-k}.$$

b) *Differential relations (Appell polynomial sequences). For* $n, j \geq 0$ *with* $0 \leq j \leq n$*, we have*

$$\left[E_n^{[m-1]}(x)\right]^{(j)} = \frac{n!}{(n-j)!} E_{n-j}^{[m-1]}(x).$$

c) *Inversion formula. For every* $n \geq 0$,

$$2^m x^n = \sum_{k=0}^{n} \binom{n}{k} (1 + a_{k,m}) E_{n-k}^{[m-1]}(x), \quad \textit{where} \quad a_{k,m} = \begin{cases} 1, & 0 \leq k < m, \\ 0, & k \geq m. \end{cases}$$

Consequently, the set $\left\{ E_0^{[m-1]}(x), E_1^{[m-1]}(x), \ldots, E_n^{[m-1]}(x) \right\}$ *is a basis for* $\mathbb{P}_n$ *(cf., [26]).*

d) *Integral formulas*

$$\int_{x_0}^{x_1} E_n^{[m-1]}(x) dx = \frac{1}{n+1} \left[E_{n+1}^{[m-1]}(x_1) - E_{n+1}^{[m-1]}(x_0) \right]$$

$$= \sum_{k=0}^{n} \frac{1}{n-k+1} \binom{n}{k} E_k^{[m-1]} \left((x_1)^{n-k+1} - (x_0)^{n-k+1} \right).$$

$$E_n^{[m-1]}(x) = n \int_0^x E_{n-1}^{[m-1]}(t) dt + E_n^{[m-1]}.$$

e) *Recurrence relation. For any* $m \geq 2$ *and* $n \geq 0$, *the following recurrence relation holds*

$$E_{n+1}^{[m-1]}(x) = \left(2x E_n^{[m-2]} - E_n^{[m-1]} \right) + \frac{1}{2^{m-1}} \sum_{k=1}^{n} \left[\left(\binom{n}{k} \right) \left(2x E_{n-k}^{[m-2]} - E_{n-k}^{[m-1]} \right) \right.$$
$$\left. -2 \left(\binom{n}{k-1} \right) E_{n-k+1}^{[m-2]} \right] E_k^{[m-1]}(x).$$

f) *Differential equation. For any* $m \geq 2$, *the hypergeometric Euler polynomials* $E_n^{[m-1]}(x)$ *satisfy the differential equation*

$$0 = \left[\frac{2}{n!} \left(E_n^{[m-2]} - 1 \right) + \frac{2x E_{n-1}^{[m-2]} - E_{n-1}^{[m-1]}}{(n-1)!} \right] y^{(n)}$$
$$+ \left[\frac{2}{(n-1)!} \left(E_{n-1}^{[m-2]} - 2 \right) + \frac{2x E_{n-2}^{[m-2]} - E_{n-2}^{[m-1]}}{(n-2)!} \right] y^{(n-1)}$$
$$+ \cdots + \left[2^{m-1}(1-x) - n + 1 + E_2^{[m-2]} \right] y'' + \left[2^{m-1}(x-2) - 2n \right] y' - n 2^{m-1} y.$$

3. DEGENERATE VERSIONS OF HYPERGEOMETRIC BERNOULLI AND EULER POLYNOMIALS

In this section, we introduce degenerate versions of the hypergeometric Bernoulli and Euler polynomials. We demonstrate that they form Δ_λ-Appell sets and present some of their algebraic properties, including inversion formulas, as well as the associated matrix formulation.

Definition 1. *For* $\lambda \in \mathbb{R} \setminus \{0\}$ *and a fixed* $m \in \mathbb{N}$, *the degenerate hypergeometric of Bernoulli and Euler polynomials are defined by means of the following generating function and series*

$$\frac{t^m \, e_\lambda^x(t)}{e_\lambda(t) - \sum_{l=0}^{m-1} (1)_{l,\lambda} \frac{t^l}{l!}} = \sum_{n=0}^{\infty} B_{n,\lambda}^{[m-1]}(x) \frac{t^n}{n!}, \quad |t| < \min \left\{ 2\pi, \frac{1}{|\lambda|} \right\} \tag{10}$$

and

$$\frac{2^m \, e_\lambda^x(t)}{e_\lambda(t) + \sum_{l=0}^{m-1} (1)_{l,\lambda} \frac{t^l}{l!}} = \sum_{n=0}^{\infty} E_{n,\lambda}^{[m-1]}(x) \frac{t^n}{n!}, \quad |t| < \min \left\{ \pi, \frac{1}{|\lambda|} \right\}. \tag{11}$$

The substitution $x = 0$ into (10) and (11) yields, respectively, the degenerate hypergeometric Bernoulli and Euler numbers, i.e., $B^{[m-1]}_{n,\lambda} := B^{[m-1]}_{n,\lambda}(0)$ and $E^{[m-1]}_{n,\lambda} := E^{[m-1]}_{n,\lambda}(0)$. Also, the following implicit summation formulas hold

$$B^{[m-1]}_{n,\lambda}(x+y) = \sum_{k=0}^{n}\binom{n}{k}(y)_{n-k,\lambda}B^{[m-1]}_{k,\lambda}(x) = \sum_{k=0}^{n}\binom{n}{k}B^{[m-1]}_{k,\lambda}(y)(x)_{n-k,\lambda},$$

$$E^{[m-1]}_{n,\lambda}(x+y) = \sum_{k=0}^{n}\binom{n}{k}(y)_{n-k,\lambda}E^{[m-1]}_{k,\lambda}(x) = \sum_{k=0}^{n}\binom{n}{k}E^{[m-1]}_{k,\lambda}(y)(x)_{n-k,\lambda}.$$

In particular,

$$B^{[m-1]}_{n,\lambda}(x) = \sum_{k=0}^{n}\binom{n}{k}(x)_{n-k,\lambda}B^{[m-1]}_{k,\lambda}, \quad E^{[m-1]}_{n,\lambda}(x) = \sum_{k=0}^{n}\binom{n}{k}(x)_{n-k,\lambda}E^{[m-1]}_{k,\lambda}.$$

Furthermore, due to the generating functions (10) and (11), the identity

$$\frac{\partial}{\partial x}e^{x}_{\lambda}(t) = \frac{\ln(1+\lambda t)}{\lambda}e^{x}_{\lambda}(t),$$

and the well-known Maclaurin series

$$\ln(1+\lambda t) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\lambda^{n}}{n}t^{n}, \quad |\lambda t| < 1,$$

it is not difficult to appropriately use the Cauchy product of series to deduce that the derivatives of degenerate hypergeometric Bernoulli and Euler polynomials satisfy the following identities:

$$\frac{d}{dx}B^{[m-1]}_{n,\lambda}(x) = \sum_{k=0}^{n}\binom{n}{k}b_{k,\lambda}B^{[m-1]}_{n-k,\lambda}(x), \tag{12}$$

$$\frac{d}{dx}E^{[m-1]}_{n,\lambda}(x) = \sum_{k=0}^{n}\binom{n}{k}b_{k,\lambda}E^{[m-1]}_{n-k,\lambda}(x), \tag{13}$$

where

$$b_{n,\lambda} = \begin{cases} 0, & \text{if} \quad n = 0, \\ (-1)^{n+1}(n+1)!\lambda^{n-1}, & \text{if} \quad n \neq 0. \end{cases}$$

Thus, from (12) and (13), these polynomials do not satisfy an Appell condition.

Theorem 1. *For a fixed $m \in \mathbb{N}$ the sequences $\left\{B^{[m-1]}_{n,\lambda}(x)\right\}_{n\geq 0}$ and $\left\{E^{[m-1]}_{n,\lambda}(x)\right\}_{n\geq 0}$ are Δ_λ-Appell sequences.*

Proof. Since $\deg\left(B^{[m-1]}_{n,\lambda}(x)\right) = \deg\left(E^{[m-1]}_{n,\lambda}(x)\right) = n$, it is suffices to prove that

$$\Delta_\lambda\left[B^{[m-1]}_{n,\lambda}\right](x) = \lambda n B^{[m-1]}_{n-1,\lambda}(x), \tag{14}$$

$$\Delta_\lambda\left[E^{[m-1]}_{n,\lambda}\right](x) = \lambda n E^{[m-1]}_{n-1,\lambda}(x), \tag{15}$$

whenever $n \geq 1$.

In order to prove the identity (14), we proceed as follows. We consider the generating function $\frac{t^m e^{x}_{\lambda}(t)}{e_\lambda(t) - \sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{t^l}{l!}}$, as a function depending on the variable x, i.e.,

$$G^{[m-1]}_{t,\lambda}(x) = \frac{t^m e^{x}_{\lambda}(t)}{e_\lambda(t) - \sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{t^l}{l!}},$$

and then, applying the finite difference operator Δ_λ to $G_{t,\lambda}^{[m-1]}$ and using (5), we get

$$\Delta_\lambda\left[G_{t,\lambda}^{[m-1]}\right](x)=\lambda t G_{t,\lambda}^{[m-1]}(x)=\lambda t\left[\frac{t^m e_\lambda^x(t)}{e_\lambda(t)-\sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{t^l}{l!}}\right].$$

Or, equivalently, $\Delta_\lambda\left[G_{t,\lambda}^{[m-1]}\right]=\lambda t G_{t,\lambda}^{[m-1]}$. Therefore,

$$\sum_{n=0}^{\infty}\Delta_\lambda\left[B_{n,\lambda}^{[m-1]}\right](x)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\lambda n B_{n-1,\lambda}^{[m-1]}(x)\frac{t^n}{n!}. \tag{16}$$

Consequently, comparing the coefficients on both sides of (16), equality (14) follows.

A similar reasoning applied to the generating function $\frac{2^m e_\lambda^x(t)}{e_\lambda(t)+\sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{t^l}{l!}}$, as a function depending on the variable x, yields (15). □

Theorem 2. *For a fixed $m\in\mathbb{N}$ the sequences $\left\{B_{n,\lambda}^{[m-1]}(x)\right\}_{n\geq 0}$ and $\left\{E_{n,\lambda}^{[m-1]}(x)\right\}_{n\geq 0}$ satisfy, respectively, the following inversion formulas*

$$(x)_{n,\lambda}=\sum_{k=0}^{n}\binom{n}{k}\frac{k!(1)_{k+m,\lambda}}{(k+m)!}B_{n-k,\lambda}^{[m-1]}(x), \tag{17}$$

$$(x)_{n,\lambda}=\frac{1}{2^m}\sum_{k=0}^{n}\binom{n}{k}a_{k,\lambda}E_{n-k,\lambda}^{[m-1]}(x), \tag{18}$$

where

$$a_{n,\lambda}=\begin{cases}2(1)_{n,\lambda}, & \text{if} \quad 0\leq n<m,\\ (1)_{n,\lambda}, & \text{if} \quad n\geq m.\end{cases}$$

Consequently, the sets

$$\left\{B_{0,\lambda}^{[m-1]}(x),B_{1,\lambda}^{[m-1]}(x),\ldots,B_{n,\lambda}^{[m-1]}(x)\right\} \quad \text{and} \quad \left\{E_{0,\lambda}^{[m-1]}(x),E_{1,\lambda}^{[m-1]}(x),\ldots,E_{n,\lambda}^{[m-1]}(x)\right\}$$

are bases for $\mathbb{P}_n$.

Proof. Using (10), we can deduce that

$$t^m e_\lambda^x(t)=\left(e_\lambda(t)-\sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{n=0}^{\infty}B_{n,\lambda}^{[m-1]}(x)\frac{t^n}{n!}\right)$$
$$=\sum_{l=m}^{\infty}(1)_{l,\lambda}\frac{t^l}{l!}\sum_{n=0}^{\infty}B_{n,\lambda}^{[m-1]}(x)\frac{t^n}{n!}=t^m\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(1)_{k+m,\lambda}}{(k+m)!(n-k)!}t^n.$$

Hence,

$$\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{k!(1)_{k+m,\lambda}}{(k+m)!}B_{n-k,\lambda}^{[m-1]}(x)\frac{t^n}{n!}, \tag{19}$$

and comparing the coefficients on both sides of (19), we obtain (17).

Analogously, (11) implies that

$$2^m e^{xt}=\left(e_\lambda(t)+\sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{n=0}^{\infty}E_{n,\lambda}^{[m-1]}(x)\frac{t^n}{n!}\right)=\sum_{n=0}^{\infty}a_{n,\lambda}\frac{t^n}{n!}\sum_{n=0}^{\infty}E_{n,\lambda}^{[m-1]}(x)\frac{t^n}{n!},$$

where

$$a_{n,\lambda}=\begin{cases}2(1)_{n,\lambda}, & \text{if} \quad 0\leq n<m,\\ (1)_{n,\lambda}, & \text{if} \quad n\geq m.\end{cases}$$

Therefore,

$$\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}=\frac{1}{2^m}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}a_{k,\lambda}E_{n-k,\lambda}^{[m-1]}(x)\frac{t^n}{n!}, \tag{20}$$

and comparing the coefficients on both sides of (20), equality (18) follows.

To conclude, by combining the fact that the set $\{1, x, x(x-\lambda), \ldots, (x)_{n,\lambda}\}$ forms a specific Newton basis for $\mathbb{P}_n$ with the inversion formulas (17) and (18), we complete the proof. □

Theorem 2 offers an alternative approach to compute degenerate hypergeometric Bernoulli and Euler polynomials using inversion formulas instead of the determinant method (6). For instance, by referring to (17), one can readily obtain and explicit representation of the first four degenerate hypergeometric Bernoulli polynomials as follows:

$$B_{0,\lambda}^{[m-1]}(x)=\frac{m!}{(1)_{m,\lambda}},\quad B_{1,\lambda}^{[m-1]}(x)=\frac{m!}{(1)_{m,\lambda}}\left[x-\frac{(1-m\lambda)}{(m+1)}\right],$$

$$B_{2,\lambda}^{[m-1]}(x)=\frac{m!}{(1)_{m,\lambda}}\left[(x)_{2,\lambda}-\frac{2(1)_{m+1,\lambda}}{(m+1)(1)_{m,\lambda}}x\right.$$
$$\left.+\frac{2(1)_{m+1,\lambda}(1-m\lambda)}{(m+1)^2(1)_{m,\lambda}}-\frac{2(1)_{m+2,\lambda}}{(m+2)(m+1)(1)_{m,\lambda}}\right],$$

$$B_{3,\lambda}^{[m-1]}(x)=\frac{m!}{(1)_{m,\lambda}}\left[(x)_{3,\lambda}-\frac{3(1)_{m+1}}{(m+1)(1)_{m,\lambda}}(x)_{2,\lambda}\right.$$
$$+\left(\frac{6(1)_{2+m,\lambda}}{(m+2)(m+1)(1)_{m,\lambda}}-\frac{6(1)_{m+1}^2}{(m+1)^2(1)_{m,\lambda}}\right)x$$
$$+\left(\frac{6(1)_{m+1,\lambda}^2}{(m+1)^3(1)_{m,\lambda}^2}-\frac{6(1)_{m+2,\lambda}}{(m+2)(m+1)^2(1)_{m,\lambda}}\right)(1-m\lambda)$$
$$\left.+\left(\frac{6(1)_{m+3,\lambda}}{(m+3)(m+2)(m+1)(1)_{m,\lambda}}-\frac{6(1)_{m+1,\lambda}(1)_{m+2,\lambda}}{(m+1)^2(m+2)(1)_{m,\lambda}^2}\right)\right].$$

Similarly, by referring to (18), one can readily obtain and explicit representation of the first four degenerate hypergeometric Euler polynomial as follows:

$$E_{0,\lambda}^{[m-1]}(x)=2^{m-1},\quad E_{1,\lambda}^{[m-1]}(x)=2^{m-1}(x-1),\quad E_{2,\lambda}^{[m-1]}(x)=2^{m-1}\left[(x-1)^2-\lambda(x-1)\right],$$
$$E_{3,\lambda}^{[m-1]}(x)=2^{m-1}\left[(x-1)^3-2\lambda^2(x-1)-6x^2+3\lambda(x+1)-3\lambda(1+\lambda)+(6x+3\lambda-3)\right].$$

From a matrix perspective, Theorem 2 yields the following corollaries.

Corollary 1. *For a fixed $m\in\mathbb{N}$ and any $n\in\mathbb{N}_0$, the matrix $\mathbf{T}_\lambda(x)=\begin{pmatrix}1 & (x)_{1,\lambda} & \cdots & (x)_{n,\lambda}\end{pmatrix}^T$ admits two representations*

$$\mathbf{T}_\lambda(x)=\mathbf{M}_\lambda^{[m-1]}\mathbf{B}_\lambda^{[m-1]}(x),\quad \mathbf{T}_\lambda(x)=\frac{1}{2^m}\mathbf{N}_\lambda^{[m-1]}\mathbf{E}_\lambda^{[m-1]}(x),$$

where $\mathbf{M}_\lambda^{[m-1]}$ and $\mathbf{N}_\lambda^{[m-1]}$ are $(n+1)\times(n+1)$ lower triangular matrices defined as

$$\mathbf{M}_\lambda^{[m-1]}=\begin{pmatrix}\frac{(1)_{m,\lambda}}{m!} & 0 & 0 & \cdots & 0\\ \frac{(1)_{1+m,\lambda}}{(1+m)!} & \frac{(1)_{m,\lambda}}{m!} & 0 & \cdots & 0\\ 2\frac{(1)_{2+m,\lambda}}{(2+m)!} & 2\frac{(1)_{1+m,\lambda}}{(1+m)!} & \frac{(1)_{m,\lambda}}{m!} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ n!\frac{(1)_{n+m,\lambda}}{(n+m)!} & n!\frac{(1)_{n+m-1,\lambda}}{(n+m-1)!} & \frac{n!}{2!}\frac{(1)_{n+m-2,\lambda}}{(n+m-2)!} & \cdots & \frac{(1)_{m,\lambda}}{m!}\end{pmatrix},$$

$$\mathbf{N}_{\lambda}^{[m-1]} = \begin{pmatrix} a_{0,\lambda} & 0 & 0 & \cdots & 0 \\ a_{1,\lambda} & a_{0,\lambda} & 0 & \cdots & 0 \\ a_{2,\lambda} & 2a_{1,\lambda} & a_{0,\lambda} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,\lambda} & \binom{n}{n-1}a_{n-1,\lambda} & \binom{n}{n-2}a_{n-2,\lambda} & \cdots & a_{0,\lambda} \end{pmatrix},$$

$$\mathbf{B}_{\lambda}^{[m-1]}(x) = \left(B_{0,\lambda}^{[m-1]}(x) \ \ B_{1,\lambda}^{[m-1]}(x) \ \ \cdots \ \ B_{n,\lambda}^{[m-1]}(x)\right)^T,$$

and

$$\mathbf{E}_{\lambda}^{[m-1]}(x) = \left(E_{0,\lambda}^{[m-1]}(x) \ \ E_{1,\lambda}^{[m-1]}(x) \ \ \cdots \ \ E_{n,\lambda}^{[m-1]}(x)\right)^T.$$

Under the hypotheses of Theorem 2, it is evident that the matrix $\mathbf{N}_{\lambda}^{[m-1]}$ is invertible. Consequently,

Corollary 2. *For a fixed $m \in \mathbb{N}$ and any $n \in \mathbb{N}_0$, the following holds*

$$\mathbf{E}_{\lambda}^{[m-1]}(x) = 2^m \left(\mathbf{N}_{\lambda}^{[m-1]}\right)^{-1} \mathbf{M}_{\lambda}^{[m-1]} \mathbf{B}_{\lambda}^{[m-1]}(x).$$

4. ASSOCIATED MONOMIALITY PRINCIPLE

It is well known that the monomiality principle is based on an abstract definition of the concept of derivative and multiplicative operators which allows us to treat different families of special polynomials as ordinary monomials. The procedure underlines a generalization of the Heisenberg–Weyl group, and many relevant properties of a broad classes of special polynomials can be conveniently framed within the context of the monomiality principle (see, for instance [8, 10]). This principle is essentially a Giuseppe Datolli modern formulation of a point of view, not only tracing back to Steffensen [27–29], but even to older studies by H.M. Jeffery (cf. [7–9] and the references therein).

The rules underlying monomiality are fairly simple and can be formulated as follows.

Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. If a couple of operators $\hat{\mathcal{D}}, \hat{\mathcal{M}}$ are such that

(a) They do exist along with a differential realization, cf. [9].

(b) They can be embedded to form Weyl algebra, namely, if the commutator is such that $[\hat{\mathcal{D}}, \hat{\mathcal{M}}] := \hat{\mathcal{D}}\hat{\mathcal{M}} - \hat{\mathcal{M}}\hat{\mathcal{D}} = \hat{1}$.

(c) It is possible to univocally define a polynomial set such that

$$p_0(x) = 1, \quad \hat{\mathcal{D}}p_0(x) = 0, \quad p_n(x) = \hat{\mathcal{M}}^n p_0(x) = \hat{\mathcal{M}}^n 1,$$

then it follows that

$$\hat{\mathcal{M}}p_n(x) = \hat{\mathcal{M}}^{n+1}1 = p_{n+1}(x), \tag{21}$$

$$\hat{\mathcal{D}}p_n(x) = \hat{\mathcal{D}}\hat{\mathcal{M}}^n 1 = np_{n-1}(x), \tag{22}$$

and the polynomials $\{p_n(x)\}_{n\geq 0}$ are called quasi-monomials.

A consequence of (21) and (22), we have that $p_n(x)$ satisfies the differential equation $\hat{\mathcal{M}}\hat{\mathcal{D}}\{p_n(x)\} = np_n(x)$, if $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ have differential realizations.

The primary objective of the monomiality principle is to identify operators for multiplication and differentiation. Additionally, in the context of the monomiality principle, we establish the following outcomes to characterize the degenerate hypergeometric Bernoulli polynomials $B_{n,\lambda}^{[m-1]}(x)$ and the degenerate hypergeometric Euler $E_{n,\lambda}^{[m-1]}(x)$ polynomials.

In what follows, we determine the derivative and multiplicative operators associated to degenerate hypergeometric Bernoulli/Euler polynomials. In order to do that we have consider some ideas from [17].

Theorem 3. *For a fixed $m \in \mathbb{N}$ the polynomials $\left\{B_{n,\lambda}^{[m-1]}(x)\right\}_{n\geq 0}$ and $\left\{E_{n,\lambda}^{[m-1]}(x)\right\}_{n\geq 0}$ are quasi-monomials with respect to the following derivative and multiplicative operators*

$$\hat{\mathcal{D}}_{\lambda}^{[m-1]} = \frac{e^{\lambda\frac{\partial}{\partial x}}-1}{\lambda}, \tag{23}$$

$$\hat{M}_{\lambda}^{[m-1]} = \frac{\lambda m}{e^{\lambda\frac{\partial}{\partial x}}-1} + \frac{x}{e^{\lambda\frac{\partial}{\partial x}}} - \frac{(e^{\frac{\partial}{\partial x}})^{1-\lambda} - \sum_{l=0}^{m-2}(1)_{l+1,\lambda}\frac{(e^{\lambda\frac{\partial}{\partial x}}-1)^l}{\lambda^l l!}}{e^{\frac{\partial}{\partial x}} - \sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{(e^{\lambda\frac{\partial}{\partial x}}-1)^l}{\lambda^l l!}}, \tag{24}$$

and

$$\hat{N}_{\lambda}^{[m-1]} = \frac{x}{e^{\lambda\frac{\partial}{\partial x}}} - \frac{(e^{\frac{\partial}{\partial x}})^{1-\lambda} + \sum_{l=0}^{m-2}(1)_{l+1,\lambda}\frac{(e^{\lambda\frac{\partial}{\partial x}}-1)^l}{\lambda^l l!}}{e^{\frac{\partial}{\partial x}} + \sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{(e^{\lambda\frac{\partial}{\partial x}}-1)^l}{\lambda^l l!}}. \tag{25}$$

Proof. Since

$$e^{\lambda\frac{\partial}{\partial x}\left(e_\lambda^x(t)\right)} = e^{\lambda\frac{\partial}{\partial x}\left(\ln(1+\lambda t)^{x/\lambda}\right)} = e^{\lambda\frac{\partial}{\partial x}\left(x\ln(1+\lambda t)^{1/\lambda}\right)} = e^{\lambda\ln(1+\lambda t)^{1/t}} = e^{\ln(1+\lambda t)} = 1+\lambda t,$$

we have $\frac{e^{\lambda\frac{\partial}{\partial x}(e_\lambda^x(t))}-1}{\lambda} = t$, which implies that $\left(\frac{e^{\lambda\frac{\partial}{\partial x}(e_\lambda^x(t))}-1}{\lambda}\right)e_\lambda^x(t) = te_\lambda^x(t)$. Or, equivalently,

$$\left(\frac{e^{\lambda\frac{\partial}{\partial x}}-1}{\lambda}\right)e_\lambda^x(t) = te_\lambda^x(t). \tag{26}$$

Let us define the derivative operator $\hat{\mathcal{D}}_{\lambda}^{[m-1]}$ as follows: $\hat{\mathcal{D}}_{\lambda}^{[m-1]} = \frac{e^{\lambda\frac{\partial}{\partial x}}-1}{\lambda}$. Thus, from (10), (11), (26), and direct calculations, it follows that

$$\hat{\mathcal{D}}_{\lambda}^{[m-1]}\,B_{n,\lambda}^{[m-1]}(x) = nB_{n-1,\lambda}^{[m-1]}(x), \quad \hat{\mathcal{D}}_{\lambda}^{[m-1]}\,E_{n,\lambda}^{[m-1]}(x) = nE_{n-1,\lambda}^{[m-1]}(x),$$

whenever $n \geq 1$.

In order to prove (24), we proceed as follows. Let us consider the generating function

$$B_{\lambda}^{[m-1]}(x,t) = \frac{t^m e_\lambda^x(t)}{e_\lambda(t) - \sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{t^l}{l!}}.$$

Then, differentiation of $B_{\lambda}^{[m-1]}(x,t)$ with respect to t, yields

$$\frac{\partial}{\partial t}B_{\lambda}^{[m-1]}(x,t) = \left(\frac{m}{t} + \frac{x}{1+\lambda t} - \frac{e_\lambda^{1-\lambda}(t) - \sum_{l=0}^{m-2}(1)_{l+1,\lambda}\frac{t^l}{l!}}{e_\lambda(t) - \sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{t^l}{l!}}\right)B_{\lambda}^{[m-1]}(x,t). \tag{27}$$

So, from differentiation with respect to t on the right hand side of (10) and (27), we can deduce that

$$\left(\frac{m}{t} + \frac{x}{1+\lambda t} - \frac{e_\lambda^{1-\lambda}(t) - \sum_{l=0}^{m-2}(1)_{l+1,\lambda}\frac{t^l}{l!}}{e_\lambda(t) - \sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{t^l}{l!}}\right)B_{\lambda}^{[m-1]}(x,t) = \sum_{n=0}^{\infty}B_{n+1,\lambda}^{[m-1]}(x)\frac{t^n}{n!}.$$

We define the multiplicative operator as follows:

$$\hat{M}_{\lambda}^{[m-1]} = \frac{m}{t} + \frac{x}{1+\lambda t} - \frac{(1+\lambda t)^{\frac{1-\lambda}{\lambda}} - \sum_{l=0}^{m-2}(1)_{l+1,\lambda}\frac{t^l}{l!}}{(1+\lambda t)^{\frac{1}{\lambda}} - \sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{t^l}{l!}}$$

$$= \frac{\lambda m}{e^{\lambda\frac{\partial}{\partial x}}-1} + \frac{x}{e^{\lambda\frac{\partial}{\partial x}}} - \frac{(e^{\frac{\partial}{\partial x}})^{1-\lambda} - \sum_{l=0}^{m-2}(1)_{l+1,\lambda}\frac{(e^{\lambda\frac{\partial}{\partial x}}-1)^l}{\lambda^l l!}}{e^{\frac{\partial}{\partial x}} - \sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{(e^{\lambda\frac{\partial}{\partial x}}-1)^l}{\lambda^l l!}}.$$

Hence, by using (10), (27), and the last formulas, we obtain

$$\sum_{n=0}^{\infty}\hat{M}_{\lambda}^{[m-1]}\,B_{n,\lambda}^{[m-1]}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}B_{n+1,\lambda}^{[m-1]}(x)\frac{t^n}{n!}, \tag{28}$$

and by comparing the coefficients on both sides of (28), we conclude that

$$\hat{M}_{\lambda}^{[m-1]}\,B_{n,\lambda}^{[m-1]}(x) = B_{n+1,\lambda}^{[m-1]}(x).$$

Finally, a similar argument can be applied for the proof of (25). □

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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