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RESEARCH ARTICLE

Geometric analysis of ruled surfaces constructed from integral curves in three-dimensional Euclidean space

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ABSTRACT

Ruled surfaces, defined by the motion of a straight line along a space curve, represent a fundamental class of surfaces in differential geometry with significant applications in engineering design, architectural modeling, and computer graphics. Despite their classical nature, the construction of ruled surfaces from integral curves, solutions to differential systems derived from Frenet frames, remains relatively unexplored in the literature. This paper presents a detailed geometric study of a new class of ruled surfaces constructed from integral curves associated with the Frenet frame of regular space curves with positive curvature. We focus on surfaces whose base curves are given by the integral binormal and integral normal curves of a given spatial curve. Explicit expressions for the fundamental forms, curvature properties, and striction curves are derived for six distinct types of surfaces. Necessary and sufficient conditions under which these surfaces are minimal or developable are established. A numerical example illustrates the theoretical results, highlighting potential applications in geometric modeling. This work extends the theory of ruled surfaces in differential geometry by introducing families based on integral curves and providing a complete geometric characterization via fundamental forms and curvature analysis.

1. Introduction

The geometric study of surfaces in three-dimensional Euclidean space $\mathbb{E}^3$ forms a fundamental pillar of classical differential geometry, with far-reaching implications across mathematics, physics, engineering, and computational visualization. Since the pioneering work of Do Carmo[1] researchers have systematically investigated surface properties through curvature metrics, fundamental forms, and parametric representations. Among the diverse surface classifications, ruled surfaces stand out for their elegant construction and practical utility in architectural design, mechanical engineering, and computer-aided geometric modeling.[2]

A ruled surface emerges from the continuous motion of a straight line (called the ruling or generator) along a space curve (known as the directrix or base curve). Mathematically, we can express such surfaces parametrically as:

$$\Sigma(t,u) = \gamma(t) + u\mathrm{D}(t),$$

where $\gamma(t)$ represents the base curve and $\mathrm{D}(t)$ denotes the direction vector of the ruling.[3,4] The geometric characteristics of ruled surfaces are intimately connected to the behavior of their generators and the properties of the base curve. Particularly noteworthy are developable ruled surfaces (characterized by vanishing Gaussian curvature) and minimal surfaces (with zero mean curvature),

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which have attracted substantial attention in both Euclidean and Minkowski geometries.[5,6]

The classification and examination of ruled surfaces frequently involve analyzing special curves associated with their formation. Integral curves—solutions to differential equation systems defined by vector fields—provide a natural framework for constructing interesting base curves.[7,8] Specifically, integral curves derived from the Frenet frame (comprising the tangent, normal, and binormal vectors of a space curve) offer a powerful approach for generating and analyzing ruled surfaces with distinctive geometric properties,[9] The Frenet triad $\{\mathrm{T}, \mathrm{N}, \mathrm{B}\}$ and its associated curvature κ and torsion τ furnish a complete characterization of a curve's local geometry, making them particularly suitable for creating ruled surfaces with controlled geometric behavior.[10]

Recent investigations have explored various aspects of ruled surfaces generated by special curves. Ali and colleagues[11] examined ruled surfaces produced by special curves in Euclidean space, while Almoneef and Abdel-Baky[12] analyzed families of constant axis ruled surfaces in Minkowski space. Computational approaches to ruled surfaces have been investigated by Buse et al.[13] and Peternel et al.,[14] with geometric modeling applications explored by Andradas et al.[2] The study of rational ruled surfaces and their offsets has been advanced by Pottmann, Lu, and Ravani,[15] while Solouma et al.[16] characterized imbricate-ruled surfaces via rotation-minimizing Darboux frames in Minkowski space. Turgut and Hacisalihoglu[17] contributed to our understanding of timelike ruled surfaces in Minkowski geometry. Collision detection for ruled surfaces has been addressed by Chen et al.,[18] and visualization techniques were examined by Li et al.[19] Additional recent works on ruled surfaces can be found in[20–28]

In this work, we conduct a systematic investigation of ruled surfaces generated by integral curves in Euclidean space, concentrating specifically on integral binormal and integral normal curves serving as base curves. Our methodology extends the research of Elsharkawy and Baizeed[9] concerning integral curves and quasi-frames, generalizing their findings to the construction and examination of ruled surfaces. We utilize a new frame apparatus to derive explicit formulations for fundamental forms, Gaussian curvature, mean curvature, and geodesic torsion of these surfaces. Our findings generalize some previous work on special ruled surfaces while offering new perspectives on their geometric characteristics.

The manuscript is structured as follows: Section 2 provides essential preliminary concepts from differential geometry, including Frenet frame formulations and surface fundamental forms. Section 3 explores ruled surfaces with integral binormal curves as base curves, examining their tangent, normal, and binormal ruled surfaces. Section 4 studies ruled surfaces generated by integral normal curves, deriving their geometric invariants and special attributes. Section 5 analyzes several geometrically significant special cases where the surfaces exhibit simplified or interesting behavior. Section 6 presents a detailed numerical example illustrating the theoretical results. Finally, Section 7 summarizes our principal findings and outlines potential applications in geometric modeling and computer graphics, linking our theoretical results to practical implementations.
Our investigation advances the study of specialized surfaces in differential geometry by:

- Establishing new families of ruled surfaces based on integral curves.
- Delivering comprehensive geometric characterizations through fundamental forms and curvature analysis.
- Determining criteria for developability and minimality.
- Connecting these theoretical considerations to practical geometric modeling applications.

The findings presented here extend and consolidate previous research on ruled surfaces,[5,29] integral curves,[7,8] and their geometric properties,[30] while suggesting new avenues for future investigations in this domain. For more information about Minkowsi space[31–35] and Galilean space, see the cited references.[36–40]

2. Preliminaries

Let $\mathbb{E}^3$ represent three-dimensional Euclidean space equipped with the standard metric defined as $\langle \cdot,\cdot \rangle = dx^2 + dy^2 + dz^2$, where (x, y, z) denotes a coordinate system in $\mathbb{E}^3$. Consider a curve $\Gamma : I \subset \mathbb{R} \to \mathbb{E}^3$, where I is an interval on the real line.[1,3] A curve Γ is called a unit speed curve if parameterized by arc-length s, satisfying $\langle \Gamma'(s), \Gamma'(s)\rangle = 1$. The inner product in $\mathbb{E}^3$ is expressed as:

$$\langle \mathrm{U}, \mathrm{V}\rangle = u_1v_1 + u_2v_2 + u_3v_3, \tag{1}$$

for vectors $\mathrm{U} = (u_1, u_2, u_3)$ and $\mathrm{V} = (v_1, v_2, v_3) \in \mathbb{E}^3$.[41]

Let $\{\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s), \kappa(s), \tau(s)\}$ denote the Frenet–Serret apparatus. The collection $\{\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s)\}$ forms a moving frame along curve Γ, referred to as the TNB frame, where T(s), N(s), and B(s) represent the tangent, normal, and binormal vector fields, respectively.[1] The functions $\kappa(s)$ and $\tau(s)$ correspond to the curvature and torsion of curve Γ, respectively. These frame vectors are mutually orthogonal and normalized, satisfying:

$$\langle \mathrm{T}(s), \mathrm{T}(s)\rangle = \langle \mathrm{N}(s), \mathrm{N}(s)\rangle = \langle \mathrm{B}(s), \mathrm{B}(s)\rangle = 1,$$
$$\langle \mathrm{T}(s), \mathrm{N}(s)\rangle = \langle \mathrm{T}(s), \mathrm{B}(s)\rangle = \langle \mathrm{N}(s), \mathrm{B}(s)\rangle = 0. \tag{2}$$

These vectors are defined as:

$$\mathrm{T}(s) = \Gamma'(s), \quad \mathrm{N}(s) = \frac{\Gamma''(s)}{\|\Gamma''(s)\|}, \quad \mathrm{B}(s) = \mathrm{T}(s)\times\mathrm{N}(s), \tag{3}$$

where prime indicates differentiation with respect to s.

The Frenet–Serret equations for curve Γ are given in matrix form as:[10]

$$\begin{pmatrix}\mathrm{T}'(s)\\ \mathrm{N}'(s)\\ \mathrm{B}'(s)\end{pmatrix} = \begin{pmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{pmatrix}\begin{pmatrix}\mathrm{T}(s)\\ \mathrm{N}(s)\\ \mathrm{B}(s)\end{pmatrix}. \tag{4}$$

Curvature and torsion can be computed using:

$$\kappa(s) = \langle \mathrm{T}'(s), \mathrm{N}(s)\rangle = -\langle \mathrm{N}'(s), \mathrm{T}(s)\rangle = \|\mathrm{T}'(s)\|,$$
$$\tau(s) = \langle \mathrm{N}'(s), \mathrm{B}(s)\rangle = -\langle \mathrm{B}'(s), \mathrm{N}(s)\rangle. \tag{5}$$

A ruled surface Σ can be described as a surface generated by a straight line L moving through space. Let $\gamma(t)$ denote a smooth curve in three-dimensional Euclidean space and let D(t) represent the direction vector of line L. The parametric representation of ruled surface Σ is expressed as

$$\Sigma(t,u) = \gamma(t) + u\mathrm{D}(t),$$

where $\gamma(t)$ serves as the base curve of the surface.[4,13] The striction line and distribution parameter for ruled surface Σ are defined as:

$$\gamma^*(t) = \gamma(t) + \frac{\langle v_p(t), \mathrm{D}'(t)\rangle}{\|\mathrm{D}'(t)\|^2}\mathrm{D}(t), \tag{6}$$

and

$$\delta(t) = \frac{\det[v_p(t), \mathrm{D}(t), \mathrm{D}'(t)]}{\|\mathrm{D}'(t)\|^2}, \tag{7}$$

where $v_p(t)$ is the unit tangent vector field along curve $\gamma(t)$. The ruled surface Σ is developable if and only if $\delta(t) = 0$. If $\|\mathrm{D}'(t)\| = 0$, the ruled surface has no striction curve, indicating a cylindrical surface where the base curve can be considered a striction curve.[11,29]

The standard unit normal vector field N on surface Σ is defined by:

$$N = \frac{\Sigma_t \times \Sigma_u}{\|\Sigma_t \times \Sigma_u\|}, \tag{8}$$

where Σ_t and Σ_u denote partial derivatives of $\Sigma(t,u)$ with respect to t and u, respectively. The first fundamental form, second fundamental form, and third fundamental form of surface $\Sigma(t,u)$ are given by:

$$I = E(dt)^2 + 2F\,dt\,du + G(du)^2, \tag{9}$$
$$II = L(dt)^2 + 2M\,dt\,du + N(du)^2, \tag{10}$$
$$III = e(dt)^2 + 2f\,dt\,du + g(du)^2, \tag{11}$$

where

$$E = \langle \Sigma_t, \Sigma_t\rangle, \quad F = \langle \Sigma_t, \Sigma_u\rangle, \quad G = \langle \Sigma_u, \Sigma_u\rangle,$$
$$L = \langle \Sigma_{tt}, N\rangle, \quad M = \langle \Sigma_{tu}, N\rangle, \quad N = \langle \Sigma_{uu}, N\rangle,$$
$$e = \langle N_t, N_t\rangle, \quad f = \langle N_t, N_u\rangle, \quad g = \langle N_u, N_u\rangle.$$

Gaussian curvature K and mean curvature H are expressed as:

$$K = \frac{LN - M^2}{EG - F^2}, \quad H = \frac{EN - 2MF + GL}{2(EG - F^2)}. \tag{12}$$

The geodesic curvature κ_g, normal curvature κ_n, and torsion τ_g associated with curve $\gamma(t)$ on surface Σ are computed as:

$$\kappa_g = \langle N(t) \times v_p(t), v_p'(t)\rangle, \tag{13}$$
$$\kappa_n = \langle N(t), \gamma''(t)\rangle, \tag{14}$$
$$\tau_g = \langle N \times N', v_p'(t)\rangle, \tag{15}$$

where N indicates the unit normal vector along curve $\gamma(t)$ and v_p is the unit tangent vector of $\gamma(t)$.

Definition 1. *Let $\gamma(t)$ be a regular curve lying on a surface $\Sigma(t,u)$.*

(i) The Curve $\gamma(t)$ is classified as a geodesic curve if its geodesic curvature vanishes.

(ii) The Curve $\gamma(t)$ is considered an asymptotic line if its normal curvature vanishes.

(iii) The curve $\gamma(t)$ is called a line of curvature *(or* principal line*) if its geodesic torsion vanishes, i.e., $\tau_g = 0$. Equivalently, the tangent vector of $\gamma(t)$ coincides everywhere with a principal direction of the surface.*

Definition 2. *(i) A regular surface is termed flat (or developable) if its Gaussian curvature is identically zero.*

(ii) A regular surface is referred to as a minimal surface if its mean curvature is identically zero.

Theorem 1. *Let $\Sigma(t,u)$ be a surface, then II is the second fundamental form and $L_{ij} = \begin{pmatrix}L_{11} & L_{12}\\ L_{12} & L_{22}\end{pmatrix}$, then*

- *If $\det L_{ij} = 0$, then all points in $\Sigma(t,u)$ are parabolic points.*

- *If* $\det L_{ij} > 0$*, then all points in* $\Sigma(t,u)$ *are elliptic points.*
- *If* $\det L_{ij} < 0$*, then all points in* $\Sigma(t,u)$ *are hyperbolic points.*

Throughout this article, we make the following assumptions on the base curve $\alpha(s)$:

i The curve $\alpha(s)$ is a regular curve parameterized by arc-length.
ii The curvature satisfies $\kappa_\alpha(s) > 0$ for all s in the domain of interest.
iii Both curvature $\kappa_\alpha(s)$ and torsion $\tau_\alpha(s)$ are C^2 (twice continuously differentiable).

These assumptions ensure that:

- The Frenet frame $\{\mathrm{T}_\alpha, \mathrm{N}_\alpha, \mathrm{B}_\alpha\}$ is well-defined at every point.
- Integral curves $\Omega(s) = \int \mathrm{B}_\alpha\, ds$ and $\mu(s) = \int \mathrm{N}_\alpha\, ds$ possess meaningful geometric structure.
- All derived formulas involving ratios such as $\tau_\alpha/\kappa_\alpha$ and their derivatives are non-singular.

When we refer to limiting cases such as $\kappa_\alpha = 0$ in subsequent corollaries, we mean the limiting behavior as $\kappa_\alpha \to 0^+$, recognizing that the Frenet frame becomes undefined at $\kappa_\alpha = 0$ (corresponding to straight lines).

3. Ruled surfaces of integral binormal curves

This section examines tangent, normal, and binormal ruled surfaces derived from integral binormal curves. The construction parallels that of surfaces based on integral binormal curves, but with distinct geometric characteristics due to the different nature of the base curves. Integral binormal curves, defined as integrals of the binormal vector field of a base curve $\alpha(s)$, exhibit unique curvature properties that influence the resulting ruled surfaces. We systematically derive the fundamental forms, curvature properties, and special geometric attributes for three types of ruled surfaces: tangent, binormal, and normal surfaces. The results demonstrate how the curvature and torsion of the original curve $\alpha(s)$ propagate through the integral construction to determine the geometric invariants of the generated surfaces.

3.1. Tangent ruled surfaces of integral binormal curves

We introduce tangent ruled surfaces based on the Frenet frame with integral binormal curves in this subsection. We compute the surface's fundamental forms and examine curvature properties.

Definition 3. *Let* $\Omega(s)$ *be the integral binormal curve, and let* Ξ^1 *be a tangent ruled surface generated by integral binormal curve* $\Omega(s)$ *according to the Frenet frame.*
The parametric description of tangent ruled surface Ξ^1 *is given by:*

$$\Xi^1(s,\nu) = \Omega(s) + \nu \mathrm{T}_\Omega(s), \tag{16}$$

where $\Omega(s) = \int \mathrm{B}_\alpha\, \mathrm{d}s$ *denotes the base curve.*

Theorem 2. *If* $\Xi^1(s,\nu)$ *is a tangent ruled surface generated by integral binormal curve* $\Omega(s)$*, then the striction line and distribution parameter are given by:*

$$\Omega^*_{\Xi^1}(s) = \Omega(s),$$
$$\delta_{\Xi^1}(s) = 0.$$

Proof. Let $\Xi^1(s,\nu)$ be a tangent ruled surface generated by integral binormal curve $\Omega(s)$. We prove this theorem in two parts:

Part 1: The striction curve is defined by:

$$\Omega^*_{\Xi^1}(s) = \Omega(s) + \frac{\langle \mathrm{T}_\Omega, \mathrm{T}'_\Omega\rangle}{\|\mathrm{T}'_\Omega\|^2}\mathrm{T}_\Omega;$$

hence,

$$\Omega^*_{\Xi^1}(s) = \Omega(s) + \frac{\langle \mathrm{B}_\alpha, \tau_\alpha \mathrm{N}_\alpha\rangle}{\tau_\alpha^2}\mathrm{B}_\alpha.$$

Therefore, we obtain:

$$\Omega^*_{\Xi^1}(s) = \Omega(s).$$

Part 2: The distribution parameter is defined by:

$$\delta_{\Xi^1}(s) = \frac{\det[\mathrm{T}_\Omega, \mathrm{T}_\Omega, \mathrm{T}'_\Omega]}{\|\mathrm{T}'_\Omega\|^2}.$$

Hence, we have:

$$\delta_{\Xi^1}(s) = \frac{\det[\mathrm{B}_\alpha, \mathrm{B}_\alpha, \mathrm{B}'_\alpha]}{\tau_\alpha^2} = \frac{\det[\mathrm{B}_\alpha, \mathrm{B}_\alpha, -\tau_\alpha \mathrm{N}_\alpha]}{\tau_\alpha^2}.$$

Therefore,

$$\delta_{\Xi^1}(s) = 0.$$

Theorem 3. *Let* $\Xi^1(s,\nu)$ *be the tangent ruled surface generated by integral binormal curve* $\Omega(s)$*, then the first, second, and third fundamental forms of* $\Xi^1(s,\nu)$ *are given by:*

$$I = 1 + (\nu\tau_\alpha)^2(ds)^2 + 2\,ds\,d\nu + (d\nu)^2, \tag{17}$$
$$II = -\nu\tau_\alpha\kappa_\alpha(ds)^2, \tag{18}$$
$$III = \kappa_\alpha^2(ds)^2. \tag{19}$$

Proof. Let $\Xi^1(s,\nu) = \Omega(s) + \nu\mathrm{T}_\Omega(s)$, where $\mathrm{T}_\Omega = \mathrm{B}_\alpha$.

The partial derivatives of $\Xi^1(s,\nu)$ are given by:

$$\Xi^1_s = \Omega'(s) + \nu \mathrm{T}'_\Omega = \mathrm{B}_\alpha + \nu\tau_\alpha(-\mathrm{N}_\alpha) = -\nu\tau_\alpha \mathrm{N}_\alpha + \mathrm{B}_\alpha,$$

$$\Xi^1_\nu = \mathrm{B}_\alpha,$$

$$\begin{aligned}\Xi^1_{ss} &= \mathrm{B}'_\alpha + \nu(\tau_\alpha \mathrm{N}_\alpha)' \\ &= -\tau_\alpha \mathrm{N}_\alpha + \nu[\tau'_\alpha \mathrm{N}_\alpha + \tau_\alpha \mathrm{N}'_\alpha] \\ &= -\tau_\alpha \mathrm{N}_\alpha + \nu\tau'_\alpha(-\mathrm{N}_\alpha) + \nu\tau_\alpha(-\kappa_\alpha \mathrm{T}_\alpha + \tau_\alpha \mathrm{B}_\alpha) \\ &= \nu\tau_\alpha\kappa_\alpha \mathrm{T}_\alpha - (\tau_\alpha + \nu\tau'_\alpha)\mathrm{N}_\alpha - \nu\tau_\alpha^2 \mathrm{B}_\alpha,\end{aligned}$$

$$\Xi^1_{s\nu} = -\tau_\alpha \mathrm{N}_\alpha, \quad \Xi^1_{\nu\nu} = 0.$$

The unit normal vector can be obtained as:

$$\begin{aligned}\Xi^1_s \times \Xi^1_\nu &= (-\nu\tau_\alpha \mathrm{N}_\alpha + \mathrm{B}_\alpha) \times \mathrm{B}_\alpha = -\nu\tau_\alpha(\mathrm{N}_\alpha \times \mathrm{B}_\alpha) \\ &= -\nu\tau_\alpha \mathrm{T}_\alpha.\end{aligned}$$

Thus, ${}^{\Xi^1}N = -\mathrm{T}_\alpha$, with derivatives:

$${}^{\Xi^1}N_s = -\mathrm{T}'_\alpha = -\kappa_\alpha \mathrm{N}_\alpha, \quad {}^{\Xi^1}N_\nu = 0.$$

So, the first fundamental form coefficients are:

$$E = \langle \Xi^1_s, \Xi^1_s\rangle = \nu^2\tau_\alpha^2\langle \mathrm{N}_\alpha, \mathrm{N}_\alpha\rangle + \langle \mathrm{B}_\alpha, \mathrm{B}_\alpha\rangle = 1 + \nu^2\tau_\alpha^2,$$

$$F = \langle \Xi^1_s, \Xi^1_\nu\rangle = \langle \mathrm{B}_\alpha, \mathrm{B}_\alpha\rangle = 1,$$

$$G = \langle \Xi^1_\nu, \Xi^1_\nu\rangle = 1.$$

Hence, $I = (1 + \nu^2\tau_\alpha^2)(ds)^2 + 2\,ds\,d\nu + (d\nu)^2$.

The second fundamental form coefficients are:

$$\begin{aligned}L &= \langle \Xi^1_{ss}, N\rangle = \langle \nu\tau_\alpha\kappa_\alpha \mathrm{T}_\alpha - (\tau_\alpha + \nu\tau'_\alpha)\mathrm{N}_\alpha - \nu\tau_\alpha^2 \mathrm{B}_\alpha, -mathrm{T}_\alpha\rangle \\ &= -\nu\tau_\alpha\kappa_\alpha,\end{aligned}$$

$$M = \langle \Xi^1_{s\nu}, N\rangle = \langle -\tau_\alpha \mathrm{N}_\alpha, -\mathrm{T}_\alpha\rangle = 0,$$

$$N = \langle \Xi^1_{\nu\nu}, N\rangle = 0.$$

Therefore, $II = -\nu\tau_\alpha\kappa_\alpha(ds)^2$.

$${}^{\Xi^1}N_s = \frac{d}{ds}(-\mathrm{T}_\alpha) = -\mathrm{T}'_\alpha = -\kappa_\alpha \mathrm{N}_\alpha.$$

Since $N = -\mathrm{T}_\alpha$ depends only on s and not on ν:

$${}^{\Xi^1}N_\nu = \frac{\partial}{\partial\nu}(-\mathrm{T}_\alpha) = 0.$$

The coefficients e, f, g can be obtained by:

$$\begin{aligned}e &= \langle N_s, N_s\rangle = \langle -\kappa_\alpha \mathrm{N}_\alpha, -\kappa_\alpha \mathrm{N}_\alpha\rangle \\ &= \kappa_\alpha^2\langle \mathrm{N}_\alpha, \mathrm{N}_\alpha\rangle = \kappa_\alpha^2,\end{aligned}$$

$$f = \langle N_s, N_\nu\rangle = \langle -\kappa_\alpha \mathrm{N}_\alpha, 0\rangle = 0,$$

$$g = \langle N_\nu, N_\nu\rangle = \langle 0, 0\rangle = 0.$$

Therefore,

$$III = \kappa_\alpha^2(ds)^2.$$

Theorem 4. *If $\Xi^1(s,\nu)$ is a tangent ruled surface generated by integral binormal curve $\Omega(s)$, then mean curvature and Gaussian curvature are:*

$$H_{\Xi^1(s,\nu)} = \frac{-\kappa_\alpha}{2\nu\tau_\alpha}, \quad K_{\Xi^1(s,\nu)} = 0.$$

Proof. Using the formulas for mean and Gaussian curvature:

$$K = \frac{LN - M^2}{EG - F^2} = \frac{(-\nu\tau_\alpha\kappa_\alpha)(0) - (0)^2}{((\nu\tau_\alpha)^2 + 1)(1) - (1)^2} = 0,$$

$$\begin{aligned}H &= \frac{EN - 2MF + GL}{2(EG - F^2)} \\ &= \frac{((\nu\tau_\alpha)^2 + 1)(0) - 2(0)(1) + (1)(-\nu\tau_\alpha\kappa_\alpha)}{2(((\nu\tau_\alpha)^2 + 1)(1) - (1)^2)} \\ &= \frac{-\nu\tau_\alpha\kappa_\alpha}{2(\nu\tau_\alpha)^2} = \frac{-\kappa_\alpha}{2\nu\tau_\alpha}.\end{aligned}$$

If $\Xi^1(s,\nu)$ is a tangent ruled surface generated by integral binormal curve $\Omega(s)$, then all points in $\Xi^1(s,\nu)$ are parabolic points.

Proof. Since $\det L_{ij} = LN - M^2 = 0$, then all points are parabolic.

If $\Xi^1(s,\nu)$ is a tangent ruled surface generated by the integral binormal curve $\Omega(s)$, then the normal vector, the geodesic curvatures, and the geodesic torsion of the integral binormal curve $\Omega(s)$ on the surface are:

$$\kappa_{n_{\Xi^1}} = 0, \quad \kappa_{g_{\Xi^1}} = -\tau_\alpha, \quad \tau_{g_{\Xi^1}} = 0.$$

Proof. For the base curve $\Omega(s)$ on the surface:

$$\begin{aligned}\kappa_{n_{\Xi^1}} &= \langle N, \Omega''(s)\rangle = \langle -\mathrm{T}_\alpha, \mathrm{T}'_\Omega\rangle = \langle -\mathrm{T}_\alpha, \kappa_\Omega \mathrm{N}_\Omega\rangle \\ &= \langle -\mathrm{T}_\alpha, \tau_\alpha(-\mathrm{N}_\alpha)\rangle = \tau_\alpha\langle \mathrm{T}_\alpha, \mathrm{N}_\alpha\rangle = 0, \\ \kappa_{g_{\Xi^1}} &= \langle N \times v_p, v'_p\rangle = \langle (-\mathrm{T}_\alpha) \times \mathrm{T}_\Omega, \mathrm{T}'_\Omega\rangle \\ &= \langle -\mathrm{T}_\alpha \times \mathrm{B}_\alpha, \tau_\alpha(-\mathrm{N}_\alpha)\rangle = \tau_\alpha\langle \mathrm{N}_\alpha, -\mathrm{N}_\alpha\rangle = -\tau_\alpha, \\ \tau_{g_{\Xi^1}} &= \langle N \times N', v'_p\rangle = \langle (-\mathrm{T}_\alpha) \times (\kappa_\alpha \mathrm{N}_\alpha), \tau_\alpha(-\mathrm{N}_\alpha)\rangle \\ &= -\tau_\alpha\kappa_\alpha\langle -\mathrm{B}_\alpha, -\mathrm{N}_\alpha\rangle = -\tau_\alpha\kappa_\alpha\langle \mathrm{B}_\alpha, \mathrm{N}_\alpha\rangle = 0.\end{aligned}$$

If $\Xi^1(s,\nu)$ is a tangent ruled surface generated by integral binormal curve $\Omega(s)$, then:

- $\Xi^1(s,\nu)$ is a developable surface everywhere.
- $\Xi^1(s,\nu)$ is a minimal surface everywhere.
- The base curve $\Omega(s)$ is a geodesic curve on the surface $\Xi^1(s,\nu)$ if $\tau_\alpha = 0$, i.e., the curve $\alpha(s)$ is a plane curve.

Proof. Given that $K = 0$ everywhere, the surface is developable. Since $H = \frac{-\kappa_\alpha}{2\nu\tau_\alpha}$ and this is not identically zero, the surface is minimal only when $\kappa_\alpha = 0$.

3.2. Normal ruled surfaces of integral binormal curves

We introduce normal ruled surfaces based on the Frenet frame with integral binormal curves in this subsection.

Definition 4. *Let $\Omega(s)$ be the integral binormal curve and let Ξ^2 be a normal ruled surface generated by the integral binormal curve $\Omega(s)$ according to the Frenet frame.*
The parametric description of normal ruled surface Ξ^2 is given by:

$$\Xi^2(s,\nu) = \Omega(s) + \nu \mathrm{N}_\Omega(s), \tag{20}$$

where $\Omega(s) = \int \mathrm{B}_\alpha \, \mathrm{d}s$ denotes the base curve.

Theorem 5. *If $\Xi^2(s,\nu)$ is a normal ruled surface generated by integral binormal curve $\Omega(s)$, then the striction line and distribution parameter are given by:*

$$\Omega^*_{\Xi^2}(s) = \Omega(s) + \frac{\tau_\alpha}{\kappa_\alpha^2 + \tau_\alpha^2}\mathrm{N}_\alpha,$$
$$\delta_{\Xi^2} = \frac{\kappa_\alpha}{\kappa_\alpha^2 + \tau_\alpha^2}.$$

Proof. For the normal ruled surface $\Xi^2(s,\nu)$:

$$\begin{aligned}
\Omega^*_{\Xi^2}(s) &= \Omega(s) + \frac{\langle \mathrm{T}_\Omega, \mathrm{N}'_\alpha\rangle}{\|\mathrm{N}'_\Omega\|^2}\mathrm{N}_\Omega \\
&= \Omega(s) + \frac{\langle \mathrm{B}_\alpha, \kappa_\alpha \mathrm{T}_\alpha - \tau_\alpha \mathrm{B}_\alpha\rangle}{\kappa_\alpha^2 + \tau_\alpha^2}(-\mathrm{N}_\alpha) \\
&= \Omega(s) + \frac{\tau_\alpha}{\kappa_\alpha^2 + \tau_\alpha^2}\mathrm{N}_\alpha, \\
\delta_{\Xi^2} &= \frac{\det[\mathrm{T}_\Omega, \mathrm{N}_\Omega, \mathrm{N}'_\Omega]}{\|\mathrm{N}'_\Omega\|^2} \\
&= \frac{\det[\mathrm{B}_\alpha, -\mathrm{N}_\alpha, \kappa_\alpha \mathrm{T}_\alpha - \tau_\alpha \mathrm{B}_\alpha]}{\kappa_\alpha^2 + \tau_\alpha^2} \\
&= \frac{\kappa_\alpha}{\kappa_\alpha^2 + \tau_\alpha^2}.
\end{aligned}$$

Theorem 6. *Let $\Xi^2(s,\nu)$ be the normal ruled surface generated by integral binormal curve $\Omega(s)$, then the first, second, and third fundamental forms of $\Xi^2(s,\nu)$ are given by:*

$$I = \beta(ds)^2 + (d\nu)^2,$$
$$II = \frac{\nu\gamma(ds)^2 + 2\kappa_\alpha \, dsd\nu}{\sqrt{\beta}},$$
$$III = \frac{(\nu^2\gamma^2 + \kappa_\alpha^4\beta)(ds)^2 + 2\nu\kappa_\alpha\gamma dsd\nu + \kappa_\alpha^2(d\nu)^2}{\beta^2},$$

where

$$\beta = (\nu\kappa_\alpha)^2 + [1 - \nu\tau_\alpha]^2,$$
$$\gamma = \kappa'_\alpha + \nu\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'.$$

Proof. Let $\Xi^2(s,\nu) = \Omega(s) + \nu\mathrm{N}_\Omega(s)$, where $\mathrm{N}_\Omega = -\mathrm{N}_\alpha$.

The partial derivatives of the surface $\Xi^2(s,\nu)$ are given by:

$$\begin{aligned}
\Xi^2_s &= \mathrm{B}_\alpha + \nu\mathrm{N}'_\Omega = \mathrm{B}_\alpha + \nu(\kappa_\alpha\mathrm{T}_\alpha - \tau_\alpha\mathrm{B}_\alpha) \\
&= \nu\kappa_\alpha\mathrm{T}_\alpha + (1 - \nu\tau_\alpha)\mathrm{B}_\alpha, \\
\Xi^2_\nu &= -\mathrm{N}_\alpha, \\
\Xi^2_{ss} &= \nu\kappa'_\alpha\mathrm{T}_\alpha + \nu\kappa_\alpha\mathrm{T}'_\alpha + (1 - \nu\tau_\alpha)'\mathrm{B}_\alpha + \\
&\quad (1 - \nu\tau_\alpha)\mathrm{B}'_\alpha \\
&= \nu\kappa'_\alpha\mathrm{T}_\alpha + \nu\kappa_\alpha^2\mathrm{N}_\alpha - \nu\tau'_\alpha\mathrm{B}_\alpha - (1 - \nu\tau_\alpha)\tau_\alpha\mathrm{N}_\alpha \\
&= \nu\kappa'_\alpha\mathrm{T}_\alpha + [\nu\kappa_\alpha^2 - \tau_\alpha(1 - \nu\tau_\alpha)]\mathrm{N}_\alpha - \nu\tau'_\alpha\mathrm{B}_\alpha, \\
\Xi^2_{s\nu} &= \kappa_\alpha\mathrm{T}_\alpha - \tau_\alpha\mathrm{B}_\alpha, \quad \Xi^2_{\nu\nu} = 0.
\end{aligned}$$

The unit normal is given by:

$$\begin{aligned}
\Xi^2_s \times \Xi^2_\nu &= [\nu\kappa_\alpha\mathrm{T}_\alpha + (1 - \nu\tau_\alpha)\mathrm{B}_\alpha] \times (-\mathrm{N}_\alpha) \\
&= \nu\kappa_\alpha(\mathrm{T}_\alpha \times (-\mathrm{N}_\alpha)) + (1 - \nu\tau_\alpha)(\mathrm{B}_\alpha \times (-\mathrm{N}_\alpha)) \\
&= -\nu\kappa_\alpha\mathrm{B}_\alpha + (1 - \nu\tau_\alpha)\mathrm{T}_\alpha.
\end{aligned}$$

With norm $\|\Xi^2_s \times \Xi^2_\nu\| = \sqrt{(1 - \nu\tau_\alpha)^2 + \nu^2\kappa_\alpha^2} = \sqrt{\beta}$, we get:

$$^{\Xi^2}N = \frac{(1 - \nu\tau_\alpha)\mathrm{T}_\alpha - \nu\kappa_\alpha\mathrm{B}_\alpha}{\sqrt{\beta}}.$$

The first fundamental coefficients are:

$$\begin{aligned}
E &= \nu^2\kappa_\alpha^2 + (1 - \nu\tau_\alpha)^2 = \beta, \\
F &= \langle \nu\kappa_\alpha\mathrm{T}_\alpha + (1 - \nu\tau_\alpha)\mathrm{B}_\alpha, -\mathrm{N}_\alpha\rangle = 0, \\
G &= 1.
\end{aligned}$$

The second fundamental coefficients are:

$$\begin{aligned}
L = \langle \Xi^2_{ss}, N\rangle &= \frac{1}{\sqrt{\beta}}\langle \nu\kappa'_\alpha\mathrm{T}_\alpha \\
&\quad + [\nu\kappa_\alpha^2 - \tau_\alpha(1 - \nu\tau_\alpha)]\mathrm{N}_\alpha - \nu\tau'_\alpha\mathrm{B}_\alpha, \\
&\quad + (1 - \nu\tau_\alpha)\mathrm{T}_\alpha - \nu\kappa_\alpha\mathrm{B}_\alpha\rangle \\
&= \frac{1}{\sqrt{\beta}}[\nu\kappa'_\alpha(1 - \nu\tau_\alpha) + \nu^2\kappa_\alpha\tau'_\alpha] \\
&= \frac{\nu}{\sqrt{\beta}}[\kappa'_\alpha + \nu\kappa_\alpha^2(\tau_\alpha/\kappa_\alpha)'] = \frac{\nu\gamma}{\sqrt{\beta}}.
\end{aligned}$$

Similarly, $M = \kappa_\alpha/\sqrt{\beta}$, $N = 0$.

$$^{\Xi^2}N = \frac{(1 - \nu\tau_\alpha)\mathrm{T}_\alpha - \nu\kappa_\alpha\mathrm{B}_\alpha}{\sqrt{\beta}},$$

where $\beta = (\nu\kappa_\alpha)^2 + (1 - \nu\tau_\alpha)^2$.

$$\begin{aligned}\beta_s &= \frac{\partial}{\partial s}[(\nu\kappa_\alpha)^2 + (1-\nu\tau_\alpha)^2] \\ &= 2\nu^2\kappa_\alpha\kappa'_\alpha + 2(1-\nu\tau_\alpha)(-\nu\tau'_\alpha) \\ &= 2\nu^2\kappa_\alpha\kappa'_\alpha - 2\nu\tau'_\alpha(1-\nu\tau_\alpha),\end{aligned}$$

$$\begin{aligned}\beta_\nu &= \frac{\partial}{\partial \nu}[(\nu\kappa_\alpha)^2 + (1-\nu\tau_\alpha)^2] \\ &= 2\nu\kappa_\alpha^2 + 2(1-\nu\tau_\alpha)(-\tau_\alpha) \\ &= 2\nu\kappa_\alpha^2 - 2\tau_\alpha(1-\nu\tau_\alpha).\end{aligned}$$

The N_s can be obtained by:

$$\begin{aligned}N_s &= \frac{\partial}{\partial s}\left[\frac{(1-\nu\tau_\alpha)\mathrm{T}_\alpha - \nu\kappa_\alpha\mathrm{B}_\alpha}{\sqrt{\beta}}\right] \\ &= \frac{1}{\sqrt{\beta}}\left[\frac{\partial}{\partial s}\big((1-\nu\tau_\alpha)\mathrm{T}_\alpha - \nu\kappa_\alpha\mathrm{B}_\alpha\big)\right] \\ &\quad - \frac{(1-\nu\tau_\alpha)\mathrm{T}_\alpha - \nu\kappa_\alpha\mathrm{B}_\alpha}{2\beta^{3/2}}\beta_s.\end{aligned}$$

Therefore:

$$\begin{aligned}N_s = \frac{1}{\sqrt{\beta}}\Big\{ &-\nu\tau'_\alpha\mathrm{T}_\alpha + [(1-\nu\tau_\alpha)\kappa_\alpha + \nu\kappa_\alpha\tau_\alpha]\mathrm{N}_\alpha - \\ &\nu\kappa'_\alpha\mathrm{B}_\alpha\Big\} \\ &- \frac{\beta_s}{2\beta^{3/2}}\Big[(1-\nu\tau_\alpha)\mathrm{T}_\alpha - \nu\kappa_\alpha\mathrm{B}_\alpha\Big].\end{aligned}$$

The N_ν can be obtained by:

$$\begin{aligned}N_\nu &= \frac{\partial}{\partial \nu}\left[\frac{(1-\nu\tau_\alpha)\mathrm{T}_\alpha - \nu\kappa_\alpha\mathrm{B}_\alpha}{\sqrt{\beta}}\right] \\ &= \frac{1}{\sqrt{\beta}}[-\tau_\alpha\mathrm{T}_\alpha - \kappa_\alpha\mathrm{B}_\alpha] - \frac{\beta_\nu}{2\beta^{3/2}}[(1-\nu\tau_\alpha)\mathrm{T}_\alpha \\ &\quad - \nu\kappa_\alpha\mathrm{B}_\alpha].\end{aligned}$$

The coefficients of the third fundamental form are:

$$\begin{aligned}e &= \frac{1}{\beta}\Big[\nu^2\tau'^2_\alpha + [(1-\nu\tau_\alpha)\kappa_\alpha + \nu\kappa_\alpha\tau_\alpha]^2 + \nu^2\kappa'^2_\alpha\Big] \\ &\quad + \frac{\beta_s^2}{4\beta^3}[(1-\nu\tau_\alpha)^2 + \nu^2\kappa_\alpha^2] \\ &\quad - \frac{\beta_s}{\beta^2}\Big[\nu\tau'_\alpha(1-\nu\tau_\alpha) + \nu\kappa_\alpha\kappa'_\alpha\Big].\end{aligned}$$

Simplifying using $\gamma = \kappa'_\alpha + \nu\kappa_\alpha^2(\tau_\alpha/\kappa_\alpha)'$ and after algebraic manipulation, we obtain:

$$e = \frac{\nu^2\gamma^2 + \kappa_\alpha^4\beta}{\beta^2}.$$

Computing the inner product of N_s and N_ν and, we have:

$$f = \frac{\nu\kappa_\alpha\gamma}{\beta^2}.$$

$$\begin{aligned}g &= \frac{1}{\beta}[\tau_\alpha^2 + \kappa_\alpha^2] + \frac{\beta_\nu^2}{4\beta^3}[(1-\nu\tau_\alpha)^2 + \nu^2\kappa_\alpha^2] \\ &\quad - \frac{\beta_\nu}{\beta^2}[\tau_\alpha(1-\nu\tau_\alpha) + \nu\kappa_\alpha^2].\end{aligned}$$

After simplification:

$$g = \frac{\kappa_\alpha^2}{\beta^2}.$$

Therefore,

$$III = \frac{(\nu^2\gamma^2 + \kappa_\alpha^4\beta)(ds)^2 + 2\nu\kappa_\alpha\gamma\, ds\, d\nu + \kappa_\alpha^2(d\nu)^2}{\beta^2}.$$

Theorem 7. *If $\Xi^2(s,\nu)$ is a normal ruled surface generated by integral binormal curve $\Omega(s)$, then mean curvature and Gaussian curvature are:*

$$H_{\Xi^2(s,\nu)} = \frac{\nu\gamma}{2\beta^{\frac{3}{2}}},$$

$$K_{\Xi^2(s,\nu)} = -\left(\frac{\kappa_\alpha}{\beta}\right)^2.$$

Proof. Using the curvature formulas:

$$K = \frac{LN - M^2}{EG - F^2} = \frac{0 - (\kappa_\alpha/\sqrt{\beta})^2}{\beta\cdot 1 - 0} = -\left(\frac{\kappa_\alpha}{\beta}\right)^2,$$

$$\begin{aligned}H &= \frac{EN - 2MF + GL}{2(EG - F^2)} \\ &= \frac{\beta\cdot 0 - 2(\kappa_\alpha/\sqrt{\beta})(0) + 1\cdot(\nu\gamma/\sqrt{\beta})}{2\beta} = \frac{\nu\gamma}{2\beta^{\frac{3}{2}}}.\end{aligned}$$

If $\Xi^2(s,\nu)$ is a normal ruled surface generated by an integral binormal curve $\Omega(s)$, then all points in $\Xi^2(s,\nu)$ are hyperbolic points.

If $\Xi^2(s,\nu)$ is a normal ruled surface generated by integral binormal curve $\Omega(s)$, then normal vector, geodesic curvatures, and geodesic torsion for $\Omega(s)$ on the surface are:

$$\kappa_{n_{\Xi^2}} = 0,$$

$$\kappa_{g_{\Xi^2}} = \frac{(1-\nu\tau_\alpha)\tau_\alpha}{\sqrt{\beta}},$$

$$\tau_{g_{\Xi^2}} = \frac{-\nu\tau_\alpha\gamma}{\beta}.$$

Proof. Detailed computation using the formulas for curvatures and torsion on surfaces.

Let $\Xi^2(s,\nu)$ be a normal ruled surface generated by an integral binormal curve $\Omega(s)$, with $\kappa_\alpha(s) > 0$. Then:

- $\Xi^2(s,\nu)$ is **never developable**, since $K = -\left(\frac{\kappa_\alpha}{\beta}\right)^2 < 0$ everywhere.
- $\Xi^2(s,\nu)$ is **minimal** if and only if

$$\gamma = \kappa'_\alpha + \nu\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' = 0 \quad \text{for all } s.$$

This is satisfied when $\alpha(s)$ is a circular helix (constant κ_α and constant $\tau_\alpha/\kappa_\alpha$).

Proof. When $\kappa_\alpha = 0$, we have $K = 0$ and $H = 0$, making the surface both developable and minimal.

3.3. Binormal ruled surfaces of integral binormal curves

We introduce binormal ruled surfaces based on the Frenet frame with integral binormal curves.

Definition 5. *Let $\Omega(s)$ be the integral binormal curve and let Ξ^3 be a binormal ruled surface generated by $\Omega(s)$ according to the Frenet frame. The parametric description of the binormal ruled surface Ξ^3 is given by:*

$$\Xi^3(s,\nu) = \Omega(s) + \nu \mathrm{B}_\Omega(s), \tag{21}$$

where $\Omega(s) = \int \mathrm{B}_\alpha \, \mathrm{d}s$ denotes the base curve.

Theorem 8. *If $\Xi^3(s,\nu)$ is a binormal ruled surface generated by an integral binormal curve $\Omega(s)$, then the striction line and distribution parameter are given by:*

$$\Omega^*_{\Xi^3}(s) = \Omega(s),$$
$$\delta_{\Xi^3} = \frac{1}{\kappa_\alpha}.$$

Proof. For the binormal ruled surface $\Xi^3(s,\nu)$:

$$\begin{aligned}\Omega^*_{\Xi^3}(s) &= \Omega(s) + \frac{\langle \mathrm{T}_\Omega, \mathrm{B}'_\Omega\rangle}{\|\mathrm{B}'_\Omega\|^2}\mathrm{B}_\Omega \\ &= \Omega(s) + \frac{\langle \mathrm{B}_\alpha, -\tau_\alpha \mathrm{N}_\alpha\rangle}{\tau_\alpha^2}\mathrm{T}_\alpha = \Omega(s), \\ \delta_{\Xi^3} &= \frac{\det[\mathrm{T}_\Omega, \mathrm{B}_\Omega, \mathrm{B}'_\Omega]}{\|\mathrm{B}'_\Omega\|^2} \\ &= \frac{\det[\mathrm{B}_\alpha, \mathrm{T}_\alpha, -\tau_\alpha \mathrm{N}_\alpha]}{\tau_\alpha^2} = \frac{1}{\kappa_\alpha}.\end{aligned}$$

Theorem 9. *Let $\Xi^3(s,\nu)$ be the binormal ruled surface generated by the integral binormal curve $\Omega(s)$, then the first, second, and third fundamental forms are given by:*

$$\begin{aligned} I &= (1+\nu^2\kappa_\alpha^2)(ds)^2 + (d\nu)^2, \\ II &= \frac{\sigma(ds)^2 + 2\kappa_\alpha ds d\nu}{\sqrt{\Psi}}, \\ III &= \frac{(\kappa_\alpha^2(1+(\nu\kappa_\alpha)^2)+\sigma^2)(ds)^2 + 2\kappa_\alpha\sigma ds d\nu + \kappa_\alpha^2(d\nu)^2}{\Psi^2}, \end{aligned}$$

where

$$\sigma = \nu\kappa'_\alpha - \tau_\alpha(1+(\nu\kappa_\alpha)^2), \qquad \Psi = (1+(\nu\kappa_\alpha)^2).$$

Proof. Let $\Xi^3(s,\nu) = \Omega(s) + \nu \mathrm{B}_\Omega(s)$, where $\mathrm{B}_\Omega = \mathrm{T}_\alpha$.

The partial derivatives of the surface $\Xi^3(s,\nu)$ are given by:

$$\begin{aligned} \Xi^3_s &= \mathrm{B}_\alpha + \nu \mathrm{T}'_\alpha = \mathrm{B}_\alpha + \nu\kappa_\alpha \mathrm{N}_\alpha, \\ \Xi^3_\nu &= \mathrm{T}_\alpha, \\ \Xi^3_{ss} &= -\tau_\alpha \mathrm{N}_\alpha + \nu\kappa'_\alpha \mathrm{N}_\alpha + \nu\kappa_\alpha(-\kappa_\alpha \mathrm{T}_\alpha + \tau_\alpha \mathrm{B}_\alpha) \\ &= -\nu\kappa_\alpha^2 \mathrm{T}_\alpha + (\nu\kappa'_\alpha - \tau_\alpha)\mathrm{N}_\alpha + \nu\kappa_\alpha\tau_\alpha \mathrm{B}_\alpha, \\ \Xi^3_{s\nu} &= \kappa_\alpha \mathrm{N}_\alpha, \quad \Xi^3_{\nu\nu} = 0. \end{aligned}$$

The unit normal vector to the surface $\Xi^3(s,\nu)$ is given by:

$$\Xi^3_s \times \Xi^3_\nu = (\mathrm{B}_\alpha + \nu\kappa_\alpha \mathrm{N}_\alpha) \times \mathrm{T}_\alpha = \mathrm{N}_\alpha - \nu\kappa_\alpha \mathrm{B}_\alpha,$$

with norm $\sqrt{1+\nu^2\kappa_\alpha^2}$, giving:

$${}^{\Xi^3}N = \frac{\mathrm{N}_\alpha - \nu\kappa_\alpha \mathrm{B}_\alpha}{\sqrt{1+\nu^2\kappa_\alpha^2}}.$$

Following similar algebraic steps as in previous theorems:

$$\begin{aligned} I &= (1+\nu^2\kappa_\alpha^2)(ds)^2 + (d\nu)^2, \\ L &= \frac{\sigma}{\sqrt{1+\nu^2\kappa_\alpha^2}}, \quad M = \frac{\kappa_\alpha}{\sqrt{1+\nu^2\kappa_\alpha^2}}, \quad N = 0, \end{aligned}$$

where $\sigma = \nu\kappa'_\alpha - \tau_\alpha(1+\nu^2\kappa_\alpha^2)$ from direct computation of $\langle \Xi^3_{ss}, N\rangle$.
The terms Ψ_s and Ψ_ν are given as:

$$\Psi_s = \frac{\partial}{\partial s}[1+\nu^2\kappa_\alpha^2] = 2\nu^2\kappa_\alpha\kappa'_\alpha,$$

$$\Psi_\nu = \frac{\partial}{\partial \nu}[1+\nu^2\kappa_\alpha^2] = 2\nu\kappa_\alpha^2.$$

The derivatives of N, with respect to s and N_ν are given by:

$$\begin{aligned} N_s &= \frac{\partial}{\partial s}\left[\frac{\mathrm{N}_\alpha - \nu\kappa_\alpha \mathrm{B}_\alpha}{\sqrt{\Psi}}\right] \\ &= \frac{1}{\sqrt{\Psi}}\left[\mathrm{N}'_\alpha - \nu\kappa'_\alpha \mathrm{B}_\alpha - \nu\kappa_\alpha \mathrm{B}'_\alpha\right] - \frac{\Psi_s}{2\Psi^{3/2}}[\mathrm{N}_\alpha - \nu\kappa_\alpha \mathrm{B}_\alpha]. \end{aligned}$$

Therefore,

$$\begin{aligned} N_s = &\frac{1}{\sqrt{\Psi}}\Big[-\kappa_\alpha \mathrm{T}_\alpha + \tau_\alpha \mathrm{B}_\alpha - \nu\kappa'_\alpha \mathrm{B}_\alpha + \nu\kappa_\alpha\tau_\alpha \mathrm{N}_\alpha\Big] \\ &- \frac{\nu^2\kappa_\alpha\kappa'_\alpha}{\Psi^{3/2}}[\mathrm{N}_\alpha - \nu\kappa_\alpha \mathrm{B}_\alpha]. \end{aligned}$$

$$\begin{aligned} N_\nu &= \frac{\partial}{\partial \nu}\left[\frac{\mathrm{N}_\alpha - \nu\kappa_\alpha \mathrm{B}_\alpha}{\sqrt{\Psi}}\right] \\ &= \frac{1}{\sqrt{\Psi}}[-\kappa_\alpha \mathrm{B}_\alpha] - \frac{\Psi_\nu}{2\Psi^{3/2}}[\mathrm{N}_\alpha - \nu\kappa_\alpha \mathrm{B}_\alpha] \\ &= \frac{1}{\sqrt{\Psi}}\left[-\frac{\nu\kappa_\alpha^2}{\Psi}\mathrm{N}_\alpha + \left(-\kappa_\alpha + \frac{\nu^2\kappa_\alpha^3}{\Psi}\right)\mathrm{B}_\alpha\right]. \end{aligned}$$

The coefficients of the third fundamental form are:

$$e = \frac{1}{\Psi}\left[\kappa_\alpha^2 + \left(\nu\kappa_\alpha\tau_\alpha - \frac{\nu^2\kappa_\alpha\kappa_\alpha'}{\Psi}\right)^2 + \right.$$
$$\left.\left(\tau_\alpha - \nu\kappa_\alpha' + \frac{\nu^3\kappa_\alpha^2\kappa_\alpha'}{\Psi}\right)^2\right].$$

After simplification and using $\sigma = \nu\kappa_\alpha' - \tau_\alpha(1+(\nu\kappa_\alpha)^2)$:

$$e = \frac{\kappa_\alpha^2\Psi + \sigma^2}{\Psi^2}.$$

$$f = \frac{\kappa_\alpha\sigma}{\Psi^2}.$$

$$g = \frac{\kappa_\alpha^2}{\Psi^2}.$$

$$III = \frac{[\kappa_\alpha^2(1+(\nu\kappa_\alpha)^2)+\sigma^2](ds)^2 + 2\kappa_\alpha\sigma\, ds\, d\nu + \kappa_\alpha^2(d\nu)^2}{\Psi^2}.$$

Theorem 10. *If $\Xi^3(s,\nu)$ is a binormal ruled surface generated by integral binormal curve $\Omega(s)$, then mean curvature and Gaussian curvature are:*

$$H_{\Xi^3(s,\nu)} = \frac{\sigma}{2(1+(\nu\kappa_\alpha)^2)^{\frac{3}{2}}},$$
$$K_{\Xi^3(s,\nu)} = -\left(\frac{\kappa_\alpha}{1+(\nu\kappa_\alpha)^2}\right)^2.$$

Proof. Using the curvature formulas with the computed fundamental form coefficients.

If $\Xi^3(s,\nu)$ is a binormal ruled surface generated by an integral binormal curve $\Omega(s)$, then all points in $\Xi^3(s,\nu)$ are hyperbolic points.

If $\Xi^3(s,\nu)$ is a binormal ruled surface generated by integral binormal curve $\Omega(s)$, then normal vector, geodesic curvatures and geodesic torsion for $\Omega(s)$ on the surface are:

$$\kappa_{n_{\Xi^3}} = \frac{-\tau_\alpha}{\sqrt{1+(\nu\kappa_\alpha)^2}},$$
$$\kappa_{g_{\Xi^3}} = 0,$$
$$\tau_{g_{\Xi^3}} = \frac{-\nu\tau_\alpha\kappa_\alpha^2}{1+(\nu\kappa_\alpha)^2}.$$

If $\Xi^3(s,\nu)$ is a binormal ruled surface generated by integral binormal curve $\Omega(s)$, then:

- $\Xi^3(s,\nu)$ is a developable surface if $\kappa_\alpha = 0$ (α is a straight line).
- $\Xi^3(s,\nu)$ is a minimal surface if $\kappa_\alpha = 0$ (α is a straight line).

4. Ruled surfaces with integral normal base curves

This section examines tangent, normal, and binormal ruled surfaces derived from integral normal curves. The construction parallels that of surfaces based on integral binormal curves but has distinct geometric characteristics due to the different nature of the base curves. Integral normal curves, defined as integrals of the normal vector field of a base curve $\alpha(s)$, exhibit unique curvature properties that influence the resulting ruled surfaces. We systematically derive the fundamental forms, curvature properties, and special geometric attributes for three types of ruled surfaces: tangent, binormal, and normal surfaces. The results demonstrate how the curvature and torsion of the original curve $\alpha(s)$ propagate through the integral construction to determine the geometric invariants of the generated surfaces.

Definition 6. [42–44] *The integral normal curve $\mu(s)$ is given by:*

$$\mu(s) = \int \mathrm{N}_\alpha \, \mathrm{d}s \quad \textit{with} \ \ \mathrm{B}_\alpha = \mu'(s). \tag{22}$$

Definition 7. *If curve $\mu(s)$ is an N_α-integral curve of curve α let $\{\mathrm{T}_\mu, \mathrm{N}_\mu, \mathrm{B}_\mu\}$ be the Frenet frame of curve μ and $\{\mathrm{T}_\alpha, \mathrm{N}_\alpha, \mathrm{B}_\alpha\}$ be the Frenet frame of α, then $\{\mathrm{T}_\mu, \mathrm{N}_\mu, \mathrm{B}_\mu, \kappa_\mu\}$ and τ_μ can be written in terms of the curve α as:*

$$\begin{aligned}
\mathrm{T}_\mu &= \mathrm{N}_\alpha,\\
\mathrm{N}_\mu &= \frac{-\kappa_\alpha\mathrm{T}_\alpha + \tau_\alpha\mathrm{B}_\alpha}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}},\\
\mathrm{B}_\mu &= \frac{\tau_\alpha\mathrm{T}_\alpha + \kappa_\alpha\mathrm{B}_\alpha}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}},\\
\kappa_\mu &= \sqrt{\kappa_\alpha^2+\tau_\alpha^2},\\
\tau_\mu &= \frac{\kappa_\alpha^2}{\kappa_\alpha^2+\tau_\alpha^2}.
\end{aligned} \tag{23}$$

4.1. Tangent ruled surfaces of integral normal curves

We introduce tangent ruled surfaces based on the Frenet frame with integral normal curves.

Definition 8. *Let $\mu(s)$ be the integral normal curve and let Ξ^4 be a tangent ruled surface generated by $\mu(s)$ according to the Frenet frame. The parametric description of the tangent ruled surface Ξ^4 is given by:*

$$\Xi^4(s,\nu) = \mu(s) + \nu\mathrm{T}_\mu(s), \tag{24}$$

where $\mu(s) = \int \mathrm{N}_\alpha \, \mathrm{d}s$ denotes the base curve.

Theorem 11. *If $\Xi^4(s,\nu)$ is a tangent ruled surface generated by the integral normal curve $\mu(s)$,* *then the striction line and distribution parameter are given by:*

$$\mu^*_{\Xi^4}(s) = \mu(s),$$
$$\delta_{\Xi^4} = 0.$$

Proof. Similar to the proof for tangent ruled surfaces of integral binormal curves.

Theorem 12. *Let $\Xi^4(s,\nu)$ be the tangent ruled surface generated by the integral normal curve $\mu(s)$, then the fundamental forms are given by:*

$$I = (1+\nu^2(\kappa_\alpha^2+\tau_\alpha^2))(ds)^2 + 2dsd\nu + (d\nu)^2, \tag{25}$$

$$II = \frac{-\nu\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'(ds)^2}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}, \tag{26}$$

$$III = \left(\frac{\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'}{\kappa_\alpha^2+\tau_\alpha^2}\right)^2 (ds)^2. \tag{27}$$

Proof. This follows a methodology similar to the previous proofs, using the appropriate Frenet frame relationships.

Theorem 13. *If $\Xi^4(s,\nu)$ is a tangent ruled surface generated by integral normal curve $\mu(s)$, then mean curvature and Gaussian curvature are:*

$$H_{\Xi^4(s,\nu)} = \frac{-\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'}{2\nu(\kappa_\alpha^2+\tau_\alpha^2)^{\frac{3}{2}}},$$
$$K_{\Xi^4(s,\nu)} = 0.$$

If $\Xi^4(s,\nu)$ is a tangent surface ruled generated by an integral normal curve $\mu(s)$, then all points in $\Xi^4(s,\nu)$ are parabolic points.

If $\Xi^4(s,\nu)$ is a tangent ruled surface generated by integral normal curve $\mu(s)$, then normal vector, geodesic curvatures, geodesic curvatures and geodesic torsion for $\mu(s)$ on the surface are:

$$\kappa_{n_{\Xi^4}} = 0,$$
$$\kappa_{g_{\Xi^4}} = -\sqrt{\kappa_\alpha^2+\tau_\alpha^2},$$
$$\tau_{g_{\Xi^4}} = 0.$$

If $\Xi^4(s,\nu)$ is a tangent ruled surface generated by integral normal curve $\mu(s)$, then;

- $\Xi^4(s,\nu)$ is a developable surface everywhere.
- $\Xi^4(s,\nu)$ is a minimal surface everywhere.

4.2. Binormal ruled surfaces of integral normal curves

We introduce binormal ruled surfaces based on the Frenet frame with integral normal curves.

Definition 9. *Let $\mu(s)$ be the integral normal curve and Ξ^5 be a binormal ruled surface generated by $\mu(s)$ according to the Frenet frame.*
The parametric description of the binormal ruled surface Ξ^5 is given by:

$$\Xi^5(s,\nu) = \mu(s) + \nu \mathrm{B}_\mu(s), \tag{28}$$

where $\mu(s) = \int \mathrm{N}_\alpha\, \mathrm{d}s$ denotes the base curve.

Theorem 14. *If $\Xi^5(s,\nu)$ is a binormal ruled surface generated by integral normal curve $\mu(s)$, then the striction line and distribution parameter are given by:*

$$\mu^*_{\Xi^5}(s) = \mu(s),$$
$$\delta_{\Xi^5} = \frac{(\kappa_\alpha^2+\tau_\alpha^2)}{\kappa_\alpha\tau_\alpha' - \kappa_\alpha'\tau_\alpha}.$$

Theorem 15. *Let $\Xi^5(s,\nu)$ be the binormal ruled surface generated by the integral normal curve $\mu(s)$, then the fundamental forms are given by:*

$$I = \left(1+(\Omega_h)^2\right)(ds)^2 + (d\nu)^2,$$

$$II = \frac{\mathcal{Q}_\mathcal{B}(ds)^2 + 2\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' dsd\nu}{\sqrt{\mathcal{A}}},$$

$$\begin{aligned} III = {} & \frac{(\eta_1^2+\eta_2^2+\eta_3^2)(ds)^2}{\mathcal{A}^3} \\ & + \frac{2\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}{\mathcal{A}^3} \\ & \times \left[\nu\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'(\tau_\alpha\eta_3 - \kappa_\alpha\eta_1)\right. \\ & \left. -\eta_2(\kappa_\alpha^2+\tau_\alpha^2)^{3/2}\right] dsd\nu \\ & + \frac{\left[\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'(\kappa_\alpha^2+\tau_\alpha^2)\right]^2 (d\nu)^2}{\mathcal{A}^2}, \end{aligned}$$

where

$$\Omega_h = \frac{\nu\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'}{\kappa_\alpha^2+\tau_\alpha^2},$$

$$\begin{aligned} \mathcal{Q}_\mathcal{B} = {} & -(\kappa_\alpha^2+\tau_\alpha^2)^{3/2} + \nu\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'' \\ & - \frac{\nu^2\kappa_\alpha^4\left[\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'\right]^2}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}} \\ & - \frac{2\nu\tau_\alpha\kappa_\alpha^3\left[\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'\right]^2}{\kappa_\alpha^2+\tau_\alpha^2}, \end{aligned}$$

$$\mathcal{A} = \nu\kappa_\alpha^4\left[\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'\right]^2 + (\kappa_\alpha^2+\tau_\alpha^2)^2,$$

and

$$\begin{aligned}
\mathcal{C} &= -2\nu^2\kappa_\alpha^3\kappa_\alpha'\left(\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'\right)^2 \\
&\quad - (\kappa_\alpha\kappa_\alpha' + \tau_\alpha\tau_\alpha')(\kappa_\alpha^2+\tau_\alpha^2) \\
&\quad - \nu^2\kappa_\alpha^4\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'', \\
\eta_1 &= \kappa_\alpha\mathcal{C}\sqrt{\kappa_\alpha^2+\tau_\alpha^2} \\
&\quad - \mathcal{A}\Bigg[\kappa_\alpha\left(\frac{\kappa_\alpha\kappa_\alpha'+\tau_\alpha\tau_\alpha'}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}\right) \\
&\quad +\nu\kappa_\alpha^3\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' \\
&\quad +\kappa_\alpha'\sqrt{\kappa_\alpha^2+\tau_\alpha^2}\Bigg], \\
\eta_2 &= -\nu\kappa_\alpha^2\mathcal{C}\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' \\
&\quad + \mathcal{A}\Bigg[(\kappa_\alpha^2+\tau_\alpha^2)^{\frac{3}{2}} \\
&\quad -2\nu\kappa_\alpha\kappa_\alpha'\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' \\
&\quad -\nu\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)''\Bigg], \\
\eta_3 &= -\tau_\alpha\mathcal{C}\sqrt{\kappa_\alpha^2+\tau_\alpha^2} \\
&\quad - \mathcal{A}\Bigg[\tau_\alpha\left(\frac{\kappa_\alpha\kappa_\alpha'+\tau_\alpha\tau_\alpha'}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}\right) \\
&\quad +\nu\kappa_\alpha^2\tau_\alpha\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' \\
&\quad +\tau_\alpha'\sqrt{\kappa_\alpha^2+\tau_\alpha^2}\Bigg],
\end{aligned}$$

Theorem 16. *If $\Xi^5(s,\nu)$ is a binormal ruled surface generated by the integral normal curve $\mu(s)$, then the mean curvature and Gaussian curvature are given by:*

$$H_{\Xi^5(s,\nu)} = \frac{T}{2(1+(\Omega_h)^2)\sqrt{\mathcal{A}}},$$

$$K_{\Xi^5(s,\nu)} = \frac{\left(\kappa_\alpha^4\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'\right)^2}{(1+(\Omega_h)^2)\mathcal{A}}.$$

If $\Xi^5(s,\nu)$ is a binormal ruled surface generated by integral normal curve $\mu(s)$, then all points in $\Xi^5(s,\nu)$ are hyperbolic points.

If $\Xi^5(s,\nu)$ is a binormal ruled surface generated by integral normal curve $\mu(s)$, then normal vector, geodesic curvatures, and geodesic torsion for $\mu(s)$ on the surface are:

$$\begin{aligned}
\kappa_{n_{\Xi^5}} &= -\sqrt{\frac{(\kappa_\alpha^2+\tau_\alpha^2)^3}{\mathcal{A}}}, \\
\kappa_{g_{\Xi^5}} &= 0, \\
\tau_{g_{\Xi^5}} &= \sqrt{\frac{\nu^2\kappa_\alpha^8(\kappa_\alpha^2+\tau_\alpha^2)\left(\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'\right)^2}{\mathcal{A}}}.
\end{aligned}$$

If $\Xi^5(s,\nu)$ is a binormal ruled surface generated by the integral normal curve $\mu(s)$, then;

- $\Xi^5(s,\nu)$ is a developable surface if $\kappa_\alpha = 0$ (α is a straight line).
- $\Xi^5(s,\nu)$ is a minimal surface if $\kappa_\alpha = 0$ (α is a straight line).

4.3. Normal ruled surfaces of integral normal curves

We introduce normal ruled surfaces based on the Frenet frame with integral normal curves.

Definition 10. *Let $\mu(s)$ be the integral normal curve and Ξ^6 be a normal ruled surface generated by $\mu(s)$ according to the Frenet frame.*
The parametric description of the normal ruled surface Ξ^6 is given by:

$$\Xi^6(s,\nu) = \mu(s) + \nu\mathrm{N}_\mu(s), \tag{29}$$

where $\mu(s) = \int \mathrm{N}_\alpha\, \mathrm{d}s$ denotes the base curve.

Theorem 17. *If $\Xi^6(s,\nu)$ is a normal ruled surface generated by the integral normal curve $\mu(s)$, then the striction line and distribution parameter are given by:*

$$\mu^*_{\Xi^6}(s) = \mu(s) - \frac{(\kappa_\alpha^2+\tau_\alpha^2)^{\frac{5}{2}}}{\kappa_\alpha^4\left(\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'\right)^2 + (\kappa_\alpha^2+\tau_\alpha^2)^3}\mathrm{N}_\mu,$$

$$\delta_{\Xi^6} = \frac{\kappa_\alpha^2(\kappa_\alpha^2+\tau_\alpha^2)\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'}{\kappa_\alpha^4\left(\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'\right)^2 + (\kappa_\alpha^2+\tau_\alpha^2)^3}.$$

Theorem 18. *Let $\Xi^6(s,\nu)$ be the normal ruled surface generated by the integral normal curve $\mu(s)$, then the fundamental forms are given by:*

$$\begin{aligned}
I &= \frac{\Pi}{(\kappa_\alpha^2+\tau_\alpha^2)^2}(ds)^2 + (d\nu)^2, \\
II &= \frac{O(ds)^2 + 2\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' dsd\nu}{\sqrt{\Pi}}, \\
III &= (J_1^2+J_2^2+J_3^2)(ds)^2 \\
&\quad + \frac{2\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}{\Pi^{3/2}} \\
&\quad \times\Bigg[\nu\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'(\tau_\alpha J_1+\kappa_\alpha J_3) \\
&\quad +(\kappa_\alpha^2+\tau_\alpha^2)^{3/2}\Omega_{hh}J_2\Bigg] dsd\nu \\
&\quad + \left(\frac{\kappa_\alpha^2(\kappa_\alpha^2+\tau_\alpha^2)\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'}{\Pi}\right)^2(d\nu)^2,
\end{aligned}$$

where

$$\Omega_{hh} = 1 - \nu\sqrt{\kappa_\alpha^2 + \tau_\alpha^2},$$

$$\Omega_h = \frac{\nu\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'}{\kappa_\alpha^2 + \tau_\alpha^2},$$

$$\Pi = \nu\kappa_\alpha^4\left[\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'\right]^2 + [(\kappa_\alpha^2 + \tau_\alpha^2)\Omega_{hh}]^2,$$

$$O = \nu\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)''\Omega_{hh} + \frac{\nu\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'(\kappa_\alpha\kappa_\alpha' + \tau_\alpha\tau_\alpha')}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}} - \frac{2\tau_\alpha\kappa_\alpha\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'\Omega_{hh}}{\kappa_\alpha^2 + \tau_\alpha^2},$$

$$J_1 = \left(\frac{1}{\sqrt{\Pi}}\right)'Q_1 + \frac{Q_1' - \kappa_\alpha Q_2}{\sqrt{\Pi}},$$

$$J_2 = \left(\frac{1}{\sqrt{\Pi}}\right)'Q_2 + \frac{Q_2' - \kappa_\alpha Q_1 - \tau_\alpha Q_3}{\sqrt{\Pi}},$$

$$J_3 = \left(\frac{1}{\sqrt{\Pi}}\right)'Q_3 + \frac{Q_3' + \tau_\alpha Q_2}{\sqrt{\Pi}},$$

and

$$Q_1 = \tau_\alpha\left(\frac{1}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}} - \nu\right),$$

$$Q_2 = \frac{-\nu\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'}{(\kappa_\alpha^2 + \tau_\alpha^2)},$$

$$Q_3 = \kappa_\alpha\left(\frac{1}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}} - \nu\right).$$

Theorem 19. *If $\Xi^6(s,\nu)$ is a normal ruled surface generated by the integral normal curve $\mu(s)$, then mean curvature and Gaussian curvature are:*

$$H_{\Xi^6(s,\nu)} = \frac{O(\kappa_\alpha^2 + \tau_\alpha^2)^2}{2\Pi^{\frac{3}{2}}},$$

$$K_{\Xi^6(s,\nu)} = -\frac{\left(\kappa_\alpha^2(\kappa_\alpha^2 + \tau_\alpha^2)\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'\right)^2}{\Pi^2}.$$

If $\Xi^6(s,\nu)$ is a normal ruled surface generated by an integral normal curve $\mu(s)$, then all points in $\Xi^6(s,\nu)$ are hyperbolic points.

If $\Xi^6(s,\nu)$ is a normal ruled surface generated by the integral normal curve $\mu(s)$, then normal vector, geodasic curvatures, and geodesic torsion for $\mu(s)$ on the surface are given by:

$$\kappa_{n_{\Xi^6}} = 0,$$

$$\kappa_{g_{\Xi^6}} = \frac{(\kappa_\alpha^2 + \tau_\alpha^2)^{\frac{3}{2}}\Omega_{AA}}{\sqrt{\Pi}},$$

$$\tau_{g_{\Xi^6}} = \frac{-\kappa_\alpha[J_3Q_2 - J_2Q_3] + \tau_\alpha[J_2Q_1 - J_1Q_2]}{\sqrt{\Pi}},$$

where

$$\Omega_{AA} = \sqrt{\kappa_\alpha^2 + \tau_\alpha^2} - \nu(\kappa_\alpha^2 + \tau_\alpha^2),$$

$$\Pi = \left(\kappa_\alpha^4\left(\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'\right)^2 + [\kappa_\alpha^2 + \tau_\alpha^2 - \nu(\kappa_\alpha^2 + \tau_\alpha^2)^{\frac{3}{2}}]^2\right).$$

If $\Xi^6(s,\nu)$ is a normal ruled surface generated by the integral normal curve $\mu(s)$, then;

- $\Xi^6(s,\nu)$ is a developable surface if $\kappa_\alpha = 0$ (α is a straight line).
- $\Xi^6(s,\nu)$ is a minimal surface if $\kappa_\alpha = 0$ (α is a straight line).

5. Special cases and geometric significance

In this section, we examine several geometrically significant special cases where the ruled surfaces constructed in Sections 3 and 4 exhibit particularly interesting or simplified behavior.

5.1. Curves with constant curvature

Proposition 1. *Let $\alpha(s)$ be a curve with constant curvature $\kappa_\alpha' = 0$, then:*

i For tangent ruled surfaces Ξ^1 (Theorem 3):
- *The Gaussian curvature remains $K = 0$ (developable).*
- *The mean curvature simplifies to $H = \frac{-\kappa_\alpha}{2\nu\tau_\alpha}$ (constant in s).*

ii For normal ruled surfaces Ξ^2 (Theorem 6):
- *The coefficient $\gamma = \nu\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'$.*
- *If, in addition, $\tau_\alpha/\kappa_\alpha$ is constant (circular helices), then $\gamma = 0$, yielding $H = 0$.* ***The surface is minimal.***

iii For binormal ruled surfaces Ξ^3 (Theorem 9):
- *The coefficient $\sigma = -\tau_\alpha(1 + (\nu\kappa_\alpha)^2)$ (no κ_α' term).*
- *The surface is minimal only if $\tau_\alpha = 0$ (planar curves), reducing to circles.*

Proof. Direct substitution of $\kappa_\alpha' = 0$ into the formulas of Theorems 3, 6, and 9.

Remark 1. *Circular helices, characterized by constant κ_α and constant $\tau_\alpha/\kappa_\alpha$, thus generate* *minimal normal ruled surfaces. This is a classical result in differential geometry: the right helicoid (generated by a circular helix with $\tau_\alpha/\kappa_\alpha =$ constant) is a minimal surface.*

5.2. Planar curves

Proposition 2. *Let $\alpha(s)$ be a planar curve, so $\tau_\alpha = 0$ identically. Then:*

(1) For tangent ruled surfaces Ξ^1:
- *The first fundamental form reduces to $I = (ds)^2 + 2\,ds\,d\nu + (d\nu)^2$.*
- *The geodesic curvature of the base curve vanishes: $\kappa_{g_{\Xi^1}} = -\tau_\alpha = 0$, so $\Omega(s)$ is a geodesic on Ξ^1.*
- *These are classical tangent developables of planar curves.*

(2) For normal ruled surfaces Ξ^2:
- *$\beta = (\nu\kappa_\alpha)^2 + 1$ and $\gamma = \kappa'_\alpha$.*
- *Gaussian curvature: $K = -\frac{\kappa_\alpha^2}{(\nu\kappa_\alpha)^2+1}$.*
- *Mean curvature: $H = \frac{\nu\kappa'_\alpha}{2[(\nu\kappa_\alpha)^2+1]^{3/2}}$.*
- *The surface is minimal if $\kappa'_\alpha = 0$ (circles generate minimal surfaces).*

(3) For binormal ruled surfaces Ξ^3:
- *$\sigma = \nu\kappa'_\alpha$ (simplified form).*
- *The base curve is a geodesic: $\kappa_{g_{\Xi^3}} = 0$.*
- *Mean curvature: $H = \frac{\nu\kappa'_\alpha}{2(1+(\nu\kappa_\alpha)^2)^{3/2}}$.*
- *Minimal if and only if $\kappa'_\alpha = 0$ (circles).*

Remark 2. *For planar curves, both Ξ^2 and Ξ^3 are minimal surfaces when the base curve is a circle ($\kappa'_\alpha = 0$). Geometrically, these are surfaces of revolution around an axis in the plane, which are known minimal surfaces (catenoids or planes, depending on the radius).*

5.3. Generalized helices

Definition 11. *A curve $\alpha(s)$ is called a* generalized helix *(or* inclined curve*) if the ratio $\tau_\alpha/\kappa_\alpha$ is constant, i.e.,*

$$\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' = 0.$$

Equivalently, the tangent vector T_α makes a constant angle with a fixed direction in space.

Proposition 3. *Let $\alpha(s)$ be a generalized helix with $\tau_\alpha/\kappa_\alpha = c$ (constant, then:*

i For normal ruled surfaces Ξ^2:
- *The coefficient $\gamma = \kappa'_\alpha$ (a dramatic simplification).*
- *The second fundamental form: $II = \frac{\nu\kappa'_\alpha(ds)^2+2\kappa_\alpha\,ds\,d\nu}{\sqrt{\beta}}$.*
- *Gaussian curvature depends only on κ_α and ν: $K = -\left(\frac{\kappa_\alpha}{\beta}\right)^2$.*
- *The surface is minimal if $\kappa'_\alpha = 0$ (circular helices).*

ii For binormal ruled surfaces Ξ^3:
- *No simplification in $\sigma = \nu\kappa'_\alpha - \tau_\alpha(1+(\nu\kappa_\alpha)^2)$.*
- *However, if κ_α is also constant (circular helix), then $\sigma = -\tau_\alpha(1+(\nu\kappa_\alpha)^2)$, and minimality requires $\tau_\alpha = 0$, contradicting the helix assumption. Thus, non-planar circular helices never generate minimal binormal ruled surfaces.*

iii For tangent ruled surfaces Ξ^1:
- *No direct simplification, as fundamental forms depend on κ_α and τ_α separately, not their ratio.*

Example 1 (Circular helix)**.** *Consider the circular helix $\alpha(s) = (a\cos(s/c), a\sin(s/c), bs/c)$ with $c = \sqrt{a^2+b^2}$, where $a, b > 0$ are constants. Then:*

$$\kappa_\alpha = \frac{a}{a^2+b^2} = \frac{a}{c^2} \quad (constant),$$
$$\tau_\alpha = \frac{b}{a^2+b^2} = \frac{b}{c^2} \quad (constant),$$
$$\frac{\tau_\alpha}{\kappa_\alpha} = \frac{b}{a} \quad (constant).$$

For the normal ruled surface Ξ^2:
- *$\gamma = \kappa'_\alpha = 0$, so $H = 0$. The surface is minimal.*
- *This is the classical right helicoid, a well-known minimal surface.*

For the binormal ruled surface Ξ^3:
- *$\sigma = -\tau_\alpha(1+(\nu\kappa_\alpha)^2) = -\frac{b}{c^2}(1+(\nu a/c^2)^2) \neq 0$.*
- *Thus $H \neq 0$, confirming the surface is not minimal.*

5.4. Ruled surfaces at the base curve

Proposition 4. *At $\nu = 0$ (restricting to the base curve), all ruled surfaces are reduce to the geometry of the base curve:*

i Tangent ruled surfaces Ξ^1:
- *$I|_{\nu=0} = (ds)^2 + (d\nu)^2$ (standard metric).*
- *$II|_{\nu=0} = 0$ (no bending in s-direction at the curve itself).*
- *Consistency check: The base curve $\Omega(s)$ has vanishing normal curvature on the surface.*

ii ***Normal ruled surfaces Ξ^2:***
- *$\beta|_{\nu=0} = 1$, $\gamma|_{\nu=0} = \kappa'_\alpha$.*
- *$I|_{\nu=0} = (ds)^2 + (d\nu)^2$.*

- $II|_{\nu=0} = 2\kappa_\alpha\, ds\, d\nu$ *(mixed term only).*

iii ***Binormal ruled surfaces*** Ξ^3***:***

- *Similar reductions at* $\nu = 0$.
- $\sigma|_{\nu=0} = -\tau_\alpha$, *giving* $L|_{\nu=0} = -\tau_\alpha (ds)^2$.

Remark 3. *These reductions provide essential consistency checks: our formulas correctly reproduce known properties of curves when restricted to the base curve. They also show that the "interesting" geometry (non-zero Gaussian curvature, potential minimality) emerges away from the base curve, for* $\nu \neq 0$.

5.5. Asymptotic analysis for small $|\nu|$

For applications in geometric approximation, it is useful to understand the behavior of these surfaces near the base curve.

Proposition 5. *For small* $|\nu|$*, the following asymptotic expansions hold:*

i ***Normal ruled surface*** Ξ^2***:***

$$\beta = 1 + \nu^2(\kappa_\alpha^2 + \tau_\alpha^2) + \mathcal{O}(\nu^3),$$

$$\gamma = \kappa'_\alpha + \nu\kappa_\alpha^2 \left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' + \mathcal{O}(\nu^2),$$

$$K = -\kappa_\alpha^2[1 - \nu^2(\kappa_\alpha^2 + \tau_\alpha^2) + \mathcal{O}(\nu^3)],$$

$$H = \frac{\nu\kappa'_\alpha}{2} + \mathcal{O}(\nu^2).$$

ii ***Binormal ruled surface*** Ξ^3***:***

$$1 + (\nu\kappa_\alpha)^2 = 1 + \nu^2\kappa_\alpha^2 + \mathcal{O}(\nu^3),$$

$$\sigma = -\tau_\alpha + \nu(\kappa'_\alpha - \tau_\alpha\kappa_\alpha^2) + \mathcal{O}(\nu^2),$$

$$K = -\kappa_\alpha^2[1 - \nu^2\kappa_\alpha^2 + \mathcal{O}(\nu^3)],$$

$$H = \frac{-\tau_\alpha}{2} + \mathcal{O}(\nu).$$

Remark 4. *These expansions show that:*

- *Gaussian curvature is approximately* $-\kappa_\alpha^2$ *near the base curve, with corrections of order* ν^2.
- *Mean curvature for* Ξ^2 *starts at order* ν *(vanishes at* $\nu = 0$*), while for* Ξ^3 *it has a non-zero leading term* $-\tau_\alpha/2$.
- *For geometric modeling applications requiring surfaces close to a given curve, these approximations provide computationally efficient estimates.*

6. Example

This section provides a concrete illustration of the theoretical results through a detailed numerical example. We consider the canonical helix curve $\alpha(s) = (\cos s, \sin s, s)$, which has constant curvature $\kappa = 1/\sqrt{2}$ and torsion $\tau = 1/\sqrt{2}$. This choice allows for explicit computation of all geometric quantities while maintaining nontrivial curvature characteristics. We construct the integral normal curve $\mu(s)$ and generate three types of ruled surfaces: tangent (Ξ^4), normal (Ξ^5), and binormal (Ξ^6) surfaces. The example demonstrates how the theoretical formulas for fundamental forms, curvatures, and striction curves manifest in a concrete case. The visualizations in Figure 1 provide geometric intuition for the analytical results, showing the distinctive characteristics of each surface type. This example serves both to validate the theoretical derivations and to illustrate the geometric diversity achievable through integral curve constructions.

Example 2. *Let the curve* $\alpha(s)$ *be defined by:*

$$\alpha(s) = (\cos s, \sin s, s)\,.$$

The Frenet frame vectors are:

$$\mathrm{T}(s) = \frac{1}{\sqrt{2}}(-\sin s, \cos s, 1),$$

$$\mathrm{N}(s) = (-\cos s, -\sin s, 0),$$

$$\mathrm{B}(s) = \left(\frac{\sin s}{\sqrt{2}}, -\frac{\cos s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

The integral normal curve $\mu(s)$

$$\mu(s) = \int \mathrm{N}(s)\, ds = (-\sin s, \cos s, 0)$$

The tangent, the normal, and the binormal ruled surfaces are given in Figure 1 as follow:
Tangent Ruled Surface $\Xi^4(s,\nu)$

$$\Xi^4(s,\nu) = \mu(s) + \nu\mathrm{T}(s) = \left(-\sin s - \frac{\nu \sin s}{\sqrt{2}}, \cos s + \frac{\nu\cos s}{\sqrt{2}}, \frac{\nu}{\sqrt{2}}\right)$$

Normal ruled surface $\Xi^5(s,\nu)$

$$\Xi^5(s,\nu) = \mu(s) + \nu\mathrm{N}(s) = (-\sin s - \nu\cos s, \cos s - \nu \sin s, 0).$$

Binormal ruled surface $\Xi^6(s,\nu)$

$$\Xi^6(s,\nu) = \mu(s) + \nu\mathrm{B}(s) = \left(-\sin s + \frac{\nu\sin s}{\sqrt{2}}, \cos s - \frac{\nu \cos s}{\sqrt{2}}, \frac{\nu}{\sqrt{2}}\right)$$

These surfaces exhibit the geometric properties derived in our theorems, with the tangent surface being developable and minimal, while the normal and binormal surfaces typically contain hyperbolic points.

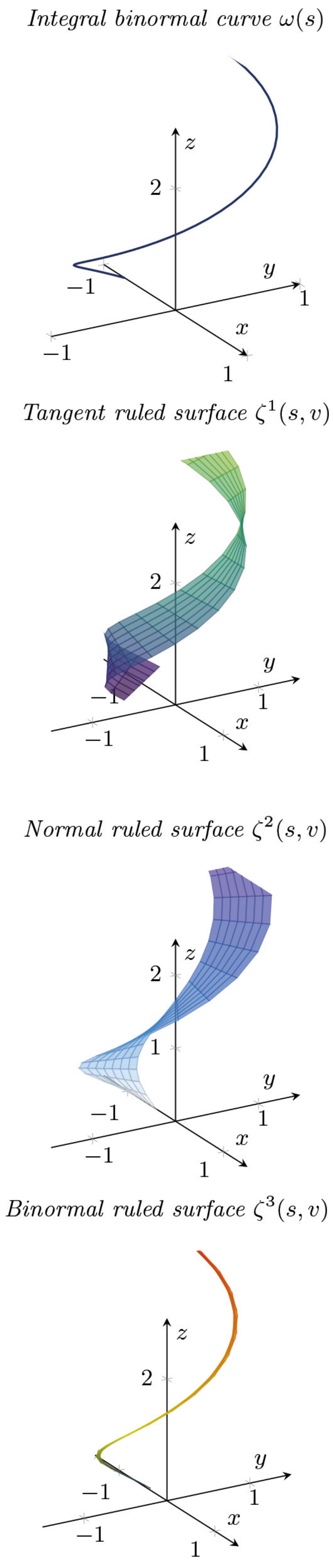

Figure 1. Ruled surfaces generated from integral binormal curve

Table 1. Special cases behavior

Condition	Ξ^1	Ξ^2	Ξ^3
$\kappa'_\alpha = 0$	Developable	Minimal if helix	Minimal if planar
$\tau_\alpha = 0$	Geodesic base	Minimal if circle	Geodesic base
Circular helix	Developable	Minimal	Not minimal

7. Conclusion

This study introduced and analyzed a new class of ruled surfaces generated from integral curves in Euclidean 3-space. By leveraging the Frenet frame, we constructed six types of ruled surfaces and derived their fundamental forms, curvature properties, and striction curves. We established conditions for these surfaces to be minimal or developable, and provided illustrative examples. The results extend existing knowledge on ruled surfaces and have potential applications in geometric modeling and computer-aided design. Future work may explore these surfaces in non-Euclidean spaces or under different framing conditions.

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Author contributions

Conceptualization: Ayman Elsharkawy, Hasnaa Baizeed
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