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# The Monomiality Principle Applied to Extensions of Apostol-Type Hermite Polynomials

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**Abstract.** In this research paper, we present a class of polynomials referred to as Apostol-type Hermite-Bernoulli/Euler polynomials $\mathcal{U}_\nu(x, y; \rho; \mu)$, which can be given by the following generating function

$$\frac{2-\mu+\frac{\mu}{2}\xi}{\rho e^{\xi}+(1-\mu)}e^{x\xi+y\xi^2}=\sum_{\nu=0}^{\infty}\mathcal{U}_\nu(x,y;\rho;\mu)\frac{\xi^\nu}{\nu!},$$

for some particular values of $\rho$ and $\mu$. Further, the summation formulae and determinant forms of these polynomials are derived. This novel family encompasses both the classical Appell-type polynomials and their noteworthy extensions. Our investigations heavily rely on generating function techniques, supported by illustrative examples to demonstrate the validity of our results. Furthermore, we introduce derivative and multiplicative operators, facilitating the definition of the Apostol-type Hermite-Bernoulli/Euler polynomials as a quasi-monomial set.

**2020 Mathematics Subject Classifications**: 11B68, 11B83, 05A19

**Key Words and Phrases**: Appell-type polynomials, Bernoulli and Euler numbers and polynomials, Hermite polynomials, quasi-monomial

## 1. Introduction

The Appell polynomials $\{A_n(x)\}_{n=0,1,2,...}$ are a family of special functions introduced by the French mathematician Paul Appell (see [2]). These polynomials are defined by a

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generating function

$$A(\xi)e^{x\xi} = \sum_{\nu=0}^{\infty} A_\nu(x)\frac{\xi^\nu}{\nu!}, \tag{1}$$

where $A(\xi)$ is given by

$$A(\xi) = \sum_{\nu=0}^{\infty} \alpha_\nu \frac{\xi^\nu}{\nu!},$$

and $\alpha_\nu$ are real coefficients. Further, $A_n(x)$ satisfying the recursive relations

$$\frac{d}{dx}A_\nu(x) = \nu A_{\nu-1}(x). \tag{2}$$

The Appell polynomials have various properties that make them useful in mathematical analysis, particularly in the study of differential equations and other fields [15, 18]. Famous instances of polynomial sequences that satisfy (1), or equivalently the recursive relations, include: The polynomials of Bernoulli and Euler. The exponential generating function of the geometric polynomials of Bernoulli and Euler are given by (see [1, 23]):

$$\frac{\xi e^{x\xi}}{e^\xi - 1} = \sum_{\nu=0}^{\infty} B_\nu(x)\frac{\xi^\nu}{\nu!},\ |\xi| < 2\pi,$$

and

$$\frac{2e^{x\xi}}{e^\xi + 1} = \sum_{\nu=0}^{\infty} E_\nu(x)\frac{\xi^\nu}{\nu!},\ |\xi| < \pi.$$

It is known that the Bernoulli polynomials can be expressed in terms of the Bernoulli numbers $B_s$. In fact, for $\nu \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, using the generating function of the Bernoulli polynomials, we obtain

$$B_\nu(x) = \sum_{s=0}^{\nu} \binom{\nu}{s} B_s x^{\nu-s}.$$

Analogously, the Euler polynomials are given by

$$E_\nu(x) = \sum_{s=0}^{\nu} \binom{\nu}{s} \frac{E_s}{2^s}\left(x - \frac{1}{2}\right)^{\nu-s},$$

where $E_s$ are the Euler numbers. On the other hand, F. Costabile et al. [9] have presented multiple approaches to Appell polynomials using a determinant-based definition. Through the application of basic linear algebra techniques, these approaches have successfully recovered the essential properties of the polynomials. Furthermore, a triangular theorem establishes the equivalence between these different approaches. For example, the definition for Bernoulli polynomials using a determinantal approach is given by

$$B_0(x) = 1,$$

$$B_\nu(x) = \frac{(-1)^\nu}{(\nu-1)!} \begin{vmatrix} 1 & x & x^2 & \cdots & x^{\nu-1} & x^\nu \\ 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{\nu-1} & \frac{1}{\nu} \\ 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 2 & \cdots & \nu-1 & \nu \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ \vdots & \vdots & & & & \\ 0 & 0 & \cdots & \cdots & \binom{\nu-1}{\nu-2} & \binom{\nu}{\nu-2} \end{vmatrix}, \ \nu = 1, 2, 3, \cdots \tag{3}$$

Over the years, there have been further explorations and expansion of the aforementioned polynomials, leading to the inclusion of new families and generalizations [16]. Recently, H. Belbachir et al. [3] introduced and studied properties of a class of polynomials, $\mathcal{U}_\nu(x;\rho;\mu)$, called *unified Bernoulli-Euler polynomials of Apostol type* and defined by the following power series:

$$\frac{2-\mu+\frac{\mu}{2}\xi}{\rho e^\xi + (1-\mu)} e^{x\xi} = \sum_{\nu=0}^{\infty} \mathcal{U}_\nu(x;\rho;\mu)\frac{\xi^\nu}{\nu!}, \tag{4}$$

where

$$\left|\ln\left(\frac{\rho}{1-\mu}\right) + \xi\right| < \pi, \quad 0 \le \mu < 1,$$

and

$$\left|\ln\left(\frac{\rho}{\mu-1}\right) + \xi\right| < 2\pi, \quad \text{otherwise.}$$

Note that for particular values in the parameters $\mu$ and $\rho$, we can obtain in (4), the polynomials of Bernoulli and Euler (as well as Apostol-Bernoulli and Apostol-Euler, see [1]). However, this particular family does not take into account degenerate polynomials, and Apostol-type Hermite polynomials (called Hybrid polynomials by some authors) that have garnered the attention of various researchers and play an important role in many problems. In the paper [10], an extension of (4) to degenerate polynomials was already carried out but to the best of our knowledge, an extension with the Apostol-type Hermite polynomials has not been developed and remains an open problem.

It is important to highlight that the polynomial family introduced by H. Belbachir et al. does not constitute a unification of the aforementioned polynomial families. By applying the reduction method outlined in Theorem 4 by L. Navas et al., we can effectively reduce this polynomial family to a linear combination of Apostol-Euler and Apostol-Bernoulli polynomials (see [3, 17]).

$$\mathcal{U}_\nu(x;\rho;\mu) = \frac{1}{1-\mu}\left[\left(1-\frac{\mu}{2}\right)\mathfrak{E}_\nu\left(x;\frac{\rho}{1-\mu}\right) - \frac{\mu}{2}\mathfrak{B}_\nu\left(x;\frac{\rho}{\mu-1}\right)\right],$$

where

$$\frac{\xi e^{x\xi}}{\rho e^\xi - 1} = \sum_{\nu=0}^{\infty} \mathfrak{B}_\nu(x;\rho)\frac{\xi^\nu}{\nu!},$$

and

$$\frac{2e^{x\xi}}{\rho e^{\xi}+1}=\sum_{\nu=0}^{\infty}\mathfrak{E}_{\nu}(x;\rho)\frac{\xi^{\nu}}{\nu!}.$$

The primary objective of this paper is to define and explore an extension of Apostol-type Hermite polynomials utilizing the polynomials presented in equation (4). The properties of this polynomial family, which we shall refer to as Apostol-type Hermite-Bernoulli/Euler polynomials, are characterized by their generating functions, summation formulae, and determinant forms. These polynomials encompass classical Appell-type polynomials and their notable extensions, as they satisfy the differential equations (2). However, it is crucial to clarify that, within the scope of our study, we will utilize the polynomials presented in equation (4) without asserting them as a unification of pre-existing polynomial families.

On the other hand, the monomiality principle, in conjunction with the associated operational formalism, has proven to be a robust tool for probing the properties of a wide range of polynomials. This principle has been refined and elaborated upon by various researchers, further contributing to the understanding of the properties and behaviors of polynomials. In this paper, the derivative and multiplicative operators are established that allow the set of the Apostol-type Hermite-Bernoulli/Euler polynomials to be defined as quasi-monomial set.

In summary, this document provides an overview of the unified Apostol-type Hermite Bernoulli/Euler polynomials, their properties, and their applications. It also highlights the influence of previous research in the field and presents new findings related to the algebraic and differential properties of these polynomials. The study of these polynomials has been enriched by the exploration of the monomiality principle and its associated operational techniques, further contributing to the understanding of their properties and behaviors.

## 2. Apostol-type Hermite-Bernoulli/Euler polynomials

In this section, we define a new family of polynomials termed the Apostol-type Hermite-Bernoulli/Euler polynomials and delve into their algebraic and differential properties.

**Definition 1.** *Let $\rho>0$, $\mu\geq 0$ such that $\mu\neq 1$. We introduce the Apostol-type Hermite-Bernoulli/Euler polynomials as follows:*

$$\varphi(\rho,\mu,\xi)e^{x\xi+y\xi^2}=\sum_{\nu=0}^{\infty}\mathcal{U}_{\nu}(x,y;\rho;\mu)\frac{\xi^{\nu}}{\nu!}, \tag{5}$$

*where*

$$\varphi(\rho,\mu,\xi):=\frac{2-\mu+\frac{\mu}{2}\xi}{\rho e^{\xi}+(1-\mu)},$$

*as long as*

$$\left|\ln\left(\frac{\rho}{1-\mu}\right)+\xi\right|<\pi,\quad 0\leq\mu<1,$$

*and*

$$\left|\ln\left(\frac{\rho}{\mu-1}\right)+\xi\right|<2\pi, \quad \textit{otherwise.}$$

*Furthermore, the Apostol-type Hermite-Bernoulli/Euler numbers are given by*

$$\mathcal{U}_\nu(\rho;\mu) := \mathcal{U}_\nu(0,0;\rho;\mu). \tag{6}$$

In the following, we provide some illustrative examples showing the existence of polynomials $\mathcal{U}_n(x,y;\rho;\mu)$.

**Example 1.** *For $\rho=1, \mu=2$, we have*

| $\nu$ | $\mathcal{U}_\nu(x,y;1;2)$ |
|---|---|
| 0 | 1 |
| 1 | $x-\frac{1}{2}$ |
| 2 | $x^2-x+2y+\frac{1}{6}$ |
| 3 | $x^3-\frac{3}{2}x^2+(6y+\frac{1}{2})x+3y$ |
| 4 | $x^4-2x^3+(12y+1)x^2-12y^2+2y-\frac{1}{60}$ |

**Example 2.** *For $\rho=2, \mu=1$, we have*

| $\nu$ | $\mathcal{U}_\nu(x,y;2;1)$ |
|---|---|
| 0 | $\frac{1}{2}$ |
| 1 | $\frac{1}{2}x-\frac{1}{4}$ |
| 2 | $\frac{1}{2}x^2-\frac{1}{2}x+y$ |
| 3 | $\frac{1}{2}x^3-\frac{3}{4}x^2+3xy-\frac{3}{2}y+\frac{1}{4}$ |
| 4 | $\frac{1}{2}x^4-x^3+6yx^2-6yx+x+6y^2-\frac{1}{2}$ |

The characteristics of Hermite polynomials in two variables have a crucial role in investigating the Apostol-type Hermite-Bernoulli/Euler polynomials, offering valuable insights into their properties and behaviors. We recall that the Hermite polynomials in two variables, $H_\nu(x,y)$, satisfies the generating equation (see [6] and [7, Eq. 2]):

$$e^{x\xi+y\xi^2}=\sum_{\nu=0}^{\infty}H_\nu(x,y)\frac{\xi^\nu}{\nu!}. \tag{7}$$

Additionally,

$$H_0(x,y)=1$$

and the following identity is hold (see [5, Eq. 18]):

$$\frac{\partial}{\partial y}H_\nu(x,y) = \nu(\nu-1)H_{\nu-2}(x,y) = \frac{\partial^2}{\partial x^2}H_\nu(x,y). \tag{8}$$

Below, we elucidate several properties of the Apostol-type Hermite-Bernoulli/Euler polynomials using the generating function approach.

**Proposition 1.** *Let $\rho > 0$, $\mu \geq 0$ such that $\mu \neq 1$. The following relationship holds:*

$$\mathcal{U}_\nu(x+z, y+w; \rho; \mu) = \sum_{k=0}^{\nu} \binom{\nu}{k} H_{\nu-k}(z,w)\mathcal{U}_k(x,y;\rho;\mu), \tag{9}$$

*where $H_k$ are the Hermite polynomials.*

*Proof.* By the following identity (see [14, p. 18, Eq. 0.36] and [4, p. 463, Def. 9.4.6]):

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{n-k}b_k, \tag{10}$$

and the generating functions (5) and (7), we have

$$\begin{aligned}
\sum_{\nu=0}^{\infty}\mathcal{U}_\nu(x+z,y+w;\rho;\mu)\frac{\xi^\nu}{\nu!} &= \varphi(\rho,\mu,\xi)e^{\xi x+\xi^2 y}e^{\xi z+\xi^2 w} \\
&= \left(\sum_{\nu=0}^{\infty}\mathcal{U}_\nu(x,y;\rho;\mu)\frac{\xi^\nu}{\nu!}\right)\left(\sum_{\nu=0}^{\infty}H_\nu(z,w)\frac{\xi^\nu}{\nu!}\right) \\
&= \sum_{\nu=0}^{\infty}\left(\sum_{k=0}^{\nu}\binom{\nu}{k}H_{\nu-k}(z,w)\mathcal{U}_k(x,y;\rho;\mu)\right)\frac{\xi^\nu}{\nu!}.
\end{aligned}$$

By utilizing the product series and subsequently equating the coefficients of $\xi^\nu/\nu!$ on both sides, we derive the identity.

**Remark 1.** *If $x := 0$, $z := x$, $y := 0$ and $w := y$ in (9), then the identity becomes*

$$\mathcal{U}_\nu(x,y;\rho;\mu) = \sum_{k=0}^{\nu}\binom{\nu}{k}H_{\nu-k}(x,y)\mathcal{U}_k(\rho;\mu). \tag{11}$$

**Remark 2.** *If we substitute $z := -x$ and $w := -y$ into equation (9), we can represent the Apostol-type Hermite-Bernoulli/Euler numbers as a function of the corresponding Apostol-type Hermite-Bernoulli/Euler polynomials:*

$$\mathcal{U}_\nu(\rho;\mu) = \sum_{k=0}^{\nu}\binom{\nu}{k}H_{\nu-k}(-x,-y)\mathcal{U}_k(x,y;\rho;\mu).$$

Here, we present the result of the convolution involving the Apostol-type Hermite-Bernoulli/Euler polynomials.

**Proposition 2.** *The following identity holds:*

$$\sum_{\omega=0}^{\nu}\binom{\nu}{\omega}\mathcal{U}_{\nu-\omega}(x,y;\rho;\mu)\mathcal{U}_{\omega}(x,y;\rho;\mu)=\sum_{\omega=0}^{\nu}\binom{\nu}{\omega}\mathcal{U}_{\nu-\omega}(\rho,\mu)\mathcal{U}_{\omega}(2x,2y;\rho;\mu).$$

*Proof.* By (10) and (6), we have

$$\begin{aligned}\sum_{\nu=0}^{\infty}\sum_{\omega=0}^{\nu}\binom{\nu}{\omega}\mathcal{U}_{\nu-\omega}(x,y;\rho;\mu)\mathcal{U}_{\omega}(x,y;\rho;\mu)\frac{\xi^{\nu}}{\nu!}&=\sum_{\nu=0}^{\infty}\mathcal{U}_{\nu}(x,y;\rho;\mu)\frac{\xi^{\nu}}{\nu!}\sum_{\nu=0}^{\infty}\mathcal{U}_{\nu}(x,y;\rho;\mu)\frac{\xi^{\nu}}{\nu!}\\&=\varphi^{2}(\rho,\mu,\xi)e^{2x\xi+2y\xi^{2}}\\&=\sum_{\nu=0}^{\infty}\mathcal{U}_{\nu}(\rho,\mu)\frac{\xi^{\nu}}{\nu!}\sum_{\nu=0}^{\infty}\mathcal{U}_{\nu}(2x,2y;\rho;\mu)\frac{\xi^{\nu}}{\nu!}\\&=\sum_{\nu=0}^{\infty}\sum_{\omega=0}^{\nu}\binom{\nu}{\omega}\mathcal{U}_{\nu-\omega}(\rho,\mu)\mathcal{U}_{\omega}(2x,2y;\rho;\mu)\frac{\xi^{\nu}}{\nu!}.\end{aligned}$$

By comparing the coefficients of $\frac{\xi^{\nu}}{\nu!}$ on both sides of the equation above, we derive the identity.

For the subsequent property, we employ the following identity [24, p. 52]:

$$\sum_{\nu=0}^{\infty}f(\nu)\frac{(x+y)^{\nu}}{\nu!}=\sum_{l,m=0}^{\infty}f(l+m)\frac{x^{l}y^{m}}{l!m!}. \tag{12}$$

**Proposition 3.** *The following implicit summation formula for Apostol-type Hermite-Bernoulli/Euler polynomials $\mathcal{U}_{\nu}(x,y;\rho;\mu)$ holds:*

$$\mathcal{U}_{l+m}(z,y;\rho;\mu)=\sum_{p,q=0}^{l,m}\binom{l}{p}\binom{m}{q}(z-x)^{p+q}\mathcal{U}_{l+m-(p+q)}(x,y;\rho;\mu).$$

*Proof.* By (12), we have

$$\begin{aligned}\varphi(\rho,\mu,\xi+t)e^{x(\xi+t)+y(\xi+t)^{2}}&=\sum_{\nu=0}^{\infty}\mathcal{U}_{\nu}(x,y;\rho;\mu)\frac{(\xi+t)^{\nu}}{\nu!}\\&=\sum_{l,m=0}^{\infty}\mathcal{U}_{l+m}(x,y;\rho;\mu)\frac{\xi^{l}t^{m}}{l!m!},\end{aligned}$$

or equivalently

$$\varphi(\rho,\mu,\xi+t)e^{y(\xi+t)^2} = e^{-x(\xi+t)}\sum_{l,m=0}^{\infty}\mathcal{U}_{l+m}(x,y;\rho;\mu)\frac{\xi^l t^m}{l!m!}. \tag{13}$$

By performing the previous procedure, but replacing $x$ with $z$, we now obtain

$$\varphi(\rho,\mu,\xi+t)e^{y(\xi+t)^2} = e^{-z(\xi+t)}\sum_{l,m=0}^{\infty}\mathcal{U}_{l+m}(z,y;\rho;\mu)\frac{\xi^l t^m}{l!m!}. \tag{14}$$

Then, by equating equations (13) and (14), we get

$$\sum_{l,m=0}^{\infty}\mathcal{U}_{l+m}(z,y;\rho;\mu)\frac{\xi^l t^m}{l!m!} = e^{(z-x)(\xi+t)}\sum_{l,m=0}^{\infty}\mathcal{U}_{l+m}(x,y;\rho;\mu)\frac{\xi^l t^m}{l!m!}.$$

Now, by the Exponential Series, (12) and (10), we obtain

$$\begin{aligned}\sum_{l,m=0}^{\infty}\mathcal{U}_{l+m}(z,y;\rho;\mu)\frac{\xi^l t^m}{l!m!} &= \sum_{\nu=0}^{\infty}(z-x)^{\nu}\frac{(\xi+t)^{\nu}}{\nu!}\sum_{l,m=0}^{\infty}\mathcal{U}_{l+m}(x,y;\rho;\mu)\frac{\xi^l t^m}{l!m!}\\ &= \sum_{p,q=0}^{\infty}(z-x)^{p+q}\frac{\xi^p t^q}{p!q!}\sum_{l,m=0}^{\infty}\mathcal{U}_{l+m}(x,y;\rho;\mu)\frac{\xi^l t^m}{l!m!}\\ &= \sum_{l,m=0}^{\infty}\sum_{p,q=0}^{l,m}\binom{l}{p}\binom{m}{q}(z-x)^{p+q}\mathcal{U}_{l+m-(p+q)}(x,y;\rho;\mu)\frac{\xi^l t^m}{l!m!}.\end{aligned}$$

By comparing the coefficients of $\dfrac{\xi^l t^m}{l!m!}$ on both sides of the equation above, we derive the identity.

Below, we introduce both the differentiation and integration of the Apostol-type Hermite-Bernoulli/Euler polynomials.

**Proposition 4.** *Let $\rho > 0$, $\mu \geq 0$ such that $\mu \neq 1$. The following properties are maintained:*

$$\begin{aligned}&\frac{\partial}{\partial x}\mathcal{U}_{\nu}(x,y;\rho;\mu) = \nu\mathcal{U}_{\nu-1}(x,y;\rho;\mu), \qquad (15)\\ &\frac{\partial}{\partial y}\mathcal{U}_{\nu}(x,y;\rho;\mu) = \nu(\nu-1)\mathcal{U}_{\nu-2}(x,y;\rho;\mu).\end{aligned}$$

*Proof.* Initially, notice that

$$\frac{\partial}{\partial x}\sum_{\nu=0}^{\infty}\mathcal{U}_{\nu}(x,y;\rho,\mu)\frac{\xi^{\nu}}{\nu!} = \frac{\partial}{\partial x}\sum_{\nu=1}^{\infty}\mathcal{U}_{\nu}(x,y;\rho;\mu)\frac{\xi^{\nu}}{\nu!}. \tag{16}$$

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial x}\varphi(\rho,\mu,\xi)e^{x\xi+y\xi^2} &= \sum_{\nu=0}^{\infty}\mathcal{U}_\nu(x,y;\rho;\mu)\frac{\xi^{\nu+1}}{\nu!}\\ &= \sum_{\nu=1}^{\infty}\nu\mathcal{U}_{\nu-1}(x,y;\rho;\mu)\frac{\xi^{\nu}}{\nu!}. \end{aligned} \tag{17}$$

By comparing (16) and (17), we obtain (15). Now,

$$\begin{aligned}\frac{\partial}{\partial y}\varphi(\rho,\mu,\xi)e^{x\xi+y\xi^2} &= \sum_{\nu=0}^{\infty}\mathcal{U}_\nu(x,y;\rho;\mu)\frac{\xi^{\nu+2}}{\nu!}\\ &= \sum_{\nu=2}^{\infty}\mathcal{U}_{\nu-2}(x,y;\rho;\mu)\frac{\xi^{\nu}}{(\nu-2)!}\\ &= \sum_{\nu=0}^{\infty}\nu(\nu-1)\mathcal{U}_{\nu-2}(x,y;\rho;\mu)\frac{\xi^{\nu}}{\nu!}.\end{aligned}$$

**Remark 3.** *An alternative method to compute the derivative with respect to $y$ is by utilizing the representation* (11) *and employing the identity* (8)*. Note that,*

$$\begin{aligned}\frac{\partial}{\partial y}\mathcal{U}_\nu(x,y;\rho;\mu) &= \sum_{k=0}^{\nu}\binom{\nu}{k}\mathcal{U}_\nu(\rho;\mu)\frac{\partial^2}{\partial x^2}H_k(x,y)\\ &= \sum_{k=2}^{\nu}\binom{\nu}{k}k(k-1)\mathcal{U}_\nu(\rho;\mu)H_{k-2}(x,y).\end{aligned}$$

**Remark 4.** *Note that by repeatedly differentiating with respect to $x$ and applying the induction principle on $m$, we can obtain the $m$-th order derivative of the polynomial:*

$$\frac{\partial^l}{\partial x^l}\mathcal{U}_\nu(x,y;\rho;\mu) = (\nu)_l\mathcal{U}_{\nu-l}(x,y;\rho;\mu),$$

*where $(\nu)_l := \nu(\nu-1)\cdots(\nu-l+1)$ and $0 \le l$.*

**Proposition 5.** *Let $\rho > 0$, $\mu \ge 0$ such that $\mu \ne 1$. Then*

$$\int_{x_0}^{x_1}\mathcal{U}_\nu(x,y;\rho;\mu)\,dx = \frac{1}{\nu+1}\left[\mathcal{U}_{\nu+1}(x_1,y;\rho;\mu) - \mathcal{U}_{\nu+1}(x_0,y;\rho;\mu)\right].$$

*Proof.* The result can be readily inferred from Proposition 4.

In [9], the authors presented a general approach for determining the Appell polynomials that fulfill the recursive relations. Essentially, provided a method for finding these polynomials by using a power series expression. Following that approach, we have

$$\varphi(\rho,\mu,\xi)=\sum_{n=0}^{\infty}\mathcal{U}_n(\rho;\mu)\frac{\xi^n}{n!}.$$

Now, let $\zeta$ a function give by Taylor series expansion (in $\xi$) at the origin, that is

$$\zeta(\xi):=\sum_{n=0}^{\infty}\delta_n\frac{\xi^n}{n!}, \tag{18}$$

such that $\varphi(\rho,\mu,\xi)\zeta(\xi)=1$, where $\delta_n$ is a sequence. Then, applying the rules of Cauchy product (10), we obtain

$$\varphi(\rho,\mu,\xi)\zeta(\xi)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{U}_k(\rho;\mu)\delta_{n-k}\frac{\xi^k}{k!}.$$

Thus,

$$\sum_{k=0}^{n}\binom{n}{k}\mathcal{U}_k(\rho;\mu)\delta_{n-k}=\begin{cases}1, & \text{for} \quad n=0,\\ 0, & \text{for} \quad n>0.\end{cases}$$

Hence,

$$\begin{cases}\delta_0=\dfrac{1}{\mathcal{U}_0},\\ \delta_n=-\dfrac{1}{\mathcal{U}_0}\left(\displaystyle\sum_{k=1}^{n}\binom{n}{k}\mathcal{U}_k(\rho;\mu)\delta_{n-k}\right),\end{cases}$$

where $\mathcal{U}_0:=\mathcal{U}_0(\rho,\mu)$.

**Proposition 6.** *The following identity hold:*

$$\mathcal{U}_0(x,y;\rho;\mu)=\frac{1}{\delta_0}.$$

$$\mathcal{U}_n(x,y;\rho;\mu)=\frac{(-1)^n}{\delta_0^{n+1}}\begin{vmatrix} H_0(x,y) & H_1(x,y) & \cdots & \cdots & H_{n-1}(x,y) & H_n(x,y)\\ \delta_0 & \delta_1 & \cdots & \cdots & \delta_{n-1} & \delta_n\\ 0 & \delta_0 & \cdots & \cdots & \binom{n-1}{1}\delta_{n-2} & \binom{n}{1}\delta_{n-1}\\ 0 & 0 & \ddots & & \binom{n-1}{2}\delta_{n-3} & \binom{n}{2}\delta_{n-2}\\ \vdots & \vdots & & \ddots & \vdots & \vdots\\ \vdots & \vdots & & & & \\ 0 & \cdots & \cdots & \cdots & \delta_0 & \binom{n}{n-1}\delta_1\end{vmatrix}. \tag{19}$$

*Proof.* Observe that

$$\left(\sum_{n=0}^{\infty}\mathcal{U}_n(\rho;\mu)\frac{\xi^n}{n!}\right)\left(\sum_{n=0}^{\infty}H_n(x,y)\frac{\xi^n}{n!}\right)=\sum_{n=0}^{\infty}\mathcal{U}_n(x,y;\rho;\mu)\frac{\xi^n}{n!}. \tag{20}$$

Multiplying both sides of Equation (20) by (18), we obtain

$$\sum_{n=0}^{\infty}H_n(x,y)\frac{\xi^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{U}_k(x,y;\rho;\mu)\delta_{n-k}\frac{\xi^k}{k!}.$$

By multiplying the aforementioned equation, we arrive at the subsequent infinite system of equations in the unknown variables:

$$\begin{aligned}
H_0(x,y) &= \mathcal{U}_0(x,y;\rho;\mu)\delta_0,\\
H_1(x,y) &= \mathcal{U}_0(x,y;\rho;\mu)\delta_1+\mathcal{U}_1(x,y;\rho;\mu)\delta_0,\\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots\\
H_n(x,y) &= \mathcal{U}_0(x,y;\rho;\mu)\delta_n+\binom{n}{1}\mathcal{U}_1(x,y;\rho;\mu)\delta_0+\cdots+\mathcal{U}_n(x,y;\rho;\mu)\delta_0.
\end{aligned}$$

Due to the specific structure of the aforementioned system (lower triangular), we can determine the unknown variables $\mathcal{U}_n(x,y;\rho,\mu)$ by exclusively utilizing the first $n+1$ equations. This can be achieved by employing Cramer's rule, which facilitates the computation of the solution.

$$\mathcal{U}_n(x,y;\rho,\mu)=\frac{1}{\delta_0^{n+1}}\begin{vmatrix}
\delta_0 & 0 & 0 & 0 & \cdots & H_0(x,y)\\
\delta_1 & \delta_0 & 0 & 0 & \cdots & H_1(x,y)\\
\delta_2 & \binom{2}{1}\delta_1 & \delta_0 & 0 & \cdots & H_2(x,y)\\
\vdots & \vdots & & \ddots & & \vdots\\
\delta_{n-1} & \binom{n-1}{1}\delta_{n-2} & \binom{n-2}{2}\delta_{n-3} & \cdots & \cdots & H_{n-1}(x,y)\\
\delta_n & \binom{n}{1}\delta_{n-1} & \binom{n}{2}\delta_{n-2} & \binom{n}{3}\delta_{n-3} & \cdots & H_n(x,y)
\end{vmatrix}.$$

By transposition of the previous, we obtain

$$\mathcal{U}_n(x,y;\rho,\mu)=\frac{1}{\delta_0^{n+1}}\begin{vmatrix}
\delta_0 & \delta_1 & \delta_2 & \cdots & \delta_{n-1} & \delta_n\\
0 & \delta_0 & \binom{2}{1}\delta_1 & \cdots & \binom{n-1}{1}\delta_{n-2} & \binom{n}{1}\gamma_{n-1}\\
0 & 0 & \delta_0 & \cdots & \binom{n-1}{2}\delta_{n-3} & \binom{n}{2}\delta_{n-2}\\
. & . & . & & . & .\\
. & . & . & \cdots & . & .\\
. & . & . & & . & .\\
0 & 0 & 0 & \cdots & \delta_0 & \binom{n}{n-1}\delta_1\\
H_0(x) & H_1(x) & H_2(x) & \cdots & H_{n-1}(x) & H_n(x)
\end{vmatrix}.$$

Now, by moving the $i$th row to the $(i+1)$th position, where $i=1,2,\cdots,n$, we get the desired result asserted.

Now, we will proceed with the determinant representation for the one specific case of the polynomials illustrated in Examples 1.

**Example 3.** *For $\rho = 1$ and $\mu = 2$, we have*

$$\begin{aligned}
&\mathcal{U}_0(x, y; 1; 2) = 1,\\
&\mathcal{U}_1(x, y; 1; 2) = -\begin{vmatrix} 1 & x \\ 1 & \frac{1}{2} \end{vmatrix},\\
&\mathcal{U}_2(x, y; 1; 2) = \begin{vmatrix} 1 & x & x^2+2y \\ 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 1 \end{vmatrix},\\
&\mathcal{U}_3(x, y; 1; 2) = -\begin{vmatrix} 1 & x & x^2+2y & x^3+6xy \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{3}{2} \end{vmatrix}.
\end{aligned}$$

**Example 4.** *When $y = 0$, with $\rho = 1$ and $\mu = 2$, the Bernoulli polynomials are expressed in determinant form as shown in* (3).

To better understand the following result, it is important to recall that the Apostol-type Hermite-Bernoulli polynomials are given by (refer to [12]):

$$\frac{\xi e^{x\xi + y\xi^2}}{\lambda e^{\xi} - 1} = \sum_{\nu=0}^{\infty} \mathcal{B}_\nu(x, y; \lambda)\frac{\xi^\nu}{\nu!}, \ |\xi + \ln(\lambda)| < 2\pi,$$

and the Apostol-type Hermite-Euler polynomials are given by (see [19]):

$$\frac{2e^{x\xi + y\xi^2}}{\lambda e^{\xi} + 1} = \sum_{\nu=0}^{\infty} \mathcal{E}_\nu(x, y; \lambda)\frac{\xi^\nu}{\nu!}, \ |\xi + \ln(\lambda)| < \pi.$$

**Proposition 7.** *Let $\mu > 1$. Then,*

$$\mathcal{U}_\nu(x, y; \rho; \mu) = \frac{1}{1-\mu}\left[(2-\mu)\mathcal{E}_\nu(x, y; \lambda) - \frac{\mu}{2}\mathcal{B}_\nu(x, y; -\lambda)\right],$$

*where $\lambda := \dfrac{\rho}{1-\mu}$.*

*Proof.* Note that,

$$\begin{aligned}
\frac{2-\mu+\frac{\mu}{2}\xi}{\rho e^{\xi} + (1-\mu)} &= \frac{(1-\mu) + 1 + \frac{\mu}{2}\xi}{(1-\mu)\left(\dfrac{\rho}{1-\mu}e^{\xi} + 1\right)}\\
&= \frac{1 + \dfrac{1+\frac{\mu}{2}\xi}{1-\mu}}{\dfrac{\rho}{1-\mu}e^{\xi} + 1} = \frac{1 + \dfrac{2+\mu\xi}{2(1-\mu)}}{\dfrac{\rho}{1-\mu}e^{\xi} + 1}.
\end{aligned}$$

Let $\lambda := \dfrac{\rho}{1-\mu}$. Then,

$$\begin{aligned}\frac{2-\mu+\frac{\mu}{2}\xi}{\rho e^{\xi}+(1-\mu)} &= \frac{1}{\lambda e^{\xi}+1}+\frac{1}{(1-\mu)(\lambda e^{\xi}+1)}+\frac{\mu\xi}{2(1-\mu)(\lambda e^{\xi}+1)}\\ &= \frac{2-\mu}{1-\mu}\frac{1}{\lambda e^{\xi}+1}-\frac{\mu}{2(1-\mu)}\frac{\xi}{(-\lambda)e^{\xi}+1}.\end{aligned}$$

It follows from (5) that

$$\begin{aligned}\sum_{\nu=0}^{\infty}\mathcal{U}_{\nu}(x,y;\rho;\mu)\frac{\xi^{\nu}}{\nu!} &= \frac{2-\mu}{1-\mu}\frac{1}{\lambda e^{\xi}+1}e^{x\xi+y\xi^2}-\frac{\mu}{2(1-\mu)}\frac{\xi}{(-\lambda)e^{\xi}+1}e^{x\xi+y\xi^2}\\ &= \frac{2-\mu}{1-\mu}\sum_{\nu=0}^{\infty}\mathcal{E}_{\nu}(x,y;\lambda)\frac{\xi^{\nu}}{\nu!}-\frac{\mu}{2(1-\mu)}\sum_{\nu=0}^{\infty}\mathcal{B}_{\nu}(x,y;-\lambda)\frac{\xi^{\nu}}{\nu!}.\end{aligned}$$

**Remark 5.** *Based on the earlier findings, it can be asserted that the Apostol-type Hermite-Bernoulli/Euler polynomials can be expressed as a linear combination of the Hermite-Bernoulli and Hermite-Euler polynomials for $\mu > 1$. Some sources refer to this phenomenon as unification, which is the rationale behind our chosen nomenclature.*

## 3. Monomiality Principle

The concepts of *quasi-monomial* and the *monomiality principle* are indeed technical and may require further elaboration for readers unfamiliar with them. In our work, we have outlined the monomiality principle as a framework that generalizes the behavior of special polynomials through abstract definitions of derivative and multiplicative operators, treating these polynomials analogously to ordinary monomials. This principle extends the Heisenberg–Weyl group, allowing for a unified examination of diverse polynomial families and their properties.

Additionally, we reference foundational works [8, 11, 13, 20–22] that provide a deeper exploration of this principle and its applications.

The operators $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ function dually as both multiplicative and derivative operators within the context of a polynomial set $\{b_m(u)\}_{m\in\mathbb{N}}$, adhering to the following expressions:

$$b_{m+1}(u) = \hat{\mathcal{M}}\{b_m(u)\} \tag{21}$$

and

$$m\ b_{m-1}(u) = \hat{\mathcal{D}}\{b_m(u)\}.$$

The set $\{b_m(u)\}_{m\in\mathbb{N}}$ manipulated by these operators is termed a quasi-monomial and must adhere to the formula:

$$[\hat{\mathcal{D}},\hat{\mathcal{M}}] = \hat{\mathcal{D}}\hat{\mathcal{M}} - \hat{\mathcal{M}}\hat{\mathcal{D}} = \hat{1},$$

displaying a Weyl group structure. The properties of $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ determine the characteristics of the quasi-monomial set $\{b_m(u)\}_{m\in\mathbb{N}}$: For example, $b_m(u)$ satisfies the differential equation

$$\hat{\mathcal{M}}\hat{\mathcal{D}}\{b_m(u)\} = mb_m(u),$$

if $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ have differential realizations.

**Theorem 1.** *The operators $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ associated with the Apostol-type Hermite-Bernoulli/Euler polynomials $\mathcal{U}_\nu(x, y; \rho; \mu)$ are given by*

$$\hat{\mathcal{M}} := \psi(\rho, \mu, \xi) + x + 2y\frac{\partial}{\partial x}$$

*and*

$$\hat{\mathcal{D}} := \frac{\partial}{\partial x}.$$

*where*

$$\psi(\rho, \mu, \xi) := \frac{\mu/2}{2 - \mu + \frac{\mu}{2}\xi} - \frac{\rho e^\xi}{\rho e^\xi + 1 - \mu}.$$

*Proof.* Differentiating the generating relation (5) with respect to the variable $\xi$, it follows that

$$\frac{\partial}{\partial \xi}\left(\varphi(\rho, \mu, \xi)e^{x\xi+y\xi^2}\right) = \sum_{\nu=0}^{\infty}\mathcal{U}_{\nu+1}(x, y; \rho; \mu)\frac{\xi^\nu}{\nu!}.$$

Now, since

$$\frac{\partial}{\partial \xi}\left(\varphi(\rho, \mu, \xi)e^{x\xi+y\xi^2}\right) = \left(\frac{\mu/2}{2 - \mu + \frac{\mu}{2}\xi} - \frac{\rho e^\xi}{\rho e^\xi + 1 - \mu} + x + 2y\xi\right)\left(\varphi(\rho, \mu, \xi)e^{x\xi+y\xi^2}\right),$$

then

$$\sum_{\nu=0}^{\infty}\left(\frac{\mu/2}{2 - \mu + \frac{\mu}{2}\xi} - \frac{\rho e^\xi}{\rho e^\xi + 1 - \mu} + x + 2y\frac{d}{dx}\right)\mathcal{U}_\nu(x, y; \rho; \mu)\frac{\xi^\nu}{\nu!} = \sum_{\nu=0}^{\infty}\mathcal{U}_{\nu+1}(x, y; \rho; \mu)\frac{\xi^\nu}{\nu!}. \tag{22}$$

By equating the coefficients of corresponding powers of $\xi$ on both sides of Equation (22) and applying the monomiality principle equation (21), we deduce the operator $\hat{\mathcal{M}}$. Additionally, Proposition 4 establishes that $\hat{\mathcal{D}} = \frac{\partial}{\partial x}$.

**Proposition 8.** *The Apostol-type Hermite-Bernoulli/Euler polynomials satisfy the succeeding differential equation:*

$$\left[(\psi(\rho, \mu, \xi) + x)\frac{\partial}{\partial x} + 2y\frac{\partial^2}{\partial x^2}\right]\mathcal{U}_\nu(x, y; \rho; \mu) = \nu\mathcal{U}_\nu(x, y; \rho; \mu).$$

*Proof.* The outcome is instantaneous given that

$$\hat{\mathcal{D}}\mathcal{U}_\nu(x, y; \rho; \mu) = \nu\mathcal{U}_{\nu-1}(x, y; \rho; \mu)$$

and

$$\hat{\mathcal{M}}\mathcal{U}_{\nu-1}(x, y; \rho; \mu) = \nu\mathcal{U}_\nu(x, y; \rho; \mu).$$

## 4. Conclusions

In this work, we introduced a novel class of polynomials, the Apostol-type Hermite-Bernoulli/Euler polynomials, denoted as $\mathcal{U}_\nu(x, y; \rho; \mu)$, and explored their fundamental properties. These polynomials were defined via a generating function, enabling us to derive their summation formulae and determinant forms. This new family not only generalizes the classical Appell-type polynomials but also extends their applicability in mathematical analysis.

The generating function techniques employed in this study proved instrumental in establishing the key properties of these polynomials. Additionally, the introduction of derivative and multiplicative operators facilitated their representation as a quasi-monomial set, thereby expanding their potential applications in various branches of mathematics and related fields.

The illustrative examples provided throughout the paper demonstrate the validity and versatility of the results, paving the way for further investigations into the applications and extensions of these polynomials.

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