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High-frequency magnetohydrodynamics

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A B S T R A C T

We consider the case that the *displacement current* is not neglected in the classical MHD equations, as it is usually done. This amounts to cast them in the relativistic framework of a finite speed of light. We show some consequences in describing magnetic reconnection phenomena and for hydromagnetic waves. In the first case, the equation for the magnetic induction is changed from (formally) parabolic to (formally) hyperbolic, in the second case both, the perturbed magnetic field and the particle velocity, obey to a certain third-order in time partial differential equation, rather than to the classical wave equation. We stress the role of two typically small but nonzero parameters, the magnetic diffusivity, η (corresponding to large values of the Lundquist number), and $\varepsilon := c^{-2}$.

1. Introduction

Within the set of the magnetohydrodynamics (MHD) equations, the *displacement current* is usually neglected in the Maxwell equations [6, Ch. 4, § 4.1, p. 112]. The ensuing results would be rather different in several respects in comparison to when this assumption is not made. These changes include the description of *high frequency* phenomena, relevant, in particular, in space and laser plasmas in ultrarelativistic limits and in astrophysics; see, e.g., [3,8], where the role of the displacement current is important. In this paper, we will show how the equations governing magnetic reconnection as well as some typical hydromagnetic waves will change if the displacement current is retained. We use the wording "high-frequency MHD" theory to refer to the case when the contribution due to the displacement current is not neglected (clearly related to the relativistic framework of $c < \infty$), since it is likely that such a term will be important in a high frequencies regime, or perhaps with high phase velocities.

In case of magnetic reconnection, which is a mechanism considered responsible of a number of phenomena such as, e.g., solar flares and gamma ray bursts in astrophysics, and a certain nuclear fusion dynamical behavior in plasma laboratory, when a sudden change in magnetic topology transforms magnetic into kinetic energy and then particle acceleration is observed, the (formally) parabolic equation satisfied by the magnetic field becomes (formally) hyperbolic. Moreover, in case of hydromagnetic waves, both, the perturbed magnetic field and the particle velocity obey now to a third-order in time partial differential equation, rather than to the classical wave equation. Third-order in time partial differential equations are encountered elsewhere, in mathematical physics, for instance in linear and nonlinear acoustics. These are for instance the Moore-Gibson-Thomson equation and the Jordan-Moore-Gibson-Thomson, which govern the acoustic velocity potential [13,20]. Another example is given by the Jeffreys-type equations modeling heat conduction [9, sec. III, p. 44, eq. (3.1)].

Several papers on the Moore-Gibson-Thomson and the Jordan-Moore-Gibson-Thomson equations have appeared even very recently, to the point that this subject can now be considered an active research subject. We mention [11], where the limiting behavior of solutions is studied as the sound diffusivity vanishes, and many relevant previous works are quoted. These include, in particular, the pioneering works by Moore and Gibson, Thomson, Morrison, and Jordan. In [12], some general linear third-order in time equations, extending the classical wave equations, were studied asymptotically as the relaxation time, which multiplies the highest derivatives, vanishes; in [10,2], the limiting behavior of solutions for vanishing relaxation time was considered in case of nonlinear acoustics.

In this paper, we consider two examples where the aforementioned inclusion of the displacement current term in the Maxwell equations within the MHD model, plays a role. The first one is of interest in describing magnetic reconnection phenomena, where the formally "parabolic" equation for the magnetic field, **B**, be-

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comes (also formally) “hyperbolic” (section 2), the second one shows the emergence of a modified Alfvén equation, which has the form of a third-order in time equation (section 3). A short summary section (section 4) concludes the paper.

2. Magnetic reconnection in a high-frequency regime

Adopting *SI units*, the Maxwell equations read

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \tag{2}$$

being $\mathbf{B} = \mu_0 \mathbf{H}$ and $\mathbf{D} = \varepsilon_0 \mathbf{E}$ the displacement vector. We differentiate both sides of equation (1) with respect to time, obtaining

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = -\nabla \times \frac{\partial \mathbf{E}}{\partial t} = -c^2\, \nabla \times \nabla \times \mathbf{B} + c^2\, \mu_0\, \nabla \times \mathbf{J}. \tag{3}$$

Then, we use the generalized Ohm's law,

$$\mathbf{J} = \sigma\, (\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{f}, \tag{4}$$

where σ denotes the electrical conductivity of the medium, and for an ion-electron plasma we assume $\sigma = \frac{n_e e^2}{m_e \nu_c}$, where n_e and m_e are the electron number density and (rest) mass, e is the electric charge, and the ν_c the Coulomb collision frequency. We have collected in the vector $\mathbf{f}$ all forcing terms (the driving factors) that there may enter such a generalized physical law. For instance, these might be additional terms, such as pressure gradients (hence, possibly density and/or temperature gradients), Hall, or inertial terms; see [4, eq. (IV.7), p. 459], and [1, Ch. 6, eq. (6.19), p. 204], e.g., but for more general two-fluids models. Thus, we obtain

$$\begin{aligned}\frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c^2 \left[\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}\right] + c^2\, \mu_0\, \sigma\, \nabla \times (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\ &\quad + c^2\, \mu_0\, \nabla \times \mathbf{f} \\ &= c^2\, \nabla^2 \mathbf{B} - c^2\, \mu_0\, \sigma\, \frac{\partial \mathbf{B}}{\partial t} + c^2\, \mu_0\, \sigma\, \nabla \times (\mathbf{v} \times \mathbf{B}) \\ &\quad + c^2\, \mu_0\, \nabla \times \mathbf{f},\end{aligned}$$

since $\nabla \cdot \mathbf{B} = 0$ and we used again equation (1). Therefore, we have

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} + c^2\, \mu_0\, \sigma\, \frac{\partial \mathbf{B}}{\partial t} = c^2\, \nabla^2 \mathbf{B} + c^2\, \mu_0\, \sigma\, \nabla \times (\mathbf{v} \times \mathbf{B}) + c^2\, \mu_0\, \nabla \times \mathbf{f},$$

and finally, noticing that $c^2 \mu_0 = 1/\varepsilon_0$, being $c = 1/\sqrt{\varepsilon_0\, \mu_0}$ the speed of light, we end up with

$$\frac{\eta}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} + \frac{\partial \mathbf{B}}{\partial t} = \eta\, \nabla^2 \mathbf{B} + \nabla \times (\mathbf{v} \times \mathbf{B}) + \mathbf{g}, \tag{5}$$

where we set $\mathbf{g} := \sigma^{-1}\, \nabla \times \mathbf{f}$ and introduced the *magnetic diffusivity*

$$\eta := \frac{1}{\mu_0 \sigma}, \tag{6}$$

(measured in m^2/s, in SI units), which is inversely proportional to the electrical conductivity of the medium. For an ion-electron plasma we assume $\sigma = \frac{n_e e^2}{m_e \nu_c}$, where n_e and m_e are the electron number density and (rest) mass, e is the electric charge, and the ν_c the Coulomb collision frequency. Hence, η is proportional to the collision frequency, ν_c.

Note that equation (5) is a [formally] *hyperbolic* equation for $\mathbf{B}$, which reduces to the well-known [formally] *parabolic* equation

$$\frac{\partial \mathbf{B}}{\partial t} = \eta\, \nabla^2 \mathbf{B} + \nabla \times (\mathbf{v} \times \mathbf{B}), \tag{7}$$

found within the classical resistive MHD theory, obtained when the displacement current vector is neglected in (2), and $\mathbf{f} = \mathbf{0}$ (for simplicity).

Clearly, the parameter $\eta > 0$ may be small, even very small ($\eta \to 0$ as $\sigma \to \infty$, in case of ideal nonresistive, that is, perfectly collisionless MHD, and both equations, (5) and (7), look *singularly perturbed* as $\eta \to 0^+$. For any fixed $\eta > 0$, however, in (5) we face a singularly perturbed problem as $c \to \infty$, that is in the “nonrelativistic limit”. Referring to time-harmonic fields, so that $\partial/\partial t \sim -i\omega$, the possibly small parameter will be $\eta\, \omega^2/c^2$, rather than $1/c^2$, to be compared to ω, see (5).

That singularly perturbed problems imply, in general, the existence of a boundary (actually, here, an initial) layer in order to match the solutions of the “hyperbolic” to that of the “parabolic” equation for $\mathbf{B}$, would deserve a separate investigation. This is relevant to establish whether the solution to the former equation converges uniformly or not to that of the latter equation, thus allowing trivial a straightforward approximation or not. In two cases, that of Cattaneo-Maxwell-Vernotte and that of the hyperbolic Schrödinger equations, discussed respectively in [7] and [16], it was shown that no initial layer is required for the initial-value problem. This behavior cannot be generalized at once since the unknowns there were scalar, while here $\mathbf{B}$ is a vector, hence here we actually face a linear *system*.

Even though equation (5) (as well as (7)) is coupled with other quantities, such as $\mathbf{v}$ and those in $\mathbf{f}$, one may expect that the speed of variation of $\mathbf{B}$, hence perhaps the magnetic reconnection rate, will be affected and be somewhat different from that described by the diffusive model equation (7). We discuss this issue in the next subsection.

2.1. *Speed of magnetic reconnection in diffusive, Sweet-Parker, and the present model*

In [14], the speed, says V_D (“D” for diffusion), of merging of the magnetic lines of force estimated by a purely diffusive model, such as $\frac{\partial \mathbf{B}}{\partial t} = \frac{c^2}{4\pi\sigma}\, \nabla^2 \mathbf{B}$ (in CGS units), was compared to that, V_{SP} (“SP” for Sweet-Parker), obtained including the velocity term $\nabla \times (\mathbf{v} \times \mathbf{B})$, based on the so-called Sweet-Parker mechanism [18,19,14]. The result was

$$V_D = c\left(\frac{c}{\sigma L}\right), \tag{8}$$

where L denotes a spatial scale (e.g., the size of some sunspot, on the Sun), and all equalities here and below are intended as approximated (typically up to some multiplicative constant), as order of magnitude, and

$$V_{SP} = c\left(\frac{v_A}{\sigma L}\right)^{1/2}, \tag{9}$$

where

$$v_A := \frac{B_0}{\sqrt{4\pi \rho_0}} \tag{10}$$

is the Alfvén velocity, ρ_0 being the mass density (in CGS units). The ratio

$$\frac{V_{SP}}{V_D} = \left(\frac{\sigma L}{c}\right)\left(\frac{v_A}{c}\right) = S^{1/2}, \tag{11}$$

where S is the *Lundquist number*, defined as

$$S := \frac{L v_A}{\eta}, \tag{12}$$

where L denotes a typical space length scale and assumed that it is large (small values of η, i.e., weakly collisional plasmas), can be very large.

The Lundquist number is a dimensionless parameter that can be considered as a special case of the magnetic Reynolds numbers, and is useful to compare the timescale of Alfvén waves and that of resistive diffusion. It important in connection to the magnetic reconnection, since high Lundquist numbers characterize highly conducting plasmas, while low values identify somewhat more resistive plasmas. Typical values for S in laboratory plasmas range from 10^2 to 10^8, while they be larger than 10^{20} in astrophysical plasmas. Therefore, from (11) follows that the Sweet-Parker model predicts a much faster reconnection than the purely diffusive model. For this reason, the Sweet-Parker model is considered as providing a "fast reconnection". Indeed, in [14] the example is reported that two oppositely directed sunspot fields with scales of 10^4 Km (10^9 cm), accordingly to the Sweet-Parker model would merge in about two weeks, while it would take 600 years on the basis of pure diffusion. Nevertheless, the times predicted by the Sweet-Parker model are judged to be too long in several known instances.

According to the induction equation in (5) (with $\mathbf{g}=\mathbf{0}$), the motion of the magnetic lines of force should obey a wave equation, resulting in a phenomenon traveling with a speed, V_w, equal to the speed of light, hence its ratio with V_{SP},

$$\frac{V_w}{V_{SP}} = \frac{c}{c\left(\frac{v_A}{\sigma L}\right)} = \left(\frac{\sigma L}{v_A}\right)^{1/2} = \left(\frac{c}{v_A}\right)^{1/2} S^{1/2}, \tag{13}$$

is definitely very large. Numerical examples show that this is likely to be too much, to be realistic. However, the model equation (5) (with $\mathbf{g}=\mathbf{0}$) shows that the wave equation there is very strongly damped, leading to kill any solution even on a very short time (or length). Only for extremely large (likely not realistic) frequencies, one may expect that the first term (the second derivative term) dominates on the second one on the left-hand side of (5). In summary, the present modification dictated by the inclusion of the displacement current in the classical resistive MHD equations, does not seem to change appreciably what is predicted by the Sweet-Parker model. Since, in spite of the apparent singular perturbation affecting equation (5), one can expect no initial layer to appear in case of purely initial-value problems (and in some initial-boundary-value problem, as pointed out in [7,16], the main difference between the solutions to equations (7) and (5) will be an uniform error of order of η/c^2.

3. Hydromagnetic waves in the high-frequency regime

In this section, we will see that when the displacement current term is kept in the MHD equations, an equation of the *third-order* in time, describing hydromagnetic waves, replaces the Alfvén equation. Third-order ordinary and partial differential equations are encountered elsewhere, in the mathematical physics' literature. See for instance, in linear acoustics, the Moore-Gibson-Thompson equation, which governs the acoustic velocity potential, ψ,

$$\tau\frac{\partial^3\psi}{\partial t^3} + \frac{\partial^2\psi}{\partial t^2} - C^2\nabla^2\psi - (\tau C^2+\delta)\nabla^2\left(\frac{\partial\psi}{\partial t}\right) = 0 \tag{14}$$

[13,20,11]. Another case is that of the Jeffreys-type equations modeling heat conduction,

$$\frac{\partial^2\theta}{\partial t^2} + \frac{1}{\tau}\frac{\partial\theta}{\partial t} - C^2\nabla^2\theta - \kappa\nabla^2\left(\frac{\partial\theta}{\partial t}\right) = 0, \tag{15}$$

where θ denotes the absolute temperature, [9, sec. III, p. 44, eq. (3.1)]. Here, τ, C, and δ are appropriate physical constants (not the same in the two equations above).

We use the well known identity

$$\nabla\times(\mathbf{v}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{B} + \mathbf{v}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{v}) \tag{16}$$

in equation (5) (with ($\mathbf{g}=\mathbf{0}$), recalling that $\nabla\cdot\mathbf{B}=0$ and assuming

$$\nabla\cdot\mathbf{v} = 0 \tag{17}$$

(incompressible fluid). Setting $\mathbf{B}=\mathbf{B}_0+\mathbf{b}$ with $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ along the z-axis, with B_0 *constant* and $|\mathbf{b}| \ll B_0$ in (16), equation (5) thus becomes, upon *linearization*,

$$\frac{\eta}{c^2}\frac{\partial^2\mathbf{b}}{\partial t^2} + \frac{\partial\mathbf{b}}{\partial t} = \eta\nabla^2\mathbf{b} + B_0\frac{\partial\mathbf{v}}{\partial z}. \tag{18}$$

Note that linearizing, we have neglected, in particular, the term $(\mathbf{v}\cdot\nabla)\mathbf{B} = (\mathbf{v}\cdot\nabla)\mathbf{b}$.

The motion equation,

$$\rho_0\frac{\partial\mathbf{v}}{\partial t} = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{b}, \tag{19}$$

where $\rho_0 = \sum_\alpha n_\alpha m_\alpha$ ($\approx n_i m_i$ for a plasma of electrons and one species ions) is the background (constant) mass density and p is the pressure, can be simplified proceeding as in [15, Ch. IV, § 4.4, pp. 100-102]. Since we chose $\mathbf{B_0} = (0,0,B_0)$, (19) becomes

$$\rho_0\frac{\partial\mathbf{v}}{\partial t} = \frac{B_0}{\mu_0}\frac{\partial\mathbf{b}}{\partial z} - \frac{\partial}{\partial z}\left(p + \frac{B^2}{2\mu_0}\right)\hat{\mathbf{z}}. \tag{20}$$

Assuming a space dependence only on z, looking in practice for plane wave solutions, we set $\partial/\partial x = \partial/\partial y = 0$. From $\nabla\cdot\mathbf{B}=0$ and (17) then follows that

$$\frac{\partial v_z}{\partial z} = 0, \quad \frac{\partial b_z}{\partial z} = 0, \tag{21}$$

hence v_z and b_z can be taken identically zero, since in so doing we only neglect trivial (i.e., not wave) solutions [15, Ch. IV, § 4.4, p. 101]. Consequently, equation (20) implies that

$$\frac{\partial}{\partial z}\left(p + \frac{B^2}{2\mu_0}\right) \equiv 0, \tag{22}$$

and then it reduces to

$$\rho_0\frac{\partial\mathbf{v}}{\partial t} = \frac{B_0}{\mu_0}\frac{\partial\mathbf{b}}{\partial z}, \tag{23}$$

cf. [15, Sec. 4.4,p. 101].

Differentiating with respect to time both sides of equation (18) and inserting (23), we obtain the decoupled third-order in time equation for $\mathbf{b}$ alone,

$$\frac{\eta}{c^2}\frac{\partial^3\mathbf{b}}{\partial t^3} + \frac{\partial^2\mathbf{b}}{\partial t^2} = \eta\frac{\partial^2}{\partial z^2}\left(\frac{\partial\mathbf{b}}{\partial t}\right) + v_A^2\frac{\partial^2\mathbf{b}}{\partial z^2}, \tag{24}$$

where $v_A := B_0/\sqrt{\mu_0\rho_0}$ is the Alfv́en velocity in SI units. Equation (24) shows that each component of $\mathbf{b}$ (but $b_z \equiv 0$), satisfies an equation like the Moore-Gibson-Thompson equation (14) of linear acoustics in one space variable.

It is noteworthy that such equation can also be written as

$$\frac{\eta}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial^2\mathbf{b}}{\partial t^2} - c^2\frac{\partial^2\mathbf{b}}{\partial z^2}\right) + \left(\frac{\partial^2\mathbf{b}}{\partial t^2} - v_A^2\frac{\partial^2\mathbf{b}}{\partial z^2}\right) = 0, \tag{25}$$

in the form analyzed in [12]; see [12, Sect 2, eq.s (2.1a), (2.2)], and note that also the here obvious condition $c > v_A$ agrees with that the authors state being satisfied in the dissipative physical problems that they consider there, that is, in viscoelastic wave propagation and in propagation of waves in real gases.

In case of an ideal fluid, incompressible and with *infinite conductivity*, $\sigma = \infty$ (or $\eta = 0$), equation (24) reduces to the customary Alfvén equation,

$$\frac{\partial^2 \mathbf{b}}{\partial t^2} = v_A^2 \frac{\partial^2 \mathbf{b}}{\partial z^2}, \tag{26}$$

and this happens whether the displacement current term is neglected or not.

In case of resistive MHD (finite conductivity, $\sigma < \infty$, i.e., $0 < \eta < \infty$), instead, equation (24) should be considered in place of (26), whether the displacement current term plays a role or not. In fact, if the latter is ignored, $c \to \infty$ but $\eta \neq 0$, equation (24) (formally) simplifies to

$$\frac{\partial^2 \mathbf{b}}{\partial t^2} = \eta \frac{\partial^2}{\partial z^2}\left(\frac{\partial \mathbf{b}}{\partial t}\right) + v_A^2 \frac{\partial^2 \mathbf{b}}{\partial z^2}, \tag{27}$$

which, however, is still a third order equation.

In the ideal case of infinite conductivity, it is known that the particle velocity, $\mathbf{v}$, obeys precisely the same Alfvén equation satisfied by $\mathbf{b}$; see [15, Ch. IV, Sec. 4.4, p. 101, (4-77)-(4-78)]. Here, we can derive the corresponding equation for $\mathbf{v}$, in case of finite conductivity and including high-frequency effects or not (that is, those of the displacement current). Starting from equation (18) (with $\nabla^2 = \partial^2/\partial z^2$), and differentiating both sides of equation (23) with respect to z and to t, we have, respectively,

$$\frac{\partial^2 \mathbf{b}}{\partial z^2} = \frac{\mu_0 \rho_0}{B_0} \frac{\partial^2 \mathbf{v}}{\partial t \partial z} \tag{28}$$

and

$$\frac{\partial^2 \mathbf{b}}{\partial z \partial t} = \frac{\mu_0 \rho_0}{B_0} \frac{\partial^2 \mathbf{v}}{\partial t^2} \tag{29}$$

(assuming regularity as everywhere in the paper, to ensure invertibility of the order of differentiation). Further differentiating both sides of (29) with respect to t, we obtain

$$\frac{\partial^3 \mathbf{b}}{\partial z \partial t^2} = \frac{\mu_0 \rho_0}{B_0} \frac{\partial^3 \mathbf{v}}{\partial t^3}, \tag{30}$$

and inserting (28) in the right-hand side of (18) (in 1D), we have

$$\frac{\eta}{c^2} \frac{\partial^2 \mathbf{b}}{\partial t^2} + \frac{\partial \mathbf{b}}{\partial t} = \frac{\rho_0}{\sigma B_0} \frac{\partial^2 \mathbf{v}}{\partial z \partial t} + B_0 \frac{\partial \mathbf{v}}{\partial z}. \tag{31}$$

We then differentiate again with respect to z, obtaining

$$\frac{\varepsilon_0}{\sigma} \frac{\partial^3 \mathbf{b}}{\partial z \partial t^2} + \frac{\partial^2 \mathbf{b}}{\partial t \partial z} = \frac{\rho_0}{\sigma B_0} \frac{\partial^3 \mathbf{v}}{\partial z^2 \partial t} + B_0 \frac{\partial^2 \mathbf{v}}{\partial z^2}, \tag{32}$$

and finally, inserting here (30) and (29), we have

$$\frac{\rho_0}{B_0 c^2 \sigma} \frac{\partial^3 \mathbf{v}}{\partial t^3} + \frac{\mu_0 \rho_0}{B_0} \frac{\partial^2 \mathbf{v}}{\partial t^2} = \frac{\rho_0}{\sigma B_0} \frac{\partial^2}{\partial z^2}\left(\frac{\partial \mathbf{v}}{\partial t}\right) + B_0 \frac{\partial^2 \mathbf{v}}{\partial z^2}, \tag{33}$$

that is

$$\frac{\eta}{c^2} \frac{\partial^3 \mathbf{v}}{\partial t^3} + \frac{\partial^2 \mathbf{v}}{\partial t^2} = \eta \frac{\partial^2}{\partial z^2}\left(\frac{\partial \mathbf{v}}{\partial t}\right) + v_A^2 \frac{\partial^2 \mathbf{v}}{\partial z^2}, \tag{34}$$

recalling that $\mu_0 c^2 = 1/\varepsilon_0$ and $\eta := 1/\mu_0 \sigma$. This equation, that can also be written as

$$\frac{\eta}{c^2} \frac{\partial}{\partial t}\left(\frac{\partial^2 \mathbf{v}}{\partial t^2} - c^2 \frac{\partial^2 \mathbf{v}}{\partial z^2}\right) + \left(\frac{\partial^2 \mathbf{v}}{\partial t^2} - v_A^2 \frac{\partial^2 \mathbf{v}}{\partial z^2}\right) = 0, \tag{35}$$

coincides with the equation (24) satisfied by $\mathbf{b}$, as it happens in the simpler case when the displacement current is dropped from the Maxwell's equations [15, Ch. IV, § 4.4], hence again with the Moore-Gibson-Thompson equation for each component of $\mathbf{v}$ (but $v_z \equiv 0$). It reduces to the usual Alfvén equation when $\sigma \to \infty$ ($\eta \to 0$), including or not the displacement current. When, instead, $\sigma < \infty$, equation (34) yields two different third-order equations (as it happens above, for the magnetic field $\mathbf{b}$) whether the displacement current effects are included or not ($c \to \infty$).

Equation (24) can be studied by Fourier analyzing it in space and time. Looking for plane wave solutions $\mathbf{b} \sim \exp\{i(kz - \omega t)\}$ yields the *dispersion relation*

$$i\eta\, \varepsilon\, \omega^3 - \omega^2 - i\eta k^2 \omega + v_A^2 k^2 = 0, \tag{36}$$

where we set $\varepsilon := c^{-2}$.

Note that in collisionless plasmas $\eta = 0$, and the solutions will be unaffected in both, a classical model ($c = \infty$) and relativistic models ($c < \infty$). This means that relativistic effects here are coupled to the collisional motion. However, in even weakly collisional models, η being proportional to the electrical resistivity, hence to the collision frequency, a difference when $c = \infty$ or $c < \infty$ does exist for any fixed value of $\eta > 0$. Indeed, high-frequency effects may be relevant whenever $\eta\varepsilon\omega^3 = \eta\omega^3/c^2$ is comparable or higher than ω. Consider that, for instance, the measured mean values of η in the solar photosphere are roughly in the range $(1 \div 20) \times 10^6$ $m^2 s^{-1}$, but elsewhere they can be of order of $(70 \div 600) \times 10^6$ $m^2 s^{-1}$, obtained with high-resolution magnetograms [5]. We have that the previous condition requires frequencies $f \gtrsim c/(2\pi\sqrt{\eta}) \approx 3 \times 10^8/(2\pi(1 \div 10) \times 10^3)$ i.e., of order of $10^4 \div 10^5$ Hz.

The algebraic equation (36) can be solved expanding in powers of ε, keeping in mind that one faces a singularly perturbed problem. The regular expansion in powers of ε yields the two roots

$$\omega = \omega^{\pm} := \pm k v_A (1 - r) \mp i \frac{k^2 \eta}{2} + \mathcal{O}(\eta\varepsilon), \tag{37}$$

where we defined the dimensionless parameter

$$r := \frac{k^2 \eta^2}{4 v_A^2}, \tag{38}$$

assuming $r < 1$. This parameter can be related to Lundquist number, defined in (12). Thus, $r = \frac{k^2 L^2}{4} \frac{1}{S}$. If $r \ll 1$, i.e., essentially in case of large Lundquist numbers, we have the approximation

$$\omega = \omega^{\pm} \approx \pm k v_A \left(1 - \frac{r}{2}\right) \mp i \frac{k^2 \eta}{2} + \mathcal{O}(\eta\varepsilon) \tag{39}$$

Note also that $(k^2\eta/2)/(k v_A) = (kL/2)(1/S)$ will be small, so that the imaginary correction to the frequency is also small.

The third root of the dispersion relation (36) is lost, being located at infinity. Setting $\omega := \Omega/\varepsilon$, a scaling suggested by the so-called "dominating balance", leads to the equation

$$i\eta\Omega^3 - \Omega^2 - ik^2\eta\varepsilon\Omega + k^2 v_A^2 \varepsilon^2 = 0, \tag{40}$$

Expanding Ω in powers of ε reveals the existence of the "singular" root $\Omega_s \approx -i/\eta$ that is the third root of the dispersion relation (36). More precisely, we obtain

$$\omega_s = -\frac{i}{\eta\varepsilon} + ik^2\eta + \mathcal{O}(\eta\varepsilon), \tag{41}$$

where we considered $\eta > 0$ fixed.

For solutions $\sim \exp\{ikz - i\omega_s t\}$, i.e., in particular depending on time as $\sim \exp\{-i\omega_s t\}$, this third solution typically represents a *strongly damped* wave, since $-i\omega_s = -1/(\eta\varepsilon)$ is real negative and typically large ($1/(\eta\varepsilon)$ is the damping rate). In fact, the algebraic equation (36) is singularly perturbed by the small parameter ε.

We conclude that three waves exist, two of them being essentially Alfvén waves with frequency slightly affected by collisional effects and also slightly damped in time, and a third (new) hydromagnetic wave, strongly damped in time according to $e^{-t/(\eta\varepsilon)}$.

Remark 3.1. Solving the dispersion equation in (36) is strictly related to solving equation (34) by Fourier transforming it in both, space and time. If we Laplace transform in time an initial-value problem for equation (34), after Fourier transforming it in space, denoting by a hat the latter (with respect to z), and with a tilde the former (with respect to t), setting $\widehat{\mathbf{b}}(k,t) := \mathcal{F}_z[\mathbf{b}(z,t)]$ and $\widetilde{\mathbf{b}}^\varepsilon(k,s) := \mathcal{L}_t[\mathcal{F}_z[\mathbf{b}(z,t)]]$, we obtain after a little algebra

$$\widetilde{\mathbf{b}}^\varepsilon - \widetilde{\mathbf{b}}^0 = \frac{k^2 v_A^2 s^2 \widehat{\mathbf{b}}(k,0) + (\eta s + v_A^2) k^2 s \widehat{\mathbf{b}}_t(k,0) + (s^2 + k^2\eta s + k^2 v_A^2)\widehat{\mathbf{b}}_{tt}(k,0)}{(\eta\varepsilon s^3 + s^2 + k^2\eta s + k^2 v_A^2)(s^2 + k^2\eta s + k^2 v_A^2)}\,\eta\varepsilon. \tag{42}$$

Thus, $|\widetilde{\mathbf{b}}^\varepsilon - \widetilde{\mathbf{b}}^0| = \mathcal{O}(\varepsilon)$, so that $\widetilde{\mathbf{b}}^\varepsilon \to \widetilde{\mathbf{b}}^0$, at least if $|k|$ is bounded (band-limited or low-pass solutions), and upon inverse transforming, we can (formally) infer that also $\mathbf{b}^\varepsilon \to \mathbf{b}^0$, and no "initial layer" appears, so that this convergence is uniform. A similar behavior was found for the solution to the Cattaneo-Maxwell-Vernotte equation (converging uniformly to the solution to the Fourier heat equation) in [7] and for the solution to the hyperbolic Schrödinger equation (converging to the solution to the classical Schrödinger equation) in [16].

It may be interesting to observe that the *dissipation* leading to a third-order equation in the present case of hydromagnetic waves, as, on the other hand, in acoustics and elsewhere [12,2,10], may be related to the what appears in relativistic effects in magnetic reconnection. In [3], the author states that such effects, important in space and laser plasmas, manifest themselves in the *displacement current* effects playing a role of "*dissipation*" in the ultrarelativistic limit. In other words, here, in the present analysis, as there, the displacement current plays a dissipative role. Indeed, its action was termed "dynamic dissipation" of the magnetic field, whose energy is transferred to the kinetic energy [8], by S.I. Syrovatskii [17], since in the ultrarelativistic regime, due to the limitation of the conduction current by the relativistic constraint on the particles velocity [8], the variation of the magnetic field has to be mostly sustained by the displacement current.

4. Summary

Some new results observable when the displacement current term is included in the MHD equations have been established. Retaining the displacement current term amounts to cast them in the MHD equations in relativistic framework of a finite speed of light. The typical equation for the magnetic induction, used to study magnetic reconnection phenomena, becomes formally hyperbolic instead of parabolic, and the equation describing hydromagnetic (Alfvén) waves for the perturbed magnetic field as well as for the particle velocity both obey now a third-order in time partial differential equation (coinciding with the Moore-Gibson-Thomson equation of linear acoustics), rather than the classical wave equation. In both examples, the higher derivatives are multiplied by a small parameter, but no initial layer is necessarily needed. From the physical point of view, the new speed of magnetic reconnection will be slightly different, while an additional third wave, strongly damped in time, will emerge from the new hydromagnetic (Alfvén) equation.

CRediT authorship contribution statement

Renato Spigler: Writing – review & editing, Writing – original draft, Supervision, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Renato Spigler reports equipment, drugs, or supplies was provided by Roma Tre University. Renato Spigler reports a relationship with Roma Tre University that includes: non-financial support. Renato Spigler has patent No patents exist pending to No patents. No conflict of interest exists to my knowledge.

Data availability

No data was used for the research described in the article.

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