



An efficient hybrid Chebyshev spectral method for multidimensional time-fractional partial integro-differential equations

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Abstract

The aim of this paper is to implement a high-order Chebyshev spectral collocation method for solving partial integro-differential equations (PIDEs) in multiple dimensions. Time marching methods and transform-based methods (non-time marching) can be used to discretize temporal terms. The combination of a small time step size in time marching methods and a high-order scheme in space with many degrees of freedom requires a considerable amount of computational time. To address this limitation, we propose an algorithm based on a combination of the Laplace transform and Chebyshev spectral collocation for solving second-order time-fractional PIDEs with weakly singular kernels. Unlike time-marching approaches, the transform-based method has a much lower computational complexity and can take advantage of parallel computing. Numerical experiments in one, two, and three dimensions validate the accuracy, efficiency, and applicability of the proposed approach.

Keywords Fractional derivative · Partial integro-differential equation · Laplace transform · Modified Talbot's method · Chebyshev spectral collocation scheme

1 Introduction

Fractional calculus (FC) is a modified version of ordinary integer-order calculus that allows for non-integer-order integrals and derivatives. This generalization offers a natural framework for modeling complex systems' high-order dynamics and memory effects, where standard integer-order models often fall short. Furthermore, FC has found numerous applications in engineering and sciences, including the mathematical modeling of physical systems [1], diffusion [2, 3], food engineering [4], signal

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processing [5], robotics [6], and control theory [7]. Ongoing advances in the theory and computation of fractional operators continue to broaden its scope and utility [8–11].

In recent years, time-fractional partial integro-differential equations (TF-PIDEs) have attracted significant attention across diverse fields. The nonlocal nature of the fractional derivative enables these equations to capture memory and hereditary effects in physical phenomena such as heat conduction [12], nuclear reactor dynamics [13], and finance [14] that cannot be accurately modeled by integer-order models. However, obtaining analytical solutions to TF-PIDEs remains challenging. Approaches like the homotopy analysis [15] and Adomian decomposition [16, 17] are often restricted by the algebraic complicity and problem-specific assumptions, underscoring the need for robust numerical techniques.

Numerous numerical methods have been developed for TF-PIDEs, such as the authors of [18] developed a finite difference scheme for PIDEs with a weakly singular kernel. In [19], the authors utilized the compact difference scheme for solving the time-fractional PIDEs. Alavi and Aminikhah [20] used the parametric finite difference and orthogonal spline approximation for solving the weakly singular nonlinear fractional order PIDEs. Chen et al. [21] developed a finite element technique for the parabolic IDEs with a weakly-singular kernel. Samadyar and Mirzaee [22] solved singular fractional PIDE via orthonormal Bernoulli polynomials. Saadatmandi and Dehghan [23] developed a Legendre collocation scheme for fractional PIDEs. In [24], the author coupled the shifted Legendre polynomials and a semi-analytic approach for solving one dimensional PIDEs. In [25] a fully spectral Galerkin method was developed for solving one and two dimensional fractional fourth order PIDEs with a weakly singular kernel. The author of [26] utilized Galerkin spectral method and finite difference method for solving fractional fourth order PIDEs with a weakly singular kernel. Sadri et al. [27] utilized bivariate Chebyshev polynomials of the fifth kind for variable order fractional order PIDEs. Atta and Youssri [28] studied the numerical solution of fractional order PIDEs using advanced shifted first-kind Chebyshev collocation approach. In [29], the authors utilized a high-order local meshfree method for solving 2D multi-term time-fractional PIDEs. The authors of [30] developed two high order numerical method based on Lagranges polynomials for solving distributed order PIDEs. In [31] a finite block method was developed for nonlinear time-fractional PIDEs. Arshed [32] utilized the B-spline technique for solving fractional order PIDEs. Long et al. [33] studied the numerical solution of a class of PIDEs using Quasi-wavelet based numerical method. The authors of [34] coupled the Laplace transform with local discontinuous Galerkin method for solving fourth order time-fractional PIDEs. Zaeri et al. [35] combined a fractional integration operator and a second-order difference approximation for numerical approximation of the fractional partial-integro diffusion heat equation. Santra [36] examined a higher-order method for the solution of a multi-term TF-PIDE with multi-term weakly singular kernels. Despite these efforts, numerous existing numerical methods struggle to handle the nonlocality of the fractional derivative and the Volterra integral efficiently, particularly in the presence of weak singularities. Moreover, they often require expensive computational resources due to memory effects in time-stepping or lack a unified technique that maintains high accuracy without compromising stability or convergence.

To address these limitations, the present work proposes a hybrid numerical technique for TF-PIDEs that combines the strengths of Chebyshev spectral collocation scheme (ChSCS) for spatial discretization with the Laplace transform (LT) for temporal discretization. The ChSCS offers spectral accuracy for sufficiently smooth solutions in space, on the other hand the LT converts the temporal fractional derivative and the Volterra integral operator into simple algebraic relations in the frequency domain, efficiently circumventing the costly history dependence inherent in time-stepping methods for time-fractional operators. The time domain solution is then recovered using the improved Talbot method [37].

The novelty of the proposed approach is threefold. First, unlike existing Laplace-spectral methods that are limited to 1D or 2D problems, our framework provides a unified treatment for 1D, 2D, and 3D TF-PIDEs through a systematic Kronecker product formulation. Second, the proposed method completely eliminates the convolution memory cost and time-stepping stability constraints by transforming the fractional derivative and Volterra integral into algebraic relations in the Laplace domain, enabling fully parallel computation across frequency nodes. Third, we provide a rigorous d -dimensional error analysis for the ChSCS, demonstrating exponential convergence in space, which, combined with the exponential convergence of the modified Talbot quadrature, yields a comprehensive error bound for the fully discrete scheme. The validity of the Talbot contour for the resulting fractional problems is ensured by verifying that all singularities of the Laplace-domain solution are confined to the left half-plane. To the best of our knowledge, this is the first work to present such a comprehensive hybrid framework for multidimensional fractional PIDEs with rigorous error analysis and exponential convergence in both space and time.

The rest of the article is organized as follows. In Section 2 some definitions, lemmas, and theorems are provided. In Section 3, we present the proposed numerical method. Section 5 presents the numerical results. The conclusions are provided in Section 6.

2 Basic notations

This section presents basic definitions, lemmas and theorems related to the study.

Definition 2.1 [8] A real function $u(t)$, $t > 0$ is said to be in the space C_ε , if for $\varepsilon \in \mathbb{R}$, a real number $\eta > \varepsilon$ exists such that $u(t) = t^\eta u_1(t)$, where $u_1(t) \in C[0, \infty)$ and it is said to be in the space C_ε^n if $u^{(n)} \in C_\varepsilon$, $n \in \mathbb{N} \cup \{0\}$.

Definition 2.2 [9] The Riemann-Liouville fractional integral of $u(t)$ of order β is as follows

$$\mathfrak{I}_t^\beta u(t) = \begin{cases} \frac{1}{\Gamma(\beta)} \int_0^t (t-\zeta)^{\beta-1} u(\zeta) d\zeta; & t > 0, \beta > 0, \\ u(t); & \text{for } \beta = 0. \end{cases}$$

The operator $\mathfrak{I}_t^\beta$ with $\varepsilon \geq -1$, $\varpi, \beta \geq 0$ and $\gamma > -1$ has the properties as follows

$$1. \mathfrak{I}_t^\beta \mathfrak{I}_t^\varpi u(t) = \mathfrak{I}_t^{\beta+\varpi} u(t),$$

2. $\mathfrak{I}_t^\beta \mathfrak{I}_t^\alpha u(t) = \mathfrak{I}_t^\alpha \mathfrak{I}_t^\beta u(t),$
3. $\mathfrak{I}_t^\beta t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(\gamma+1+\beta)} t^{\beta+\gamma}.$

Definition 2.3 [9] The Riemann-Liouville's derivative $\frac{\partial^\beta u(\bar{x}, t)}{\partial t^\beta}$ of fractional orders β is defined as

$$\frac{\partial^\beta u(\bar{x}, t)}{\partial t^\beta} = \frac{1}{\Gamma(n-\beta)} \frac{\partial^n}{\partial t^n} \int_0^t (t-q)^{n-\beta-1} u(\bar{x}, q) dq, \quad n-1 < \beta < n.$$

Definition 2.4 [9] For every smallest integer n , which exceeds β , the Liouville-Caputo's derivatives $\frac{\partial^\beta u(\bar{x}, t)}{\partial t^\beta}$ is as follows

$$\frac{\partial^\beta u(\bar{x}, t)}{\partial t^\beta} = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{\partial^n u(\bar{x}, \zeta)}{\partial \zeta^n} (t-\zeta)^{n-\beta-1} d\zeta; & n-1 < \beta < n, \\ \frac{\partial^n u(\bar{x}, t)}{\partial t^n}; & \beta = n \in \mathbb{N}. \end{cases} \quad (1)$$

Lemma 2.5 [9] The basic properties of the Liouville-Caputo's time fractional derivative are given as

1. $\frac{\partial^\beta u(\bar{x}, t)}{\partial t^\beta} = 0, \quad u = \text{constant},$
2. $\frac{\partial^\beta t^p}{\partial t^\beta} = \frac{\Gamma(p+1)}{\Gamma(p-\beta+1)} t^{p-\beta}, \quad p > -1, \quad \beta > 0.$

Definition 2.6 [9] The Laplace transform (LT) of a piecewise continuous function $u(\bar{x}, t)$, which is of exponential order 'a', is given as follows

$$\widehat{u}(\bar{x}, s) = \int_0^\infty \exp(-st) u(\bar{x}, t) dt, \quad \text{Re}(s) > a.$$

Lemma 2.7 [8] The LT of $\frac{\partial^\beta u(\bar{x}, t)}{\partial t^\beta}$ of the Liouville-Caputo's fractional order $\beta \in (n-1, n]$ is given as

$$\mathcal{L} \left\{ \frac{\partial^\beta u(\bar{x}, t)}{\partial t^\beta} \right\} = s^\beta \widehat{u}(\bar{x}, s) - \sum_{j=0}^{n-1} s^{\beta-j-1} u^{(j)}(\bar{x}, 0).$$

3 Numerical scheme

The proposed methodology comprises three main steps. First, for the discretization of time variable of the proposed model, the LT is employed; then, secondly, the Chebyshev spectral method is utilized for spatial discretization, and the solution is obtained in Laplace space. Finally, the numerical inverse LT methods (NILTMs) are utilized to get back the temporal domain solution. For inversion of LT, the modified Talbot's method is used.

We consider the Laplace-transformed Chebyshev spectral collocation scheme for the approximation of TF-PIDE with Dirichlet's boundary condition of the following form

$$\frac{\partial^\beta u(\bar{x}, t)}{\partial t^\beta} = c\mathcal{L}u(\bar{x}, t) + \frac{1}{\Gamma(\alpha)} \int_0^t k(t, \zeta) \mathcal{L}u(\bar{x}, \zeta) d\zeta + h(\bar{x}, t), \quad \bar{x} \in \Psi, t \in [0, T], \quad (2)$$

subject to the initial and boundary conditions

$$u(\bar{x}, 0) = u_0, \quad \bar{x} \in \Psi, \quad (3)$$

$$\mathcal{B}u(\bar{x}, t) = g(\bar{x}, t), \quad \bar{x} \in \Psi, t \in [0, T], \quad (4)$$

where $0 < \beta, \alpha < 1$, $c \geq 0$, $\bar{x} \in \Psi \subseteq \mathbb{R}^d$, $d = 1, 2, 3$, $\bar{x} = x$ for 1D case, $\bar{x} = (x, y)$ for 2D case, $\bar{x} = (x, y, z)$ for 3D case, $t \in [0, T]$, $h(\bar{x}, t)$, $g(\bar{x}, t)$ are given continuous functions, $k(t, \zeta) = (t - \zeta)^{\alpha-1}$ is the kernel, $\mathcal{L}$ is the laplacian, $\mathcal{B}$ is the linear boundary operator and $\frac{\partial^\beta u(\bar{x}, t)}{\partial t^\beta}$ is the Liouville-Caputo's derivative defined in (1).

3.1 Laplace transform

In this section, we apply LT to the proposed model (2)-(4) and obtain a time-independent problem in the frequency domain

$$s^\beta \hat{u}(\bar{x}, s) - s^{\beta-1} \hat{u}(\bar{x}, 0) = c\mathcal{L}\hat{u}(\bar{x}, s) + s^{-\alpha} \mathcal{L}\hat{u}(\bar{x}, s) + \hat{h}(\bar{x}, s), \quad (5)$$

$$\mathcal{B}\hat{u}(\bar{x}, s) = \hat{g}(\bar{x}, s), \quad (6)$$

by simplifying the terms, we get,

$$(s^\beta I - c\mathcal{L} - s^{-\alpha} \mathcal{L})\hat{u}(\bar{x}, s) = \hat{X}(\bar{x}, s) \quad (7)$$

$$\mathcal{B}\hat{u}(\bar{x}, s) = \hat{g}(\bar{x}, s), \quad (8)$$

with

$$\hat{X}(\bar{x}, s) = s^{\beta-1} u_0 + \hat{h}(\bar{x}, s),$$

where I , $\mathcal{L}$, and $\mathcal{B}$ are the identity, the linear differential, and the boundary differential operators respectively. In our numerical scheme, the next step is to employ the Chebyshev spectral method for discretizing the operators $\mathcal{L}$ and $\mathcal{B}$, then, we solve the system defined in (7) and (8), for each s , and finally we use the improved Talbot's method.

3.2 Chebyshev spectral collocation scheme

In ChSCS, the unknown solution of the model is expanded through the global Lagrange interpolant polynomial (LIP), which is based on appropriate nodal points, i.e., Chebyshev collocation nodes. The spatial derivative in the proposed problem is transformed

into a discrete operator called the differentiation matrix (DM) [38]. In ChSCS, naturally the solution is considered over the spatial domain $[-1, 1]$, the following transformation is used to map the Chebyshev points for spatial domain $[a, b]$ as follows:

$$\Gamma_{N_c} = \frac{b-a}{2}x_k + \frac{b+a}{2},$$

consider the 1D spatial domain $[-1, 1]$ and interpolated at the points $\{(x_k, \hat{u}(x_k))\}$ by the LIP, denoted $\mathcal{P}_k(x)$ of degree $\leq N_c$ [39].

$$I_{N_c}(x) = \sum_{k=0}^{N_c} \mathcal{P}_k(x) \hat{u}_k,$$

where $\hat{u}_k = \hat{u}(x_k)$, and $\mathcal{P}_k(x)$ is the LIP defined at the nodes $\{x_k\}_{k=0}^{N_c}$ as

$$\mathcal{P}_k(x) = \frac{(x-x_0) \cdots (x-x_{k-1})(x-x_{k+1}) \cdots (x-x_{N_c})}{(x_k-x_0) \cdots (x_k-x_{k-1})(x_k-x_{k+1}) \cdots (x_k-x_{N_c})}. \quad (9)$$

In ChSCS, the Chebyshev nodes $\{x_k\}_{k=0}^{N_c}$ are utilized defined in [39] for discretizing the one-dimensional spatial domain $\Psi = [-1, 1]$ as

$$x_k = \cos\left(\frac{k\pi}{N_c}\right), \quad k = 0, 1, \dots, N_c. \quad (10)$$

The first-order derivative of $\hat{u}$ with respect to spatial variable x is discretized by ChSCS as

$$\partial_x \hat{u}(x) \approx \mathbf{D}_{N_c} \hat{u},$$

where $\mathbf{D}_{N_c}$ is the differentiation matrix [39] whose entries are given by

$$[\mathbf{D}_{N_c}]_{n,k} = \mathcal{P}'_k(x_n), \quad n, k = 1, 2, \dots, N_c.$$

The entries of $\mathbf{D}_{N_c}$ other than diagonal entries are

$$[\mathbf{D}_{N_c}]_{n,k} = \frac{\gamma_k}{\gamma_n(x_n - x_k)}, \quad n \neq k,$$

where $\gamma_k^{-1} = \prod_{n \neq k} (x_n - x_k)$, and the diagonal entries for $n, k \in \{0, 1, \dots, N_c\}$ are given by

$$[\mathbf{D}_{N_c}]_{n,n} = -\sum_{\substack{k=0 \\ k \neq n}}^{N_c} [\mathbf{D}_{N_c}]_{n,k}.$$

The ϱ -th order elements of $\mathbf{D}_{N_c}^{(\varrho)}$ for $n, k \in \mathbb{N}$ are computed as

$$[\mathbf{D}_{N_c}^{(\varrho)}]_{n,k} = \mathcal{P}_k^{(\varrho)}(x_n).$$

Welfert [38] derived an easy to use relation for $\mathbf{D}_{N_c}$ is given as follows:

$$[\mathbf{D}_{N_c}^{(\varrho)}]_{n,k} = \frac{\varrho}{x_n - x_k} \left\{ \frac{\gamma_k}{\gamma_n} [\mathbf{D}_{N_c}^{(\varrho-1)}]_{n,n} - [\mathbf{D}_{N_c}^{(\varrho-1)}]_{n,k} \right\}, \quad n \neq k.$$

Next, we consider the 2D domain $\Psi = [-1, 1]^2$ with the Chebyshev points $\bar{\mathbf{x}}_{nk}$ with $n, k \in \mathbb{N}$ given by

$$\bar{\mathbf{x}}_{nk} = \left(\cos\left(\frac{n\pi}{N_c}\right), \cos\left(\frac{k\pi}{N_c}\right) \right).$$

The corresponding LIP for the 2D nodes $\bar{\mathbf{x}}_{nk}$ is of the form

$$\mathcal{P}_{nk}(\bar{\mathbf{x}}) = \mathcal{P}_n(x)\mathcal{P}_k(y), \quad (11)$$

where $\mathcal{P}_{nk}(\bar{\mathbf{x}}_{nk}) = \{\phi_{nk}\}_{n,k=0}^{N_c}$ utilizing the property of Kronecker product. The 2^{nd} order derivative of the LIP given in (11) are as follows:

$$\begin{aligned} \partial_x^2 \mathcal{P}_{nk}(\bar{\mathbf{x}}_{rs}) &= \mathcal{P}_n''(x_r)\mathcal{P}_k(y_s) = [\mathbf{D}_{N_c}^{(2)}]_{rn}\phi_{ks}, \\ \partial_y^2 \mathcal{P}_{nk}(\bar{\mathbf{x}}_{rs}) &= \mathcal{P}_n(x_r)\mathcal{P}_k''(y_s) = \phi_{rn}[\mathbf{D}_{N_c}^{(2)}]_{sk}, \end{aligned}$$

where $\mathbf{D}_{N_c}^{(2)}$ denotes the second-order differentiation matrix based on Chebyshev collocation points. Next, applying the operator $\mathcal{L}$ to (11) at $\bar{\mathbf{x}}_{rs}$, we obtain

$$\mathcal{L}(\mathcal{P}_{nk}(\bar{\mathbf{x}}_{rs})) = [\mathbf{D}_{N_c}^{(2)}]_{rn}\phi_{ks} + \phi_{rn}[\mathbf{D}_{N_c}^{(2)}]_{sk}. \quad (12)$$

The approximated linear operator $\mathcal{L}$ by ChSCS is

$$\mathcal{L}_{disc} = \mathbf{D}_{N_c}^{(2)} \otimes \mathbf{I}_{N_c} + \mathbf{I}_{N_c} \otimes \mathbf{D}_{N_c}^{(2)}, \quad (13)$$

where $\mathbf{I}_{N_c}$ is the identity matrix of size $(N_c + 1) \times (N_c + 1)$. In three dimensions, the linear operator $\mathcal{L}$ can be approximated using ChSCS as

$$\mathcal{L}_{disc} = \mathbf{I}_N \otimes \mathbf{I}_N \otimes \mathbf{D}_{N_c}^{(2)} + \mathbf{I}_N \otimes \mathbf{D}_{N_c}^{(2)} \otimes \mathbf{I}_N + \mathbf{D}_{N_c}^{(2)} \otimes \mathbf{I}_N \otimes \mathbf{I}_N, \quad (3D). \quad (14)$$

Remark The discrete Laplacian operator $\mathcal{L}_{disc}$ formed via Kronecker products inherits the conditioning properties of the 1D Chebyshev second-derivative matrix $\mathbf{D}_{N_c}^{(2)}$. For Chebyshev collocation, the condition number of $\tilde{\mathbf{D}}_{N_c}^{(2)}$ grows as $\mathcal{O}(N_c^4)$ [39] (Chapter 10). Consequently, the condition number of $\mathcal{L}_{disc}$ in d dimensions also scales as $\mathcal{O}(N_c^4)$, since its eigenvalues are sums of the 1D eigenvalues. To mitigate conditioning issues for large N_c , the following strategies are recommended: (i) using explicit formulas or recurrences for $\mathbf{D}_{N_c}^{(2)}$ instead of squaring $\mathbf{D}_{N_c}$ [38]; (ii) employing integration preconditioners to reduce condition numbers; and (iii) for large 3D problems, utilizing iterative solvers (e.g., preconditioned GMRES) with finite difference preconditioners

[39]. In three dimensions, the matrix dimension grows as $(N_c - 1)^3$, and the CPU times reported in Tables 1–3 reflect this cubic growth.

By substituting (14) into (7), we obtain

$$\left(s^\beta \mathbf{I} - c\mathcal{L}_{disc} - s^{-\alpha} \mathcal{L}_{disc} \right) \widehat{\mathbf{u}}(\bar{\mathbf{x}}, s) = \widehat{X}(\bar{\mathbf{x}}, s). \quad (15)$$

We assemble $\mathcal{L}_{disc}$ at all nodes $\bar{\mathbf{x}}$ to employ the boundary-conditions in (8). The rows of $\mathcal{L}_{disc}$ corresponding to boundary nodal points are replaced with unit vectors, placing unity at the diagonal positions. The $\mathcal{B}\widehat{\mathbf{u}}(\bar{\mathbf{x}}, s) = \widehat{h}(\bar{\mathbf{x}}, s)$ from (8) is directly applied [39]. After reordering the corresponding columns and rows of the operator $\mathcal{L}_{disc}$, we have

$$\mathcal{L}_\Psi = \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},$$

here the order of $\mathbf{A}$ (square matrix) is $(N_c - N_{CB}) \times (N_c - N_{CB})$ corresponding to interior nodes, and $\mathbf{I}$ is the identity matrix of order $(N_{CB} \times N_{CB})$, with N_{CB} denoting the number of boundary nodal points. The system (7)–(8) is solved by solving the following system

$$\mathcal{L}_\Psi \widehat{\mathbf{u}}(\bar{\mathbf{x}}, s) = \begin{bmatrix} \widehat{X}(\bar{\mathbf{x}}, s) \\ \widehat{g}(\bar{\mathbf{x}}, s) \end{bmatrix},$$

where the corresponding interior and boundary nodes are combined by $\widehat{X}(\bar{\mathbf{x}}, s)$ and $\widehat{g}(\bar{\mathbf{x}}, s)$. Next, the NILTMs are used to obtain the solution of the proposed problem (2)–(4).

3.3 Inversion of solution from Laplace space into time domain via contour integration method (CIM)

In this step, the inversion of the solution from Laplace space into the time domain via contour integration methods is discussed. The CIM is employed to numerically invert the LT. In CIM the $u(\bar{\mathbf{x}}, t)$ is computed as follows:

$$u(\bar{\mathbf{x}}, t) = \frac{1}{2\pi i} \int_{\eta - i\infty}^{\eta + i\infty} \exp(st) \widehat{\mathbf{u}}(\bar{\mathbf{x}}, s) ds = \frac{1}{2\pi i} \int_C \exp(st) \widehat{\mathbf{u}}(\bar{\mathbf{x}}, s) ds, \quad t > 0, \quad \eta > \eta_0, \quad (16)$$

here C is a selected contour in the complex plane linking $\eta - i\infty$ to $\eta + i\infty$. The above integral is known as the Bromwich integral [40]. Many authors developed numerical methodologies to calculate the Bromwich integral. The complex function $\exp(st)$ on η diminishes very slowly, therefore making the Bromwich integral very tedious to calculate. The deformation of η is another way that can handle $\exp(st)$. The line $C : (\eta + i\infty, \eta - i\infty)$ is deformed to the contour proposed by Hankel [41] which is initiated and terminated inside the half plane that will be to the left of C and also $\text{Re}(s) \rightarrow -\infty$ at both ends, so the $\exp(st)$ converges very quickly, so the complex integral is easy to evaluate by applying the trapezoidal or mid point rule. The Hankel's deformation of the contour is easily verified via Cauchy's theorem

subject to the condition that C will be present in the neighborhood where the $\widehat{u}(\bar{x}, s)$ is differentiable everywhere in the given domain. The modified Talbot's technique will easily fail if there are some singularities in the imaginary region where $\widehat{u}(\bar{x}, s)$ lies. Various CIMs are formulated to approximate the inverse LT numerically [42]. In this work, the well-known Talbot's method, improved in [37] is utilized. The parametric form of the modified Talbot's contour is as follows:

$$C : s = s(v), \quad -\pi \leq v \leq \pi, \quad \operatorname{Re}(s(\pm\pi)) = -\infty, \quad (17)$$

we have

$$s(v) = \frac{M_t}{t} \delta(v), \quad \delta(v) = -\alpha_1 + \alpha_2 v \cot(\alpha_3 v) + \alpha_4 i v, \quad (18)$$

here, M_t denotes the number of quadrature points employed in the modified Talbot contour for the numerical inversion of the LT and α_1 , α_2 , α_3 , and α_4 , must be precisely chosen.

Remark The accuracy of the Talbot method requires that the singularities of $\widehat{u}(\bar{x}, s)$ be confined to the negative real axis (or left half-plane). For the d -dimensional Laplacian for the problems studied in this work, the singularities satisfy

$$s^{\beta+\alpha} + c|\lambda|s^\alpha + |\lambda| = 0,$$

where $\lambda = -\sum_{i=1}^d \frac{\pi n_i}{2}$ are the eigenvalues of the discrete Laplacian. Since $\lambda < 0$, all poles lie in the left half-plane, and the modified Talbot contour from [37] is designed to avoid these singularities. Substituting (18) in (16), we have

$$u(\bar{x}, t) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} e^{s(v)t} \widehat{u}(\bar{x}, s(v)) s'(v) dv. \quad (19)$$

The midpoint quadrature rule with $\eta = \frac{2\pi}{M_t}$ as uniform step size is used to calculate the (19) as follows

$$u(\bar{x}, t) \approx \frac{1}{M_t i} \sum_{j=1}^{M_t} \exp(s(v_j)t) \widehat{u}(\bar{x}, s(v_j)) s'(v_j), \quad v_j = -\pi + \left(\frac{2j-1}{2} \right) \eta. \quad (20)$$

3.4 Error analysis of the proposed method

In this section, the analysis of error of LT-ChSCS is discussed. By the application of the LT, the first step is free of error. In the second step, the ChSCS is employed for the approximation of the transformed problem. Here we formulate the error bound for the ChSCS in d dimensions.

For Chebyshev nodes and LIP defined in (9) and (10), the interpolant is defined as [43]:

$$I_{N_c} : C(\Psi) \rightarrow P_{N_c}, \quad I_{N_c}(\widehat{u}) = \sum_{k=0}^{N_c} \mathcal{P}_k(x) \widehat{u}(x_k).$$

The bound of error for the ChSCS is calculated by using the literature in [44]. Let a constant K_{N_c} , here the estimate of stability is

$$\|I_{N_c}(\hat{u})\|_\infty \leq K_{N_c} \|\hat{u}\|_\infty, \quad \text{for all } \hat{u} \in C[-1, 1], \quad (21)$$

and

$$I_{N_c}(\hat{u}) = \hat{u}, \quad \text{for all } \hat{u} \in P_{N_c}. \quad (22)$$

We can show for the LIP based on Chebyshev nodes

$$K_{N_c} = \frac{\ln(N_c + 1)}{(\frac{\pi}{2})} + 1 \leq (N_c + 1). \quad (23)$$

The stability constant depends on N_c very slowly. The error-bound (approximation) [44] for $\hat{u} \in C^{(N_c+1)}$ is

$$\|\hat{u} - I_{N_c}(\hat{u})\|_\infty \leq \frac{2^{-N_c}}{(N_c + 1)!} \|\hat{u}^{(N_c+1)}\|_\infty. \quad (24)$$

Theorem 1 [44] Let $\hat{u} \in C^{(N_c+1)}$ and if (21) and (24) holds with $l \in \mathbb{N}$, then

$$\|\hat{u}^{(l)} - I_{N_c}(\hat{u})^{(l)}\|_\infty \leq \frac{2(K_{N_c}^{(l)} + 1)}{2^{(N_c-l+1)}(N_c - l + 1)!} \|\hat{u}^{(N_c+1)}\|_\infty \quad (25)$$

which depends on the stability constant

$$K_{N_c}^{(l)} = \frac{K_{N_c}}{l!} \left(\frac{N_c!}{(N_c - l)!} \right).$$

For the d -dimensional tensor-product interpolation operator on the domain $\Psi = [-1, 1]^d$, the stability and approximation estimates generalize as follows. Let $\Lambda_{N_c} \leq N_c + 1$ denote the 1D Lebesgue constant. Then the d -dimensional stability estimate is

$$\|I_{N_c}(\hat{u})\|_\infty \leq \Lambda_{N_c}^d \|\hat{u}\|_\infty, \quad \Lambda_{N_c} \leq N_c + 1. \quad (26)$$

The interpolation error for $\hat{u} \in C^{N_c+1}(\Psi)$ in d dimensions is

$$\|\hat{u} - I_{N_c}(\hat{u})\|_\infty \leq \frac{2d \Lambda_{N_c}^{d-1}}{(N_c + 1)!} \left(\frac{\text{diam}(\Psi)}{4} \right)^{N_c+1} \max_{1 \leq i \leq d} \|\partial_i^{N_c+1} \hat{u}\|_\infty. \quad (27)$$

For the second derivatives, using Theorem 3.17 of [44], we have

$$\|\partial_i^2(\hat{u} - I_{N_c}(\hat{u}))\|_\infty \leq \frac{2(\Lambda_{N_c}^{(2)} + 1)}{(N_c - 1)!} \left(\frac{\text{diam}(\Psi)}{4} \right)^{N_c-1} \max_{1 \leq i \leq d} \|\partial_i^{N_c+1} \hat{u}\|_\infty, \quad (28)$$

where $\Lambda_{N_c}^{(2)} \leq (N_c + 1)^2$.

By using the estimates (26), (27), and (28) for the TF-PIDE defined in (2) with the operator $\mathcal{L}\hat{u} = \sum_{i=1}^d \partial_{x_i}^2 \hat{u}$, we find the error estimate as follows

$$\begin{aligned} E &= \|(s^\beta I - c\mathcal{L} - s^{-\alpha}\mathcal{L})\hat{u} - (s^\beta I - c\mathcal{L} - s^{-\alpha}\mathcal{L})I_{N_c}\hat{u}\|_\infty \\ &= \|s^\beta(\hat{u} - I_{N_c}\hat{u}) - c\mathcal{L}(\hat{u} - I_{N_c}\hat{u}) - s^{-\alpha}\mathcal{L}(\hat{u} - I_{N_c}\hat{u})\|_\infty \\ &\leq |s^\beta| \|\hat{u} - I_{N_c}\hat{u}\|_\infty + (|c| + |s^{-\alpha}|) \sum_{i=1}^d \|\partial_i^2(\hat{u} - I_{N_c}\hat{u})\|_\infty. \end{aligned}$$

Substituting the estimates from (27) and (28), we obtain

$$E \leq r_0(d) \max_{1 \leq i \leq d} \|\partial_i^{N_c+1} \hat{u}\|_\infty,$$

where the constant

$$r_0(d) = \frac{2d \Lambda_{N_c}^{d-1}}{(N_c + 1)!} \left(\frac{\text{diam}(\Psi)}{4} \right)^{N_c+1} \left(|s^\beta| + d(|c| + |s^{-\alpha}|) \frac{(N_c + 1)(\Lambda_{N_c}^{(2)} + 1)}{(\text{diam}(\Psi)/4)^2} \right)$$

depends on the dimension d . For the 1D case ($d = 1$), this reduces to

$$E \leq r_0(1) \|\hat{u}^{(N_c+1)}\|_\infty,$$

which is consistent with the classical Chebyshev spectral convergence result. The error decays exponentially as $\mathcal{O}(2^{-N_c})$ for sufficiently smooth solutions, confirming the spectral accuracy of the proposed method.

The final step of the proposed numerical method involves the inversion of the Laplace transform in (16). To accomplish this, we utilize the enhanced Talbot method proposed in [37], which provides exponential convergence rates with the correct selection contour parameters. The convergence behavior is significantly affected by the contour parameter C and the optimization of method-specific parameters. The improved Talbot approach delineates a parabolic contour in the complex plane that contains the singularities of the Laplace domain solution. Using the parameterizations from [37], the midpoint/trapezoidal rule is used to approximate the Bromwich contour defined in (16) along this contour. The following theorem offers a theoretical foundation for error estimation in the improved Talbot method:

Theorem 2 [37] *Let $v_j = -\pi + \left(\frac{2j-1}{2}\right)\eta$, and suppose $f : A \rightarrow \mathbb{C}$ is analytic in the strip region*

$$A = \{v \in \mathbb{C} : \text{Re}(v) \in (-\pi, \pi), \text{ and } \text{Im}(v) \in (-c_1, c_2)\},$$

where c_1, c_2 are positive constants. The quadrature error fulfills

$$\int_{-\pi}^{\pi} f(v) dv - \frac{2\pi}{M_t} \sum_{j=1}^{M_t} f(v_j) = \delta_-(\gamma) + \delta_+(\zeta),$$

with

$$\delta_+(\gamma) = \frac{1}{2} \left(\int_{-\pi}^{-\pi+i\gamma} + \int_{i\gamma-\pi}^{i\gamma+\pi} + \int_{i\gamma+\pi}^{\pi} \right) \left(1 + i \tan \left(\frac{M_t v}{2} \right) \right) f(v) dv,$$

and

$$\delta_-(\zeta) = \frac{1}{2} \left(\int_{-\pi}^{-\pi-i\zeta} + \int_{-\pi-i\zeta}^{\pi-i\zeta} + \int_{\pi-i\zeta}^{\pi} \right) \left(1 - i \tan \left(\frac{M_t v}{2} \right) \right) f(v) dv,$$

$\forall \gamma \in (0, c_1)$, $\zeta \in (0, c_2)$, and $M_t \in \{2, 4, 6, \dots\}$. When $M_t \in \{1, 3, \dots\}$, substitute $\tan \left(\frac{M_t v}{2} \right)$ with $-\cot \left(\frac{M_t v}{2} \right)$. If $f(v)$ meets the symmetry requirements $f(\bar{v}) = \overline{f(v)}$ and $\gamma = \zeta$, then

$$\delta(\zeta) = \delta_+(\zeta) + \delta_-(\zeta) = \operatorname{Re} \int_{-\pi+i\zeta}^{\pi+i\zeta} \left(1 + i \tan \left(\frac{M_t v}{2} \right) \right) f(v) dv.$$

Using the decay characteristic of the tangent function produces the following error bound:

$$|\delta(\zeta)| \leq \frac{4b\pi}{e^{c_1 M_t} - 1},$$

where b and c_1 are positive constants that represent the analyticity strip of $f(v)$.

The exactness of the improved Talbot approach is highly dependent on the best parameter values. Following a thorough investigation by Dingfelder and Weideman [37], we use the most effective parameter set:

$$\alpha_1 = \frac{6122}{10000}, \quad \alpha_2 = \frac{5017}{10000}, \quad \alpha_3 = \frac{6407}{10000}, \quad \alpha_4 = \frac{2645}{10000}.$$

The above set of parameters optimize the convergence rate while retaining numerical stability. The obtained error estimate shows exponential convergence:

$$\text{error}^{\text{est}} = |u(\bar{x}, t) - u_k(\bar{x}, t)| = \mathcal{O}(e^{-1.3580 M_t}).$$

This shows that the error decreases exponentially with M_t .

The proposed LT-ChSCS method exhibits exponential convergence in both space and time for sufficiently smooth solutions. The error is rapidly reduced by increasing either the number of Talbot points M_t or the number of Chebyshev nodes N_c . The method is accurate and efficient because it does not suffer from the stability constraints typical of standard time-stepping approaches.

Let $u(\bar{x}, t)$ denote the exact solution of (2) and let $u_k(\bar{x}, t)$ denote the fully discrete LT-ChSCS approximation. The total error can be decomposed into two components:

$$\|u - u_k\|_{\infty} \leq \|u - u_{LT}\|_{\infty} + \|u_{LT} - u_k\|_{\infty},$$

where u_{LT} denotes the solution after Laplace inversion with exact spatial discretization. The first term represents the error from the Talbot quadrature (Theorem 4.1), and the second term represents the error from the Chebyshev spectral collocation in space.

Assuming the solution is sufficiently smooth so that the Chebyshev interpolation estimate (24) holds, and that the Laplace-transformed solution remains analytic in a strip of positive width about the Talbot parameter interval, there exist positive constants C_1 (independent of N_c) and C_2 (independent of M_t) such that

$$\|u - u_k\|_\infty \leq C_1 \max_{1 \leq i \leq d} \|\partial_i^{N_c+1} \hat{u}\|_\infty + C_2 e^{-1.3580 M_t}. \quad (29)$$

For the 1D case, this simplifies to

$$\|u - u_k\|_\infty \leq C_1 \|\hat{u}^{(N_c+1)}\|_\infty + C_2 e^{-1.3580 M_t}. \quad (30)$$

The constant C_1 depends on the dimension d and the spatial domain, while C_2 depends on the analyticity strip width of the Laplace-domain solution. The first term decays exponentially with N_c for analytic functions, i.e.,

$$\|\hat{u}^{(N_c+1)}\|_\infty = \mathcal{O}(2^{-N_c}),$$

and the second term decays exponentially with M_t . Therefore, the overall error satisfies

$$\|u - u_k\|_\infty = \mathcal{O}(2^{-N_c}) + \mathcal{O}(e^{-1.3580 M_t}). \quad (31)$$

This confirms that the LT-ChSCS method achieves exponential convergence in both space and time simultaneously, with no restrictive stability constraints, making it highly efficient for solving multidimensional time-fractional partial integro-differential equations. A brief overview of the proposed numerical method is presented in Algorithm 1.

4 Stability of the numerical scheme

To evaluate the stability of our proposed method, we formulate the system (7)–(8) in discretized structure as follows:

$$S\hat{u} = b, \quad (32)$$

where S represents the differentiation derived via the ChSCS. The stability constant for the system (32) is expressed as follows:

$$\mathcal{E} = \sup_{\hat{u} \neq 0} \frac{\|\hat{u}\|}{\|S\hat{u}\|}, \quad (33)$$

The constant $\mathcal{E}$ remains finite under any discrete norms $\|\cdot\|$ on $\mathbb{R}^n$. From (33), we have

$$\|S\|^{-1} \leq \frac{\|\hat{u}\|}{\|S\hat{u}\|} \leq \mathcal{E}. \quad (34)$$

Algorithm 1 Algorithm for the LT-ChSCS.

Input: Spatial domain $\Psi = [-1, 1]^d$ ($d = 1, 2, 3$); final time T ; fractional orders $\beta, \alpha \in (0, 1)$; constant $c \geq 0$; initial condition $u(\bar{x}, 0) = u_0$; source term $h(\bar{x}, t)$; boundary data $g(\bar{x}, t)$; number of Chebyshev nodes N_c per dimension; number of quadrature points M_t for Laplace inversion; modified Talbot parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ (e.g., from Dingfelder & Weideman [37]).

Step 1: Apply Laplace Transform.

Transform the time-fractional PIDE (2) using the Laplace transform of the Caputo derivative (Lemma 2.7).

Obtain the elliptic problem (parameter dependent) in Laplace space ((7)-(8)):

$$\left(s^\beta I - c\mathcal{L} - s^{-\alpha}\mathcal{L}\right)\widehat{u}(\bar{x}, s) = \widehat{X}(\bar{x}, s),$$

where $\widehat{X}(\bar{x}, s) = s^{\beta-1}u_0 + \widehat{h}(\bar{x}, s)$.

Step 2: Spatial Discretization via Chebyshev Spectral Collocation.

Discretize the spatial domain $[-1, 1]^d$ using Chebyshev nodes (10):

$$x_k = \cos\left(\frac{k\pi}{N_c}\right), \quad k = 0, 1, \dots, N_c.$$

Construct the 1D spectral differentiation matrix $\mathbf{D}_{N_c}^{(2)}$ for the second derivative (Sec. 3.2).

Form the discrete Laplacian operator $\mathcal{L}_{disc}$ via Kronecker products:

$$\mathcal{L}_{disc} = \mathbf{D}_{N_c}^{(2)} \otimes \mathbf{I}_{N_c} + \mathbf{I}_{N_c} \otimes \mathbf{D}_{N_c}^{(2)} \quad (2D),$$

$$\mathcal{L}_{disc} = \mathbf{I}_N \otimes \mathbf{I}_N \otimes \mathbf{D}_{N_c}^{(2)} + \mathbf{I}_N \otimes \mathbf{D}_{N_c}^{(2)} \otimes \mathbf{I}_N + \mathbf{D}_{N_c}^{(2)} \otimes \mathbf{I}_N \otimes \mathbf{I}_N \quad (3D).$$

Apply boundary conditions (8) by modifying rows of $\mathcal{L}_{disc}$ at boundary nodes N_{cB} .

Step 3: Solve in Laplace Domain.

For each Laplace parameter $s(\vartheta_j)$ on the modified Talbot contour (18):

$$s(\vartheta) = \frac{M_t}{t}\delta(\vartheta), \quad \delta(\vartheta) = -\alpha_1 + \alpha_2\vartheta \cot(\alpha_3\vartheta) + \alpha_4 i\vartheta, \quad -\pi \leq \vartheta \leq \pi,$$

solve the discretized linear system:

$$\left(s^\beta \mathbf{I} - c\mathcal{L}_{disc} - s^{-\alpha}\mathcal{L}_{disc}\right)\widehat{u}(s) = \widehat{X}(s),$$

where $\widehat{X}(s)$ incorporates transformed initial data and source term.

Store the solution vector $\widehat{u}(s)$ for each quadrature point.

Step 4: Numerical Inversion of Laplace Transform.

Reconstruct the temporal domain solution at time t using the modified Talbot's method (20):

$$u_{App}(\bar{x}, t) \approx \frac{1}{M_t i} \sum_{j=1}^{M_t} \exp(s(\vartheta_j)t) \widehat{u}(\bar{x}, s(\vartheta_j)) s'(\vartheta_j),$$

where $\vartheta_j = -\pi + \left(j - \frac{1}{2}\right)\eta$, $\eta = \frac{2\pi}{M_t}$, and $s'(\vartheta)$ is the derivative of the contour.

Step 5: Output.

The approximate solution in the time domain: $u_{App}(\bar{x}, t)$ at the desired time t .

Using the pseudoinverse $S^\dagger$ of the matrix S , we have

$$\|S^\dagger\| = \sup_{v \neq 0} \frac{\|S^\dagger v\|}{\|v\|}. \quad (35)$$

Consequently, we derive the following bound:

$$\|S^\dagger\| \geq \sup_{v=S\hat{u} \neq 0} \frac{\|S^\dagger S\hat{u}\|}{\|S\hat{u}\|} = \sup_{\hat{u} \neq 0} \frac{\|\hat{u}\|}{\|S\hat{u}\|} = \mathcal{E}. \quad (36)$$

Equations (34) and (36) define bounds for $\mathcal{E}$. While direct computation of the pseudo-inverse is computationally intensive, it offers a theoretical assurance of the system stability. For square matrices in practical applications, MATLAB's `condest` function enables estimation of the inverse norm $\|S^{-1}\|_\infty$. This facilitates an approximation of $\mathcal{E}$ as:

$$\mathcal{E} \approx \frac{\text{condest}(S')}{\|S\|_\infty}. \quad (37)$$

This technique yields effective results for our differentiation matrix S , demanding minimal computational effort. The bounds of the stability constant $\mathcal{E}$ for the system (7)–(8) corresponding to Problem 1 are illustrated in Fig. 1. It is clear that $1 \leq \mathcal{E} \leq 1.045$, which represents the stability constant $\mathcal{E}$ is bounded by numbers not very large, which suggest that the LT-ChSCS is numerically stable.

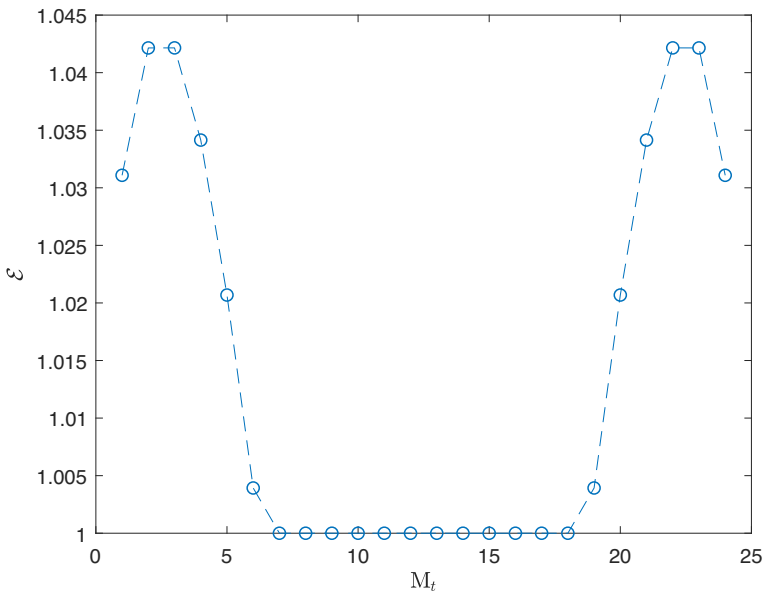


Fig. 1 The plot of $\mathcal{E}$ for Problem 1, with $N_c = 1200$, $M_t = 24$, $\alpha = 0.6$, $\beta = 0.5$ at $t = 1$

5 Numerical results and validation

This section presents numerical results that validate the performance of the proposed method. The 1D, 2D, and 3D problems are solved. We conducted our numerical results in MATLAB R2019a on a Windows 10(64 bit) personal computer equipped with an Intel(R) Core(TM) i5-3317U CPU @ 1.704 GHz with 4 GB of RAM. Solution errors are quantified using four error norms: the absolute error $\mathbf{E}_{ab}$, the relative error $\mathbf{E}_2$, the maximum absolute error $\mathbf{E}_\infty$, and the root-mean-square error $\mathbf{E}_{rms}$. These are defined as follows:

$$\begin{aligned}\mathbf{E}_{ab} &= |u(\bar{x}_i, t) - u_k(\bar{x}_i, t)|, \\ \mathbf{E}_2 &= \sqrt{\frac{\sum_{i=1}^{N_c} (u(\bar{x}_i, t) - u_k(\bar{x}_i, t))^2}{\sum_{i=1}^{N_c} (u(\bar{x}_i, t))^2}}, \\ \mathbf{E}_\infty &= \max_{1 \leq i \leq N_c} |u(\bar{x}_i, t) - u_k(\bar{x}_i, t)|, \\ \mathbf{E}_{rms} &= \sqrt{\frac{\sum_{j=1}^{N_c} (u(\bar{x}_i, t) - u_k(\bar{x}_i, t))^2}{N_c}},\end{aligned}$$

where $u(\bar{x}_i, t)$ and $u_k(\bar{x}_i, t)$ denote the analytic solution and numerical solution respectively.

Problem 1

We consider a one dimensional model [47] defined in (2) with analytical solution $u(x, t) = t^3 \sin \pi x$, $c = 1$, $(x, t) \in [0, 1] \times [0, 1]$ and $0 < \alpha, \beta < 1$ as follows

$$\frac{\partial^\beta u(x, t)}{\partial t^\beta} - \frac{\partial^2 u(x, t)}{\partial x^2} - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \zeta)^{\alpha-1} \frac{\partial^2 u(x, \zeta)}{\partial x^2} d\zeta = \sin(\pi x) \left[\frac{6t^{3-\beta}}{\Gamma(4-\beta)} + \pi^2 t^3 + \frac{6\pi^2 \Gamma(1-\alpha) t^{4-\alpha}}{\Gamma(5-\beta)} \right],$$

the initial and boundary conditions are extracted from the analytical solution. Table 1 presents the numerical values $\mathbf{E}_2$, $\mathbf{E}_\infty$ and $\mathbf{E}_{rms}$ obtained by the LT-ChSCS for varying values of N_c , M_t , α with $\beta = 0.5$ and $t = 1$. Table 2 presents the $\mathbf{E}_2$, $\mathbf{E}_\infty$ and $\mathbf{E}_{rms}$ error norms for $t \in \{0.1, 0.2, 0.3, \dots, 1\}$ with $N_c = 1200$, $M_t = 28$, $\alpha = 0.6$ and $\beta = 0.5$. The profiles of numerical solution and $\mathbf{E}_{ab}$ for $t = \{0, 0.1, 0.2, 0.4, 0.6, 0.8, 1\}$ are presented in Fig. 2a and b respectively with $N_c = 200$, $M_t = 42$, $\alpha = 0.8$, and $\beta = 0.5$. The plots of numerical solution and $\mathbf{E}_{ab}$ are represented in Fig. 3a and b, respectively, with $N_c = 200$, $M_t = 42$, $\alpha = 0.8$, and $\beta = 0.5$. We compared our results with the local Sinc method based on a single exponential and finite difference method [46]. The proposed method is also compared with another hybrid numerical method based on Legendre wavelets and a new operational matrix of fractional integration method [47]. After comparison, it is observed that our method produces precise and stable results.

Table 1 The numerical values of the error norms for $\beta = 0.5$ and $t = 1$ for 1D problem

α	N_c	M_t	E_2	E_∞	E_{rms}	CPU(sec.)
0.49	500	30	2.8431×10^{-12}	2.8682×10^{-12}	1.6752×10^{-12}	0.6655
	600		2.6521×10^{-12}	2.5342×10^{-12}	1.5629×10^{-12}	0.9261
	700		1.6454×10^{-12}	1.5697×10^{-12}	9.6979×10^{-13}	1.3688
	800		7.4266×10^{-12}	7.3378×10^{-12}	4.3776×10^{-12}	1.8695
	900		4.3546×10^{-12}	5.3288×10^{-12}	2.5670×10^{-12}	2.5401
	1000		3.2177×10^{-12}	4.1734×10^{-12}	1.8969×10^{-12}	3.3170
0.80	1200	32	3.7785×10^{-12}	4.5675×10^{-12}	2.2277×10^{-12}	5.4115
		34	6.3227×10^{-12}	6.9429×10^{-12}	3.7277×10^{-12}	6.2022
		36	3.5376×10^{-12}	3.2900×10^{-12}	2.0857×10^{-12}	6.2917
		38	3.6145×10^{-12}	3.3556×10^{-12}	2.1310×10^{-12}	6.7184
		40	3.1569×10^{-12}	4.0946×10^{-12}	1.8612×10^{-12}	6.9152
		42	3.6488×10^{-12}	4.5077×10^{-12}	2.1512×10^{-12}	7.7597
[46] $\alpha = 0.8$			---	2.953×10^{-5}	---	---
[47] $\alpha = 0.6$	1.0926×10^{-15} (E_{ab})					

Table 2 The numerical values of the error norms at $t = 1$ with $N_c = 1200$, $M_t = 28$, $\alpha = 0.6$, $\beta = 0.5$ for 1D problem

t	E_2	E_∞	E_{rms}	CPU Time(s)
0.1	2.2142×10^{-11}	2.3557×10^{-14}	1.3054×10^{-14}	4.978748
0.2	2.4601×10^{-11}	2.1114×10^{-13}	1.1603×10^{-13}	6.918656
0.3	2.0354×10^{-11}	5.8654×10^{-13}	3.2401×10^{-13}	5.906572
0.4	2.4979×10^{-11}	1.6759×10^{-12}	9.4250×10^{-13}	5.125324
0.5	2.3195×10^{-11}	3.0743×10^{-12}	1.7094×10^{-12}	5.557087
0.6	2.3831×10^{-11}	5.3863×10^{-12}	3.0348×10^{-12}	5.109422
0.7	2.4780×10^{-11}	8.8027×10^{-12}	5.0110×10^{-12}	5.305663
0.8	2.1132×10^{-11}	1.1375×10^{-11}	6.3788×10^{-12}	5.093508
0.9	2.3820×10^{-11}	1.8195×10^{-11}	1.0238×10^{-11}	4.860032
1.0	2.6139×10^{-11}	2.6454×10^{-11}	1.5411×10^{-11}	5.033078

Problem 2

We consider a 2D model [48] defined in (2) with the analytical solution $u(x, y, t) = \sin(\pi x) \sin(\pi y) t^{\frac{4}{5}}$, $c = 0$, $(x, y, t) \in [-1, 1]^2 \times [0, 1]$ and $0 < \alpha, \beta < 1$ as follows

$$\begin{aligned} \frac{\partial^\beta u(x, y, t)}{\partial t^\beta} - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \zeta)^{\alpha-1} \left[\frac{\partial^2 u(x, y, \zeta)}{\partial x^2} + \frac{\partial^2 u(x, y, \zeta)}{\partial y^2} \right] d\zeta \\ = \Gamma\left(\frac{9}{5}\right) \left[\frac{t^{\frac{4}{5}-\beta}}{\Gamma\left(\frac{9}{5}-\beta\right)} + \frac{2\pi^2 t^{\frac{4}{5}+\alpha}}{\Gamma\left(\frac{9}{5}+\alpha\right)} \right] \sin(\pi x) \sin(\pi y), \end{aligned}$$

the initial and boundary conditions are extracted from the exact solution. The errors obtained using LT-ChSCS are displayed in Table 3. The results presented in the table demonstrate high accuracy and numerical stability. It is observed that with an increasing number of spatial and quadrature nodes, the error decreases. Figures 4a–b present

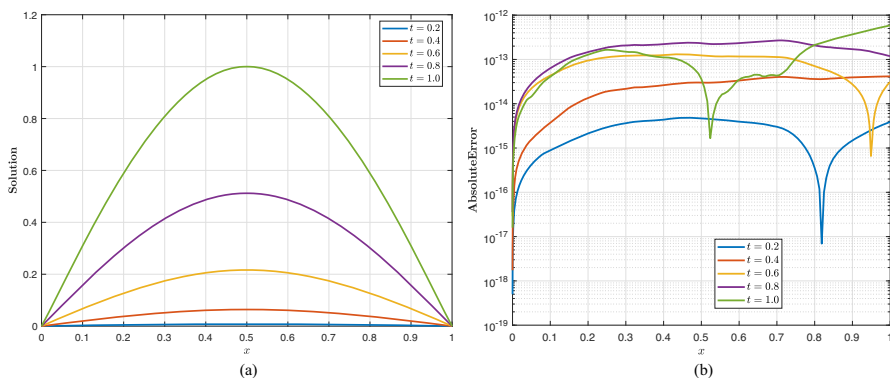


Fig. 2 (a) Numerical solution of 1D TF-PIDE. (b) The plot of E_{abs} of 1D TF-PIDE

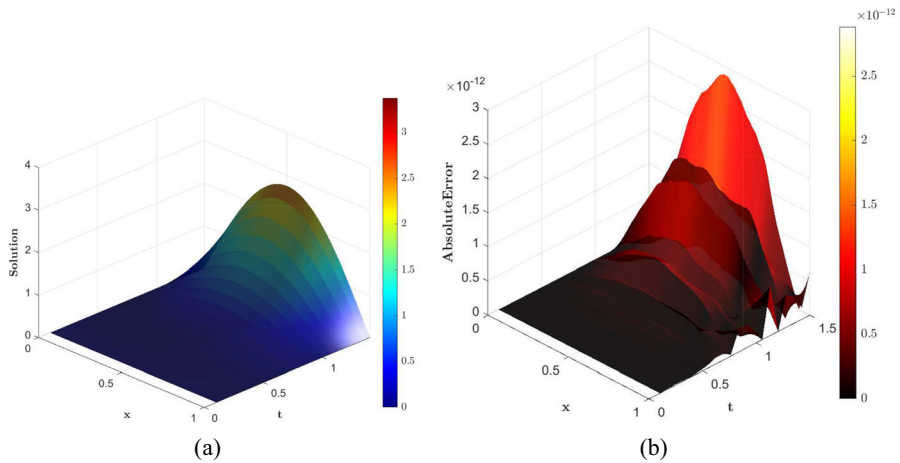


Fig. 3 (a) The plot of numerical solution computed with $N_c = 200$, $M_t = 42$, $\alpha = 0.8$, and $\beta = 0.5$. (b) The space-time plot of E_{ab} computed with $N_c = 200$, $M_t = 42$, $\alpha = 0.8$, and $\beta = 0.5$

the surface plot and the contour plot of the numerical solution, respectively. Similarly, Fig. 5a–b present the surface plot and the contour plot of the E_{ab} , respectively; both figures demonstrate high accuracy of the proposed numerical scheme. Figures 6a–b plot the E_2 , E_∞ , and E_{rms} as functions of the quadrature nodes M_t and the spatial nodes N_c respectively. All the figures demonstrate high accuracy. In the first two figures, we observe that with an increasing number of quadrature nodes M_t and spatial nodes N_c , the error decreases. A comparison with alternating direction implicit numerical approach [48] demonstrates that the proposed method has superior accuracy. Overall, the results confirm the superior accuracy of the method.

Problem 3

In our third problem, we consider a 2D model [49] defined in (2) with the exact solution $u(x, y, t) = (1 - x^2)(1 - y^2)t^{\frac{3}{2}}$, $c = 0$, $(x, y, t) \in [-1, 1]^2 \times [0, 1]$ and $0 < \alpha, \beta < 1$ as follows

$$\begin{aligned} \frac{\partial^\beta u(x, y, t)}{\partial t^\beta} - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \zeta)^{\alpha-1} \left[\frac{\partial^2 u(x, y, \zeta)}{\partial x^2} + \frac{\partial^2 u(x, y, \zeta)}{\partial y^2} \right] d\zeta \\ = \frac{3\sqrt{\pi}}{4} \frac{t^{\frac{3}{2}-\beta}}{\Gamma\left(\frac{5}{2}-\beta\right)} (1-x^2)(1-y^2) + \frac{3\sqrt{\pi}}{2} \frac{t^{\alpha+\frac{3}{2}}}{\Gamma\left(\alpha+\frac{5}{2}\right)} (2-x^2-y^2), \end{aligned}$$

the initial and boundary conditions are extracted from the exact solution. The errors obtained using the proposed method for different combinations of β and α and various values of N_c and M_t are displayed in Table 4. For this problem also, the results presented in the table demonstrate high accuracy and numerical stability. Figures 7a–b present

Table 3 The E_2 , E_∞ , E_{rms} calculated by the proposed method at $t = 1$ for Problem 2

	$N_c(\text{total nodes})$	M_t	E_2	E_∞	E_{rms}	CPU(sec.)
$\beta = 0.15$ $\alpha = 0.25$	10(121)	24	2.6966×10^{-6}	2.3214×10^{-6}	9.5571×10^{-7}	0.0770
	12(169)		3.6191×10^{-8}	3.0676×10^{-8}	1.3024×10^{-8}	0.0641
	14(225)		3.6893×10^{-10}	3.2576×10^{-10}	1.3424×10^{-10}	0.0914
	16(289)		3.3928×10^{-12}	3.1688×10^{-12}	1.2449×10^{-12}	0.1677
	18(361)		9.5109×10^{-13}	9.6489×10^{-13}	3.5128×10^{-13}	0.3007
$\beta = 0.75$ $\alpha = 0.50$	20(441)		9.5851×10^{-13}	9.5013×10^{-13}	3.5589×10^{-13}	0.4029
	22(529)		9.7066×10^{-13}	9.5268×10^{-13}	3.6197×10^{-13}	0.5994
	24(625)		9.8407×10^{-13}	9.7344×10^{-13}	3.6831×10^{-13}	0.8972
	16(289)	18	9.0275×10^{-11}	8.6280×10^{-11}	3.3124×10^{-11}	0.1964
		20	2.4349×10^{-11}	2.2758×10^{-11}	8.9345×10^{-12}	0.1509
[48] $\alpha = \frac{1}{256}$, $\beta = 0.75$		22	3.2501×10^{-12}	3.1875×10^{-12}	1.1926×10^{-12}	0.1493
		24	2.8954×10^{-12}	2.6813×10^{-12}	1.0624×10^{-12}	0.1625
		26	2.9734×10^{-12}	2.7645×10^{-12}	1.0910×10^{-12}	0.1794
		28	2.9613×10^{-12}	2.7462×10^{-12}	1.0866×10^{-12}	0.1811
		30	2.9794×10^{-12}	2.8031×10^{-12}	1.0932×10^{-12}	0.2004
		32	2.9779×10^{-12}	2.7940×10^{-12}	1.0927×10^{-12}	0.2177
		—	—	6.0393×10^{-4}	—	3.1000

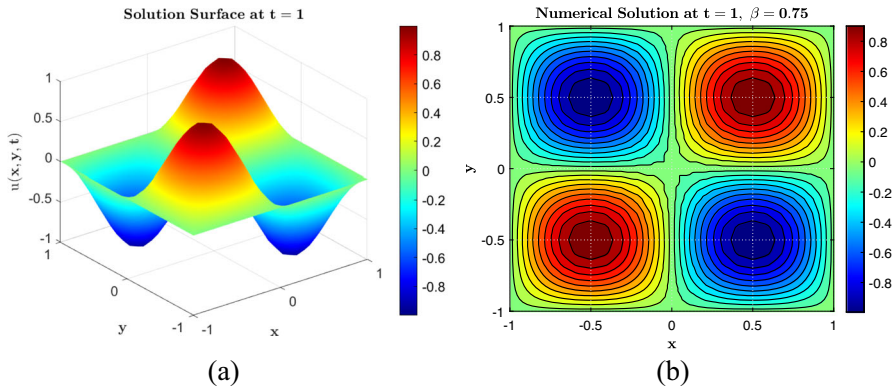


Fig. 4 (a) The numerical solution for problem 2 obtained at $t = 1$ with $M_t = 36$, $N_c = 30$, $\beta = 0.75$, $\alpha = 0.25$. (b) The contour plot of problem 2 obtained at $t = 1$ with $M_t = 36$, $N_c = 30$, $\beta = 0.75$, $\alpha = 0.25$

the surface plot and the contour plot of the numerical solution, respectively. Similarly, Fig. 8a–b present the surface plot and the contour plot of the E_{ab} , respectively; both figures demonstrate high accuracy of the proposed numerical scheme. Figures 9a–b plot the E_2 , E_∞ , and E_{rms} as functions of the quadrature nodes M_t and the spatial nodes N_c respectively. All the figures demonstrate high accuracy. For this problem, we observe that with increasing number of quadrature nodes M_t , the error shows the same decreasing trend; however, for the increasing spatial nodes N_c , the E_2 error increases slightly while the other two error norms remain nearly constant. This slight increase in the error with increasing N_c is attributed to the $\mathcal{O}(N_c^4)$ conditioning of the Chebyshev differentiation matrix, which amplifies rounding errors. Additionally, the presence of “outlier” eigenvalues of order N_c^4 localized near boundaries [39] (Chapter 10) further contaminates the solution when rounding errors become significant. This is not a loss of spectral accuracy but a consequence of finite-precision arithmetic.

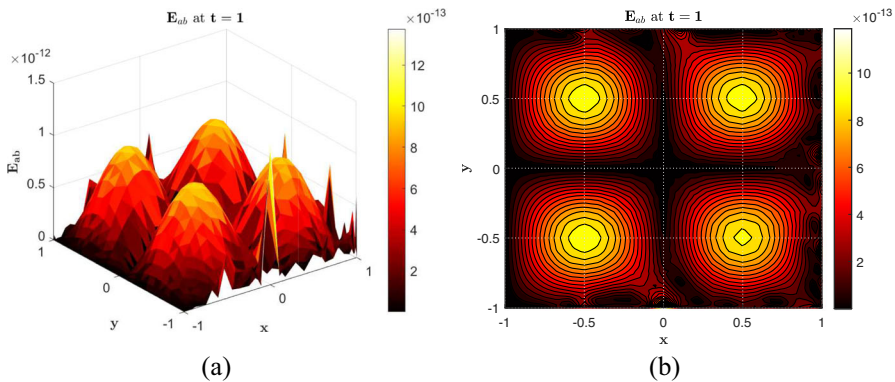


Fig. 5 (a) The surface plot of E_{ab} obtained at $t = 1$ with $M_t = 36$, $N_c = 30$, $\beta = 0.75$, $\alpha = 0.25$. (b) The figure shows the contour plot of E_{ab} obtained at $t = 1$ with $M_t = 36$, $N_c = 30$, $\beta = 0.75$, $\alpha = 0.25$

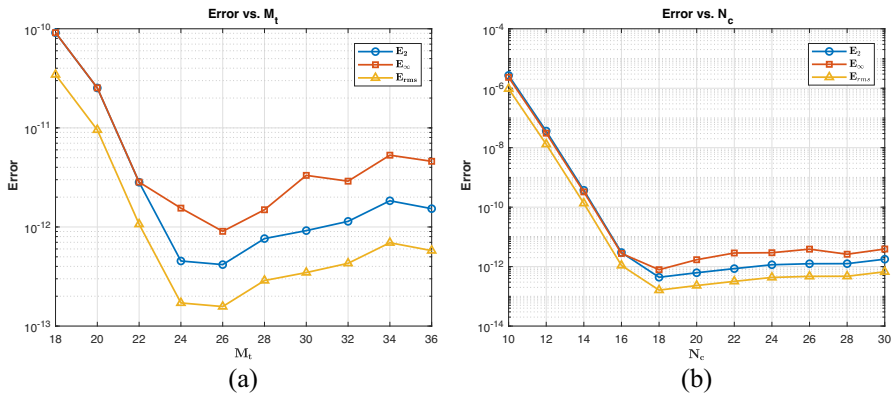


Fig. 6 (a) The comparison plot of E_2 , E_∞ , and E_{rms} as functions of quadrature nodes M_t at $t = 1$ with $N_c = 30$, $\beta = 0.75$, $\alpha = 0.25$. (b) The comparison plot of E_2 , E_∞ , and E_{rms} as functions of collocation nodes N_c at $t = 1$ with $M_t = 36$, $\beta = 0.75$, $\alpha = 0.25$

Problem 4

In our fourth problem, we consider a 3D model [48] defined in (2) with the exact solution $u(x, y, z, t) = \sin(\pi x) \sin(\pi y) \sin(\pi z) t^{\frac{4}{5}}$, $c = 1$, $(x, y, z, t) \in [-1, 1]^3 \times [0, 1]$ and $0 < \alpha, \beta < 1$ as follows

$$\begin{aligned} \frac{\partial^\beta u(x, y, z, t)}{\partial t^\beta} - \frac{\partial^2 u(x, y, z, t)}{\partial x^2} - \frac{\partial^2 u(x, y, z, t)}{\partial y^2} - \frac{\partial^2 u(x, y, z, t)}{\partial z^2} - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \zeta)^{\alpha-1} \left[\frac{\partial^2 u(x, y, z, \zeta)}{\partial x^2} \right. \\ \left. + \frac{\partial^2 u(x, y, z, \zeta)}{\partial y^2} + \frac{\partial^2 u(x, y, z, \zeta)}{\partial z^2} \right] d\zeta = \left[\frac{\Gamma\left(\frac{9}{5}\right)}{\Gamma\left(\frac{9}{5} - \beta\right)} t^{\frac{4}{5} - \beta} + 3\pi^2 t^{\frac{4}{5}} \right. \\ \left. + \frac{3\pi^2 \Gamma\left(\frac{9}{5}\right)}{\Gamma\left(\alpha + \frac{9}{5}\right)} t^{\alpha + \frac{4}{5}} \right] \sin(\pi x) \sin(\pi y) \sin(\pi z), \end{aligned}$$

Table 4 The E_2 , E_∞ , E_{rms} calculated by the proposed method at $t = 1$ for Problem 3

	N_c	M_t	E_2	E_∞	E_{rms}	CPU(sec.)
$\beta = 0.75$	4(25)	38	1.1669×10^{-15}	1.2331×10^{-15}	3.5008×10^{-16}	0.0479
$\alpha = 0.50$	5(36)		2.7575×10^{-15}	3.7108×10^{-15}	8.6173×10^{-16}	0.0217
	6(49)		7.3219×10^{-15}	1.1326×10^{-14}	2.3535×10^{-15}	0.0156
	7(64)		4.5275×10^{-15}	6.3955×10^{-15}	1.4856×10^{-15}	0.0197
	8(81)		9.5068×10^{-15}	1.4450×10^{-14}	3.1689×10^{-15}	0.0311
$\beta = 0.75$	16(289)	16	4.7221×10^{-7}	4.7221×10^{-7}	1.6666×10^{-7}	0.1437
$\alpha = 0.50$		20	2.9175×10^{-9}	2.9175×10^{-9}	1.0297×10^{-9}	0.1372
		24	1.7003×10^{-11}	1.7002×10^{-11}	6.0009×10^{-12}	0.1542
		28	1.0379×10^{-13}	1.0070×10^{-13}	3.6632×10^{-14}	0.1816
		32	3.8757×10^{-14}	7.2020×10^{-14}	1.3679×10^{-14}	0.2196
Tang et al. [49]	8(81)	—	—	1.76×10^{-4}	—	—

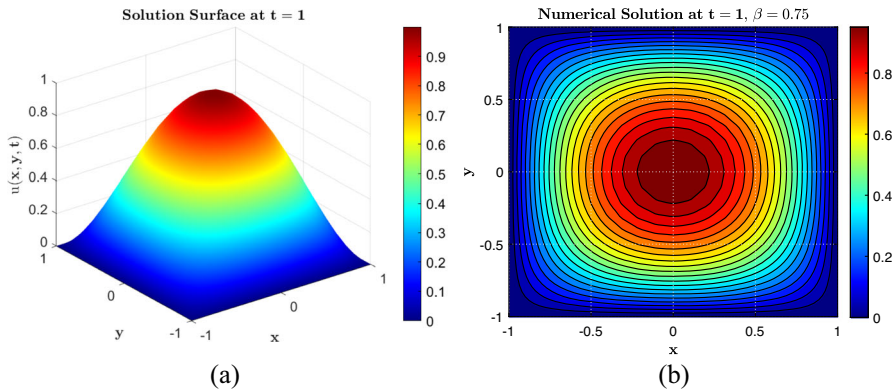


Fig. 7 (a) The numerical solution obtained at $t = 1$ with $M_t = 38$, $N_c = 32$, $\beta = 0.75$, $\alpha = 0.25$. (b) The contour plot obtained at $t = 1$ with $M_t = 38$, $N_c = 32$, $\beta = 0.75$, $\alpha = 0.25$

the initial and boundary conditions are extracted from the exact solution. The numerical results for this problem are displayed in Table 5. For this 3D problem, the condition number of $\mathcal{L}_{disc}$ scales as $\mathcal{O}(N_c^4)$, and the matrix dimension grows as $(N_c - 1)^3$. For $N_c \geq 15$, the dense matrix becomes ill-conditioned, requiring careful linear algebra treatment. The CPU times reported in Table 3 reflect this cubic growth in matrix dimension. Like the previous two problems, the method produced highly accurate results. It is also observed that the error decreases with increasing numbers of quadrature and spatial nodes. Figures 10a–b depict the slice plots of the numerical solution and $\mathbf{E}_{ab}$ demonstrating a very high accuracy for this complex 3D problem. Figure 11a depicts the isosurfaces of the numerical solution, which clearly demonstrate the spherical symmetry of the solution. Figure 11b shows the contour slice plot of $\mathbf{E}_{ab}$ demonstrating high accuracy. Figures 12a–b show the plots of $\mathbf{E}_2$, $\mathbf{E}_\infty$, and $\mathbf{E}_{rms}$ as functions of the quadrature nodes M_t and spatial nodes N_c , respectively. All the figures demonstrate high accuracy. For this problem, we observe that with the increasing number

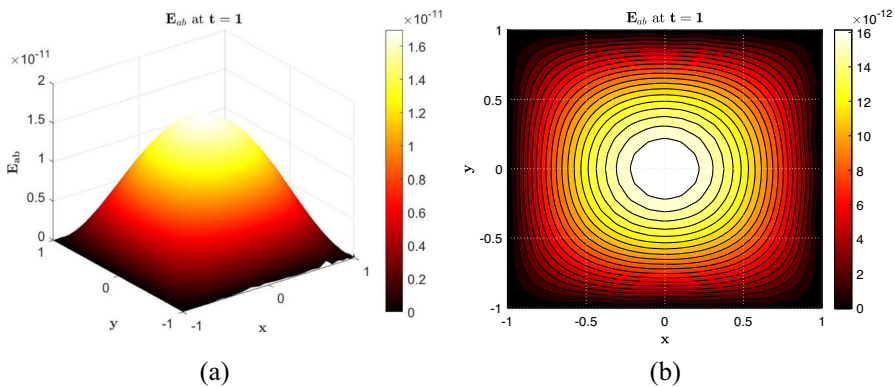


Fig. 8 (a) The surface plot of $\mathbf{E}_{ab}$ obtained at $t = 1$ with $M_t = 38$, $N_c = 32$, $\beta = 0.75$, $\alpha = 0.25$. (b) The contour plot of $\mathbf{E}_{ab}$ obtained at $t = 1$ with $M_t = 38$, $N_c = 32$, $\beta = 0.75$, $\alpha = 0.25$

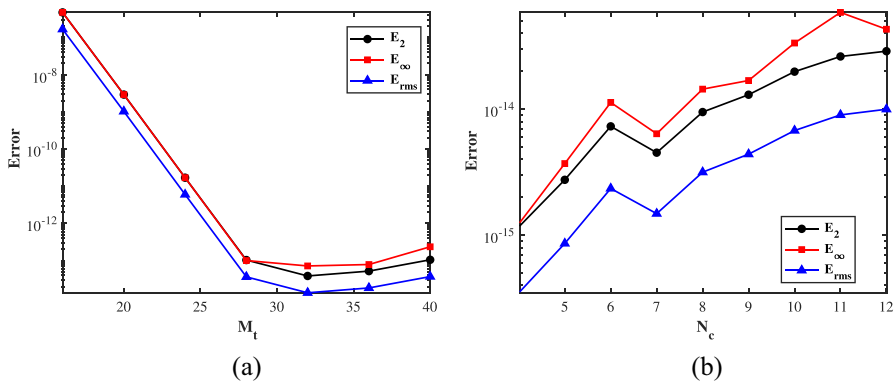


Fig. 9 (a) The comparison plot of E_2 , E_∞ , and E_{rms} as functions of quadrature nodes M_t at $t = 1$ with $N_c = 16$, $\beta = 0.75$, $\alpha = 0.50$. (b) The comparison plot of E_2 , E_∞ , and E_{rms} as functions of collocation nodes N_c at $t = 1$ with $M_t = 38$, $\beta = 0.75$, $\alpha = 0.50$

of quadrature nodes M_t and the spatial nodes, the error shows the same decreasing trend. The results are compared with the compact finite difference method [48] and better performance is observed. Overall, the results confirm the superior accuracy of the method.

Table 5 The E_2 , E_∞ , E_{rms} calculated by the proposed method at $t = 1$ for Problem 4

	N_c (total nodes)	M_t	E_2	E_∞	E_{rms}	CPU(sec.)
$\beta = 0.75$	4(125)	40	1.7546×10^{-1}	8.8391×10^{-2}	2.2361×10^{-2}	0.1070
$\alpha = 0.50$	6(343)		5.9047×10^{-3}	3.3548×10^{-3}	1.1368×10^{-3}	0.3619
	8(729)		1.7139×10^{-4}	1.0953×10^{-4}	3.4964×10^{-5}	1.9668
	10(1331)		3.1783×10^{-6}	2.7397×10^{-6}	6.7061×10^{-7}	9.1511
	12(2197)		4.2862×10^{-8}	3.2185×10^{-8}	9.2532×10^{-9}	3.4695
	14(3375)		4.3858×10^{-10}	3.2286×10^{-10}	9.6265×10^{-11}	1.1612
	16(4913)	10	7.9258×10^{-12}	2.1523×10^{-11}	1.7616×10^{-12}	3.2651
$\beta = 0.15$	15(4096)		7.9695×10^{-5}	7.9695×10^{-5}	1.7610×10^{-5}	45.8900
$\alpha = 0.50$			6.1306×10^{-6}	6.1306×10^{-6}	1.3547×10^{-6}	54.6540
			4.6162×10^{-7}	4.6160×10^{-7}	1.0200×10^{-7}	64.4700
			3.4221×10^{-8}	3.4208×10^{-8}	7.5617×10^{-9}	73.2410
			2.5098×10^{-9}	2.4960×10^{-9}	5.5458×10^{-10}	83.0730
			1.8528×10^{-10}	1.7120×10^{-10}	4.0941×10^{-11}	90.9750
			1.9733×10^{-11}	1.8822×10^{-11}	4.3604×10^{-12}	100.3100
		24	1.2107×10^{-11}	1.1983×10^{-11}	2.6752×10^{-12}	109.1000
Guo et al. [48]	16(4913)	—	—	2.672×10^{-3}	—	44.4200

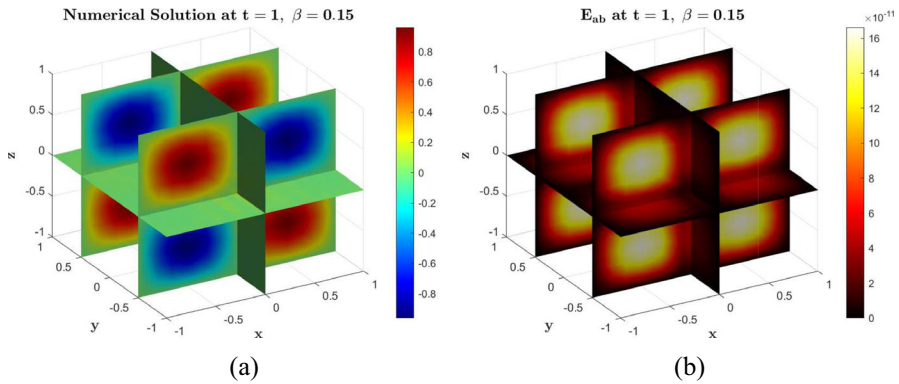


Fig. 10 (a) The slice plot of numerical solution at $t = 1$ with $N_c = 15$, $M_t = 20$, $\beta = 0.15$, $\alpha = 0.50$. (b) The slice plot of E_{ab} at $t = 1$ with $N_c = 15$, $M_t = 20$, $\beta = 0.15$, $\alpha = 0.50$

6 Conclusion

The work develops a hybrid method, the LT-ChSCS method, for approximating solutions to FPIDE including the Caputo derivative. The proposed methodology proceeds in three steps: first, the LT is applied to transform the considered time dependent model into an inhomogeneous time independent problem in Laplace domain. This step eliminates the complex convolution integrals inherent to fractional derivatives and removes the stability issues of time stepping methods. Second, the ChSCS is employed to discretize the spatial derivatives of the transformed problem, achieving exponential convergence with relatively few spatial nodes. Finally, the time domain solution is recovered using the improved Talbot method, which offers high accuracy with fewer quadrature nodes as compared to other hyperbolic contour integration method. A key

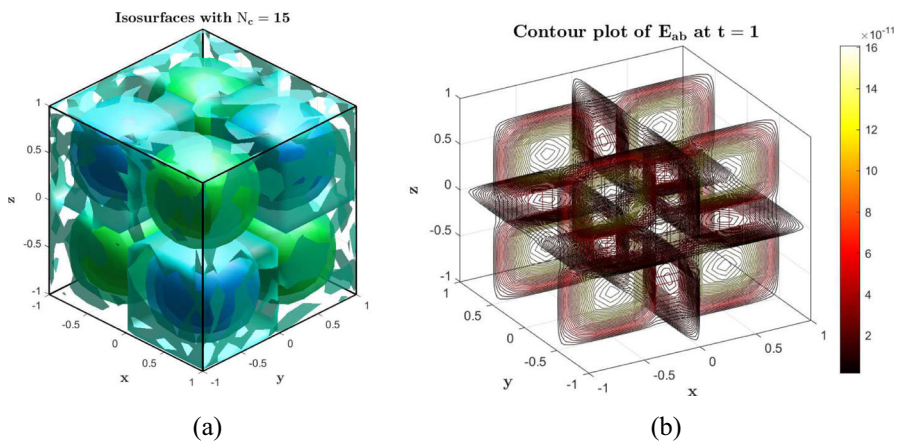


Fig. 11 (a) The iso surfaces of numerical solution at $t = 1$ with $N_c = 15$, $M_t = 20$, $\beta = 0.15$, $\alpha = 0.50$. (b) The contour slice plot of E_{ab} at $t = 1$ with $N_c = 15$, $M_t = 20$, $\beta = 0.15$, $\alpha = 0.50$

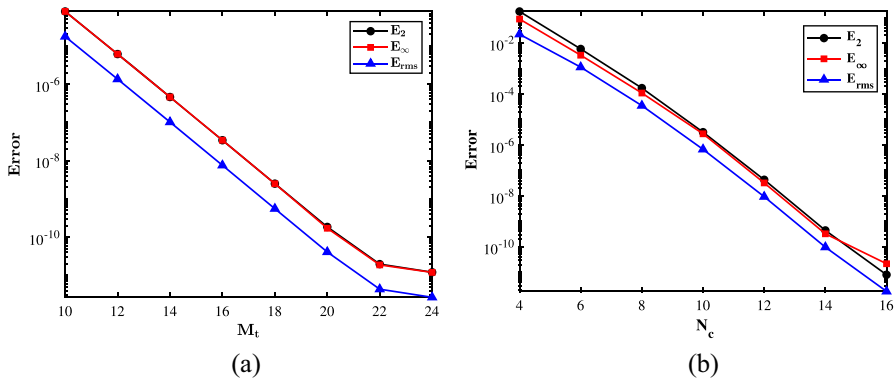


Fig. 12 (a) The comparison plot of E_2 , E_∞ , and E_{rms} as functions of quadrature nodes M_t at $t = 1$ with $N_c = 15$, $\beta = 0.15$, $\alpha = 0.50$. (b) The comparison plot of E_2 , E_∞ , and E_{rms} as functions of collocation nodes N_c at $t = 1$ with $M_t = 40$, $\beta = 0.75$, $\alpha = 0.50$

observation from the numerical experiments is that while the method achieves high accuracy in 1D, 2D, and 3D, the conditioning of the Chebyshev differentiation matrix, which scales as $\mathcal{O}(N_c^4)$, limits the maximum usable spatial resolution due to rounding error amplification. This is particularly evident in the 3D case, where the matrix dimension grows cubically and the condition number deteriorates rapidly. For large-scale problems, preconditioning strategies and iterative solvers are recommended to mitigate these effects. Consequently, this hybrid approach proves to be a highly efficient and robust tool for numerical simulation of complex physical phenomena and systems arising in engineering and science. This approach can be extended to more challenging problems, such as delayed or coupled FPIDEs.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

Conflict of Interest The authors declare no conflict of interest.

Consent to Participate declaration N/A

Consent to Publish declaration N/A

Ethics declaration N/A

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