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On Apostol-Type Hermite Degenerated Polynomials

Clemente Cesarano [1,*,†], William Ramírez [1,2,*,†], Stiven Díaz [2,†], Adnan Shamaoon [3,†] and Waseem Ahmad Khan [4,†]

1 Section of Mathematics, Universitá Telematica Internazionale Uninettuno, 00186 Rome, Italy
2 Departamento de Ciencias Naturales y Exactas, Universidad de la Costa, Barranquilla 00928-1345, Colombia
3 Syed Babar Ali School of Science and Engineering, Lahore University of Management Sciences, Lahore 54792, Pakistan
4 Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University, P.O. Box 1664, Al Khobar 31952, Saudi Arabia
* Correspondence: c.cesarano@uninettunouniversity.net (C.C.); wramirez4@cuc.edu.co (W.R.)
† These authors contributed equally to this work.

Abstract: This article presents a generalization of new classes of degenerated Apostol–Bernoulli, Apostol–Euler, and Apostol–Genocchi Hermite polynomials of level m. We establish some algebraic and differential properties for generalizations of new classes of degenerated Apostol–Bernoulli polynomials. These results are shown using generating function methods for Apostol–Euler and Apostol–Genocchi Hermite polynomials of level m.

Keywords: Hermite polynomials; Apostol-type polynomials; degenerate Apostol-type polynomials

MSC: 11B68; 11B83; 11B39; 05A19

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1. Introduction

In this document, the customary conventions of mathematical notation are employed, where $\mathbb{N} := \{1, 2, \ldots\}$; $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$; $\mathbb{Z}$ refers to a set of integers; $\mathbb{R}$ refers to a set of real numbers; and $\mathbb{C}$ refers to a set of complex numbers.

There have been numerous studies in the literature that have focused on Apostol–Bernoulli, Apostol–Euler, and Apostol–Genocchi Hermite polynomials, as well as their extensions and relatives. These studies include works in [1–15]. In recent years, several researchers have explored degraded versions of well-known polynomials, such as Bernoulli, Euler, falling factorial, and Bell polynomials, by utilizing generating functions, umbral calculus, and p-adic integrals. Examples of such studies can be found in [16–18].

The generalization of two-variable Hermite polynomials introduced by Kampé de Fériet is given by [19]:

$$H_\omega(\xi, \eta) = \omega! \sum_{\nu=0}^{[\frac{\omega}{2}]} \frac{\eta^\nu \xi^{\omega-2\nu}}{\nu!(\omega-2\nu)!}.$$

It is to be noted that [20]

$$H_\omega(2\xi, -1) = H_\omega(\xi).$$

These polynomials satisfy the following generating equation:

$$e^{\xi\tau+\eta\tau^2} = \sum_{\omega=0}^{\infty} H_\omega(\xi, \eta) \frac{\tau^\omega}{\omega!}. \tag{1}$$

Two-variable degenerate Hermite polynomials $H_n(\xi, \eta; \mu)$ ([21], p. 65) are defined by means of the generating function

$$(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}} = \sum_{\omega=0}^{\infty} H_\omega(\xi, \eta; \mu)\frac{\tau^\omega}{\omega!}. \quad (2)$$

We note that

$$\lim_{\mu \to 0} H_\omega(\xi, \eta; \mu) = H_\omega(\xi, \eta).$$

The first and second kind of Stirling numbers are given, respectively, by (see [22]):

$$\frac{1}{\nu!}[\ln(1+\tau)]^\nu = \sum_{\omega=\nu}^{\infty} S(\omega, \nu)\frac{\tau^\omega}{\omega!}$$

and

$$\frac{1}{\nu!}(e^\tau - 1)^\nu = \sum_{\omega=\nu}^{\infty} S(\omega, \nu)\frac{\tau^\omega}{\omega!}.$$

The generalized falling factorial $(\xi|\mu)_\omega$ with increment μ is defined by (see [18], Definition 2.3):

$$(\xi|\mu)_\omega = \prod_{\nu=0}^{\omega-1}(\xi - \mu\nu),$$

for positive integer ω, with the convention $(\xi|\mu)_0 = 1$. Furthermore,

$$(\xi|\mu)_\omega = \sum_{\nu=0}^{\omega} S(\omega, \nu)\mu^{\omega-\nu}\xi^\nu.$$

From the Binomial Theorem, we have

$$(1+\mu\tau)^{\frac{\xi}{\mu}} = \sum_{\omega=0}^{\infty}(\xi|mu)_\omega\frac{\tau^\omega}{\omega!}.$$

Khan [14] introduced degenerate Hermite–Bernoulli polynomials of the second kind, defined by

$$\frac{\log(1+\mu\tau)^{\frac{1}{\mu}}}{(1+\mu\tau)^{\frac{1}{\mu}} - 1}(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}} = \sum_{\omega=0}^{\infty} {}_H\mathcal{B}_\omega(\xi, \eta; \mu)\frac{\tau^\omega}{\omega!}.$$

For $\lambda, u \in \mathbb{C}$, and $\alpha \in \mathbb{N}$, with $u \neq 1$, the generalized degenerate Apostol-type Frobenius Euler–Hermite polynomials of order α are given by a generating function (see [15], p. 569):

$$\left(\frac{1-u}{\lambda(1+\mu\tau)^{\frac{1}{\mu}} - u}\right)^\alpha (1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}} = \sum_{\omega=0}^{\infty} {}_H h_\omega(\xi, \eta; \mu; \lambda; u)\frac{\tau^\omega}{\omega!}. \quad (3)$$

Taking $u = -1$ and $\alpha = 1$ in (3), the degenerate Hermite–Euler polynomials are obtained (see [7], p. 3, Equation (17)):

$$\frac{2}{\lambda(1+\mu\tau)^{\frac{1}{\mu}} + 1}(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}} = \sum_{\omega=0}^{\infty} {}_H\mathcal{E}_\omega(\xi, \eta; \mu; \lambda)\frac{\tau^\omega}{\omega!}.$$

Clemente et al. [23] introduced and studied new families of Apostol-type degenerated polynomials by means of the following generating functions:

$$\tau^{m\alpha}[\sigma(\lambda;\mu,b;\tau)]^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}} = \sum_{\omega=0}^{\infty} \mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}, \tag{4}$$

$$2^{m\alpha}[\psi(\lambda;\mu,b;\tau)]^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}} = \sum_{\omega=0}^{\infty} \mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!} \tag{5}$$

and

$$(2\tau)^{m\alpha}[\psi(\lambda;\mu,b;\tau)]^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}} = \sum_{\omega=0}^{\infty} \mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!} \tag{6}$$

where,

$$\sigma(\lambda;a,b;\tau) = \left(\lambda(1+\mu\tau)^{\frac{1}{\mu}} - \sum_{l=0}^{m-1}\frac{(\tau\log b)^{l}}{l!}\right)^{-1}$$

and

$$\psi(\lambda;\mu,b;\tau) = \left(\lambda(1+\mu\tau)^{\frac{1}{\mu}} + \sum_{l=0}^{m-1}\frac{(\tau\log b)^{l}}{l!}\right)^{-1}.$$

If $\xi = 0$, in (4)–(6), we obtain the Apostol-type degenerated numbers of order α and level m:

$$\tau^{m\alpha}[\sigma(\lambda;\mu,b;\tau)]^{\alpha} = \sum_{\omega=0}^{\infty} \mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!},$$

$$2^{m\alpha}[\psi(\lambda;\mu,b;\tau)]^{\alpha} = \sum_{\omega=0}^{\infty} \mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!},$$

$$(2\tau)^{m\alpha}[\psi(\lambda;\mu,b;\tau)]^{\alpha} = \sum_{\omega=0}^{\infty} \mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}.$$

The past few years have seen significant advancements in the generalizations of special functions used in mathematical physics. These developments provide an analytical foundation for many exact solutions to problems in mathematical physics and have practical applications in various fields. One important area of development is the introduction of one- and double-variable special functions, which have been recognized for their significance in both pure mathematical and applied contexts. Multi-index and multi-variable special functions are also necessary for solving problems in several branches of mathematics, such as partial differential equations and abstract group theory. Hermite polynomials, developed by Hermite [24–27], are an example of such special functions, which are important in combinatorics, numerical analysis, and physics. They are associated with the quantum harmonic oscillator and are utilized in solving the Schrödinger equation for the oscillator. This article aims to introduce new families of Hermite–Apostol-type degenerated polynomials. Some algebraic properties and relations for these polynomials are derived. These results extend certain relations and identities of the related polynomials.

2. Generalizations of New Classes of Degenerated Apostol–Bernoulli, Apostol–Euler, and Apostol–Genocchi Hermite Polynomials of Level *m*

In this section, based on (2) and (4)–(6), we define new families of Hermite–Apostol-type degenerated polynomials.

Definition 1. *For arbitrary real or complex parameter α and for $\mu, b \in \mathbb{R}^{+}$, the generalizations degenerate the Apostol–Bernoulli Hermite polynomials ${}_{H}\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)$, the generalizations degenerate Apostol–Euler Hermite polynomials ${}_{H}\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)$, and the generalizations degenerate Apostol–Genocchi Hermite polynomials ${}_{H}\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$ of* *order α and level m, are defined, in a suitable neighborhood of $t = 0$, by means of the generating functions:*

$$\tau^{m\alpha}[\sigma(\lambda;\mu,b;\tau)]^{\alpha}(1+a\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}} = \sum_{\omega=0}^{\infty} {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}, \tag{7}$$

$$2^{m\alpha}[\psi(\lambda;\mu,b;\tau)]^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}} = \sum_{\omega=0}^{\infty} {}_H\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!} \tag{8}$$

and

$$(2\tau)^{m\alpha}[\psi(\lambda;\mu,b;\tau)]^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}} = \sum_{\omega=0}^{\infty} {}_H\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}, \tag{9}$$

where

$$\sigma(\lambda;\mu,b;\tau) = \left(\lambda(1+\mu\tau)^{\frac{1}{\mu}} - \sum_{l=0}^{m-1}\frac{(\tau\log b)^l}{l!}\right)^{-1}$$

and

$$\psi(\lambda;\mu,b;\tau) = \left(\lambda(1+\mu\tau)^{\frac{1}{\mu}} + \sum_{l=0}^{m-1}\frac{(\tau\log b)^l}{l!}\right)^{-1}.$$

Note that for $\alpha = 1$, $\lambda = 1$, and $b = e$ in (7), we have

$$\begin{aligned}\sum_{\omega=0}^{\infty}\lim_{\mu\to 0}\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!} &= \lim_{\mu\to 0}\left(\frac{\tau^m}{\lambda(1+\mu\tau)^{\frac{1}{\mu}} - \sum\limits_{l=0}^{m-1}\frac{(\tau\log b)^l}{l!}}\right)^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}}\\ &= \left(\frac{\tau^m}{e^{\tau} - \sum\limits_{l=0}^{m-1}\frac{\tau^l}{l!}}\right)e^{\xi\tau+\eta\tau^2}\\ &= \sum_{\omega=0}^{\infty}\mathfrak{B}_{\omega}^{[m-1]}(\xi,\eta)\frac{\tau^{\omega}}{\omega!},\end{aligned}$$

where $\mathfrak{B}_{\omega}^{[m-1]}(\xi)$ are called generalized Hermite–Bernoulli polynomials (see [28], Equation (6)).

Analogously,

$$\begin{aligned}\sum_{\omega=0}^{\infty}\lim_{\mu\to 0}\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!} &= \sum_{\omega=0}^{\infty}\mathfrak{E}_{\omega}^{[m-1]}(\xi,\eta)\frac{\tau^{\omega}}{\omega!},\\ \sum_{\omega=0}^{\infty}\lim_{\mu\to 0}\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!} &= \sum_{\omega=0}^{\infty}\mathfrak{G}_{\omega}^{[m-1]}(\xi,\eta)\frac{\tau^{\omega}}{\omega!}.\end{aligned}$$

where $\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi)$ and $\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi)$ are called generalized Hermite–Euler polynomials and generalized Hermite–Genocchi polynomials, respectively.

If $\xi = 0$ and $\eta = 0$, in Definition 1, we obtain the generalizations of degenerate Apostol–Bernoulli Hermite numbers, generalizations of degenerate Apostol–Euler Hermite numbers, and generalizations of degenerate Apostol–Genocchi Hermite numbers of order α and level m.

$$\tau^{m\alpha}[\sigma(\lambda;\mu,b;\tau)]^{\alpha} = \sum_{\omega=0}^{\infty} {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!},$$

$$2^{m\alpha}[\psi(\lambda;\mu,b;\tau)]^{\alpha} = \sum_{\omega=0}^{\infty} {}_H\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!},$$

$$(2\tau)^{m\alpha}[\psi(\lambda;\mu,b;\tau)]^{\alpha} = \sum_{\omega=0}^{\infty} {}_H\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}.$$

Continuation will show the standard notation for several sub-classes of polynomials, with parameters $\lambda \in \mathbb{C}$, $\mu, b \in \mathbb{R}^+$, order $\alpha \in \mathbb{N}$, and level $m \in \mathbb{N}$ (see [12,29–31] and the references therein).

ω-th generalized Bernoulli polynomial of level m	$B_\omega^{[m-1]}(\xi) := \lim_{\mu\to 0^+} {}_H\mathfrak{B}_\omega^{[m-1,1]}(\xi, 0; \mu, e; 1)$
ω-th generalized Euler polynomial of level m	$E_\omega^{(\alpha)}(\xi) := \lim_{\mu\to 0^+} {}_H\mathfrak{B}_\omega^{[m-1,1]}(\xi, 0; \mu, e; 1)$
ω-th generalized Genocchi polynomial of level m	$G_\omega^{(\alpha)}(\xi) := \lim_{\mu\to 0^+} {}_H\mathfrak{G}_\omega^{[m-1,1]}(\xi, 0; \mu, e; 1)$
ω-th generalized Apostol–Genocchi Hermite polynomial	$\mathcal{G}_\omega^{(\alpha)}(\xi;\lambda) := \lim_{\mu\to 0^+} {}_H\mathfrak{G}_\omega^{[0,\alpha]}(\xi, 0; \mu, b; \lambda)$
ω-th Apostol–Bernoulli polynomial	$B_\omega(\xi;\lambda) := \lim_{\mu\to 0^+} {}_H\mathfrak{B}_\omega^{[0,1]}(\xi, 0; \mu, b; \lambda)$
ω-th Apostol–Euler polynomial	$\mathcal{E}_\omega(\xi;\lambda) := \lim_{\mu\to 0^+} {}_H\mathfrak{E}_\omega^{[0,1]}(\xi, 0; \mu, b; \lambda)$
ω-th Apostol–Genocchi Hermite polynomial	$\mathcal{G}_\omega(\xi;\lambda) := \lim_{\mu\to 0^+} {}_H\mathfrak{G}_\omega^{[0,1]}(\xi, 0; \mu, b; \lambda)$
ω-th generalized Bernoulli polynomial	$B_\omega^{(\alpha)}(\xi) := \lim_{\mu\to 0^+} {}_H\mathfrak{B}_\omega^{[0,\alpha]}(\xi, 0; \mu, b; 1)$
ω-th generalized Euler polynomial	$E_\omega^{(\alpha)}(\xi) := \lim_{\mu\to 0^+} {}_H\mathfrak{E}_\omega^{[0,\alpha]}(\xi, 0; \mu, b; 1)$
ω-th generalized Genocchi polynomial	$G_\omega^{(\alpha)}(\xi) := \lim_{\mu\to 0^+} {}_H\mathfrak{G}_\omega^{[0,\alpha]}(\xi, 0; \mu, b; 1)$
ω-th Bernoulli polynomial	$B_\omega(\xi) := \lim_{\mu\to 0^+} {}_H\mathfrak{B}_\omega^{[0,1]}(\xi, 0; \mu, b; 1)$
ω-th Euler polynomial	$E_\omega(\xi) := \lim_{\mu\to 0^+} {}_H\mathfrak{E}_\omega^{[0,1]}(\xi, 0; \mu, b; 1)$
ω-th Genocchi polynomial	$G_\omega(\xi) := \lim_{\mu\to 0^+} {}_H\mathfrak{G}_\omega^{[0,1]}(\xi, 0; \mu, b; 1)$

Theorem 1. *For $m \in \mathbb{N}$ and the new families of Hermite–Apostol-type degenerated polynomials in invariable x, with parameters $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{Z}$, order $\alpha \in \mathbb{N}_0$ and level m, the following relationship holds*

$$ {}_H\mathfrak{B}_\omega^{[m-1,\alpha]}(\xi+\gamma, \eta+w; \mu, b; \lambda) = \sum_{k=0}^{\omega} \binom{\omega}{k} {}_H\mathfrak{B}_{\omega-k}^{[m-1,\alpha]}(\xi, \eta; \mu, b; \lambda) H_k(\gamma, w; \mu), \tag{10} $$

$$ {}_H\mathfrak{E}_\omega^{[m-1,\alpha]}(\xi+\gamma, \eta+w; \mu, b; \lambda) = \sum_{k=0}^{\omega} \binom{\omega}{k} {}_H\mathfrak{E}_{\omega-k}^{[m-1,\alpha]}(\xi, \eta; \mu, b; \lambda) H_k(\gamma, w; \mu), \tag{11} $$

$$ {}_H\mathfrak{G}_\omega^{[m-1,\alpha]}(\xi+\gamma, \eta+w; \mu, b; \lambda) = \sum_{k=0}^{\omega} \binom{\omega}{k} {}_H\mathfrak{G}_{\omega-k}^{[m-1,\alpha]}(\xi, \eta; \mu, b; \lambda) H_k(\gamma, w; \mu). \tag{12} $$

Proof. By (7) and (2), we have

$$ \begin{aligned} \sum_{\omega=0}^{\infty} {}_H\mathfrak{B}_\omega^{[m-1,\alpha]}(\xi+\gamma, \eta+w; \mu, b; \lambda)\frac{\tau^\omega}{\omega!} &= \tau^{m\alpha}[\sigma(\lambda;\mu,b;\tau)]^\alpha (1+\mu\tau)^{\frac{\xi+\gamma}{\mu}}(1+\mu\tau^2)^{\frac{\eta+w}{\mu}} \\ &= \tau^{m\alpha}[\sigma(\lambda;\mu,b;\tau)]^\alpha (1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}}(1+\mu\tau)^{\frac{\gamma}{\mu}}(1+\mu\tau^2)^{\frac{w}{\mu}} \\ &= \left(\sum_{\omega=0}^{\infty} {}_H\mathfrak{B}_\omega^{[m-1,\alpha]}(\xi, \eta; \mu, b; \lambda)\frac{\tau^\omega}{\omega!}\right)\left(\sum_{\omega=0}^{\infty} H_\omega(\gamma, w; \mu)\frac{\tau^\omega}{\omega!}\right) \\ &= \sum_{\omega=0}^{\infty}\left(\sum_{\nu=0}^{\omega}\binom{\omega}{\nu} {}_H\mathfrak{B}_{\omega-\nu}^{[m-1,\alpha]}(\xi, \eta; \mu, b; \lambda) H_k(\gamma, w; \mu)\right)\frac{\tau^\omega}{\omega!}. \end{aligned} $$

In view of the above equation, we get the result (10). The proofs of (11) and (12) are given analogously. □

Theorem 2. *For $m \in \mathbb{N}$ and the new families of Hermite–Apostol-type degenerated polynomials in invariable x, with parameters $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{Z}$, order $\alpha \in \mathbb{N}_0$ and level m, the argument addition theorem holds*

$$\begin{aligned} {}_H\mathfrak{B}_\omega^{[m-1,\alpha+\beta]}(\xi+\eta,\gamma+w;\mu,b;\lambda) &= \sum_{\nu=0}^{\omega}\binom{\omega}{\nu}{}_H\mathfrak{B}_\nu^{[m-1,\beta]}(\eta,w;\mu,b;\lambda) \\ &\quad\times{}_H\mathfrak{B}_{\omega-k}^{[m-1,\alpha]}(\xi,\gamma;\mu,b;\lambda), \end{aligned} \tag{13}$$

$$\begin{aligned} {}_H\mathfrak{E}_\omega^{[m-1,\alpha+\beta]}(\xi+\eta,\gamma+w;\mu,b;\lambda) &= \sum_{\nu=0}^{\omega}\binom{\omega}{\nu}{}_H\mathfrak{E}_\nu^{[m-1,\beta]}(\eta,w;\mu,b;\lambda) \\ &\quad\times{}_H\mathfrak{E}_{\omega-\nu}^{[m-1,\alpha]}(\xi,\gamma;\mu,b;\lambda), \end{aligned} \tag{14}$$

$$\begin{aligned} {}_H\mathfrak{G}_\omega^{[m-1,\alpha+\beta]}(\xi+\eta,\gamma+w;\mu,b\lambda) &= \sum_{\nu=0}^{\omega}\binom{\omega}{\nu}{}_H\mathfrak{G}_\nu^{[m-1,\beta]}(\eta,w;\mu,b;\lambda) \\ &\quad\times{}_H\mathfrak{G}_{\omega-\nu}^{[m-1,\alpha]}(\xi,\gamma;\mu,b;\lambda). \end{aligned} \tag{15}$$

Proof. Observe that,

$$\begin{aligned} \sum_{\omega=0}^{\infty}{}_H\mathfrak{B}_\omega^{[m-1,\alpha+\beta])}(\xi+\eta,\gamma+w;\mu,b;\lambda)\frac{\tau^\omega}{\omega!} &= (\tau^m\sigma(\lambda;\mu,b;\tau))^{\alpha+\beta}(1+\mu\tau)^{\frac{\xi+\eta}{\mu}}(1+\mu\tau^2)^{\frac{\gamma+w}{\mu}} \\ &= \left(\sum_{\omega=0}^{\infty}{}_H\mathfrak{B}_\omega^{[m-1,\alpha]}(\xi,\gamma;\mu,b;\lambda)\frac{\tau^\omega}{\omega!}\right) \\ &\quad\times\left(\sum_{\omega=0}^{\infty}{}_H\mathfrak{B}_\omega^{[m-1,\beta]}(\eta,w;\mu,b;\lambda)\frac{\tau^\omega}{\omega!}\right) \\ &= \sum_{\omega=0}^{\infty}\left(\sum_{\nu=0}^{\omega}\binom{\omega}{\nu}{}_H\mathfrak{B}_{\omega-\nu}^{[m-1,\alpha]}(\xi,\gamma;\mu,b;\lambda)\right. \\ &\quad\left.\times{}_H\mathfrak{B}_\nu^{[m-1,\beta]}(\eta,w;\mu,b;\lambda)\right)\frac{\tau^\omega}{\omega!}. \end{aligned}$$

Therefore, by the above equation, we obtain result (13). The proofs of (14) and (15) are given analogously. □

Theorem 3. *For $m \in \mathbb{N}$ and the new families of Hermite–Apostol-type degenerated polynomials in invariable x, with parameters $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{Z}$, order $\alpha \in \mathbb{N}_0$ and level m, the following relationships are obeyed:*

$${}_H\mathfrak{B}_\omega^{[m-1,\alpha]}(\xi,\eta;\mu;\lambda) = {}_H\mathfrak{B}_\omega^{[m-1,\alpha]}(\xi+\mu,\eta;\mu,b;\lambda)-\mu\omega{}_H\mathfrak{B}_{\omega-1}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda), \tag{16}$$

$${}_H\mathfrak{E}_\omega^{[m-1,\alpha]}(\xi,\eta;\mu;\lambda) = {}_H\mathfrak{E}_\omega^{[m-1,\alpha]}(\xi+\mu,\eta;\mu,b;\lambda)-\mu\omega{}_H\mathfrak{E}_{\omega-1}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda), \tag{17}$$

$${}_H\mathfrak{G}_\omega^{[m-1,\alpha]}(\xi,\eta;\mu;\lambda) = {}_H\mathfrak{G}_\omega^{[m-1,\alpha]}(\xi+\mu,\eta;\mu,b;\lambda)-\mu\omega{}_H\mathfrak{G}_{\omega-1}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda). \tag{18}$$

Proof. From generating function (8), we have

$$\begin{aligned} (\tau^m\sigma(\lambda;\mu,b;\tau))^\alpha(1+\mu\tau)^{\frac{\xi+\mu}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{a}} &= (1+\mu\tau)\sum_{\omega=0}^{\infty}{}_H\mathfrak{E}_\omega^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^\omega}{\omega!} \\ \sum_{\omega=0}^{\infty}{}_H\mathfrak{E}_\omega^{[m-1,\alpha]}(\xi+\mu,\eta;\mu,b;\lambda)\frac{\tau^\omega}{\omega!} &= \sum_{\omega=0}^{\infty}{}_H\mathfrak{E}_\omega^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^\omega}{\omega!} \\ &\quad+\mu\tau\sum_{\omega=0}^{\infty}{}_H\mathfrak{E}_\omega^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^\omega}{\omega!}. \end{aligned}$$

Then,

$$\begin{aligned} \sum_{\omega=0}^{\infty}{}_H\mathfrak{E}_\omega^{[m-1,\alpha]}(\xi+\mu,\eta;\mu,b;\lambda)\frac{\tau^\omega}{\omega!} &= \sum_{\omega=0}^{\infty}{}_H\mathfrak{E}_\omega^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^\omega}{\omega!} \\ &\quad+\sum_{\omega=0}^{\infty}\omega\,{}_H\mathfrak{E}_{\omega-1}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\mu\omega}}{\omega!}. \end{aligned}$$

Thus, we have

$$\sum_{\omega=0}^{\infty} {}_H\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi+\mu,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!} = \sum_{\omega=0}^{\infty}\Big[{}_H\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda) + \mu\omega\,{}_H\mathfrak{E}_{\omega-1}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\Big]\frac{\tau^{\omega}}{\omega!}.$$

In view of the above equation, the result is

$${}_H\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda) = {}_H\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi+\mu,\eta;\mu,b;\lambda) - \mu\omega\,{}_H\mathfrak{E}_{\omega-1}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda).$$

Therefore, we obtain (17). The proofs of (16) and (18) are analogous to the previous procedure. □

Theorem 4. *For $m \in \mathbb{N}$, the new families of Hermite–Apostol-type degenerated polynomials in invariable x, with parameters $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{Z}$, order $\alpha \in \mathbb{N}_0$ and level m comply with the following relationships:*

$${}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu;\lambda) = {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta+\mu;\mu,b;\lambda) - \mu\omega(\omega-1)\,{}_H\mathfrak{B}_{\omega-2}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda), \tag{19}$$

$${}_H\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu;\lambda) = {}_H\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi,\eta+\mu;\mu,b;\lambda) - \mu\omega(\omega-1)\,{}_H\mathfrak{E}_{\omega-2}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda), \tag{20}$$

$${}_H\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu;\lambda) = {}_H\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta+\mu;\mu,b;\lambda) - \mu\omega(\omega-1)\,{}_H\mathfrak{G}_{\omega-2}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda). \tag{21}$$

Proof. From generating function (9), we have

$$\begin{aligned}((2\tau)^m\sigma(\lambda;\mu,b;\tau))^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta+\mu}{\mu}} &= (1+\mu\tau^2)\sum_{\omega=0}^{\infty}{}_H\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}\\ \sum_{\omega=0}^{\infty}{}_H\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta+\mu;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!} &= \sum_{\omega=0}^{\infty}{}_H\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}\\ &\quad+\mu\tau^2\sum_{\omega=0}^{\infty}{}_H\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}.\end{aligned}$$

Then,

$$\sum_{\omega=0}^{\infty}{}_H\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta+\mu;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!} = \sum_{\omega=0}^{\infty}{}_H\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!} + \sum_{\omega=0}^{\infty}{}_H\mathfrak{G}_{\omega-2}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\mu\omega(\omega-1)\frac{\tau^{\omega}}{\omega!}.$$

Thus, we have

$$\sum_{\omega=0}^{\infty}{}_H\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta+\mu;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!} = \sum_{\omega=0}^{\infty}\Big[{}_H\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda) + \mu\omega(\omega-1)\,{}_H\mathfrak{G}_{\omega-2}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\Big]\frac{\tau^{\omega}}{\omega!}.$$

Comparing the coefficients of τ^{ω} on both sides of the equation, we obtain the result (21). The proofs of (19) and (20) are analogous to the previous procedure. □

Theorem 5. *For $m \in \mathbb{N}$, for the new families of Hermite–Apostol-type degenerated polynomials in invariable x, with parameters $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{Z}$, order $\alpha \in \mathbb{N}_0$ and level m, the following properties are maintained:*

$$\frac{\partial\,{}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)}{\partial\xi,} = \sum_{k=0}^{\omega-1}\omega(-1)^k\mu^k\frac{k!}{k+1}\binom{\omega-1}{k}{}_H\mathfrak{B}_{\omega-1-k}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda), \tag{22}$$

$$\frac{\partial\,{}_H\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)}{\partial\xi,} = \sum_{k=0}^{\omega-1}\omega(-1)^k\mu^k\frac{k!}{k+1}\binom{\omega-1}{k}{}_H\mathfrak{E}_{\omega-1-k}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda), \tag{23}$$

$$\frac{\partial_H \mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)}{\partial\xi,} = \sum_{k=0}^{\omega-1} \omega(-1)^k \mu^k \frac{k!}{k+1} \binom{\omega-1}{k} {}_H\mathfrak{G}_{\omega-1-k}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda). \tag{24}$$

Proof. Partially differentiating (7) with respect to ξ, we have

$$\begin{aligned}
\sum_{\omega=0}^{\infty} \frac{\partial}{\partial\xi} {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!} &= \tau^{m\alpha}[\sigma(\lambda;\mu,b;\tau)]^{\alpha} \frac{\partial}{\partial\xi}(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}}, \\
&= \tau^{m\alpha}[\sigma(\lambda;\mu,b;\tau)]^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}} \ln(1+\mu\tau)\frac{1}{\mu} \\
&= \left(\sum_{\omega=0}^{\infty} {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}\right) \\
&\quad \times \left(\sum_{\omega=0}^{\infty} \frac{(-1)^{\omega}}{\omega+1}\mu^{\omega+1}\tau^{\omega+1}\frac{1}{\mu}\right) \\
&= \sum_{\omega=0}^{\infty}\sum_{k=0}^{\omega} {}_H\mathfrak{B}_{\omega-k}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda) \\
&\quad \times (-1)^k\mu^k\binom{\omega}{k}\frac{k!}{k+1}\frac{\tau^{\omega+1}}{\omega!}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\sum_{\omega=0}^{\infty} \frac{\partial}{\partial\xi} {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!} &= \sum_{\omega=0}^{\infty}\sum_{k=0}^{\omega-1} {}_H\mathfrak{B}_{\omega-1-k}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda) \\
&\quad \times (-1)^k\mu^k\omega\binom{\omega-1}{k}\frac{k!}{k+1}\frac{\tau^{\omega}}{\omega!}.
\end{aligned}$$

Comparing the coefficients of τ^{ω} on both sides of the equation, the result is

$$\frac{\partial_H \mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)}{\partial\xi} = \sum_{k=0}^{\omega-1} \omega(-1)^k \mu^k \frac{k!}{k+1} \binom{\omega-1}{k} {}_H\mathfrak{B}_{\omega-1-k}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda).$$

□

The proofs of (23) and (24) are analogous to (22).

Theorem 6. *For $m \in \mathbb{N}$, for the new families of Hermite–Apostol-type degenerated polynomials in invariable x, with parameters $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{Z}$, order $\alpha \in \mathbb{N}_0$ and level m, the following properties are maintained:*

$$\frac{\partial_H \mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)}{\partial\eta} = \sum_{k=0}^{\omega-k} \omega(\omega-1)(-1)^k \mu^k \frac{2k!}{k+1} \binom{\omega-2}{2k} {}_H\mathfrak{B}_{\omega-2k-2}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda), \tag{25}$$

$$\frac{\partial_H \mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)}{\partial\eta} = \sum_{k=0}^{\omega-k} \omega(\omega-1)(-1)^k a^k \frac{2k!}{k+1} \binom{\omega-2}{2k} {}_H\mathfrak{E}_{\omega-2k-2}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda), \tag{26}$$

$$\frac{\partial_H \mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)}{\partial\eta} = \sum_{k=0}^{\omega-k} \omega(\omega-1)(-1)^k a^k \frac{2k!}{k+1} \binom{\omega-2}{2k} {}_H\mathfrak{G}_{\omega-2k-2}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda). \tag{27}$$

Proof. Partially differentiating (7) with respect to η, we have

$$\sum_{\omega=0}^{\infty} \frac{\partial}{\partial \eta} {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!} = \tau^{m\alpha}[\sigma(\lambda;\mu,b;\tau)]^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}}\frac{\partial}{\partial\eta}(1+\mu\tau^2)^{\frac{\eta}{\mu}}$$
$$= \tau^{m\alpha}[\sigma(\lambda;\mu,b;\tau)]^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}}\ln(1+\mu\tau^2)\frac{1}{\mu}$$
$$= \left(\sum_{\omega=0}^{\infty} {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}\right)\left(\sum_{\omega=0}^{\infty}\frac{(-1)^{\omega}}{\omega+1}\mu^{\omega+1}\tau^{2n+2}\frac{1}{\mu}\right)$$
$$= \sum_{\omega=0}^{\infty}\sum_{k=0}^{\omega} {}_H\mathfrak{B}_{\omega-k}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{(-1)^k}{k+1}\mu^k\frac{\tau^{\omega+k+2}}{(\omega-k)!}.$$

Thus, we have

$$\sum_{\omega=0}^{\infty} \frac{\partial}{\partial \eta} {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!} = \sum_{\omega=0}^{\infty}\sum_{k=0}^{\omega-k} {}_H\mathfrak{B}_{\omega-2-2k}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda) \times(-1)^k\mu^k\omega(\omega-1)\binom{\omega-2}{2k}\frac{2k!}{k+1}\frac{\tau^{\omega}}{\omega!}.$$

Comparing the coefficients of τ^{ω} on both sides of the equation, the result is

$$\frac{\partial {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)}{\partial\eta} = \sum_{k=0}^{\omega-k}\omega(\omega-1)(-1)^k\mu^k\frac{2k!}{k+1}\binom{\omega-2}{2k} {}_H\mathfrak{B}_{\omega-2k-2}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda).$$

□

The proofs of (26) and (27) are analogous to (25).

Theorem 7. *For $m \in \mathbb{N}$, the new families of Hermite–Apostol-type degenerated polynomials in invariable x, with parameters $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{Z}$, order $\alpha \in \mathbb{N}_0$ and level m comply with the following relationships:*

$$\sum_{k=0}^{\omega} {}_H\mathfrak{B}_{\omega-k}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\,{}_H\mathfrak{B}_{k}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda) = \sum_{k=0}^{\omega}\binom{\omega}{k} {}_H\mathfrak{B}_{\omega-k}^{[m-1,\alpha]}(\mu,b;\lambda) \times {}_H\mathfrak{B}_{k}^{[m-1,\alpha]}(2\xi,2\eta;\mu,b;\lambda), \tag{28}$$

$$\sum_{k=0}^{\omega} {}_H\mathfrak{E}_{\omega-k}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\,{}_H\mathfrak{E}_{k}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda) = \sum_{k=0}^{\omega}\binom{\omega}{k} {}_H\mathfrak{E}_{\omega-k}^{[m-1,\alpha]}(\mu,b;\lambda) \times {}_H\mathfrak{E}_{k}^{[m-1,\alpha]}(2\xi,2\eta;\mu,b;\lambda), \tag{29}$$

$$\sum_{k=0}^{\omega} {}_H\mathfrak{G}_{\omega-k}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\,{}_H\mathfrak{G}_{k}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda) = \sum_{k=0}^{\omega}\binom{\omega}{k} {}_H\mathfrak{G}_{\omega-k}^{[m-1,\alpha]}(\mu,b;\lambda) \times {}_H\mathfrak{G}_{k}^{[m-1,\alpha]}(2\xi,2\eta;\mu,b;\lambda). \tag{30}$$

Proof. Consider the following expressions:

$$\tau^{m\alpha}[\sigma(\lambda;\mu,b;\tau)]^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}} = \sum_{\omega=0}^{\infty} {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!} \tag{31}$$

and

$$\tau^{m\alpha}[\sigma(\lambda;\mu,b;\tau)]^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}} = \sum_{r=0}^{\infty} {}_H\mathfrak{B}_{r}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}. \tag{32}$$

From (31) and (32), we have

$$[\tau^{m\alpha}[\sigma(\lambda;\mu,b;\tau)]]^{2\alpha}(1+\mu\tau)^{\frac{2\xi}{\mu}}(1+\mu\tau^2)^{\frac{2\eta}{\mu}} = \sum_{\omega=0}^{\infty} {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}\sum_{r=0}^{\infty} {}_H\mathfrak{B}_{r}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}$$

$$\sum_{\omega=0}^{\infty} {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}\sum_{r=0}^{\infty} {}_H\mathfrak{B}_{r}^{[m-1,\alpha]}(2\xi,2\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!} = \sum_{\omega=0}^{\infty} {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}\sum_{r=0}^{\infty} {}_H\mathfrak{B}_{r}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}$$

$$\sum_{\omega=0}^{\infty}\sum_{k=0}^{\omega}\binom{\omega}{k} {}_H\mathfrak{B}_{\omega-k}^{[m-1,\alpha]}(\mu,b;\lambda)\,{}_H\mathfrak{B}_{k}^{[m-1,\alpha]}(2\xi,2\eta,\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!} = \sum_{\omega=0}^{\infty}\sum_{k=0}^{\omega}\binom{\omega}{k} {}_H\mathfrak{B}_{\omega-k}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\,{}_H\mathfrak{B}_{k}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}.$$

Hence, we get contention (28). □

The proofs of (29) and (30) are comparable to (28).

Theorem 8. *For $m \in \mathbb{N}$, the new families of Hermite–Apostol-type degenerated polynomials in invariable x, with parameters $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{Z}$, order $\alpha \in \mathbb{N}_0$ and level m comply with the following relationships:*

$$ {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;-\lambda) = \frac{(-1)^{\alpha}\omega!}{(2)^{m\alpha}(\omega-m\alpha)!}\,{}_H\mathfrak{E}_{\omega-m\alpha}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda), \tag{33}$$

$$ {}_H\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;-\lambda) = \frac{(-2)^{m\alpha}\omega!}{(n+m\alpha)!}\,{}_H\mathfrak{B}_{n+m\alpha}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda). \tag{34}$$

Proof. Proof of (33). Considering the generating function (7):

$$\begin{aligned}\tau^{m\alpha}[\sigma(-\lambda;\mu,b;\tau)]^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}} &= \sum_{\omega=0}^{\infty} {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;-\lambda)\frac{\tau^{\omega}}{\omega!}\\ \frac{(-1)^{\alpha}2^{m\alpha}}{2^{m\alpha}}\tau^{m\alpha}[\psi(\lambda;\mu,b;\tau)]^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}} &= \sum_{\omega=0}^{\infty} {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;-\lambda)\frac{\tau^{\omega}}{\omega!},\end{aligned}$$

we have

$$\begin{aligned}\sum_{\omega=0}^{\infty} {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;-\lambda)\frac{\tau^{\omega}}{\omega!} &= \frac{(-1)^{\alpha}}{2^{m\alpha}}\sum_{\omega=0}^{\infty} {}_H\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{n+m\alpha}}{\omega!}\\ \sum_{\omega=0}^{\infty} {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;-\lambda)\frac{\tau^{\omega}}{\omega!} &= \frac{(-1)^{\alpha}}{2^{m\alpha}}\sum_{\omega=0}^{\infty} {}_H\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{(\omega-m\alpha)!}.\end{aligned}$$

Therefore, by the above equation, we obtain the result. □

Proof. Proof of (34). Considering the generating function (8):

$$\begin{aligned}2^{m\alpha}[\psi(\lambda;\mu,b;\tau)]^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}} &= \sum_{\omega=0}^{\infty} {}_H\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}\\ \frac{(-1)^{\alpha}2^{m\alpha}}{\tau^{m\alpha}}\tau^{m\alpha}[\sigma(\lambda;\mu,b;\tau)]^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}} &= \sum_{\omega=0}^{\infty} {}_H\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;-\lambda)\frac{\tau^{\omega}}{\omega!},\end{aligned}$$

we have

$$\begin{aligned}\sum_{\omega=0}^{\infty} {}_H\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;-\lambda)\frac{\tau^{\omega}}{\omega!} &= (-2)^{m\alpha}\sum_{\omega=0}^{\infty}\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{(\omega-m\alpha)}}{\omega!}\\ \sum_{\omega=0}^{\infty} {}_H\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;-\lambda)\frac{\tau^{\omega}}{\omega!} &= (-2)^{m\alpha}\sum_{\omega=0}^{\infty} {}_H\mathfrak{B}_{n+m\alpha}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{(n+m\alpha)!}.\end{aligned}$$

In view of the above equation, we obtain the result. □

Theorem 9. *For $m \in \mathbb{N}$, the new families of Hermite–Apostol-type degenerated polynomials in invariable x, with parameters $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{Z}$, order $\alpha \in \mathbb{N}_0$ and level m comply with the following relationships:*

$$\begin{aligned}{}_H\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;-\lambda) &= (-2)^{m\alpha}{}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda), &(35)\\ {}_H\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda) &= \frac{\omega!}{(\omega-m\alpha)!}{}_H\mathfrak{E}_{\omega-m\alpha}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda). &(36)\end{aligned}$$

Proof. Proof of (35). Taking into account the generating function (7), we can observe that

$$\begin{aligned}\tau^{m\alpha}[\sigma(\lambda;\mu,b;\tau)]^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{\mu}} &= \sum_{\omega=0}^{\infty} {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}\\ 2^{m\alpha}\tau^{m\alpha}[\psi(-\lambda;\mu,b;\tau)]^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{a}} &= (-2^{m})^{\alpha}\sum_{\omega=0}^{\infty} {}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}. \quad (37)\end{aligned}$$

Therefore, from (9) and (37), we obtain

$$\sum_{\omega=0}^{\infty} {}_H\mathfrak{G}_{\omega}^{[m-1,\alpha]}(x;\mu,b;-\lambda)\frac{\tau^{\omega}}{\omega!} = \sum_{\omega=0}^{\infty}(-2)^{m\alpha}{}_H\mathfrak{B}_{\omega}^{[m-1,\alpha]}(x;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}.$$

In view of the above equation, we obtain the result. □

Proof. Proof of (36). From (9), we have:

$$\begin{aligned}2^{m\alpha}\tau^{m\alpha}[\psi(\lambda;\mu,b;\tau)]^{\alpha}(1+\mu\tau)^{\frac{\xi}{\mu}}(1+\mu\tau^2)^{\frac{\eta}{a}} &= \sum_{\omega=0}^{\infty} {}_H\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}\\ \sum_{\omega=0}^{\infty} {}_H\mathfrak{E}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{n+m\alpha}}{\omega!} &= \sum_{\omega=0}^{\infty} {}_H\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!},\end{aligned}$$

then,

$$\sum_{\omega=0}^{\infty} {}_H\mathfrak{E}_{\omega-m\alpha}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{(\omega-m\alpha)!} = \sum_{\omega=0}^{\infty} {}_H\mathfrak{G}_{\omega}^{[m-1,\alpha]}(\xi,\eta;\mu,b;\lambda)\frac{\tau^{\omega}}{\omega!}.$$

Therefore, by the above equation, we obtain the result. □

3. Conclusions

In recent years, Apostol-type polynomials have become the subject of intensive research due to their diverse range of applications, while Bernoulli, Euler, Genocchi, and Hermite polynomials are well-known families of polynomials with many applications in areas such as numerical analysis, asymptotic approximation, and special function theory, which have led to a wide range of uses in engineering and applied sciences [20]. Due to the importance of these application areas, many extensions of Apostol-type polynomials have been studied, such as degenerate Apostol-type polynomials in [19], Hermite-based Apostol-type polynomials in [2], Laguerre-based Apostol-type polynomials in [3,24,32], and truncated-exponential-based Apostol-type polynomials, especially in the last decade. In the literature, extensions of several structures are considered essential if the extension unifies existing structures. Unification focuses researchers on investigating advanced properties rather than just studying modified families that have similar properties to the existing area.

The objective of this paper is to examine new families of Hermite–Apostol-type degenerated polynomials, specifically the Apostol–Bernoulli, Apostol–Euler, and Apostol–Genocchi Hermite polynomials of level m. These polynomials have significant applications in the areas of applied mathematics, physics, and engineering. The properties of these polynomials are established based on classical special functions. The theorems presented in this study demonstrate the usefulness of the series rearrangement technique for the treatment of special functions theory.

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