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ON SOME DOMAINS OF CONVERGENCE OF BRANCHED CONTINUED FRACTION EXPANSIONS OF THE RATIOS OF HORN HYPERGEOMETRIC FUNCTIONS H_4

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For various conditions imposed on the parameters of the Horn hypergeometric function $H_4,$ we study different domains of convergence for the branched continued fraction expansions of the ratios of these functions.

1. Introduction

The (Gauss, Appel, Horn, Lauricella, etc.) hypergeometric functions are encountered in various problems of applied mathematics, statistics, chemistry, biology, mathematical physics, and engineering sciences. They have been extensively studied for the last two centuries (see, e.g., [6, 9, 11, 25–30, 33–35]).

In 1931, Horn listed 34 different convergent hypergeometric series with two variables [32]. All these 34 Horn functions can be split into 14 full hypergeometric functions $[F_1–F_4$ (Appel functions), $G_1–G_3,$ and $H_1–H_7]$ and 20 degenerate hypergeometric functions $(\Phi_1–\Phi_3,\ \Psi_1,\ \Psi_2,\ \Xi_1,\ \Xi_2,\ \Gamma_1,\ \Gamma_2,$ and $\mathrm{H}_1–\mathrm{H}_{11});$ see [27, pp. 224–227].

The Horn hypergeometric function H_4 can be represented in the form of a double power series as follows:

$$H_4(a,b;c,d;\mathbf{z})=\sum_{r,s=0}^{\infty}\frac{(a)_{2r+s}(b)_s}{(c)_r(d)_s}\frac{z_1^r}{r!}\frac{z_2^s}{s!},\quad |z_1|<p,\quad |z_2|<l,$$

where $a,\ b,\ c,$ and d are complex constants; moreover, c and d are not equal to a nonpositive integer; p and l are positive numbers such that $4p=(l-1)^2$ and $l\neq 1;$ $(\cdot)_k$ is the Pochhammer symbol defined, for any complex number α and a nonnegative integer $n,$ as follows: $(\alpha)_0=1$ and $(\alpha)_n=\alpha(\alpha+1)\ldots(\alpha+n-1),$ and $\mathbf{z}=(z_1,z_2)\in\mathbb{C}^2.$

In the present paper, we continue the investigation of convergence of the branched continued fraction expansions of the ratios of Horn hypergeometric functions H_4 originated in [12, 24].

The convergence of the branched continued fraction expansions of the ratios of the hypergeometric functions F_1 and F_3 was investigated in [2, 3] and [4, 20], respectively; the convergence of the hypergeometric function $F_4(1,2;2,2;z_1,z_2)$ was investigated in [31]; the convergence of the ratios of Horn hypergeometric functions H_3 was investigated in [7, 13], and the convergence of the ratios of Horn degenerate hypergeometric

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functions H_6 and H_7 was investigated in [10] and [8], respectively. The problem of convergence of the branched continued fraction expansions for the ratios of the hypergeometric functions F_2 and, in the general case, F_4 constructed in [19] and [21], respectively, remains open.

2. Convergence of Branched Continued Fraction Expansion

By virtue of Theorem 1 in [12], under the conditions that $b = d + 1$ and $(ij)_0 = (1, 2),$ we get the following assertion:

Theorem 1. *The ratio*

$$\frac{H_4(a, d+1; c, d; \mathbf{z})}{H_4(a+1, d+1; c, d+1; \mathbf{z})} \tag{1}$$

has a formal branched continued fraction expansion of the form

$$1 - \frac{d-a}{d} z_2 - \cfrac{h_1 z_1}{1 - z_2 - \cfrac{h_2 z_1}{1 - z_2 - \cfrac{h_3 z_1}{1 - \ddots}}}, \tag{2}$$

where

$$h_1 = \frac{2(a+1)}{c}, \qquad h_k = \frac{(2c - a + k - 3)(a+k)}{(c+k-2)(c+k-1)}, \quad k \geq 2. \tag{3}$$

Remark 1. The branched continued fraction (2) has the form of a continued fraction. As a specific feature of this case, we can mention different definitions of the approximants of these fractions. Namely, the sequence of approximants of a continued fraction for a branched continued fraction is a sequence of so-called figured approximants [1, p. 18]. For the results of investigation of convergence obtained for different figured approximants, see, e.g., [5, 14, 15, 17, 18].

Theorem 1 in [23] immediately yields the following corollary:

Corollary 1. *Let a and d be complex constants such that $d \neq 0$ and let $g_{0,k},\ k \geq 1,$ be real numbers such that $0 < g_{0,k} \leq 1$ for all $k \geq 1$. Then the branched continued fraction*

$$1 - \frac{d-a}{d} z_{1,0} - \cfrac{g_{0,1} z_{0,1}}{1 - (1 - g_{0,1}) z_{1,1} - \cfrac{g_{0,2}(1 - g_{0,1}) z_{0,2}}{1 - (1 - g_{0,2}) z_{1,2} - \cfrac{g_{0,3}(1 - g_{0,2}) z_{0,3}}{1 - \ddots}}}$$

converges if $|z_{1,k}| \leq 1/2$ and $|z_{0,k+1}| \leq 1/2$ for all $k \geq 0$.

The proof of Lemma 4.41 in [33] yields the following result:

Corollary 2. *If $x \geq c > 0$ and $v^2 \leq 4u + 4$, where $u, v \in \mathbb{R}$, then*

$$\min_{-\infty < y < +\infty} \operatorname{Re}\left(\frac{u + iv}{x + iy}\right) = -\frac{\sqrt{u^2 + v^2} - u}{2x}.$$

The following theorem is true:

Theorem 2. *Let $a,$ $c,$ and d be complex constants such that*

$$|h_k| + \mathrm{Re}(h_k) \leq pq(1-q), \quad k \geq 1, \tag{4}$$

where $h_k,$ $k \geq 1,$ are given by (3), p is a positive number, $0 < q < 1,$ and $d \neq 0.$ Then the branched continued fraction (2) converges to a function $f(\mathbf{z})$ holomorphic in the domain

$$\Omega_{p,q} = \left\{ \mathbf{z} \in \mathbb{C}^2 : |z_1| < \frac{1 + \cos(\arg(z_1))}{2p},\ \mathrm{Re}\left(z_2 e^{-(i/2)\arg(z_1)}\right) < \frac{q}{2}\cos\left(\frac{\arg(z_1)}{2}\right) \right\}. \tag{5}$$

Furthermore, it is also uniformly convergent on every compact subset of the domain $\Omega_{p,q}$.

Proof. Let

$$F_n^{(n)}(\mathbf{z}) = 1, \quad n \geq 1, \tag{6}$$

and

$$F_k^{(n)}(\mathbf{z}) = 1 - z_2 - \cfrac{h_{k+1} z_1}{1 - z_2 - \cfrac{h_{k+2} z_1}{1 - \ddots - z_2 - \cfrac{h_{n-1} z_1}{1 - z_2 - h_n z_1}}}, \quad 1 \leq k \leq n-1, \quad n \geq 2.$$

Then

$$F_k^{(n)}(\mathbf{z}) = 1 - z_2 - \frac{h_{k+1} z_1}{F_{k+1}^{(n)}(\mathbf{z})}, \quad 1 \leq k \leq n-1, \quad n \geq 2, \tag{7}$$

and the nth approximant of the branched continued fraction (2) can be represented in the form

$$f_n(\mathbf{z}) = 1 - \frac{d-a}{d} z_2 - \frac{h_1 z_1}{F_1^{(n)}(\mathbf{z})}. \tag{8}$$

We now show that each approximant $f_n(\mathbf{z})$ is a holomorphic function in domain (5). To this end, it suffices to show that $F_1^{(n)}(\mathbf{z}) \neq 0$ for all $n \geq 1$ and $\mathbf{z} \in \Omega_{p,q}.$

We set $\arg(z_1) = \alpha.$ Let n be an arbitrary natural number and let $\mathbf{z}$ be any fixed point of domain (5). By induction on $k,$ we can prove the following inequalities:

$$\mathrm{Re}(F_k^{(n)}(\mathbf{z}) e^{-i\alpha/2}) > (1-q)\cos(\alpha/2) \geq c > 0, \quad 1 \leq k \leq n, \quad n \geq 1. \tag{9}$$

Since $\mathbf{z}$ is an arbitrary fixed point of domain (5), for any its neighborhood, one can find $\delta > 0$ such that $|\alpha/2| \leq \pi/2 - \delta$ and, hence,

$$(1-q)\cos(\alpha/2) \geq (1-q)\cos(\pi/2 - \delta) = (1-q)\sin(\delta) = c > 0.$$

We now show that the first inequality in (9) is true. For $k = n,$ this inequality is obvious. We assume that the first inequality in (9) holds for $k = r + 1 \leq n$ and prove it for $k = r.$ In view of relation (7), we get

$$F_r^{(n)}(\mathbf{z})e^{-i\alpha/2} = e^{-i\alpha/2} - z_2 e^{-i\alpha/2} - \frac{h_{r+1} z_1 e^{-i\alpha}}{F_{r+1}^{(n)}(\mathbf{z})e^{-i\alpha/2}}. \tag{10}$$

Applying Corollary 2, inequality (4), the inequalities in (5), and the induction assumption to relation (10), we obtain

$$\operatorname{Re}(F_r^{(n)}(\mathbf{z})e^{-i\alpha/2}) \geq \cos(\alpha/2) - \operatorname{Re}(z_2 e^{-i\alpha/2}) - \frac{|h_{r+1}| + \operatorname{Re}(h_{r+1})}{2\operatorname{Re}(F_{r+1}^{(n)}(\mathbf{z})e^{-i\alpha/2})}|z_1|$$

$$> \cos(\alpha/2) - \frac{q\cos(\alpha/2)}{2} - \frac{pq(1-q)}{2(1-q)\cos(\alpha/2)} \frac{1+\cos(\alpha)}{2p} = (1-q)\cos(\alpha/2).$$

Thus, $F_1^{(n)}(\mathbf{z}) \neq 0$ for all $n \geq 1$ and $\mathbf{z} \in \Omega_{p,q}.$ This means that each approximant (8) is a holomorphic function in domain (5).

Let Ξ be an arbitrary compact subset of domain (5). Then there exists an open bidisk

$$\Gamma_R = \{\mathbf{z} \in \mathbb{C}^2 : |z_k| < R,\ k = 1, 2\}, \quad R > 0,$$

such that $\Xi \subset \Gamma_R$ and, for any $n \geq 1$ and $\mathbf{z} \in \Omega_{p,q} \cap \Gamma_R,$ it follows from (8) that

$$|f_n(\mathbf{z})| \leq 1 + \left|\frac{d-a}{d}\right| R + \frac{|h_1| R}{\operatorname{Re}(F_1^{(n)}(\mathbf{z})e^{-i\alpha/2})} < 1 + \left|\frac{d-a}{d}\right| R + \frac{|h_1| R}{(1-q)\cos(\alpha/2)} = C(\Xi).$$

This means that the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded in each compact subset of the domain $\Omega_{p,q}.$

Since $\lim_{k\to+\infty} h_k = 1,$ there exists a constant $M > 0$ such that

$$|h_k| \leq M \quad \text{for all} \quad k \geq 1. \tag{11}$$

It is clear that, for any l such that $0 < l < \min\{1/4, 1/(8M), 1/p, q/2\},$ the domain

$$\Upsilon_l = \{\mathbf{z} \in \mathbb{R}^2 : 0 < z_k < l,\ k = 1, 2\}$$

is contained in $\Omega_{p,q}$ and, in particular, $\Upsilon_{l/2} \subset \Omega_{p,q}.$

By using inequality (11), for any $k \geq 1$ and $\mathbf{z} \in \Upsilon_l,\ \Upsilon_l \subset \Omega_{p,q},$ we obtain

$$|z_2| < 1/4, \quad |h_k z_1| < 1/8.$$

This means that the elements of the branched continued fraction (2) satisfy the conditions of Corollary 1, where $g_{0,k} = 1/2,\ k \geq 1.$ According to this corollary, the branched continued fraction (2) converges in the domain $\Upsilon_l,\ \Upsilon_l \subset \Omega_{p,q},$ and, hence, by virtue of Theorem 2.17 in [1] (see also [16, Theorem 7] and [35, Theorem 24.2]), uniformly converges on every compact subset of domain (5) to the function $f(\mathbf{z})$ holomorphic in $\Omega_{p,q}.$

Theorem 2 is proved.

By using Theorem 2, we obtain the following result:

Theorem 3. *Let d be a nonzero complex constant and let a and c be real constants such that $h_k < 0$ for all $k \geq 1,$ where $h_k,\ k \geq 1,$ are given by (3). Then the branched continued fraction (2) converges to a function $f(\mathbf{z})$ holomorphic in the domain*

$$\Omega_q = \left\{ \mathbf{z} \in \mathbb{C}^2 : |\arg(z_1)| < \pi,\ \operatorname{Re}(z_2 e^{-(i/2)\arg(z_1)}) < \frac{q}{2}\cos\left(\frac{\arg(z_1)}{2}\right) \right\}, \tag{12}$$

where $0 < q < 1.$ Furthermore, it is uniformly convergent on each compact subset of the domain $\Omega_q.$

Proof. If $0 < q < 1$ and $h_k < 0$ for all $k \geq 1,$ then it is clear that inequality (4) is true for all $p > 0.$ Let Ξ be an arbitrary compact subset of domain (12). Then the inclusions $\Xi \subseteq \Omega_{p,q} \subseteq \Omega_q$ hold for a certain sufficiently small p for which the set $\Omega_{p,q}$ is, in fact, domain (5). Thus, this theorem is a direct corollary of Theorem 2.

Remark 2. In Theorems 2 and 3, the set

$$\operatorname{Re}(z_2 e^{-(i/2)\arg(z_1)}) < (q/2)\cos((\arg(z_1))/2)$$

can be also rewritten in the form $z_2 \notin [q/2, +\infty).$

Reasoning as in the proof of Theorem 2 (see also [22]), we obtain the following result:

Theorem 4. *Suppose that d is a nonzero complex constant, a and c are real constants such that*

$$0 < h_k < r \quad \text{for all} \quad k \geq 1, \tag{13}$$

where $h_k,\ k \geq 1,$ are given by (3), and r is a positive number. Then the branched continued fraction (2) converges to a function $f(\mathbf{z})$ holomorphic in the domain

$$\Theta_r = \bigcup_{-\pi/2<\varphi<\pi/2} \Theta_{r,\varphi},$$

where

$$\Theta_{r,\varphi} = \left\{ \mathbf{z} \in \mathbb{C}^2 : |z_1| + \operatorname{Re}(z_1 e^{-2i\varphi}) < \frac{\cos^2(\varphi)}{4r},\ \operatorname{Re}(z_2 e^{-i\varphi}) < \frac{\cos(\varphi)}{4} \right\}.$$

Furthermore, it is uniformly convergent on each compact subset of the domain $\Theta_r.$

Remark 3. In Theorem 4, the domain Θ_r can be also rewritten in the form

$$\Theta_r = \left\{ \mathbf{z} \in \mathbb{C}^2 : z_1 \notin [1/(8r), +\infty),\ z_2 \notin [1/4, +\infty) \right\}.$$

By using Theorem 2 in [12], we get the following result:

Theorem 5. *Suppose that d is a nonzero complex constant, a and c are real constants satisfying inequalities (13), where $h_k,\ k \geq 1,$ are given by (3), and r is a positive number. Then the branched continuous fraction (2)* *converges to a function* $f(\mathbf{z})$ *holomorphic in the domain*

$$\Pi_r = \left\{ \mathbf{z} \in \mathbb{C}^2 : z_k \notin [1/(4(1+r)), +\infty),\ k = 1, 2 \right\}.$$

Furthermore, it is also uniformly convergent on each compact subset of the domain Π_r.

The proof of this theorem is similar to the proof of Part (A) of Theorem 3 in [12].

Remark 4. Results similar to Theorems 2–4 can be also obtained for the other two branched continued fraction expansions of the ratios of Horn hypergeometric functions H_4 obtained from Theorem 1 in [12] under the conditions $b = d,\ (ij)_0 = (1,1)$ and $b = d + 1,\ (ij)_0 = (2,2)$. For $b = d + 1$ and $(ij)_0 = (2,2)$, we can also obtain a result similar to Theorem 5. In the general case, the problem of convergence of all three expansions remains open.

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