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Oscillation Criteria for Qusilinear Even-Order Differential Equations

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Abstract: In this study, we extended and improved the oscillation criteria previously established for second-order differential equations to even-order differential equations. Some examples are given to demonstrate the significance of the results accomplished.

Keywords: oscillation criteria; even-order; quasilinear; differential equation

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1. Introduction

Various real-world application models incorporate oscillation phenomena; we refer to the works [1,2] for models from mathematical biology where oscillation and/or delay actions may be expressed using cross-diffusion terms. This paper examined the study of nonlinear functional differential equations since these equations are relevant to a number of practical issues, including non-Newtonian fluid theory and the turbulent flow of a polytrophic gas in a porous media; see, e.g., the papers [3–11] for more details. Therefore, we were interested in the oscillatory criteria of the quasilinear differential equation of even-order

$$y^{(n)}(s) + p(s)|y(\phi(s))|^{\beta-1}y(\phi(s)) = 0, \quad s \in [s_0, \infty),\ s_0 \geq 0, \tag{1.1}$$

where $n \geq 2$ is an even integer, $y^{(j)}(s) := \left(y^{(j-1)}\right)'(s)$, $j = 1, 2, \ldots, n$ with $y^{(0)}(s) := y(s)$, $\beta > 0$, $p(s)$ and $\phi(s)$ are positive continuous functions on $[s_0, \infty)$, satisfying $\lim_{s\to\infty} \phi(s) = \infty$, and $\varphi(s) := \min\{s, \phi(s)\}$ is nondecreasing on $[s_0, \infty)$. By a solution of Equation (1.1), we mean a nontrivial real-valued function $y \in C^1[T, \infty)$ with $T \in [s_0, \infty)$ such that $y^{(j)} \in C^1[T, \infty)$, $j = 1, 2, \ldots, n-1$ and $y(s)$ satisfies Equation (1.1) on $[T, \infty)$. We consider only those solutions $y(s)$ of Equation (1.1), which satisfy $\sup\{|y(s)| : s \geq T\} > 0$ for all $T \in [s_0, \infty)$. We shall not investigate solutions that vanish in the neighborhood of infinity. A solution $y(s)$ of Equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is said to be non-oscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory, see [12]. In the following, we present some oscillation criteria for differential equations that will be relevant to our oscillation criteria for (1.1) and expound the fundamental contributions of this paper. Fite [13] constructed an oscillatory criterion of the linear equation of second-order

$$y''(s) + p(s)y(s) = 0, \tag{1.2}$$

and proved that, if

$$\int_{s_0}^{\infty} p(\mu)\, \mathrm{d}\mu = \infty, \tag{1.3}$$

then (1.2) is oscillatory. This result was also established by Wintner [14] without making the assumption that $p(s) > 0$. Hille [15] improved criterion (1.3) and obtained that if

$$\liminf_{s\to\infty} s \int_{s}^{\infty} p(\mu)\, \mathrm{d}\mu > \frac{1}{4}, \tag{1.4}$$

then (1.2) is oscillatory. Nehari [16] presented the oscillatory behavior of Equation (1.2) and obtained that if

$$\liminf_{s\to\infty} \frac{1}{s} \int_{s_0}^{s} \mu^2 p(\mu)\, \mathrm{d}\mu > \frac{1}{4}, \tag{1.5}$$

then (1.2) is oscillatory. Erbe [17] generalized the Hille-type criterion (1.4) to the delay equation

$$y''(s) + p(s)y(\phi(s)) = 0, \quad \phi(s) \le s, \tag{1.6}$$

and showed that if

$$\liminf_{s\to\infty} s \int_{s}^{\infty} \frac{\phi(\mu)}{\mu} p(\mu)\, \mathrm{d}\mu > \frac{1}{4}, \tag{1.7}$$

then (1.6) is oscillatory. Ohriska [18] proved that, if

$$\limsup_{s\to\infty} s \int_{s}^{\infty} \frac{\phi(\mu)}{\mu} p(\mu)\, \mathrm{d}\mu > 1, \tag{1.8}$$

then (1.6) is oscillatory.

We direct the reader to the relevant results [19–35] and the references cited there. It should be noted that the contributions of Fite [13], Hille [15], Ohriska [18], and Wintner [14] strongly motivated the research in this paper. The aim of this paper was to extend some oscillation criteria for even-order quasilinear functional differential Equation (1.1) in the cases when $\beta \ge 1$, $\beta \le 1$, $\phi(s) \le s$, and $\phi(s) \ge s$. All subsequent inequalities are implicitly supposed to eventually hold. In other words, they are fulfilled for all sufficiently large s.

2. Main Results

This section begins with the subsequent preliminary lemmas. The following essential lemma is attributed to Kiguradze [36].

Lemma 1 (see [36]). *Let $y(s)$ be a function whose derivatives up to order $(n-1)$ inclusive are all absolutely continuous and have a constant sign. Assume that $y^{(n)}(s)$ is eventually of one sign and not identically zero. Then, there is an integer $m \in \{0, 1, \dots, n-1\}$ with $m+n$ odd for $y^{(n)}(s) \le 0$, or with $m+n$ even for $y^{(n)}(s) \ge 0$ such that*

$$y^{(h)}(s) > 0 \quad \text{for } h = 0, 1, \dots, m, \tag{2.1}$$

and

$$(-1)^{m+h} y^{(h)}(s) > 0 \quad \text{for } h = m, m+1, \dots, n, \tag{2.2}$$

eventually.

Lemma 2. *If* (1.1) *has an eventually positive solution* $y(s)$ *and* $m \in \{1, 3, \ldots, n-1\}$ *is offered as in Lemma 1 such that* (2.1) *and* (2.2) *are satisfied for* $s \in [s_0, \infty)$*, then for* $u, v \in [s_0, \infty)$ *and* $l = 0, 1, \ldots, m$, $\dfrac{y^{(m-l)}(v)}{(v-u)^l}$ *is strictly decreasing for* $v \in (u, \infty)$ *and*

$$y^{(m-l)}(v) \geq y^{(m)}(v)\frac{(v-u)^l}{l\,!} \qquad \textit{for } v \in [u, \infty). \tag{2.3}$$

Proof. From (2.1) and (2.2), we obtain for $v \geq u \geq s_0$,

$$y^{(m-1)}(v) = y^{(m-1)}(u) + \int_u^v y^{(m)}(\mu)\,\mathrm{d}\mu,$$

which implies that

$$y^{(m-1)}(v) \geq y^{(m)}(v)(v-u). \tag{2.4}$$

By replacing v by μ in (2.4) and integrating with respect to μ from u to v, we arrive at

$$y^{(m-2)}(v) \geq y^{(m-2)}(u) + \int_u^v y^{(m)}(\mu)(\mu-u)\,\mathrm{d}\mu \geq y^{(m)}(v)\frac{(v-u)^2}{2\,!}.$$

Continuing with this approach, one can easily achieve the desired inequality (2.3). By virtue of (2.4), we have $\dfrac{y^{(m-1)}(v)}{v-u}$ is strictly decreasing for $v > u \geq s$. Therefore,

$$y^{(m-2)}(v) \geq y^{(m-2)}(u) + \int_u^v \frac{y^{(m-1)}(\mu)}{\mu-u}(\mu-u)\mathrm{d}\mu \geq \frac{(v-u)}{2}y^{(m-1)}(v).$$

Consequently, $\dfrac{y^{(m-2)}(v)}{(v-u)^2}$ is strictly decreasing for $v > u \geq s$. Continuing with this approach, one can reasonably conclude that $\dfrac{y^{(m-l)}(v)}{(v-u)^l}$ is strictly decreasing for $v > u \geq s$. The proof is complete. □

Following that, we present the following notations:

$$\gamma := \begin{cases} 1, & \text{if} \quad 0 < \beta \leq 1, \\ \beta, & \text{if} \quad \beta \geq 1, \end{cases} \tag{2.5}$$

and for any $s \in [s_0, \infty)$ and for $m \in \{1, 3, \ldots, n-1\}$, the functions $p_j(s)$, $j = n-1, n-2, \ldots, m$, are defined by the following recurrence formula:

$$p_j(s) := \begin{cases} p(s), & j = n, \\ \int_s^\infty p_{j+1}(\mu)\,\mathrm{d}\mu, & j = 1, 2, \ldots, n-1, \end{cases} \tag{2.6}$$

provided that the improper integrals converge.

Lemma 3. *If* (1.1) *has an eventually positive solution* $y(s)$ *and* $m \in \{1, 3, \ldots, n-1\}$ *is offered as in Lemma 1, such that* (2.1) *and* (2.2) *are satisfied for* $s \in [s_0, \infty)$*, then for* $s \in [s_0, \infty)$ *and* $l = m, m+1, \ldots, n-1$,

$$p_l(s) < \infty \quad \textit{and} \quad (-1)^{l+1}y^{(l)}(s) \geq p_l(s)y^\beta(\varphi(s)). \tag{2.7}$$

Proof. By using Lemma 1, we obtain that $y(s)$ is strictly increasing on $[s_0, \infty)$. Hence, from (1.1) we get for $s \in [s_0, \infty)$,

$$-y^{(n)}(s) = p(s)y^\beta(\phi(s)) \;\geq\; p_n(s)\,y^\beta(\varphi(s)). \tag{2.8}$$

Replacing s by μ in (2.8), integrating from s to $v \in [s, \infty)$, and by (2.2), we have

$$\begin{aligned} y^{(n-1)}(s) &\geq -y^{(n-1)}(v) + y^{(n-1)}(s) \geq \int_s^v p_n(\mu)\, y^\beta(\varphi(\mu))\, \mathrm{d}\mu \\ &\geq y^\beta(\varphi(s)) \int_s^v p_n(\mu)\, \mathrm{d}\mu. \end{aligned}$$

Therefore, let $v \to \infty$; we can deduce that

$$y^{(n-1)}(s) \geq y^\beta(\varphi(s)) \int_s^\infty p_n(\mu)\, \mathrm{d}\mu = p_{n-1}(s) y^\beta(\varphi(s)),$$

which implies $p_{n-1}(s) = \int_s^\infty p_n(\mu)\, \mathrm{d}\mu < \infty$. Integrating again from s to v, and using (2.1) and (2.2), we get

$$\begin{aligned} -y^{(n-2)}(s) &\geq y^{(n-2)}(v) - y^{(n-2)}(s) \geq \int_s^v p_{n-1}(\mu)\, y^\beta(\varphi(\mu))\, \mathrm{d}\mu \\ &\geq y^\beta(\varphi(s)) \int_s^v p_{n-1}(\mu)\, \mathrm{d}\mu. \end{aligned}$$

Hence, as $v \to \infty$, we have

$$-y^{(n-2)}(s) \geq p_{n-2}(s) y^\beta(\varphi(s)),$$

which implies $p_{n-2}(s) = \int_s^\infty p_{n-1}(\mu)\ \mathrm{d}\mu < \infty$. Continuing with this approach, one can easily achieve the desired inequality (2.7). Therefore, the conclusion holds. □

The first theorem is a Fite–Wintner-type oscillation criterion for the Equation (1.1).

Theorem 1. *If*

$$\int_{s_0}^\infty p(\mu)\, \mathrm{d}\mu = \infty, \tag{2.9}$$

then (1.1) *is oscillatory.*

Proof. Assume that (1.1) has a non-oscillatory solution y on $[s_0, \infty)$. Without loss of generality, let $y(s) > 0$ and $y(\phi(s)) > 0$ on $[s_0, \infty)$. From Lemma 1, it follows that there is an odd integer $m \in \{1, 3, \ldots, n-1\}$ such that (2.1) and (2.2) are satisfied for $s \in [s_1, \infty)$ for some $s_1 \in [s_0, \infty)$. In view of Lemma 3 with $l = n-1$, we see that $p_{n-1}(s) = \int_{s_0}^\infty p(\mu)\, \mathrm{d}\mu < \infty$ on $[s_1, \infty)$. This contradicts (2.9); therefore, the proof is complete. □

Example 1. *Consider the quasilinear differential equation of even-order* (1.1) *with* $p(s) = \dfrac{1}{s^\alpha}$, $\alpha \leq 1$. *It is easy to see that* (2.9) *holds. Therefore, by Theorem 1,* (1.1) *is oscillatory if* $\alpha \leq 1$.

In the next results, we will assume that the improper integrals are convergent. Otherwise, we see that (1.1) oscillates in accordance with the preceding theorem.

Theorem 2. *If for each an odd integer* $m \in \{1, 3, \ldots, n-1\}$,

$$\limsup_{s \to \infty} s^m \int_s^\infty \left(\frac{\varphi^\beta(\mu)}{\mu^\gamma} \right)^m p_{m+1}(\mu)\, \mathrm{d}\mu > m\,!, \tag{2.10}$$

then (1.1) *is oscillatory.*

Proof. Assume that (1.1) has a non-oscillatory solution y on $[s_0, \infty)$. Without loss of generality, let $y(s) > 0$ and $y(\phi(s)) > 0$ on $[s_0, \infty)$. From Lemma 1, it follows that there is an odd integer $m \in \{1, 3, \ldots, n-1\}$ such that (2.1) and (2.2) are satisfied for $s \in [s_1, \infty)$ for some $s_1 \in [s_0, \infty)$. In view of Lemma 3 with $l = m+1$, we obtain that for $s \in [s_1, \infty)$,

$$y^{(m+1)}(s) \leq -p_{m+1}(s)\, y^{\beta}(\varphi(s)). \tag{2.11}$$

Integrating (2.11) from s to v, we obtain

$$\int_s^v p_{m+1}(\mu)\, y^{\beta}(\varphi(\mu))\, \mathrm{d}\mu \leq y^{(m)}(s) - y^{(m)}(v) \leq y^{(m)}(s). \tag{2.12}$$

From Lemma 2 with $l = m$, $v = s$, and $u = s_1$, we have that $\dfrac{y(s)}{(s-s_1)^m}$ is strictly decreasing on $[s_2, \infty)$ for some $s_2 \in (s_1, \infty)$. If $\beta \leq 1$, we get for $s \in [s_2, \infty)$,

$$\begin{aligned}
\frac{y^{\beta}(\varphi(s))}{y(s)} &= \left[\frac{y(\varphi(s))}{y(s)}\right]^{\beta} y^{\beta-1}(s) \\
&\geq \left(\left[\frac{\varphi(s)-s_1}{s-s_1}\right]^{m}\right)^{\beta} y^{\beta-1}(s) \\
&= \left(\frac{(\varphi(s)-s_1)^{\beta}}{s-s_1}\right)^{m} \left(\frac{y(s)}{(s-s_1)^m}\right)^{\beta-1} \\
&\geq \left(\frac{(\varphi(s)-s_1)^{\beta}}{s}\right)^{m} \left(\frac{y(s_2)}{(s_2-s_1)^m}\right)^{\beta-1},
\end{aligned}$$

whereas if $\beta \geq 1$, using $y'(s) > 0$ on $[s_2, \infty)$, we get for $s \in [s_2, \infty)$,

$$\begin{aligned}
\frac{y^{\beta}(\varphi(s))}{y(s)} &\geq \left[\left(\frac{\varphi(s)-s_1}{s-s_1}\right)^{m}\right]^{\beta} y^{\beta-1}(s) \\
&\geq \left[\left(\frac{\varphi(s)-s_1}{s}\right)^{m}\right]^{\beta} y^{\beta-1}(s_2).
\end{aligned}$$

Now, setting $l = m$, $v = s$, and $u = s_1$ in (2.3), we have for $s \in [s_2, \infty)$,

$$y(s) \geq \frac{(s-s_1)^m}{m\,!} y^{(m)}(s).$$

Let $0 < \varsigma < 1$ be arbitrary. There exists a sufficiently large $s_\varsigma \in [s_2, \infty)$ such that for $s \in [s_\varsigma, \infty)$,

$$\frac{y^{\beta}(\varphi(s))}{y(s)} \geq \varsigma \left(\frac{\varphi^{\beta}(s)}{s^{\gamma}}\right)^{m}, \tag{2.13}$$

and

$$y(s) \geq \varsigma \frac{s^m}{m\,!} y^{(m)}(s). \tag{2.14}$$

It follows from (2.13) and (2.14), and $y' > 0$ that

$$y^{\beta}(\varphi(\mu)) \geq \varsigma \left(\frac{\varphi^{\beta}(\mu)}{\mu^{\gamma}}\right)^{m} y(s) \geq \varsigma^2 \frac{s^m}{m\,!} \left(\frac{\varphi^{\beta}(\mu)}{\mu^{\gamma}}\right)^{m} y^{(m)}(s), \tag{2.15}$$

for $\mu \in [T, \infty)$ and $T \in [s_\varsigma, \infty)$. Using (2.15) in the inequality (2.12), we achieve that

$$\varsigma^2 s^m \int_s^v \left(\frac{\varphi^{\beta}(\mu)}{\mu^{\gamma}}\right)^{m} p_{m+1}(\mu)\, \mathrm{d}\mu \leq m\,!.$$

By means of $0 < \varsigma < 1$ is arbitrary, we get

$$s^m \int_s^v \left(\frac{\varphi^\beta(\mu)}{\mu^\gamma} \right)^m p_{m+1}(\mu)\, \mathrm{d}\mu \le m\,!.$$

Letting $v \to \infty$, we have

$$s^m \int_s^\infty \left(\frac{\varphi^\beta(\mu)}{\mu^\gamma} \right)^m p_{m+1}(\mu)\, \mathrm{d}\mu \le m\,!,$$

and so

$$\limsup_{s\to\infty} s^m \int_s^\infty \left(\frac{\varphi^\beta(\mu)}{\mu^\gamma} \right)^m p_{m+1}(\mu)\, \mathrm{d}\mu \le m\,!.$$

This contradicts (2.10); therefore, the proof is complete. □

The next result deals with the Hille-type oscillation criterion of (1.1).

Theorem 3. *If for each an odd integer $m \in \{1, 3, \ldots, n-1\}$,*

$$\liminf_{s\to\infty} s^m \int_s^\infty \left(\frac{\varphi^\beta(\mu)}{\mu^\gamma} \right)^m p_{m+1}(\mu)\, \mathrm{d}\mu > \frac{m\,!}{4}, \tag{2.16}$$

then (1.1) is oscillatory.

Proof. Assume that (1.1) has a non-oscillatory solution y on $[s_0, \infty)$. Without loss of generality, let $y(s) > 0$ and $y(\phi(s)) > 0$ on $[s_0, \infty)$. From Lemma 1, it follows that there is an odd integer $m \in \{1, 3, \ldots, n-1\}$ such that (2.1) and (2.2) are satisfied for $s \in [s_1, \infty)$ for some $s_1 \in [s_0, \infty)$. Define

$$w(s) := \frac{y^{(m)}(s)}{y(s)}. \tag{2.17}$$

Hence,

$$w'(s) = \frac{y^{(m+1)}(s)}{y(s)} - \frac{y^{(m)}(s) y'(s)}{y^2(s)}.$$

In view of Lemma 3 with $l = m+1$, we see that

$$y^{(m+1)}(s) \le -p_{m+1}(s)\, y^\beta(\phi(s)).$$

Hence,

$$w'(s) \le -p_{m+1}(s) \frac{y^\beta(\phi(s))}{y(s)} - w(s) \frac{y'(s)}{y(s)}. \tag{2.18}$$

Setting $l = m-1$, $v = s$ and $u = s_1$ in (2.3), we have for $s \in [s_2, \infty)$,

$$y'(s) \ge \frac{(s-s_1)^{m-1}}{(m-1)\,!} y^{(m)}(s).$$

As demonstrated in the proof of Theorem 2, for each $0 < \varsigma < 1$, there is a $s_\varsigma \in [s_1, \infty)$ such that for $s \in [s_\varsigma, \infty)$,

$$\frac{y'(s)}{y(s)} \ge \varsigma \frac{s^{m-1}}{(m-1)!} w(s), \tag{2.19}$$

and

$$\frac{y^\beta(\varphi(s))}{y(s)} \ge \varsigma \left(\frac{\varphi^\beta(s)}{s^\gamma} \right)^m. \tag{2.20}$$

Substituting (2.19) and (2.20) into (2.18), we get for $s \in [s_\varsigma, \infty)$,

$$w'(s) \leq -\varsigma \left(\frac{\varphi^\beta(s)}{s^\gamma} \right)^m p_{m+1}(s) - \varsigma \frac{s^{m-1}}{(m-1)!} w^2(s). \tag{2.21}$$

Now, for any $\epsilon > 0$, there is a $T \in [s_\varsigma, \infty)$ such that

$$\frac{s^m w(s)}{m!} \geq B - \epsilon \quad \text{for} s \in [T, \infty), \tag{2.22}$$

where

$$B := \liminf_{s \to \infty} \frac{s^m w(s)}{m!}, \quad 0 \leq B \leq 1.$$

In view of (2.21) and (2.22), we have

$$w'(s) \leq -\varsigma \left(\frac{\varphi^\beta(s)}{s^\gamma} \right)^m p_{m+1}(s) - \varsigma\, m!(B - \epsilon)^2 \frac{m}{s^{m+1}}. \tag{2.23}$$

Integrating (2.23) from s to v, we deduce that

$$w(v) - w(s) \leq -\varsigma \int_s^v \left(\frac{\varphi^\beta(\mu)}{\mu^\gamma} \right)^m p_{m+1}(\mu)\, \mathrm{d}\mu - \varsigma\, m!(B - \epsilon)^2 \int_s^v \left(\frac{-1}{\mu^m} \right)' \mathrm{d}\mu.$$

Considering the fact that $w > 0$, and taking to the limits as $v \to \infty$, we get

$$\varsigma \int_s^\infty \left(\frac{\varphi^\beta(\mu)}{\mu^\gamma} \right)^m p_{m+1}(\mu)\, \mathrm{d}\mu \leq w(s) - \varsigma\, m!(B - \epsilon)^2 \frac{1}{s^m}. \tag{2.24}$$

Multiplying both sides of (2.24) by $\frac{s^m}{m!}$, we find that

$$\varsigma \frac{s^m}{m!} \int_s^\infty \left(\frac{\varphi^\beta(\mu)}{\mu^\gamma} \right)^m p_{m+1}(\mu)\, \mathrm{d}\mu \leq \frac{s^m}{m!} w(s) - \varsigma\, (B - \epsilon)^2.$$

Taking the lim inf of the previous inequality as $s \to \infty$, we obtain

$$\frac{\varsigma}{m!} \liminf_{s \to \infty} s^m \int_s^\infty \left(\frac{\varphi^\beta(\mu)}{\mu^\gamma} \right)^m p_{m+1}(\mu)\, \mathrm{d}\mu \leq B - \varsigma\, (B - \epsilon)^2.$$

By means of $\epsilon > 0$ and $0 < \varsigma < 1$ being arbitrary, we conclude that

$$\frac{1}{m!} \liminf_{s \to \infty} s^m \int_s^\infty \left(\frac{\varphi^\beta(\mu)}{\mu^\gamma} \right)^m p_{m+1}(\mu)\, \mathrm{d}\mu \leq B - B^2.$$

We can easily achieve the desired result,

$$\liminf_{s \to \infty} s^m \int_s^\infty \left(\frac{\varphi^\beta(\mu)}{\mu^\gamma} \right)^m p_{m+1}(\mu)\, \mathrm{d}\mu \leq \frac{m!}{4}.$$

This contradicts (2.16); therefore, the proof is complete. □

As a direct result of Theorems 1, 2, and 3, we can find oscillation criteria for the Equation (1.1) when $n = 2$, i.e., for the second order equation

$$y''(s) + p(s)|y(\phi(s))|^{\beta-1} y(\phi(s)) = 0. \tag{2.25}$$

Corollary 1. *The equation (2.25) is oscillatory, provided one of the following conditions holds:*

(a) $\int_{s_0}^{\infty} p(\mu)\, \mathrm{d}\mu = \infty$;

(b) $\limsup_{s\to\infty} s \int_s^{\infty} \frac{\varphi^{\beta}(\mu)}{\mu^{\gamma}} p(\mu)\, \mathrm{d}\mu > 1$;

(c) $\liminf_{s\to\infty} s \int_s^{\infty} \frac{\varphi^{\beta}(\mu)}{\mu^{\gamma}} p(\mu)\, \mathrm{d}\mu > \frac{1}{4}$.

Example 2. *Consider a second-order quasilinear differential equation,*

$$y''(s) + \frac{\alpha}{\sqrt[3]{s^4}} \sqrt[3]{y(\lambda s)} = 0 \qquad \textit{for } s \in [s_0, \infty), \tag{2.26}$$

where $\lambda, \alpha > 0$. *Here,* $n = 2$, $\beta = \frac{1}{3}$, $p(s) = \frac{\alpha}{\sqrt[3]{s^4}}$, *and* $\phi(s) = \lambda s$. *Now,*

$$\limsup_{s\to\infty} s \int_s^{\infty} \frac{\phi^{\beta}(\mu)}{\mu^{\gamma}} p(\mu)\, \mathrm{d}\mu = \alpha \sqrt[3]{\lambda} \limsup_{s\to\infty} s \int_s^{\infty} \frac{\mathrm{d}\mu}{\mu^2} = \alpha \sqrt[3]{\lambda},$$

and

$$\limsup_{s\to\infty} s \int_s^{\infty} \mu^{\beta-\gamma} p(\mu)\, \mathrm{d}\mu = \alpha \limsup_{s\to\infty} s \int_s^{\infty} \frac{\mathrm{d}\mu}{\mu^2} = \alpha.$$

Employment of Corollary 1, Part (b) means that (2.26) is oscillatory if

$$\alpha > \begin{cases} \frac{1}{\sqrt[3]{\lambda}}, & \textit{if} \quad 0 < \lambda \le 1, \\ 1, & \textit{if} \quad \lambda \ge 1. \end{cases}$$

For the Equation (1.1) with $n \ge 4$, we get further oscillation criteria as seen below.

Corollary 2. *Let*

$$\textit{either} \qquad \int_{s_0}^{\infty} p_{n-1}(\mu)\, \mathrm{d}\mu = \infty \qquad \textit{or} \qquad \int_{s_0}^{\infty} p_{n-2}(\mu)\, \mathrm{d}\mu = \infty. \tag{2.27}$$

Then, (1.1) with $n \ge 4$ *is oscillatory provided one of the following conditions holds:*

(a) $\limsup_{s\to\infty} s^{n-1} \int_s^{\infty} \left(\frac{\varphi^{\beta}(\mu)}{\mu^{\gamma}}\right)^{n-1} p(\mu)\, \mathrm{d}\mu > (n-1)\,!$;

(b) $\liminf_{s\to\infty} s^{n-1} \int_s^{\infty} \left(\frac{\varphi^{\beta}(\mu)}{\mu^{\gamma}}\right)^{n-1} p(\mu)\, \mathrm{d}\mu > \frac{(n-1)\,!}{4}$.

Proof. Assume that (1.1) has a non-oscillatory solution y on $[s_0, \infty)$. Without loss of generality, let $y(s) > 0$ and $y(\phi(s)) > 0$ on $[s_0, \infty)$. From Lemma 1, it follows that there is an odd integer $m \in \{1, 3, \ldots, n-1\}$ such that (2.1) and (2.2) hold for $s \in [s_1, \infty)$ for some $s_1 \in [s_0, \infty)$. We claim that (2.27) yields that $m = n-1$. If $1 \le m \le n-3$, then for $s \ge s_1$

$$y^{(n)}(s) < 0,\ y^{(n-1)}(s) > 0,\ y^{(n-2)}(s) < 0,\ y^{(n-3)}(s) > 0. \tag{2.28}$$

Since $y(s)$ is strictly increasing on $[s_1, \infty)$ then for sufficiently large $s_2 \in [s_1, \infty)$, we have $y(\phi(s)) \ge y\,(\varphi(s)) \ge L > 0$ for $s \ge s_2$. Thus, the Equation (1.1) becomes

$$-y^{(n)}(s) = p(s)|y(\phi(s))|^{\beta-1} y(\phi(s)) \ge L^{\beta} p(s) = L\, p_n(s).$$

Integrating the above inequality from s to $v \in [s, \infty)$ and then letting $v \to \infty$, we get

$$y^{(n-1)}(s) \ge L^{\beta} \int_s^{\infty} p_n(\mu)\, \mathrm{d}\mu = L^{\beta} p_{n-1}(s). \tag{2.29}$$

It is known from Theorem 1 that $p_{n-1}(s) < \infty$.
Let $\int_{s_0}^{\infty} p_{n-1}(\mu)\,\mathrm{d}\mu = \infty$. By integrating (2.29) from s_2 to $s \in [s_2, \infty)$, we obtain

$$y^{(n-2)}(s) - y^{(n-2)}(s_2) > L^{\beta} \int_{s_2}^{s} p_{n-1}(\mu)\,\mathrm{d}\mu,$$

which implies that $\lim_{s\to\infty} y^{(n-2)}(s) = \infty$, which contradicts $y^{(n-2)} < 0$ on $[s_2, \infty)$.
Let $\int_{s_0}^{\infty} p_{n-2}(\mu)\,\mathrm{d}\mu = \infty$. By integrating (2.29) from s to $v \in [s, \infty)$ and letting $v \to \infty$ and using (2.2), we obtain

$$-y^{(n-2)}(s) \geq L^{\beta} \int_{s}^{\infty} p_{n-1}(\mu)\,\mathrm{d}\mu = L^{\beta} p_{n-2}(s).$$

Again integrating from s_2 to $s \in [s_2, \infty)$, we have

$$y^{(n-3)}(s_2) - y^{(n-3)}(s) \geq L^{\beta} \int_{s_2}^{s} p_{n-2}(\mu)\,\mathrm{d}\mu,$$

which implies that $\lim_{s\to\infty} y^{(n-3)}(s) = -\infty$, which contradicts $y^{(n-3)} > 0$ on $[s_2, \infty)$. This shows that if (2.27) holds, then $m = n - 1$. The remainder of the proof is the same as those for Theorems 2 and 3 when $m = n - 1$ and so can be omitted. □

Example 3. *Consider a fourth-order quasilinear delay differential equation*

$$y^{(4)}(s) + \frac{24}{s^3} \sqrt[3]{y^2(s/2)}\, \operatorname{sgn} y(s/2) = 0 \qquad \textit{for } s \in [s_0, \infty), \tag{2.30}$$

Here $n = 4$, $\beta = \dfrac{2}{3}$, $p(s) = \dfrac{24}{s^3}$, *and* $\phi(s) = \dfrac{s}{2}$. *Now*

$$\int_{s_0}^{\infty} p_{n-2}(\mu)\,\mathrm{d}\mu = \int_{s_0}^{\infty} \left(\int_{\mu}^{\infty} \frac{12}{s^2}\,\mathrm{d}s \right) \mathrm{d}\mu = \infty,$$

and

$$\liminf_{s\to\infty} s^{n-1} \int_{s}^{\infty} \left(\frac{\varphi^{\beta}(\mu)}{\mu^{\gamma}} \right)^{n-1} p(\mu)\,\mathrm{d}\mu = 6 \limsup_{s\to\infty} s^3 \int_{s}^{\infty} \frac{\mathrm{d}\mu}{\mu^4} = 2.$$

Employment of Corollary 2, Part (b) means that (2.30) is oscillatory.

3. Discussions and Conclusions

- Several Fite–Wintner–Hille–Ohriska-type criteria that can be applied to even-order quasilinear functional differential Equation (1.1) are presented in this paper. These results extend prior contributions to second-order differential equations with deviating arguments and cover the extant classical criteria for ordinary differential equations. For more details on how our findings extend known relevant thoughts to the second-order differential equations, see the details below:

(1) Condition (2.16) reduces to (1.4) in the case where $n = 2$, $\beta = 1$, and $\phi(s) = s$;
(2) Condition (2.16) reduce to (1.7) in the case when $n = 2$ and $\beta = 1$;
(3) Condition (2.10) reduces to (1.8) under the assumptions that $n = 2$ and $\beta = 1$.

- It will be important to derive the Nehari-type oscillation criterion (1.5) of the even-order differential equation (1.1).

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