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On 2-variable q-Legendre polynomials: the view point of the q-operational technique

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In this work, we exploit the methods of an operational formality and extension of quasi-monomials to describe and realize 2-variable q-Legendre polynomials. We introduce the generating function of 2-variable q-Legendre polynomials with a context of 0^{th} order q-Bessel Tricomi functions and obtain their properties such as series definition and q-differential equations. Also, we establish the q-multiplicative and q-derivative operators of these polynomials. The operational representations of 2-variable q-Legendre polynomials are obtained.

Key words and phrases: quantum calculus, Legendre polynomial, extension of quasi-monomiality, q-dilatation operator.

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Introduction

The operational techniques were utilized to create some special polynomials and new families of special polynomials with several applications in applied sciences, providing versatile and easy-to-understand solutions to boundary value problems. It has also resulted in the establishment of new computer languages, such as umbral calculus and symbolic interpretation. G. Dattoli et. al. [4] used operational formalism to define 2-variable Legendre polynomials, which have drawn the attention of several mathematicians due to their wide applications in various fields of mathematics and physics (see, for example, [5,18,28,32]).

The Legendre polynomials $S_n(x,y)$ and $R_n(x,y)/n!$ are defined [4] by means of the following generating functions:

$$e^{yt}C_0(-xt^2) = \sum_{n=0}^{\infty} S_n(x,y)\frac{t^n}{n!} \tag{1}$$

and

$$C_0(xt)\,C_0(-yt) = \sum_{n=0}^{\infty} \frac{R_n(x,y)}{n!}\frac{t^n}{n!}. \tag{2}$$

Quantum calculus is a relatively subfield in the realm of scientific study. This area is a generalization of ordinary calculus for the case where $q \to 1^-$ exists. It has been demonstrated that it is helpful in the investigation of various problems, which arise in various branches

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of the sciences, mathematics, statistics, quantum mechanics and quantum physics. Recently, several researchers, working in the field of q-special functions, introduced and studied several q-special functions and their characteristics (see, for example, [2, 12, 19, 25, 29, 30]).

In this quick recap, we take a look at a few definitions and notations associated with the quantum calculus [7].

The q-factorial is defined as

$$[n]_q! = \begin{cases} \prod_{k=1}^{n} [k]_q, & 0 < q < 1, \; n \geq 1, \\ 1, & n = 0. \end{cases}$$

The two q-exponential functions are defined as:

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \quad 0 < q < 1, \tag{3}$$

and

$$E_q(x) = \sum_{n=0}^{\infty} q^{\binom{2}{n}} \frac{x^n}{[n]_q!}, \quad 0 < q < 1,$$

which satisfy the following rule

$$e_q(x)E_q(-x) = 1. \tag{4}$$

We recall some identities of q-derivatives:

$$\hat{D}_{q,x}\, x^n = [n]_q x^{n-1}, \tag{5}$$

$$\hat{D}_{q,x} e_q(\alpha x) = \alpha e_q(\alpha x), \tag{6}$$

$$\hat{D}_{q,x}(f(x)g(x)) = f(x)\hat{D}_{q,x}g(x) + g(qx)\hat{D}_{q,x}f(x). \tag{7}$$

The theory of q-Bessel function was studied by many mathematicians and physicists. This theory grew to include two variables and generalized q-Bessel functions. The most well-known forms are two related q-Bessel functions $J_n^1(x;q)$ and $J_n^2(x;q)$ [15]. The q-Bessel function $J_n^1(x;q)$ is introduced and studied by F.H. Jackson [17]. Later, W. Hahn and H. Exton created a third form of q-Bessel function [8, 9, 14], which is studied by T.H. Koornwinder and R. Swarttouw [22]. Recently, M. Fadel et. al. [10] presented new properties and characterize q-Bessel functions of the first kind.

The series definition of q-Bessel functions of the first kind qBF $J_n(x;q)$ is given [27] by

$$J_n(x;q) = \frac{1}{(q;q)_n} \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{(q;q)_k (q^{n+1};q)_k} = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{[n+k]_q! [k]_q!}, \tag{8}$$

which converges absolutely for $|x| < 2$.

The series definition of nth order q-Tricomi Bessel functions is defined [27] by

$$C_{n,q}(x) = \frac{1}{(q;q)_n} \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{(q;q)_k (q^{n+1};q)_k} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{[k]_q! [n+k]_q!}, \tag{9}$$

which converges absolutely for all values of x.

In view of equations (8) and (9), it is clear that the nth order q-Bessel Tricomi functions of first kind $C^1_{n,q}(x) := C_{n,q}(x)$ is related with qBF $J^1_n(x;q)$ in the following manner

$$C_{n,q}(x) = x^{-x/2} J_n^{(1)}(2\sqrt{x};q). \tag{10}$$

Also, for $n = 0$, equation (10) gives the 0^{th} order q-Bessel Tricomi function $C_{0,q}(x)$, namely

$$C_{0,q}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{([k]_q!)^2}, \tag{11}$$

which converges absolutely for all values of x.

The relation between 0^{th} order q-Bessel Tricomi functions and exponential function can be written [3] as

$$C_{0,q}(xt) = e_q(-\hat{D}^{-1}_{q,x} t)\{1\}, \tag{12}$$

where

$$\hat{D}^{-1}_{q,x} f(x) := \int_0^x f(\zeta) d_q \zeta \tag{13}$$

and

$$(\hat{D}^{-1}_{q,x})^n \{1\} = \frac{x^n}{[n]_q!}, \quad n \in \mathbb{N} \cup \{0\}. \tag{14}$$

In view of equations (11) and (12), the q-partial derivatives of 0^{th} order q-Bessel Tricomi functions are realized [3] as

$$\hat{D}_{q,x} x \hat{D}_{q,x} C_{0,q}(xt) = \frac{\partial_q}{\partial_q D^{-1}_{q,x}} C_{0,q}(xt) = -t C_{0,q}(xt) \tag{15}$$

and

$$\hat{D}_{q,y} y \hat{D}_{q,y} C_{0,q}(-yt) = \frac{\partial_q}{\partial_q D^{-1}_{q,y}} C_{0,q}(-yt) = t C_{0,q}(-yt), \tag{16}$$

respectively.

The extension of monomiality can create a concept within the theory of q-special functions, generating new families of q-special polynomials and demonstrating their quasi-monomiality. This treatment provides a framework for understanding q-special polynomials as solutions to generalized forms of q-partial differential equations and q-integro-differential equations. It can also derive additional classes of q-generating functions and generalizations of q-special functions. Recently, q-special polynomials such as q-Laguerre polynomials of two variables [3] and q-Hermite-based Appell polynomials of two variables [11] were effectively achieved by extending the monomiality.

The two q-operators $\hat{M}_q$ and $\hat{P}_q$, called q-multiplicative and q-derivative operators, respectively, for a q-polynomials set $p_{n,q}(x)$, $n \in \mathbb{N}$, $x \in \mathbb{C}$, are defined [26] as

$$\hat{M}_q \{p_{n,q}(x)\} = p_{n+1,q}(x)$$

and

$$\hat{P}_q \{p_{n,q}(x)\} = [n]_q p_{n-1,q}(x). \tag{17}$$

The q-operators $\hat{M}_q$ and $\hat{P}_q$ satisfy the commutation relation

$$[\hat{P}_q, \hat{M}_q] = \hat{P}_q \hat{M}_q - \hat{M}_q \hat{P}_q. \tag{18}$$

If $\hat{M}_q$ and $\hat{P}_q$ have q-differential realization, then the q-differential equation satisfied by $p_{n,q}(x)$ is

$$\hat{M}_q \hat{P}_q \{p_{n,q}(x)\} = [n]_q \, p_{n,q}(x). \tag{19}$$

In view of equations (18) and (19), we have

$$[\hat{P}_q, \hat{M}_q] = [n+1]_q - [n]_q.$$

The q-Hermite polynomials have multiple definitions (see, for example, [1, 16, 23]). Such polynomials are used in combinatorics, quantum physics, non-commutative probability and other areas of mathematics and physics. Recently, N. Raza et. al. [25] and M. Fadel et. al. [12] introduced and studied the properties of q-Hermite polynomials with two and three variables.

The generating function of 2-variable q-Hermite polynomials is given [25] as

$$e_q(xt)\, e_q(yt^2) = \sum_{n=0}^{\infty} H_{n,q}(x,y) \frac{t^n}{[n]_q!}. \tag{20}$$

The q-partial derivative with respect to t for $e_q(yt^2)$ is given [25] by

$$D_{q,t}\, e_q(yt^2) = yt\, e_q(yt^2) + qyt\, e_q(qyt^2). \tag{21}$$

The q-dilatation operator T_z, which acts on any function of the complex variable z, is defined [13] as

$$T_z^k f(z) = f(q^k z), \quad k \in \mathbb{R}. \tag{22}$$

Many mathematicians and physicists have been explored q-Legendre polynomials. Various results related to q-difference equations, orthogonality, and temperature, corresponding to q-Legendre polynomials, have provided by T.H. Koornwinder, M. Rahman, H.T. Koelink, A.W. Van and T. Ernst (see [6, 7, 20, 21, 24, 31] for more details).

The generating function of 2 variable q-Laguerre polynomials 2VqLP $L_{n,q}(x,y)$ is given [3] by

$$C_{0,q}(xt) e_q(yt) = \sum_{n=0}^{\infty} L_{n,q}(x,y) \frac{t^n}{[n]_q!}.$$

The generating function of 2-variable mth order q-Laguerre polynomials ${}_{[m]}L_{n,q}(x,y)$ is given [3] by

$$C_{0,q}(-xt^m) e_q(yt) = \sum_{n=0}^{\infty} {}_{[m]}L_{n,q}(x,y) \frac{t^n}{[n]_q!}, \quad m \in \mathbb{N}. \tag{23}$$

We note that, for $q \to 1^-$, all the results in q-calculus lead to the corresponding results in ordinary calculus.

We motivated by the applications of operational techniques, which serve as a useful tool to simplify the study of certain results that cannot be obtained by classical methods, including the 2-variable Legendre polynomials and their applications in applied sciences, which provide ranged and relatable solutions to boundary value problems. Moreover, motivated by the applications of q-special functions in mathematical and engineering science, we present and examine the distinctive features of q-Legendre polynomials with two variables by employing the extension of the concept of monomiality principle and the method of operational formalism. In this paper, we explore the possibility of these applications to introduce the 2-variable q-Legendre polynomials from the view point of 0^{th} order q-Bessel Tricomi functions and study their properties. Also, we establish quasi-monomiality properties for these polynomials and study their characteristics.

1 2-variable q-Legendre polynomials

In this section, we introduce the 2-variable q-Legendre polynomials 2VqLP $S_{n,q}(x,y)$ and $R_{n,q}(x,y)/[n]_q!$ by means of generating function involving 0^{th} order q-Bessel Tricomi functions and obtain their series definitions, q-differential equations and operational identities.

In view of equations (1), (2) and (11), we define the 2-variable q-Legendre polynomials 2VqLP $S_{n,q}(x,y)$ and $R_{n,q}(x,y)/[n]_q!$ by means of the following generating functions

$$e_q(yt)\, C_{0,q}(-xt^2) = \sum_{n=0}^{\infty} S_{n,q}(x,y) \frac{t^n}{[n]_q!} \tag{24}$$

and

$$C_{0,q}(xt)\, C_{0,q}(-yt) = \sum_{n=0}^{\infty} \frac{R_{n,q}(x,y)}{[n]_q!} \frac{t^n}{[n]_q!}, \tag{25}$$

where $C_{0,q}(xt)$ denotes the 0^{th} order q-Bessel Tricomi functions, defined by equation (11).

Expanding the left hand side of equation (24), using equations (3) and (11) and then using the series rearrangement technique, we get

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{x^k y^{n-2k} t^n}{([k]_q!)^2 [n-2k]_q!} = \sum_{n=0}^{\infty} S_{n,q}(x,y) \frac{t^n}{[n]_q!}.$$

Comparing the coefficients of equal powers of t from both sides of the above equation, we get the following explicit form

$$S_{n,q}(x,y) = [n]_q! \sum_{k=0}^{[n/2]} \frac{x^k y^{n-2k}}{([k]_q!)^2 [n-2k]_q!} \tag{26}$$

of $S_{n,q}(x,y)$ polynomials.

Similarly, expanding the left hand side of equation (25) and using the same approaches in acquiring (26), we obtain the following explicit form

$$\frac{R_{n,q}(x,y)}{[n]_q!} = [n]_q! \sum_{k=0}^{n} \frac{(-1)^k x^k y^{n-k}}{([k]_q!)^2 ([n-k]_q!)^2} \tag{27}$$

of $R_{n,q}(x,y)/[n]_q!$.

From equations (26) and (27), we get

$$S_{n,q}(x,-y) = (-1)^n S_{n,q}(x,y) \quad \text{and} \quad R_{n,q}(-x,-y) = (-1)^n R_{n,q}(x,y),$$

respectively. Moreover, taking $x = 0$ and $y = 0$ one by one in equations (26) and (27), we get the following boundary conditions

$$S_{n,q}(0,y) = y^n, \qquad S_{n,q}(x,0) = \begin{cases} [n]_q!(x)_q^{[n/2]}/([n/2]_q!)^2, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

and

$$R_{n,q}(x,0) = (-x)^n, \qquad R_{n,q}(0,y) = y^n,$$

respectively.

The operational identity allows the representation of $S_{n,q}(x,y)$ as q-Laguerre and q-Hermite polynomials of two variables by using equations (20) and (24), equation (23), for $m = 2$, as follows

$$S_{n,q}(x,y) = {}_{[2]}L_{n,q}(x,y) = H_{n,q}(y, D_{q,x}^{-1})\{1\}.$$

Similarly, the polynomials $R_{n,q}(x,y)/[n]_q!$ can be represented as 2-variable q-Laguerre polynomials by using the operational identity and equations (12) and (25) as follows

$$\frac{R_{n,q}(x,y)}{[n]_q!} = L_{n,q}(x, D_{q,y}^{-1})\{1\}.$$

As we noted earlier, the partial differential equations of ordinary 2-variable Legendre polynomials give rise to real applied problems in areas such as heat condition equations and other kinds of heat diffusion equations. To this end, we demonstrate the q-partial differential equations of 2VqLP $S_{n,q}(x,y)$ and $R_{n,q}(x,y)/[n]_q!$.

Theorem 1. *The 2VqLP $S_{n,q}(x,y)$ and $R_{n,q}(x,y)/[n]_q!$ satisfy the following q-partial differential equations:*

$$ {}_L\hat{D}_{q,x} S_{n,q}(x,y) = \hat{D}_{q,y}^2 S_{n,q}(x,y), \tag{28}$$

or, alternatively,

$$(qx\hat{D}_{q,x}^2 - \hat{D}_{q,y}^2 + \hat{D}_{q,x}) S_{n,q}(x,y) = 0 \tag{29}$$

and

$$ {}_L\hat{D}_{q,x} R_{n,q}(x,y) = -{}_L\hat{D}_{q,y} R_{n,q}(x,y), \tag{30}$$

or, alternatively,

$$(q(x\hat{D}_{q,x}^2 + y\hat{D}_{q,y}^2) + \hat{D}_{q,x} + \hat{D}_{q,y}) R_{n,q}(x,y) = 0, \tag{31}$$

where

$$ {}_L\hat{D}_{q,x} := \hat{D}_{q,x} x \hat{D}_{q,x}. \tag{32}$$

Proof. Operating ${}_L\hat{D}_{q,x}$ on both sides of equation (24) and using equations (32) and (15) in the left hand side, we get

$$t^2 e_q(yt) C_{0,q}(-xt^2) = \sum_{k=0}^{\infty} {}_L\hat{D}_{q,x} S_{n,q}(x,y) \frac{t^n}{[n]_q!}. \tag{33}$$

Furthermore, operating $\hat{D}_{q,y}$ on both sides of equation (24), by using equation (6), we receive

$$t e_q(yt) C_{0,q}(-xt^2) = \sum_{n=0}^{\infty} \hat{D}_{q,y} S_{n,q}(x,y) \frac{t^n}{[n]_q!}. \tag{34}$$

Using equation (24) in the left hand side of equations (33) and (34), then comparing the coefficients of equal powers of t from both sides of the resultant equation, we get

$$ {}_L\hat{D}_{q,x} S_{n,q}(x,y) = [n]_q[n-1]_q S_{n-2,q}(x,y), \quad n \geq 2, \tag{35}$$

and

$$\hat{D}_{q,y} S_{n,q}(x,y) = [n]_q S_{n-1,q}(x,y), \quad n \geq 1. \tag{36}$$

Equations (35) and (36), give assertion (28).

In view of equations (5) and (7), for any function $f_q(t)$, we have the following equivalent form

$$ {}_L\hat{D}_{q,t} f_q(t) = (qt\hat{D}_{q,t}^2 + \hat{D}_{q,t}) f_q(t). \tag{37} $$

In view of equation (37), the q-partial differential equation (28) gives assertion (29).

Similarly, operating ${}_L\hat{D}_{q,x}$ and ${}_L\hat{D}_{q,y}$ one by one on both sides of equation (25) and using equations (15) and (25) in the left hand side of the resultant equations, then comparing the coefficients of equal powers of t from both sides of the resultant equations, we find

$$ -{}_L\hat{D}_{q,x} R_{n,q}(x,y) = [n]_q R_{n-1,q}(x,y), \quad n \geq 1, \tag{38} $$

and

$$ {}_L\hat{D}_{q,y} R_{n,q}(x,y) = [n]_q R_{n-1,q}(x,y), \quad n \geq 1, \tag{39} $$

respectively.

Equations (38) and (39) give assertion (30). Therefore, q-partial differential equation (30) gives assertion (31). □

Remark 1. *In view of equations* (12), (24) *and* (25), *we have*

$$ e_q(yt)\, e_q(D_{q,x}^{-1} t^2)\{1\} = \sum_{n=0}^{\infty} S_{n,q}(x,y) \frac{t^n}{[n]_q!} $$

and

$$ e_q(-tD_{q,x}^{-1})\, e_q(tD_{q,y}^{-1})\{1\} = \sum_{n=0}^{\infty} \frac{R_{n,q}(x,y)}{[n]_q!} \frac{t^n}{[n]_q!}. $$

Using equation (15) *in equations* (24) *and* (25) *and then simplifying, we get*

$$ \frac{\partial_q}{\partial_q D_{q,x}^{-1}} S_{n,q}(x,y) = {}_L\hat{D}_{q,x} S_{n,q}(x,y), \tag{40} $$

$$ -\frac{\partial_q}{\partial_q D_{q,x}^{-1}} R_{n,q}(x,y) = {}_L\hat{D}_{q,x} R_{n,q}(x,y), \tag{41} $$

and

$$ \frac{\partial_q}{\partial_q D_{q,y}^{-1}} R_{n,q}(x,y) = {}_L\hat{D}_{q,y} R_{n,q}(x,y), \tag{42} $$

respectively.

Also, in view of equations (28) *and* (40), *we have*

$$ \frac{\partial_q}{\partial_q D_{q,x}^{-1}} S_{n,q}(x,y) = \hat{D}_{q,y}^2 S_{n,q}(x,y) $$

and in view of equations (30), (41) *and* (42), *we get*

$$ \frac{\partial_q}{\partial_q D_{q,x}^{-1}} R_{n,q}(x,y) = -\frac{\partial_q}{\partial_q D_{q,y}^{-1}} R_{n,q}(x,y). $$

In the next section, we discuss the q-monomiality characteristics and operational identities for 2VqLP $S_{n,q}(x,y)$ and $R_{n,q}(x,y)/[n]_q!$.

2 Quasi-monomiality characteristics

The extension of quasi-monomials treatment to q-special functions has provided a tool for investigating the characteristics of q-special functions and q-polynomial families, such as q-multiplication and q-derivative operators, integro-differential equations and other important identities. In this section, we establish that the 2-variable q-Legendre polynomials are quasi-monomials and obtain their operational identities.

The mathematical framework of q-Legendre polynomials can be developed by utilizing the formalism associated with the extension of the concept of quasi-monomiality. We define the following q-multiplicative $\hat{M}_{q,S}$ and q-derivative $\hat{P}_{q,S}$ operators of 2VqLP $S_{n,q}(x,y)$.

Theorem 2. *The 2VqLP $S_{n,q}(x,y)$ are quasi monomials with respect to the following q-multiplicative and q-derivative operators:*

$$\hat{M}_{q,S} = y + D_{q,x}^{-1}\hat{D}_{q,y}T_y(1+qT_x), \tag{43}$$

or, alternatively,

$$\hat{M}_{q,S} := yT_x + D_{q,x}^{-1}\hat{D}_{q,y}(1+qT_x) \tag{44}$$

and

$$\hat{P}_{q,qS} = \hat{D}_{q,y}. \tag{45}$$

Proof. Differentiating both sides of equation (24) partially with respect to t by using equation (7), we get

$$\sum_{n=1}^{\infty} S_{n,q}(x,y)\frac{t^{n-1}}{[n-1]_q!} = e_q(qyt)\ \hat{D}_{q,t}C_{0,q}(-xt^2) + \hat{D}_{q,t}e_q(yt)C_{0,q}(-xt^2). \tag{46}$$

In view of equations (12) and (21), we have

$$\begin{aligned}\hat{D}_{q,t}C_{0,q}(-xt^2) = \hat{D}_{q,t}\, e_q(D_{q,x}^{-1}t^2)\{1\} &= D_{q,x}^{-1}t\, e_q(D_{q,x}^{-1}t^2)\{1\} + qD_{q,x}^{-1}t\, e_q(qD_{q,x}^{-1}t^2)\{1\} \\ &= D_{q,x}^{-1}t\ C_{0,q}(-xt^2) + qD_{q,x}^{-1}t\ C_{0,q}(-qxt^2).\end{aligned} \tag{47}$$

Using equations (6) and (47) in the right hand side of equation (46), we get

$$\sum_{n=0}^{\infty}\frac{S_{n+1,q}(x,y)t^n}{[n]_q!} = D_{q,x}^{-1}t\ e_q(qyt)C_{0,q}(-xt^2) + qD_{q,x}^{-1}t\ e_q(qyt)C_{0,q}(-qxt^2) + ye_q(yt)C_{0,q}(-xt^2).$$

In view of equation (34), using equation (24) in the right hand side of aforementioned equation, we get

$$\begin{aligned}\sum_{n=0}^{\infty} S_{n+1,q}(x,y)\frac{t^n}{[n]_q!} &= D_{q,x}^{-1}\hat{D}_{q,y}\sum_{n=0}^{\infty} S_{n,q}(x,qy)\frac{t^n}{[n]_q!} \\ &\quad + qD_{q,x}^{-1}\hat{D}_{q,y}\sum_{n=0}^{\infty} S_{n,q}(qx,qy)\frac{t^n}{[n]_q!} + y\sum_{n=0}^{\infty} S_{n,q}(x,y)\frac{t^n}{[n]_q!}.\end{aligned}$$

Using equation (22) and then comparing the coefficients of equal powers of t from both sides of the above equation, we get assertion (43).

Similarly, differentiating both sides of equation (24) with respect to t by using equation (7) for $f_q(t) = e_q(yt)$ and $g_q(t) = C_{0,q}(-xt^2)$, then using equations (22) and (24) and comparing the coefficients of equal powers of t from both sides of the resultant equation, we obtain an alternate form of q-multiplicative operator of $S_{n,q}(x,y)$, given by equation (44).

In view of equations (17) and (36), we get assertion (45). □

Moreover, we obtain the following q-multiplicative $\hat{M}_{q,R}$ and q-derivative $\hat{P}_{q,R}$ operators of 2VqLP $R_{n,q}(x,y)/[n]_q!$.

Theorem 3. *The 2VqLP $R_{n,q}(x,y)/[n]_q!$ are quasi-monomials with respect to the following q-multiplicative and q-derivative operators:*

$$\hat{M}_{q,R} = -D_{q,x}^{-1}T_y + D_{q,y}^{-1}, \tag{48}$$

or, alternatively,

$$\hat{M}_{q,R} := -D_{q,x}^{-1} + D_{q,y}^{-1}T_x \tag{49}$$

and

$$\hat{P}_{q,R} = -\hat{D}_{q,x}x\hat{D}_{q,x} = \frac{\partial_q}{\partial_q D_{q,x}^{-1}}, \tag{50}$$

or, alternatively,

$$\hat{P}_{q,R} = \hat{D}_{q,y}y\hat{D}_{q,y} = -\frac{\partial_q}{\partial_q D_{q,y}^{-1}}. \tag{51}$$

Proof. Differentiating both sides of equation (25) with respect to t, by using equation (7), we get

$$\sum_{n=1}^{\infty}\frac{R_{n,q}(x,y)}{[n]_q!}\frac{t^{n-1}}{[n-1]_q!} = \hat{D}_{q,t}C_{0,q}(xt)C_{0,q}(-qyt) + C_{0,q}(xt)\hat{D}_{q,t}C_{0,q}(-yt).$$

Using equation (15) in the right hand side of aforementioned equation, we obtain

$$\sum_{n=0}^{\infty}\frac{R_{n+1,q}(x,y)}{[n+1]_q!}\frac{t^{n}}{[n]_q!} = -xC_{0,q}(xt)C_{0,q}(-qyt) + yC_{0,q}(xt)C_{0,q}(-yt).$$

Using equations (14) and (25) in the right hand side of the above equation, we get

$$\sum_{n=0}^{\infty}\frac{R_{n+1,q}(x,y)}{[n+1]_q!}\frac{t^{n}}{[n]_q!} = -D_{q,x}^{-1}\sum_{n=0}^{\infty}\frac{R_{n,q}(x,qy)}{[n]_q!}\frac{t^{n}}{[n]_q!} + D_{q,y}^{-1}\sum_{n=0}^{\infty}\frac{R_{n,q}(x,y)}{[n]_q!}\frac{t^{n}}{[n]_q!},$$

which on using equation (22) and then comparing the coefficients of equal powers of t from both sides, gives assertion (48).

Similarly, differentiating both sides of equation (25) with respect to t by using equation (7) for $f_q(t) = C_{0,q}(-yt)$ and $g_q(t) = C_{0,q}(xt)$, then using equations (22), (25), comparing the coefficients of equal powers of t from both sides of the resultant equation, we obtain an alternate form of q-multiplicative operator of $R_{n,q}(x,y)$, given by equation (49).

From equations (17), (38) and (40), we get assertion (50). Similarly, in view of equations (17), (39) and (41), we get assertion (51). □

In the next theorem, we will see how the q-Legendre polynomials $S_{n,q}(x,y)$ can be expressed as operational identities.

Theorem 4. *The 2-variable q-Legendre polynomials $S_{n,q}(x,y)$ satisfy the following operational identities:*

$$S_{n,q}(x,y) = e_q(D_{q,x}^{-1}\hat{D}_{q,y}^{2})\{y^n\}, \tag{52}$$

$$S_{n,q}(x,y) = C_{0,q}(-x\hat{D}_{q,y}^{2})\{y^n\}, \tag{53}$$

and

$$E_q(-\hat{D}_{q,x}^{-1}\hat{D}_{q,y}^{2})S_{n,q}(x,y) = \{y^n\}. \tag{54}$$

Proof. In view of equation (5), we have

$$\hat{D}_{q,y}^{2k}\, y^n = \frac{[n]_q!}{[n-2k]_q!} y^{n-2k}.$$

Using aforementioned equation in the right hand side of equation (26), we get

$$S_{n,q}(x,y) = \sum_{k=0}^{\infty} \frac{x^k \hat{D}_{q,y}^{2k} y^n}{([k]_q!)^2}.$$

Using equations (3) and (14) in the right hand side of the above equation, we obtain assertion (52). In view of equations (12) and (52), assertion (53) follows.

Operating $E_q(-\hat{D}_{q,x}^{-1}\hat{D}_{q,y}^2)$ on both sides of equation (52) then using equation (4) in the resultant equation, we get assertion (54). □

Now, we obtain the operational identities for $R_{n,q}(x,y)/[n]_q!$.

Theorem 5. *The 2-variable q-Legendre polynomials $R_{n,q}(x,y)$ satisfy the following operational identities:*

$$R_{n,q}(x,y) = C_{0,q}\Big(x\frac{\partial_q}{\partial_q D_{q,y}^{-1}}\Big)\{y^n\}, \tag{55}$$

or, equivalently,

$$R_{n,q}(x,y) = e_q\Big(-D_{q,x}^{-1}\frac{\partial_q}{\partial_q D_{q,y}^{-1}}\Big)\{y^n\} \tag{56}$$

and

$$E_q\Big(D_{q,x}^{-1}\frac{\partial_q}{\partial_q D_{q,y}^{-1}}\Big)R_{n,q}(x,y) = y^n, \tag{57}$$

respectively.

Proof. From equation (16), we have

$$C_{0,q}\Big(x\frac{\partial_q}{\partial_q D_{q,y}^{-1}}\Big)C_{0,q}(-yt) = C_{0,q}(xt)C_{0,q}(-yt).$$

Using equations (11) and (25) in the right hand side and then comparing the coefficients of equal powers of t from both sides of the resultant equation, we get assertion (55). In view of equations (12) and (55), assertion (56) follows. Operating

$$E_q\Big(-D_{q,x}^{-1}\frac{\partial_q}{\partial_q D_{q,y}^{-1}}\Big)$$

on both sides of equation (56) and then using equation (4) in the resultant equation, we get assertion (57). □

Applying the same technique that leads to proof of Theorem 5, we derive the following operational identities of 2-variable q-Legendre polynomials $R_{n,q}(x,y)$.

Theorem 6. *The 2-variable q-Legendre polynomials $R_{n,q}(x,y)$ satisfy the following equivalent operational identities:*

$$R_{n,q}(x,y) = C_{0,q}\Big(y\frac{\partial_q}{\partial_q D_{q,x}^{-1}}\Big)\{(-x)^n\},$$

or, equivalently,

$$R_{n,q}(x,y) = e_q\Big(-D_{q,y}^{-1}\frac{\partial_q}{\partial_q D_{q,x}^{-1}}\Big)\{(-x)^n\}$$

and

$$E_q\Big(D_{q,y}^{-1}\frac{\partial_q}{\partial_q D_{q,x}^{-1}}\Big)R_{n,q}(x,y) = (-x)^n,$$

respectively.

Now, we obtain q-integro-differential equations of q-Legendre polynomials $S_{n,q}(x,y)$ and $R_{n,q}(x,y)/[n]_q!$ by utilizing the formalism associated with the extension of the quasi-monomiality.

Theorem 7. *The following q-integro-differential equations for $S_{n,q}(x,y)$ and $R_{n,q}(x,y)/[n]_q!$ hold true:*

$$q\int_0^x \hat{D}_{q,y}^2 T_u S_{n,q}(u,y)d_q u + \int_0^x \hat{D}_{q,y}^2 S_{n,q}(u,y)d_q u = ([n]_q - yT_x\hat{D}_{q,y})S_{n,q}(x,y), \tag{58}$$

$$\begin{aligned} q\int_0^x \hat{D}_{q,u}R_{n,q}(u,y)d_q u &+ \int_0^x u\hat{D}_{q,u}^2 R_{n,q}(u,y)d_q u \\ &= [n]_q R_{n,q}(x,y) + \int_0^y (qT_x\hat{D}_{q,x} + xT_x\hat{D}_{q,x}^2)R_{n,q}(x,v)d_q v, \end{aligned} \tag{59}$$

and

$$\int_0^x \hat{D}_{q,y}y\hat{D}_{q,y}R_{n,q}(u,y)d_q u = \int_0^y \hat{D}_{q,v}v\hat{D}_{q,v}R_{n,q}(x,v)d_q v - [n]_q R_{n,q}(x,y). \tag{60}$$

Proof. In view of equations (19), (44) and (45), we have

$$(yT_x + D_{q,x}^{-1}\hat{D}_{q,y}(1+qT_x))\hat{D}_{q,y}S_{n,q}(x,y) = [n]_q S_{n,q}(x,y),$$

or, equivalently,

$$(yT_x\hat{D}_{q,y} + D_{q,x}^{-1}\hat{D}_{q,y}^2 S_{n,q}(x,y) + qD_{q,x}^{-1}\hat{D}_{q,y}^2 T_x S_{n,q}(x,y) = [n]_q S_{n,q}(x,y),$$

which, in view of equation (13), gives assertion (58).

In view of equations (19), (49) and (50), we have

$$(-\hat{D}_{q,x}^{-1} + \hat{D}_{q,y}^{-1}T_x)(-\hat{D}_{q,x}x\hat{D}_{q,x})R_{n,q}(x,y) = [n]_q R_{n,q}(x,y).$$

From equation (37) we get

$$(-\hat{D}_{q,x}^{-1} + \hat{D}_{q,y}^{-1}T_x)(-q\hat{D}_{q,x} - x\hat{D^2}_{q,x})R_{n,q}(x,y) = [n]_q R_{n,q}(x,y),$$

which, in view of equation (13), gives assertion (59).

Similarly, in view of equations (19), (48), (51), and (13), we get assertion (60). □

Remark 2. *For $q \to 1^-$, equations* (24) *and* (25) *lead to the generating functions of 2VLP $S_n(x,y)$ and $R_n(x,y)/n!$ given by equations* (1) *and* (2) *respectively. Further, for $q \to 1^-$, equations* (43)–(51) *lead to the respective multiplicative and derivative operators of 2VLP $S_n(x,y)$ and $R_n(x,y)/n!$ (see [4]).*

3 Conclusions

The approach of monomiality for q-polynomials paved the way for the development of an original notion while working within the constraints of the theory of q-special functions. This approach can be utilized in order to investigate certain characteristics for some q-special polynomials. It is astonishing that this was the spark for the discovery the monomiality properties for certain q-polynomials such two-variable q-Laguerre polynomials and two-variable q-Hermite based Appell polynomials, as well as further research into their properties [3, 11]. In this context, we have introduced the 2-variable q-Legendre polynomials from the view point of 0^{th} order q-Bessel Tricomi functions and study their properties. Also, we have established quasi-monomiality properties for these polynomials and studied their characteristics. We want to learn more about the recently introduced quasi-monomiality properties for q-polynomials in order to examine their possible applications in mathematics, science and engineering.

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Раза Н., Фадель М., Чезарано К. *Про q-поліноми Лежандра з двома змінними: погляд з точки зору q-операційної техніки* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 14–26.

У цій роботі використовуються методи операційного формалізму та розширення квазімономів для опису та реалізації поліномів q-Лежандра з 2 змінними. Введено генеруючу функцію q-поліномів Лежандра з двома змінними в контексті q-Бесселя Трікомі функцій нульового порядку та отримано їх властивості, такі як визначення ряду та q-диференціальні рівняння. Встановлено q-мультиплікативний і q-похідний оператори цих поліномів та отримано операторні зображення q-поліномів Лежандра від двох змінних.

Ключові слова і фрази: квантове числення, поліном Лежандра, розширення квазімономіальності, оператор q-дилатації.