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A Survey on Orthogonal Polynomials from a Monomiality Principle Point of View

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Definition: This survey highlights the significant role of exponential operators and the monomiality principle in the theory of special polynomials. Using operational calculus formalism, we revisited classical and current results corresponding to a broad class of special polynomials. For instance, we explore the 2D Hermite polynomials and their generalizations. We also present an integral representation of Gegenbauer polynomials in terms of Gould–Hopper polynomials, establishing connections with a simple case of Gegenbauer–Sobolev orthogonality. The monomiality principle is examined, emphasizing its utility in simplifying the algebraic and differential properties of several special polynomial families. This principle provides a powerful tool for deriving properties and applications of such polynomials. Additionally, we review advancements over the past 25 years, showcasing the evolution and extensive applicability of this operational formalism in understanding and manipulating special polynomial families.

Keywords: operational calculus; exponential operators; Hermite polynomials; Gegenbauer polynomials; monomiality principle

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1. Introduction

This monograph was brought about by the current operational methods involving exponential operators and the corresponding identities commonly used for the study of special functions and diverse classes of orthogonal polynomials, both in one and several variables.

We only present a limited sampling of the many results related with methods involving exponential operators and their connection with special functions and orthogonal polynomials, placing emphasis on some of the contributions of the last 25 years (see, for instance [1–12] and the references therein). We would like to include all results but the length of this paper would not suffice. In addition, we do not prove most of the results we quote. We hope, nonetheless, that the readers will find something here of interest.

The outline of this paper is as follows. In Section 2, we give some basic features of exponential operators. Sections 3 and 4 are devoted to showing a brief state of the art about recent and interesting results involving some generalizations of Hermite and Gegenbauer polynomials via exponential operators, as well as relevant implications of monomiality in the theory of special polynomials. Finally, we provide concluding remarks in Section 5.

2. A Look at Exponential Operators

In operational calculus, operators act on functions to produce other functions. Common operators include differentiation, integration, and various transform operators. Following the ideas of [1,6], we deal with the formalism of the exponential operators. We establish the rules relevant to the action of an exponential operator on a given function and the rules for the disentanglement of exponential operators.

Let f be a real function, which is analytic in a neighborhood of the origin; then, it is clear that f can be expanded in a Taylor series and, in particular, we can write

$$f(x+\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} f^{(n)}(x), \tag{1}$$

where λ is a continuous parameter. The identity (1) induces an operator T, whose action on the function f yields a shift of x by the parameter λ. More precisely,

$$T(f)(x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \frac{\mathrm{d}^n f}{\mathrm{d}x^n} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} f^{(n)}(x) = f(x+\lambda).$$

The operator T is usually called a shift or translation operator and denoted by $e^{\lambda \frac{\mathrm{d}}{\mathrm{d}x}}$. So, (1) becomes

$$e^{\lambda \frac{\mathrm{d}}{\mathrm{d}x}} f(x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} f^{(n)}(x) = f(x+\lambda). \tag{2}$$

The apparent trivial operational identity (2) and the use of suitable variable substitutions allow us to state other important identities, namely

$$e^{\lambda x \frac{\mathrm{d}}{\mathrm{d}x}} f(x) = f\left(e^{\lambda} x\right), \tag{3}$$

$$e^{\lambda x^2 \frac{\mathrm{d}}{\mathrm{d}x}} f(x) = f\left(\frac{x}{1-\lambda x}\right), \quad |x| < \frac{1}{\lambda}, \quad \lambda > 0, \tag{4}$$

$$e^{\lambda x^n \frac{\mathrm{d}}{\mathrm{d}x}} f(x) = f\left(\frac{x}{(1-\lambda(n-1)x^{n-1})^{\frac{1}{n-1}}}\right), \quad |x| < \left(\frac{1}{\lambda(n-1)}\right)^{\frac{1}{n-1}}, \tag{5}$$

where $\lambda > 0$ and $n > 2$.

To generalize the action of the shift operator $e^{\lambda \frac{\mathrm{d}}{\mathrm{d}x}}$, it suffices to consider the exponential operator

$$e^{\lambda q(x) \frac{\mathrm{d}}{\mathrm{d}x}},$$

where the function q satisfies

$$\varphi'(\theta) = q(\varphi(\theta)), \tag{6}$$

with φ being a real function which is invertible in a neighborhood of the origin, such that $x = \varphi(\theta)$ (for a detailed explanation see [1,6]). Then, the following identity holds:

$$e^{\lambda q(x) \frac{\mathrm{d}}{\mathrm{d}x}} f(x) = f\left(\varphi\left(\varphi^{-1}(x) + \lambda\right)\right). \tag{7}$$

Using (7), we can define the following more complicated operator form:

$$E(x;\lambda) := e^{\lambda\left(v(x) + q(x)\frac{\mathrm{d}}{\mathrm{d}x}\right)}, \tag{8}$$

where $v(x)$ is a function of x that does not contain differential operators.

In this case, we have

$$e^{\lambda\left(v(x)+q(x)\frac{\mathrm{d}}{\mathrm{d}x}\right)} x = \left(e^{\lambda[q(x)(\frac{\mathrm{d}}{\mathrm{d}x})+v(x)]} x e^{-\lambda[q(x)(\frac{\mathrm{d}}{\mathrm{d}x})+v(x)]}\right) e^{\lambda[q(x)(\frac{\mathrm{d}}{\mathrm{d}x})+v(x)]} = x(\lambda) g(\lambda). \tag{9}$$

It is easily realized that the functions $x(\lambda)$ and $g(\lambda)$ are specified by the following system of first-order differential equations:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\lambda} x(\lambda) = q(x(\lambda)), & x(0) = x_0, \\ \frac{\mathrm{d}}{\mathrm{d}\lambda} g(\lambda) = v(x(\lambda)) g(\lambda), & g(0) = 1. \end{cases} \tag{10}$$

Furthermore, the relation

$$e^{\lambda[q(x)(\frac{d}{dx})+v(x)]}x^n = (x(\lambda))^n g(\lambda)$$

implies that

$$e^{\lambda[q(x)(\frac{d}{dx})+v(x)]}f(x) = f(x(\lambda))g(\lambda). \tag{11}$$

Since the exponential of two operators $\widehat{A}$ and $\widehat{B}$ generally does not satisfy the power law

$$e^{\widehat{A}+\widehat{B}} = e^{\widehat{A}}e^{\widehat{B}},$$

there are many results which allow the calculation of the compensation between the first and second member of the above relation, by using the value of the commutator of the operators:

$$[\widehat{A},\widehat{B}] = \widehat{A}\widehat{B} - \widehat{B}\widehat{A}.$$

Using the identity (8) with $q(x) = 1$ and $v(x) = x_0$, it is not difficult to show that the system of first-order differential Equation (10) becomes

$$\begin{cases} \frac{d}{d\lambda}x(\lambda) = 1, \\ x(0) = x_0, \end{cases}$$

and the solution of this system is given by $x(\lambda) = \lambda + x_0$. Then,

$$\frac{d}{d\lambda}g(\lambda) = (\lambda + x_0)g(\lambda),$$

which gives

$$g(\lambda) = e^{\frac{\lambda^2}{2}+\lambda x_0} = e^{\frac{\lambda^2}{2}}e^{\lambda x_0}.$$

Since the independent variable x can be effectively generalized via the parameter λ, such that $x \to x(\lambda)$ and $x(0) = x$, we have

$$e^{\lambda(x+\frac{d}{dx})}f(x) = e^{\frac{\lambda^2}{2}}e^{\lambda x}f(x+\lambda) = e^{\frac{\lambda^2}{2}}e^{\lambda x}e^{\lambda\frac{d}{dx}}. \tag{12}$$

Now, taking the operators

$$\widehat{B} = \lambda\frac{d}{dx}, \quad \widehat{A} = \lambda x, \tag{13}$$

we find

$$[\widehat{A},\widehat{B}] = -\lambda^2,$$

Hence, according to (12) we can conclude that the following exponential disentangling holds:

$$e^{\widehat{A}+\widehat{B}} = e^{-k/2}e^{\widehat{A}}e^{\widehat{B}}, \tag{14}$$

for $\widehat{A}$ and $\widehat{B}$ non-commuting operators, such that

$$[\widehat{A},\widehat{B}] = k, \quad [k,\widehat{A}] = [k,\widehat{B}] = 0.$$

Equation (14) is the Weyl identity and one of the possible realizations of the Weyl group is provided by the operators (13).

It is worth mentioning that the extension of the exponential operator formalism to the multidimensional setting can be easily stated using the following identity:

$$\exp\left[\lambda\left[\sum_{j=1}^{n} q_j(x_1,\ldots,x_n)\frac{\partial}{\partial x_j} + v(x_1,\ldots,x_n)\right]\right]f(x_1,\ldots,x_n) = f(x_1(\lambda),\ldots,x_n(\lambda))g(\lambda),$$

where

$$\begin{cases} \frac{\mathrm{d}x_j(\lambda)}{\mathrm{d}\lambda} = q_j(x_1(\lambda),\dots,x_n(\lambda)), & x_j(0) = x_j, \\ \frac{\mathrm{d}g(\lambda)}{\mathrm{d}\lambda} = v(x_1(\lambda),\dots,x_n(\lambda))g(\lambda), & g(0) = 1. \end{cases}$$

This approach has been commonly exploited in order to provide novel results in the last two decades.

We finish this section showing the exponential operators containing higher-order derivatives. In order to do this, we recall the Hausdorff identity:

$$e^{\lambda\widehat{A}}\widehat{B}e^{-\lambda\widehat{A}} = \widehat{B} + \lambda[\widehat{A},\widehat{B}] + \frac{\lambda^2}{2}[\widehat{A},[\widehat{A},\widehat{B}]] + \frac{\lambda^3}{3!}[\widehat{A},[\widehat{A},[\widehat{A},\widehat{B}]]] + \cdots \tag{15}$$

where $\widehat{A}$ and $\widehat{B}$ are non-commuting operators.

Substituting $\widehat{A} = \frac{\mathrm{d}^2}{d\mathrm{d}x^2}$ and $\widehat{B} = x$ into (15) and using that

$$[\widehat{A},\widehat{B}] = 2\frac{d}{dx}, \quad 0 = [\widehat{A},[\widehat{A},\widehat{B}]] = [\widehat{A},[\widehat{A},[\widehat{A},\widehat{B}]]] = [\widehat{A},\dots,[\widehat{A},[\widehat{A},\widehat{B}]]],$$

it is possible to deduce

$$e^{\lambda\frac{\mathrm{d}^2}{\mathrm{d}x^2}}x = \left(x + 2\lambda\frac{\mathrm{d}}{\mathrm{d}x}\right)e^{\lambda\frac{\mathrm{d}^2}{\mathrm{d}x^2}},$$

and, as a consequence we obtain

$$e^{\lambda\frac{\mathrm{d}^2}{\mathrm{d}x^2}}f(x) = f\left(x + 2\lambda\frac{\mathrm{d}}{\mathrm{d}x}\right)e^{\lambda\frac{\mathrm{d}^2}{\mathrm{d}x^2}}.$$

It is not difficult to realize that the above relation can be generalized as

$$e^{\lambda\frac{\mathrm{d}^m}{\mathrm{d}x^m}}f(x) = f\left(x + m\lambda\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}\right)e^{\lambda\frac{\mathrm{d}^m}{\mathrm{d}k^m}}. \tag{16}$$

The relation (16) is known as the Crofton identity. The interested readers are referred to [6] for detailed explanations and examples of exponential operators.

3. Hermite and Gegenbauer Polynomials and Exponential Operators

Classical orthogonal polynomials admit many different definitions [13–17]. In order to deal with the formalism of the exponential operators, we are interested in the definition of monic Hermite and Gegenbauer polynomials by means of the following generating functions:

$$e^{\left(xt-\frac{t^2}{2}\right)} = \sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x), \quad |t| < \infty, \tag{17}$$

and

$$\left(1 - 2xt + t^2\right)^{-\alpha} = \sum_{n=0}^{\infty}C_n^{(\alpha)}(x)t^n, \quad |t| < 1,\ |x| \le 1,\ \alpha > 0. \tag{18}$$

Note that (17) is a suitable modification of the generating function for the Hermite polynomials $\widehat{H}_n(x)$ given by

$$e^{\left(2xt-t^2\right)} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\widehat{H}_n(x), \quad |t| < \infty.$$

The 2D Hermite polynomials can be defined by using the relation (2) as follows: first, we note that for $g(x) = x^r$, $r \ge 0$ and $\lambda = y$ the relation (2) becomes $e^{y\frac{\mathrm{d}}{\mathrm{d}x}}x^r = (x+y)^r$. The,n for $g(x) = \sum_{r=0}^{\infty}a_r x^r$ we have

$$e^{y\frac{d}{dx}}g(x) = e^{y\frac{d}{dx}}\sum_{r=0}^{\infty} a_r x^r = \sum_{r=0}^{\infty} a_r e^{y\frac{d}{dx}} x^r = \sum_{r=0}^{\infty} a_r (x+y)^r.$$

The procedure above can be generalized to exponential operators containing higher derivatives. For instance, considering the second derivative, we can generalize (2) as follows (cf. [6]):

$$e^{\lambda\frac{d^2}{dx^2}} f(x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} f^{(2n)}(x). \tag{19}$$

Since for $n \geq 0$ we have

$$\frac{d^{2n}}{dx^{2n}} x^r = \begin{cases} \frac{r!}{(r-2n)!} x^{r-2n}, & \text{if } r \geq 2n, \\ 0, & \text{otherwise,} \end{cases}$$

for $f(x) = x^r$ the identity (19) becomes

$$e^{\lambda\frac{d^2}{dx^2}} x^r = \sum_{k=0}^{\left[\frac{r}{2}\right]} \frac{\lambda^k}{k!} \frac{r!}{(r-2k)!} x^{r-2k}. \tag{20}$$

Thus, (20) shows the general action of the exponential operator $e^{\lambda\frac{d^2}{dx^2}}$ and substituting $\lambda = y$ into (20) we recover the explicit representation of the 2D Hermite polynomials of Kampé de Fériet-type [6,18]:

$$H_r^{(2)}(x,y) = e^{y\frac{d^2}{dx^2}} x^r = r! \sum_{k=0}^{\left[\frac{r}{2}\right]} \frac{y^k x^{r-2k}}{k!(r-2k)!}, \quad r \geq 0. \tag{21}$$

It is not difficult to check that

$$H_n^{(2)}(x,0) = x^n, \quad H_n^{(2)}\left(x, -\frac{1}{2}\right) = H_n(x) \quad \text{and} \quad H_n^{(2)}(2x,-1) = \widehat{H}_n(x).$$

With these ideas in mind, it is possible to introduce the following generalization of the 2D Hermite polynomials of Kampé de Fériet-type (also called Gould–Hopper polynomials) as follows [1,6,8]:

$$H_n^{(m)}(x,y) := e^{y\frac{d^m}{dx^m}} x^n = n! \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{y^k x^{n-mk}}{k!(n-mk)!}, \quad n \geq 0, \quad m \in \mathbb{N}. \tag{22}$$

It is clear that $H_n^{(1)}(x,y) := e^{y\frac{d}{dx}} x^n = (x+y)^n$, $n \geq 0$. Furthermore, the following identity holds:

$$e^{xt+yt^m} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(m)}(x,y), \quad |t| < \infty.$$

It is worth noting that the expression for the Gould–Hopper polynomials, $H_n^{(m)}(x,y)$, as defined by the operator formalism on the left-hand side of (22), exhibits an apparent asymmetry between the variables x and y. This asymmetry arises from the fact that x is treated as a polynomial variable, while y acts as a parameter governing the exponential operator. Specifically, x appears directly in the polynomial argument, while y modifies the action of the differential operator $\frac{d^m}{dx^m}$ through the exponential factor. This operator acts on the powers of x, effectively shifting the degree of the polynomial in x while the parameter y scales the contributions from higher-order differential terms. As such, x and y play distinct roles: x determines the base structure of the polynomial, while y terms. This distinction underlies the apparent asymmetry but also reflects the complementary nature of the variables in generating the full family of Gould–Hopper polynomials.

The interested readers are referred to [1,2,6,8] for detailed explanations and examples of 2D generalizations of Hermite polynomials within the context of exponential operators.

Regarding Gegenbauer polynomials (18), it is well known that they can be explicitly represented as follows (cf. [16], pp. 144–156 or [13], Section 4.7):

$$C_n^{(\alpha)}(x) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k \Gamma(\alpha+n-k)}{k!(n-2k)!} (2x)^{n-2k}, \tag{23}$$

where

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt.$$

Using (21) and the identity

$$\Gamma(\alpha+\alpha-k) = \int_0^\infty e^{-t} t^{\alpha+\alpha-k-1} dt,$$

we can deduce the following integral representation

$$\begin{aligned} C_n^{(\alpha)}(x) &= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!} \int_0^\infty e^{-t} t^{\alpha+n-k-1} dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k}{k!} \left(\frac{1}{t}\right)^k \frac{(2x)^{n-2k}}{(n-2k)!} e^{-t} t^{\alpha+n-1} dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \sum_{k=0}^{\left[\frac{n}{2}\right]} \left(-\frac{1}{t}\right)^k \frac{(2x)^{n-2k}}{k!(n-2k)!} e^{-t} t^{\alpha+n-1} dt \\ &= \frac{1}{n!\Gamma(\alpha)} \int_0^\infty H_n^{(2)}\left(2x, -\frac{1}{t}\right) e^{-t} t^{\alpha+n-1} dt \end{aligned} \tag{24}$$

In this way, the author in [1,19] demonstrates how it is possible to determine the integral representation (24) in terms of the Gould–Hopper polynomials (21).

An interesting consequence of the integral representation (24) arises when suitable Sobolev inner products are considered on the linear space of polynomials $\mathbb{P}$. More precisely, let $\langle, \cdot, \cdot, \rangle_S$ be the Gegenbauer–Sobolev inner product given by (cf. [20])

$$\langle p, q \rangle_S := \int_{-1}^1 p(x)p(x)w^{(\alpha)}(x)dx + \int_{-1}^1 p'(x)q'(x)w^{(\alpha+1)}(x)dx, \quad p, q \in \mathbb{P}, \tag{25}$$

where $w^{(\alpha)}(x) = (1-x^2)^{\alpha-\frac{1}{2}}, x \in [-1,1]$ and $\alpha > -\frac{1}{2}$.

It is not difficult to see that the orthonormal Gegenbauer polynomials $\left\{p_n^{(\alpha)}(x)\right\}_{n\geq 0}$ satisfy the following Sobolev orthogonality relation:

$$\left\langle p_n^{(\alpha)}, p_m^{(\alpha)} \right\rangle_S = (1+n(n+2\alpha))\delta_{nm}, \quad n, m \geq 0. \tag{26}$$

Hence, the polynomials $\left\{q_n^{(\alpha)}(x)\right\}_{n\geq 0}$, defined by

$$q_n^{(\alpha)}(x) = (1+n(n+2\alpha))^{-\frac{1}{2}} p_n^{(\alpha)}(x), \quad n \geq 0, \quad \alpha > -\frac{1}{2}, \tag{27}$$

are orthonormal with respect to the Gegenbauer–Sobolev inner product (25).

Since, $p_n^{(\alpha)}(x) = \frac{1}{\|C_n^{(\alpha)}\|_2} C_n^{(\alpha)}(x)$ for $\alpha > 0$ and

$$\|C_n^{(\alpha)}\|_2 = \left(\int_{-1}^1 \left(C_n^{(\alpha)}(x)\right)^2 w^{(\alpha)}(x)\right)^{\frac{1}{2}} = \frac{1}{\Gamma(\alpha)} \left(\frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)}{n!(n+\alpha)}\right)^{\frac{1}{2}},$$

from (24) we deduce the following integral representation for Gegenbauer–Sobolev polynomials $q_n^{(\alpha)}(x)$:

$$\begin{aligned} q_n^{(\alpha)}(x) &= \frac{(1+n(n+2\alpha))^{-\frac{1}{2}}}{\|C_n^{(\alpha)}\|_2} C_n^{(\alpha)}(x) \\ &= \frac{(1+n(n+2\alpha))^{-\frac{1}{2}}}{\|C_n^{(\alpha)}\|_2} \frac{1}{n!\Gamma(\alpha)} \int_0^\infty H_n^{(2)}\left(2x, -\frac{1}{t}\right) e^{-t} t^{\alpha+n-1} dt \\ &= A(n,\alpha) \int_0^\infty H_n^{(2)}\left(2x, -\frac{1}{t}\right) e^{-t} t^{\alpha+n-1} dt, \end{aligned} \tag{28}$$

where $A(n,\alpha) = \frac{2^\alpha}{\Gamma(\alpha)}\left[2\pi(n!)^3\left(\frac{1+n(n+2\alpha)}{n+\alpha}\right)\Gamma(n+2\alpha)\right]^{-\frac{1}{2}}$.

Finally, the interested readers are referred to [1,19] for detailed explanations and examples of 2D generalizations of Gegenbauer polynomials within the context of exponential operators.

4. The Monomiality Principle and Its Implications

It is well known that the monomiality principle is based on an abstract definition of the concept of derivative and multiplicative operators which allows us to treat different families of special polynomials as ordinary monomials. The procedure underlines a generalization of the Heisenberg–Weyl group, and many relevant properties of a broad classes of special polynomials can be conveniently framed within the context of the monomiality principle (see, for instance [5]). This principle is essentially a Giuseppe Datolli modern formulation of a point of view not only tracing back to J. F. Steffensen [21–23], but even to older studies by H. M. Jeffery (cf. [2–4] and the references therein).

The rules underlying monomiality are fairly simple and can be formulated as follows. Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. If a couple of operators $\widehat{\mathcal{D}}, \widehat{\mathcal{M}}$ are such that

(a) They do exist along with a differential realization, cf. [4].
(b) They can be embedded to form Weyl algebra, namely, if the commutator is such that $[\widehat{\mathcal{D}}, \widehat{\mathcal{M}}] := \widehat{\mathcal{D}}\widehat{\mathcal{M}} - \widehat{\mathcal{M}}\widehat{\mathcal{D}} = \widehat{1}$.
(c) It is possible to univocally define a polynomial set such that

$$p_0(x) = 1, \quad \widehat{\mathcal{D}} p_0(x) = 0, \quad p_n(x) = \widehat{\mathcal{M}}^n p_0(x) = \widehat{\mathcal{M}}^n 1,$$

then it follows that

$$\widehat{\mathcal{M}} p_n(x) = \widehat{\mathcal{M}}^{n+1} 1 = p_{n+1}(x), \tag{29}$$

$$\widehat{\mathcal{D}} p_n(x) = \widehat{\mathcal{D}} \hat{\mathcal{M}}^n 1 = n p_{n-1}(x), \tag{30}$$

and the polynomials $\{p_n(x)\}_{n\geq 0}$ are called quasi-monomials.

As a consequence of (29) and (30), we have that $p_n(x)$ satisfies the differential equation $\widehat{\mathcal{M}}\widehat{\mathcal{D}}\{p_n(x)\} = np_n(x)$, if $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{D}}$ have differential realizations.

The primary objective of the monomiality principle is to identify operators for multiplication and differentiation. Once these operators are identified, it is possible to simplify many algebraic and differential properties for a broad class of special polynomials, including the so-called generalized, hybrid, degenerate, and mixed special polynomials [4,5,7–12,24].

In what follows, we collect some recent and interesting results involving the monomiality principle or its alternative versions when difference operators appear.

In [9], the authors introduce so-called Sheffer–Appell polynomials by means of the following formal power series:

$$\mathcal{A}(t)A(t)e^{x\mathcal{H}(t)} = \sum_{n=0}^{\infty} {}_sA_n(x)\frac{t^n}{n!}, \quad x \in \mathbb{C}, \tag{31}$$

where $\mathcal{A}(t)$, $A(t)$ and $\mathcal{H}(t)$ are given by

$$\mathcal{H}(t) = \frac{1}{g(f^{-1}(t))}, \quad A(t) = \frac{1}{g(t)}, \quad \text{and } H(t) = f^{-1}(t),$$

with $f(t)$ being a delta series and $g(t)$ an invertible series of the following forms:

$$f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, \quad f_0 = 0,\, f_1 \neq 0,$$
$$g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!}, \quad g_0 \neq 0,$$

where $f^{-1}(t)$ denotes the compositional inverse of $f(t)$. In addition, the functions $\mathcal{A}(t)$ and $\mathcal{H}(t)$ are related via the following generating expression:

$$\mathcal{A}(t) e^{\mathcal{H}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!},$$

with $s_n(x)$ being the corresponding Sheffer A-type zero polynomial sequence.

Under these assumptions, the authors shows that Sheffer–Appell polynomials ${}_sA_n(x)$ are quasi-monomials (see [9], Theorem 2.4) with respect to the following multiplicative and derivative operators:

$$\widehat{\mathcal{M}} = x\mathcal{H}'\Big(\mathcal{H}^{-1}(D_x)\Big) + \frac{\mathcal{A}'(\mathcal{H}^{-1}(D_x))}{\mathcal{A}(\mathcal{H}^{-1}(D_x))} + \frac{A'(\mathcal{H}^{-1}(D_x))}{A(\mathcal{H}^{-1}(D_x))},$$

and

$$\widehat{\mathcal{D}} = \mathcal{H}^{-1}(D_x),$$

with $D_x \equiv \frac{\mathrm{d}}{\mathrm{d}x}$.

Using a similar methodology, the authors of [8] introduce 2D Apostol-type polynomials of order α (in short, 2VATP), ${}_p\mathcal{F}_n^{(\alpha)}(x, y; \lambda; \mu, \nu)$ as follows:

$$\left(\frac{2^{\mu} t^{\nu}}{\lambda e^t + 1}\right)^{\alpha} e^{xt} \phi(y, t) = \sum_{n=0}^{\infty} {}_p\mathcal{F}_n^{(\alpha)}(x, y; \lambda; \mu, \nu) \frac{t^n}{n!}, \tag{32}$$

where the 2D general polynomials $p_n(x, y)$ are defined by the generating function

$$e^{xt} \phi(y, t) = \sum_{n=0}^{\infty} p_n(x, y) \frac{t^n}{n!}, \quad p_0(x, y) = 1,$$

with $\phi(y, t)$ given (at least formally) by a series expansion:

$$\phi(y, t) = \sum_{n=0}^{\infty} \phi_n(y) \frac{t^n}{n!}, \quad \phi_0(y) \neq 0.$$

In this case (see [8], Theorem 2.3), the authors find that the polynomials defined by (32) are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\widehat{\mathcal{M}} = x + \frac{\phi'(y, \partial_x)}{\phi(y, \partial_x)} + \frac{\alpha v\Big(\lambda e^{\partial_x} + 1\Big) - \alpha\lambda\partial_x e^{\partial_x}}{\partial_x\left(\lambda e^{\partial_x} + 1\right)},$$
$$\widehat{\mathcal{D}} = \partial_x \equiv \frac{\partial}{\partial x}.$$

It is worth mentioning that, according to [25], some particular cases of the polynomials (32) described in [8] undergo a process of reduction.

In ([10], Theorem 2.1), first the authors derive the generating functions for the truncated exponential-based Mittag–Leffler polynomials (in short, TEMLP), denoted by ${}_{e^{(r)}}g_n(x,y)$:

$$\frac{e^{\left(x\ln\left(\frac{1+t}{1-t}\right)\right)}}{\left(1-y\left(\ln\left(\frac{1+t}{1-t}\right)\right)^r\right)}=\sum_{n=0}^{\infty}{}_{e^{(r)}}g_n(x,y)\frac{t^n}{n!},\quad |t|<1. \tag{33}$$

Secondly, the authors focus their attention on the operational correspondence satisfied by the TEMLP, which is established in ([10], Theorem 2.2) as follows:

$$\widehat{M}_{e^{(r)}g}=\left(x+ry\partial_y y\partial_x^{r-1}\right)\left(\frac{\left(e^{\partial_x}+1\right)^2}{2e^{\partial x}}\right),\ \text{and}\ \widehat{D}_{e^{(r)}g}=\frac{e^{\partial_x}-1}{e^{\partial_x}+1},\ \text{respectively.}$$

A remarkable aspect of determining the aforementioned operators is the use of an analogue of the Crofton identity (16).

Next, we summarize two interesting results concerning the multidimensional setting. In [11], the following multivariate Hermite-type polynomials are introduced.

$$(1+\vartheta)^{\xi\left(\frac{j_1+j_2\xi+j_3\xi^2+\cdots+j_n\xi^{n-1}}{\vartheta}\right)}=\sum_{n=0}^{\infty}\mathbb{H}_n^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)\frac{\xi^n}{n!}$$

or

$$(1+\vartheta)^{\frac{j_1\xi}{\vartheta}}(1+\vartheta)^{\frac{\frac{j_2}{\xi}\xi^2}{\vartheta}}(1+\vartheta)^{\frac{j_3\xi^3}{\vartheta}}\cdots(1+\vartheta)^{\frac{j\cdot\xi^r}{\vartheta}}=\sum_{n=0}^{\infty}\mathbb{H}_n^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)\frac{\xi^n}{n!}.$$

The multivariate polynomials $\mathbb{H}_n^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)$ are associated with the following multiplicative and derivative operators ([11], Theorem 2.3):

$$\begin{aligned}\widehat{\mathcal{M}}=&\left(j_1\frac{\log(1+\vartheta)}{\vartheta}+2j_2\frac{\partial}{\partial j_1}+3j_3\left(\frac{\vartheta}{\log(1+\vartheta)}\right)\frac{\partial^2}{\partial j_1^2}+\cdots+rj_r\left(\frac{\vartheta}{\log(1+\vartheta)}\right)^{r-1}\frac{\partial^{r-1}}{\partial j_1^{r-1}}\right),\\ \widehat{\mathcal{D}}=&\frac{\vartheta}{\log(1+\vartheta)}D_{j_1}.\end{aligned}$$

Another a novel hybrid family of multivariate Hermite polynomials associated with Apostol-type Frobenius–Euler polynomials was introduced in [12]. More precisely, the authors provide a comprehensive method in order to determine the generating function corresponding to the multivariate Hermite–Frobenius–Euler polynomials (in short, MHFEPs) denoted by $y\mathbf{F}_n^{[m]}(u_1,u_2,\ldots,u_m;u)$ and satisfying the identity ([12], Theorem 1)

$$\left(\frac{1-u}{e^{\xi}-u}\right)\exp\left(u_1\xi+u_2\xi^2+\cdots+u_m\xi^m\right)=\sum_{n=0}^{\infty}\gamma\mathbf{F}_n^{[m]}(u_1,u_2,\ldots,u_m;u)\frac{\xi^n}{n!}, \tag{34}$$

where $u\in\mathbb{C}$, $u\neq1$.

It is showed in ([12], Theorem 5) that the multivariate polynomials in (34) are associated with the following multiplicative and derivative operators:

$$\begin{aligned}\widehat{\mathcal{M}}=&u_1+2u_2\partial_{u_1}+3u_3\partial_{u_1}^2+\cdots+mu_m\partial_{u_1}^{m-1}-\frac{e^{\partial_{u_1}}}{e^{\partial_{u_1}}-u}\ \text{and}\\ \widehat{\mathcal{D}}=&\partial_{u_1}\equiv\frac{\partial}{\partial u_1},\ \text{respectively.}\end{aligned}$$

We conclude this section by exhibiting the operational correspondence involved when the notion of degeneracy (i.e., in the setting of an operator in differences) is incorporated, both in one-dimensional and multidimensional cases.

Recently, the authors have introduced degenerate versions of the hypergeometric Bernoulli and Euler polynomials as follows (cf. [24]):

For $\lambda \in \mathbb{R}\backslash\{0\}$ and a fixed $m \in \mathbb{N}$, the degenerate hypergeometric of Bernoulli and Euler polynomials are defined, respectively, by means of the following generating functions and series:

$$\frac{t^m e_\lambda^x(t)}{e_\lambda(t) - \sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{t^l}{l!}} = \sum_{n=0}^{\infty} B_{n,\lambda}^{[m-1]}(x)\frac{t^n}{n!}, \quad |t| < \min\left\{2\pi, \frac{1}{|\lambda|}\right\}, \tag{35}$$

$$\frac{2^m e_\lambda^x(t)}{e_\lambda(t) + \sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{t^l}{l!}} = \sum_{n=0}^{\infty} E_{n,\lambda}^{[m-1]}(x)\frac{t^n}{n!}, \quad |t| < \min\left\{\pi, \frac{1}{|\lambda|}\right\}. \tag{36}$$

The special polynomials in (35) and (36) are Δ_λ–Appell sets and quasi-monomials with respect to the following derivative and multiplicative operators ([24], Theorem 5).

$$\widehat{\mathcal{D}}_\lambda^{[m-1]} = \frac{e^{\lambda\frac{\partial}{\partial x}} - 1}{\lambda},$$

$$\widehat{\mathcal{M}}_\lambda^{[m-1]} = \frac{\lambda m}{e^{\lambda\frac{\partial}{\partial x}} - 1} + \frac{x}{e^{\lambda\frac{\partial}{\partial x}}} - \frac{(e^{\frac{\partial}{\partial x}})^{1-\lambda} - \sum_{l=0}^{m-2}(1)_{l+1,\lambda}\frac{(e^{\lambda\frac{\partial}{\partial x}}-1)^l}{\lambda^l l!}}{e^{\frac{\partial}{\partial x}} - \sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{(e^{\lambda\frac{\partial}{\partial x}}-1)^l}{\lambda^l l!}},$$

and

$$\widehat{\mathcal{N}}_\lambda^{[m-1]} = \frac{x}{e^{\lambda\frac{\partial}{\partial x}}} - \frac{(e^{\frac{\partial}{\partial x}})^{1-\lambda} + \sum_{l=0}^{m-2}(1)_{l+1,\lambda}\frac{(e^{\lambda\frac{\partial}{\partial x}}-1)^l}{\lambda^l l!}}{e^{\frac{\partial}{\partial x}} + \sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{(e^{\lambda\frac{\partial}{\partial x}}-1)^l}{\lambda^l l!}}.$$

Another family of 3D degenerate hybrid special polynomials associated with Hermite polynomials, denoted by ${}_{\Delta_h H}Q_m(u,v,w;h)$ and introduced in [26], is the class of Δ_h 3D Hermite-based Appell polynomials, which possess a generating expression of the following form:

$$\gamma(t)(1+ht)^{\frac{u}{h}}\left(1+ht^2\right)^{\frac{v}{h}}\left(1+ht^3\right)^{\frac{w}{h}} = \sum_{m=0}^{\infty} {}_{\Delta_h H}Q_m(u,v,w;h)\frac{t^m}{m!}.$$

In ([26], Theorem 7), it has been shown that Δ_h 3D Hermite-based Appell polynomials are linked with the following multiplicative and derivative operators:

$$\widehat{\mathcal{M}} = \left(\frac{u}{1+{}_u\Delta_h} + \frac{2v_u\Delta_h}{h+{}_u\Delta_h{}^2} + \frac{3w_u\Delta_h{}^2}{h^2+{}_u\Delta_h{}^3} + \frac{\gamma'\left(\frac{u\Delta_h}{h}\right)}{\gamma\left(\frac{u\Delta_h}{h}\right)}\right), \text{ and}$$

$$\widehat{\mathcal{D}} = \frac{\log(1+{}_u\Delta_h)}{mh}, \text{ respectively.}$$

Finally, the Δ_h multivariate Hermite polynomials [27] given by the generating expression

$$(1+h\tilde{\xi})^{\frac{q_1}{h}}\left(1+h\tilde{\xi}^2\right)^{\frac{q_2}{h}}\left(1+h\tilde{\xi}^3\right)^{\frac{q_3}{h}}\cdots\left(1+h\tilde{\xi}^r\right)^{\frac{q_r}{h}} = \sum_{m=0}^{\infty} {}_{\Delta_h}\mathfrak{H}_m^{[r]}(q_1,q_2,\cdots,q_r;h)\frac{\xi^m}{m!},$$

are linked with the following multiplicative and derivative operators ([27], Theorem 3):

$$\widehat{\mathcal{M}} = \left(q_1 \frac{1}{1+q_1\Delta_h} + 2q_2 \frac{q_1\Delta_h}{h+q_1{\Delta_h}^2} + 3q_3 \frac{q_1\Delta_h}{h^2+q_1{\Delta_h}^3} + \cdots + rq_r \frac{q_1\Delta_h^{r-1}}{h^{r-1}+q_1{\Delta_h}^r} \right),$$
$$\widehat{\mathcal{D}} = \frac{\log(1+q_1\Delta_h)}{mh}.$$

Enthusiastic readers are encouraged to explore the works [8–12,24,26,27] (and the references therein) to examine the advantages provided by the monomiality principle as a powerful tool in determining the OPEs or PDEs satisfied by each family of special polynomials presented in this survey.

5. Conclusions

In this survey, the profound impact of exponential operators and the monomiality principle on the theory of special polynomials is underscored. Our brief review on 2D Hermite polynomials and their generalizations, along with the integral representation of Gegenbauer polynomials in terms of Gould–Hopper polynomials, has allowed us to establish connections with a simple case of Gegenbauer–Sobolev orthogonality.

We have also seen that the monomiality principle arises as a pivotal tool for simplifying both algebraic and differential features of a broad range of special polynomial families. The evolution of these methods over the past 25 years reveals their growing importance and adaptability in several fields. By revisiting classical results, this survey contributes to a richer understanding of special polynomials, illustrating the continued relevance and potential of operational formalism in many research areas.

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References

1. Cesarano, C. Generalized Hermite Polynomials in the Description of Chebyshev-like Polynomials. Ph.D. Thesis, Universidad Complutense de Madrid, Madrid, Spain, 2015.
2. Dattoli, G. Generalized polynomials, operational identities and their applications. *J. Comput. Appl. Math.* **2000**, *118*, 111–123. [CrossRef]
3. Dattoli, G. Hermite-Bessel and Laguerre-Bessel functions: A by-product of the monomiality principle. *Adv. Spec. Funct. Appl.* **2000**, *1*, 147–164.
4. Dattoli, G.; Licciardi, S. Monomiality and a new family of Hermite polynomials. *Symmetry* **2023**, *15*, 1254. [CrossRef]
5. Dattoli, G.; Migliorati, M.; Srivastava, H.M. Sheffer polynomials, monomiality principle, algebraic methods and the theory of classical polynomials. *Math. Comput. Model.* **2007**, *45*, 1033–1041. [CrossRef]
6. Dattoli, G.; Ottaviani, P.L.; Torre, A.; Vázquez, L. Evolution operator equations: Integration with algebraic and finite-difference methods. Applications to physical problems in classical and quantum mechanics and quantum field theory. *Riv. Nuovo C.* **1997**, *20*, 3–133. [CrossRef]
7. Riyasat, M.; Shahid, A.W.; Subuhi, K. Differential and integral equations associated with some hybrid families of Legendre polynomials. *Tbilisi Math. J.* **2018**, *11*, 127–139. [CrossRef]
8. Subuhi, K.; Riyasat, M.; Yasmin, G. Certain results for the 2-variable Apostol type and related polynomials. *Comput. Math. Appl.* **2015**, *69*, 1367–1382. [CrossRef]

9. Subuhi, K.; Riyasat, M. A determinantal approach to Sheffer–Appell polynomials via monomiality principle. *J. Math. Anal. Appl.* **2015**, *421*, 806–829. [CrossRef]
10. Yasmin, G.; Subuhi, K.; Ahmad, N. Operational methods and truncated exponential–based Mittag–Leffler polynomials. *Mediterr. J. Math.* **2016**, *13*, 1555–1569. [CrossRef]
11. Zayed, M.; Shahid, A.W. Exploring the versatile properties and applications of multidimensional degenerate Hermite polynomials. *AIMS Math.* **2023**, *8*, 30813–30826. [CrossRef]
12. Zayed, M.; Shahid, A.W.; Quintana, Y. Properties of multivariate Hermite polynomials in correlation with Frobenius–Euler polynomials. *Mathematics* **2023**, *11*, 3439. [CrossRef]
13. Szegő, G. *Orthogonal Polynomials*, 4th ed.; AMS: Providence, RL, USA, 1975.
14. Andrews, L.C. *Special Functions for Engineers and Applied Mathematics*; MacMillan: New York, NY, USA, 1958.
15. Burchnall, J.L. A note on the polynomials of Hermite. *Quart. J. Math. Oxf. Ser.* **1941**, *2*, 9–11. [CrossRef]
16. Chihara, T.S. *An Introduction to Orthogonal Polynomials*; Gordon and Breach: New York, NY, USA, 1978.
17. Rainville, E.D. *Special Functions*; The MacMillam Company: New York, NY, USA, 1960.
18. Appell, P.; Kampé de Fériet, J. *Fonctions Hypergéométriques et Hypersphériques: Polynomes d'Hermite*; Gauthier-Villars: Paris, France, 1926.
19. Cesarano, C. Generalization of two-variable Chebyshev and Gegenbauer polynomials. *Int. J. Appl. Math. Stat.* **2015**, *53*, 1–7.
20. Marcellán, F.; Quintana, Y.; Urieles, A. On the Pollard decomposition method applied to some Jacobi-Sobolev expansions. *Turk. J. Math.* **2013**, *37*, 934–948. [CrossRef]
21. Steffensen, J.F. *Interpolation*; The Williams & Wilkins Company: Baltimore, MD, USA, 1927.
22. Steffensen, J.F. On the definition of the central factorial. *J. Inst. Actuar.* **1933**, *64*, 165–168. [CrossRef]
23. Steffensen, J.F. The poweroid, an extension of the mathematical notion of power. *Acta Math.* **1941**, *73*, 333–366. [CrossRef]
24. Cesarano, C.; Quintana, Y.; Ramírez, W. Degenerate versions of hypergeometric Bernoulli–Euler polynomials. *Lobachevskii J. Math.* **2024**, *in press*.
25. Navas, L.M.; Ruiz F.J.; Varona, J.L. Existence and reduction of generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. *Arch. Math.* **2019**, *55*, 157–165. [CrossRef]
26. Alyusof, R.; Shahid, A.W. Certain properties and applications of Δ_h hybrid special polynomials associated with Appell sequences. *Fractal Fract.* **2023**, *7*, 233. [CrossRef]
27. Alazman, I.; Alkahtani, B.S.; Shahid, A.W. Certain properties of Δ_h multi–variate Hermite polynomials. *Symmetry* **2023**, *15*, 839. [CrossRef]