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Lie symmetries, invariant solutions, conservation laws and stability analysis of the Camassa-Holm-Nonlinear Schrödinger equation

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Abstract
In this work, we explore the defocusing nonlinear Schrödinger equation known as the Camassa–Holm–Nonlinear Schrödinger (CH–NLS) model, which is newly derived in the sense of deformation of hierarchies of integrable systems. This equation captures important features of nonlinear wave propagation in shallow water dynamics, plasma physics, and nonlinear optics. A comprehensive Lie symmetry analysis is carried out to determine the symmetry generators and to derive similarity reductions of the equation. These reductions enable us to construct several exact group-invariant solutions. In addition, we employ Ibragimov's conservation theorem to derive associated conservation laws based on the identified symmetries. The study also includes an investigation of modulational instability using the framework of linear stability analysis to assess the stability properties of continuous wave solutions. The results provide analytical insights into the interplay between symmetry, conservation, and stability in the CH–NLS model, thereby contributing to the broader understanding of nonlinear dispersive wave phenomena.

Keywords Lie symmetries · Similarity reductions · Conservation laws · Modulational instability

Mathematics Subject Classification 35Q55 · 35C05 · 35A30 · 35B35.

1 Introduction

The Camassa–Holm (CH) Eq. [1]

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$$q_t - q_{xxt} + 2kq_x = 2q_xq_{xx} - 3qq_x + q_{xxx},$$

is a well-known one-dimensional nonlinear wave equation that is completely integrable. It has attracted considerable attention due to its soliton solutions [2], its relevance to shallow water wave dynamics [3, 4], and notable mathematical features, including long-time asymptotic behavior [5] and an associated inverse scattering theory [6]. Another central model in nonlinear wave theory is the nonlinear Schrödinger (NLS) equation [7], given by

$$iq_t + q_{xx} + 2|q|^2q = 0,$$

which also belongs to the class of completely integrable equations. The NLS equation arises naturally in multiple physical contexts, , Bose–Einstein condensation [8], and nonlinear optics [9]. More recently, a deformed version of the NLS equation, referred to as the CH-NLS equation, was introduced in [11] within the framework of deformations of integrable hierarchies. This equation extends the structure of the NLS equation similarly to how the CH equation generalizes the KdV model. The standard form of the CH–NLS equation is written as:

$$im_t + q_{xx} + 2\sigma m(|q|^2 - a^2|q_x|^2) = 0, \qquad m = q - a^2q_{xx},$$

where $q(x,t)$ is a complex-valued function, $\sigma = \pm 1$, and a is a real parameter. When $a = 0$, this equation reduces to the standard NLS equation. An alternative representation involving only q is:

$$iq_t + q_{xx} + 2\sigma q|q|^2 - ia^2q_{xxt} - 2\sigma a^2q|q_x|^2 - 2\sigma a^2q_{xx}|q|^2 + 2\sigma a^4q_{xx}|q_x|^2 = 0. \quad (1)$$

Several studies have examined different aspects of the Eq. (1). According to [12], the dynamics and interaction properties of bright solitons were studied using both analytical techniques and numerical simulations. Reference [13] discusses a variety of exact wave solutions–including dark, periodic, and singular solitons–were constructed using methods such as the generalized G'/G-expansion method, a novel mapping approach, and the modified simple equation method. In another contribution [14], traveling wave transformations combined with the extended sinh-Gordon expansion method were employed to derive new explicit solutions.

In addition, a deformation of the derivative NLS equation–termed the CH-DNLS equation–was explored in [15]. Through asymptotic multiscale techniques, this model was reduced to the mKdV and KdV equations, yielding approximate soliton solutions, including both dark and anti-dark solitary waves. Furthermore, in reference [16], the (2+1)-dimension of Eq. (1) was considered and reduced via multiscale expansion to a Kadomtsev–Petviashvili (KP)-type equation, from which approximate solitary wave solutions were derived. Despite these significant findings, a investigation covering Lie symmetry analysis, group-invariant solutions, conservation laws, and stability properties of the Eq. (1) remains unexplored. This study is therefore devoted to addressing this gap through a comprehensive analytical framework targeting the mathematical structure and physical implications of the Eq. (1).

The study of the mathematical structures underlying nonlinear integer-order models can be naturally extended to fractional-order systems within the framework of fractional cal-

culus. Fractional derivatives provide a powerful means of capturing memory and hereditary effects in physical processes, thereby offering a deeper connection between nonlinear wave phenomena and special polynomials [17–22]. In this context, the investigation of the fractional model of the Eq. (1) represents a significant and meaningful direction for future research.

2 Lie symmetry analysis

This section is devoted to the Lie symmetry analysis of the Eq. (1) using the classical symmetry method [23–25]. Let the complex-valued function $q(x,t)$ be expressed in terms of its real and imaginary components as:

$$q(x,t) = u(x,t) + iv(x,t),$$

where $u(x,t)$ and $v(x,t)$ are real-valued functions. Substituting this form into Eq. (1) and separating into real and imaginary parts, we obtain the following system:

$$\begin{aligned} &u_{xx} - v_t + a^2 v_{xxt} + 2\left(u - a^2 u_{xx}\right)\left(u^2 + v^2 - a^2(u_x^2 + v_x^2)\right)\sigma = 0, \\ &v_{xx} + u_t - a^2 u_{xxt} + 2\left(u^2 + v^2 - a^2(u_x^2 + v_x^2)\right)\left(v - a^2 v_{xx}\right)\sigma = 0. \end{aligned} \tag{2}$$

To identify the point symmetries of this system, we introduce a one-parameter Lie group of infinitesimal transformations of the form:

$$\begin{aligned} t &\to t + \delta\tau(x,t,u,v) + \mathcal{O}(\delta^2), \\ x &\to x + \delta\xi(x,t,u,v) + \mathcal{O}(\delta^2), \\ u &\to u + \delta\eta_1(x,t,u,v) + \mathcal{O}(\delta^2), \\ v &\to v + \delta\eta_2(x,t,u,v) + \mathcal{O}(\delta^2), \end{aligned}$$

where δ is a small group parameter. The associated vector field is written as:

$$\mathfrak{V} = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \eta_1(x,t,u,v)\frac{\partial}{\partial u} + \eta_2(x,t,u,v)\frac{\partial}{\partial v}.$$

The prolonged vector field up to third-order derivatives, denoted by $\mathrm{pr}^{(3)}\mathfrak{V}$, takes the form:

$$pr^{(3)}\mathfrak{V} = \mathfrak{V} + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{tt}\frac{\partial}{\partial u_{tt}} + \zeta_{xt}\frac{\partial}{\partial u_{xt}} + \zeta_{xxxt}\frac{\partial}{\partial u_{xxxt}} + \cdots.$$

Where ζ_J are the extended infinitesimals involving partial derivatives such as u_t, u_x, u_{xx}, u_{xt}, and higher-order terms like u_{xxxt}. These are computed through the total derivative operators using:

$$\begin{aligned}
\zeta_t &= D_t(\eta_1) - u_x D_t(\xi) - u_t D_t(\tau),\\
\zeta_x &= D_x(\eta_1) - u_x D_x(\xi) - u_t D_x(\tau),\\
\zeta_{xt} &= D_t(\zeta_x) - u_{xx} D_t(\xi) - u_{xt} D_t(\tau),\\
\zeta_{xx} &= D_x(\zeta_x) - u_{xx} D_x(\xi) - u_{xt} D_x(\tau),\\
\zeta_{xxx} &= D_x(\zeta_{xx}) - u_{xxx} D_x(\xi) - u_{xxt} D_x(\tau),\\
\zeta_{xxxt} &= D_t(\zeta_{xxx}) - u_{xxxx} D_t(\xi) - u_{xxxt} D_t(\tau),
\end{aligned}$$

with D_x and D_t denoting the total derivatives with respect to x and t, respectively. These are formally given by:

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots \quad i, j = 1, 2, \quad (x^1, x^2) = (t, x).$$

Applying the third prolongation operator to the system (2) and enforcing the invariance condition, we have:

$$\mathrm{pr}^{(3)}\mathfrak{V}(\Lambda)|_{\Lambda_1=0} = 0, \qquad \mathrm{pr}^{(3)}\mathfrak{V}(\Lambda)|_{\Lambda_2=0} = 0, \tag{3}$$

where

$$\begin{aligned}
\Lambda_1 &= u_{xx} - v_t + a^2 v_{xxt} + 2\left(u - a^2 u_{xx}\right)\left(u^2 + v^2 - a^2(u_x^2 + v_x^2)\right)\sigma,\\
\Lambda_2 &= v_{xx} + u_t - a^2 u_{xxt} + 2\left(u^2 + v^2 - a^2(u_x^2 + v_x^2)\right)\left(v - a^2 v_{xx}\right)\sigma.
\end{aligned}$$

By substituting the expressions for Λ_1 and Λ_2 into Eq. (3) and equating the coefficients of all linearly independent derivatives to zero, we derive an overdetermined system of determining equations. Solving this system yields the infinitesimals:

$$\tau = c_1, \qquad \xi = c_2, \qquad \eta_1 = c_3 v, \qquad \eta_2 = -c_3 u,$$

where c_1, c_2, c_3 are constants. Hence, the infinitesimal symmetry algebra of the Eq. (1) is generated by the following vector fields:

$$V_1 = \frac{\partial}{\partial t}, \qquad V_2 = \frac{\partial}{\partial x}, \qquad V_3 = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}.$$

Having determined the infinitesimal generators of the governing equation, we now utilize these symmetries to perform the symmetry reduction. This reduction transforms the original partial differential equation into ordinary differential equations by introducing appropriate similarity variables. The reduced equations are then solved to obtain invariant and exact solutions that describe the essential dynamics of the system.

3 Symmetry reduction and invariant solutions

In this section, we derive invariant solutions of the Eq. (1) using classical symmetry reduction techniques. We begin with a reduction based on a translational symmetry in time.

3.1 Reduction via $V_1 = \dfrac{\partial}{\partial t}$

The characteristic system corresponding to the generator V_1 is given by

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0} = \frac{dv}{0},$$

which yields the invariants:

$$\zeta = x,$$

$$q(x,t) = F(\zeta)e^{iG(\zeta)}, \tag{4}$$

where F and G are real-valued functions depending on the invariant variable ζ. Substituting the ansatz (4) into Eq. (1) and separating real and imaginary components, we derive the following system of ordinary differential equations:

$$\begin{aligned} &F'' - 2a^2\sigma F^2F'' + 2a^4\sigma F^2(G')^2F'' + 2\sigma F^3 - F(G')^2 - 2a^2\sigma F(F')^2 + 2a^4\sigma(F')^2F'' \\ &-2a^4\sigma F^3(G')^4 = 0, \end{aligned} \tag{5}$$

$$\begin{aligned} &FG' - 2a^2\sigma F^3G' + 2a^4\sigma F^3(G')^2G' + 2FG' - 4a^2\sigma F^2FG' + 4a^4\sigma F^2F(G')^3 + \\ &4a^4\sigma(F')^3G + 2a^4\sigma F(F')^2G' = 0. \end{aligned} \tag{6}$$

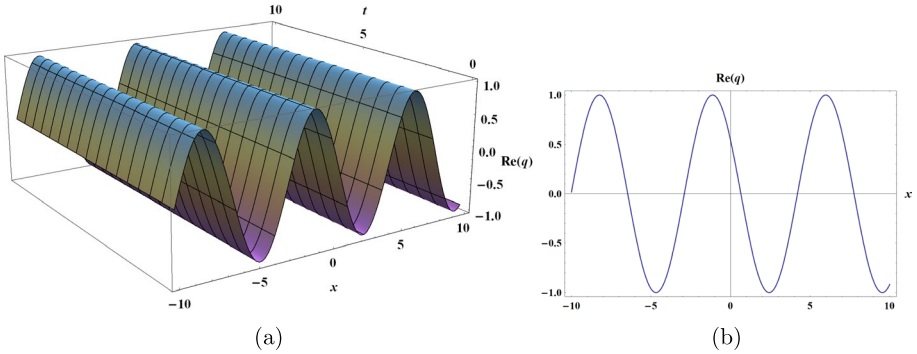
To obtain a particular solution, we assume $F(\zeta) = b$, where b is a constant. Substituting this assumption into equations (5) and (6) and solving the resulting system for $G(\zeta)$, we obtain

$$G(\zeta) = \pm\frac{1}{2}\zeta\sqrt{-\frac{1}{a^4b^2\sigma} - \frac{\sqrt{1+16a^4b^4\sigma^2}}{a^4b^2\sigma}} + c,$$

where c is an integration constant. Substituting back into (4), we get the exact invariant solution of the Eq. (1):

$$q(x,t) = b\,\exp\left(\pm i\left[\frac{1}{2}x\sqrt{-\frac{1}{a^4b^2\sigma} + \frac{\sqrt{1+16a^4b^4\sigma^2}}{a^4b^2\sigma}} + c\right]\right). \tag{7}$$

The dynamical behaviour of the plane wave solution given in Eq. (7) is illustrated in Fig. 1, which exhibits a constant amplitude and a linear phase variation characteristic of a continuous-wave profile.



(a) (b)

Fig. 1 Visualizations (3D and 2D) corresponding to the solution in Eq. (7) for parameter values $a = b = \sigma = c = 1$

3.2 Reduction via $V_2 = \dfrac{\partial}{\partial x}$

We begin by solving the characteristic system associated with the vector field:

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0} = \frac{dv}{0},$$

This implies the invariant quantities are:

$$\zeta = t,$$

$$q(x,t) = F(\zeta)e^{iG(\zeta)},$$

where ζ is the reduced independent variable and F, G are new dependent variables. Substituting this ansatz into Eq. (1) and separating into real and imaginary parts leads to the reduced ODEs:

$$F'(\zeta) = 0,$$

$$2\sigma F(\zeta)^3 - F(\zeta)G'(\zeta) = 0.$$

Solving these equations gives the constants:

$$F(\zeta) = b, \quad G(\zeta) = 2b^2\sigma\zeta + c,$$

where b and c are constants of integration. Consequently, the symmetry-invariant solution becomes:

$$q(x,t) = be^{i(2b^2\sigma t + c)}. \tag{8}$$

Fig. 2 illustrates the dynamical behaviour of the solution in Eq. (8), showing a constant-amplitude oscillation with a linearly varying phase over time.

3.3 Reduction via $V_1 + kV_2$

For this case, the characteristic system reads:

$$\frac{dt}{1} = \frac{dx}{k} = \frac{du}{0} = \frac{dv}{0},$$

leading to the invariant:

$$\zeta = x - kt,$$

and the group-invariant form of the solution:

$$q(x,t) = F(\zeta)e^{iG(\zeta)}.$$

Substitution into Eq. (1) and collecting terms by real and imaginary parts yields the following coupled system:

$$\begin{aligned}
&2\sigma F^3 - 2a^2\sigma F(F')^2 + kFG - F(G')^2 - 2a^4\sigma F(F')^2(G')^2 \\
&+ a^2kF(G')^3 - 2a^4\sigma F^3(G')^4 + F'' - 2a^2\sigma F^2F' \\
&+ 2a^4\sigma(F')^2F' - 3a^2kGF'' + 2a^4\sigma F^2(G')^2F' - 3a^2kFG'' - a^2kFG'' = 0.
\end{aligned}$$

$$\begin{aligned}
&kF + 2FG' - 4a^2\sigma F^2FG' + 4a^4\sigma(F')^3G - 3a^2kF(G')^2 \\
&+ 4a^4\sigma F^2F(G')^3 + FG' - 2a^2\sigma F^3G' + 2a^4\sigma F(F')^2G' \\
&- 3a^2kF(G')G'' + 2a^4\sigma F^3(G')^2G' + a^2kF'' = 0.
\end{aligned}$$

Choosing a constant profile $F(\zeta) = b$, the system simplifies and yields:

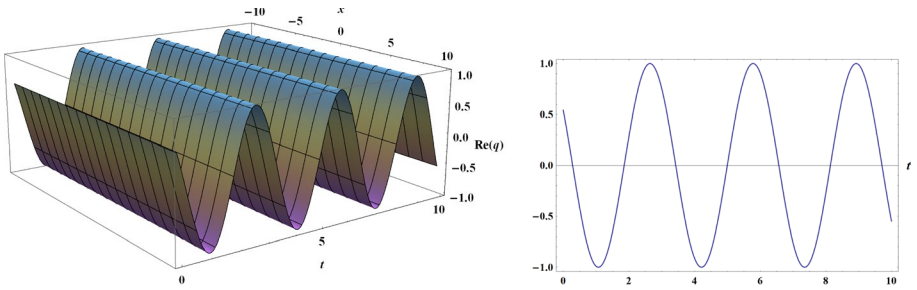


Fig. 2 Visualizations (3D and 2D) corresponding to the solution in Eq. (8) with parameter values $b = \sigma = c = 1$

$$G(\zeta) = \frac{\zeta}{4a^4b^2\sigma}\left(3a^2k \pm a^2\sqrt{9k^2 - 8b^2\sigma + 16a^2b^4\sigma^2}\right) + c,$$

with c being arbitrary. Hence, the explicit invariant solution becomes:

$$q(x,t) = b\exp\left(i\left[\frac{(x-kt)(3a^2k \pm a^2\sqrt{9k^2 - 8b^2\sigma + 16a^2b^4\sigma^2})}{4a^4b^2\sigma} + c\right]\right). \tag{9}$$

Here, k is defined as

$$k = \frac{1}{3}\left[\frac{1 - a^2b^2\sigma + 8a^4b^4\sigma^2 - 32a^6b^6\sigma^3 + \sqrt{(1 - 3a^2b^2\sigma + 8a^4b^4\sigma^2)^2(1 + 4a^2b^2\sigma + 16a^4b^4\sigma^2)}}{a^2 + 4a^4b^2\sigma}\right]^{1/2}.$$

The dynamical behaviour of the solution in Eq. (9) is illustrated in Fig. 3, showing the propagation of a plane wave whose phase varies with both space and time, reflecting a complex oscillatory motion governed by the system parameters.

3.4 Reduction via $V_1 + \gamma V_3$

Solving the corresponding characteristic system,

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{\gamma v} = \frac{dv}{-\gamma u},$$

we derive the invariant quantities as

$$\zeta = x,$$

$$q(x,t) = F(\zeta)\exp\left[i(\gamma t + G(\zeta))\right], \tag{10}$$

where F and G are functions of ζ, which is the new independent variable. Substituting expression (10) into Eq. (1) and splitting into real and imaginary components leads to the reduced system:

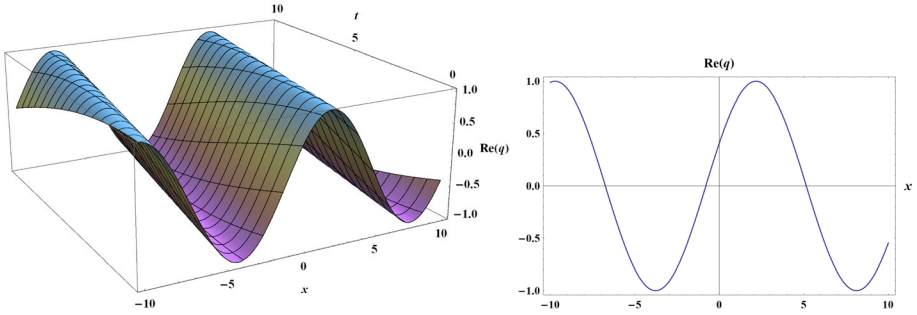


Fig. 3 Visualizations (3D and 2D) corresponding to the solution in Eq. (9) under $a = b = \sigma = c = 1$

$$-\gamma F + 2\sigma F^3 - 2a^2\sigma F(F')^2 - F(G')^2 - a^2\gamma F(G')^2 - 2a^4\sigma F(F')^2(G')^2 - 2a^4\sigma F^3(G')^4 + (1+a^2\gamma)F'' - 2a^2\sigma F^2F' + 2a^4\sigma(F')^2F' + 2a^4\sigma F^2(G')^2F' = 0.$$

$$2(1+a^2\gamma)F'G' - 4a^2\sigma F^2FG' + 4a^4\sigma(F')^3G + 4a^4\sigma F^2F(G')^3 + (1+a^2\gamma)FG' - 2a^2\sigma F^3G' + 2a^4\sigma F(F')^2G' + 2a^4\sigma F^3(G')^2G' = 0.$$

Taking a constant amplitude $F(\zeta) = b$ and solving the resulting equations yields

$$G(\zeta) = \pm\zeta\sqrt{-\frac{1}{4a^4b^2\sigma} - \frac{\gamma}{4a^2b^2\sigma} + \frac{\sqrt{(1+a^2\gamma)^2 - 8a^4b^2\sigma(\gamma - 2b^2\sigma)}}{4a^4b^2\sigma}} + c,$$

where c is a constant of integration. Substituting back gives the invariant solution:

$$q(x,t) = b\exp\left[i\left(\gamma t \pm x\sqrt{-\frac{1}{4a^4b^2\sigma} - \frac{\gamma}{4a^2b^2\sigma} + \frac{\sqrt{(1+a^2\gamma)^2 - 8a^4b^2\sigma(\gamma - 2b^2\sigma)}}{4a^4b^2\sigma}} + c\right)\right]. \quad (11)$$

Figure 4 depicts the real part of the solution in Eq. (11), showing the evolution of a plane wave whose phase varies in both space and time under the influence of system parameters.

3.5 Reduction via $V_2 + \gamma V_3$

Considering the vector field combination, we solve the associated system:

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{\gamma v} = \frac{dv}{-\gamma u},$$

leading to invariants

$$\zeta = t,$$

$$q(x,t) = F(\zeta)\exp\left[i(\gamma x + G(\zeta))\right].$$

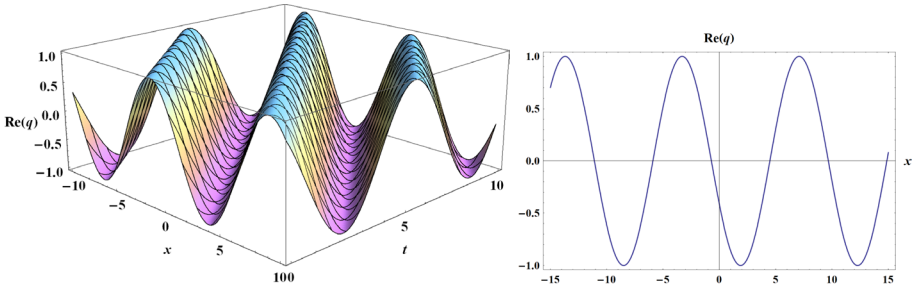


Fig. 4 Visualizations (3D and 2D) corresponding to the solution in Eq. (11) for $a = b = \sigma = c = \gamma = 1$

Substituting into Eq. (1) and splitting, we arrive at:

$$(1 + a^2\gamma^2)F'(\zeta) = 0,$$

$$F(\zeta)\left[\gamma^2 + (1 + a^2\gamma^2)G'(\zeta) - 2\sigma F(\zeta)^2(1 - a^4\gamma^4)\right] = 0.$$

This yields

$$F(\zeta) = b,$$

$$G(\zeta) = \frac{\zeta\left(-\gamma^2 + 2b^2\sigma - 2a^4b^2\gamma^4\sigma\right)}{1 + a^2\gamma^2} + c,$$

with constants b, c. The reduced solution reads:

$$q(x,t) = b\exp\left[i\left(\gamma x + \frac{t(-\gamma^2 + 2b^2\sigma - 2a^4b^2\gamma^4\sigma)}{1 + a^2\gamma^2} + c\right)\right]. \tag{12}$$

The dynamical behaviour of the plane wave solution given in Eq. (12) is illustrated in Fig. 5, where the real part of $q(x, t)$ is plotted.

3.6 Similarity reduction by $V_1 + \mu V_2 + \gamma V_3$

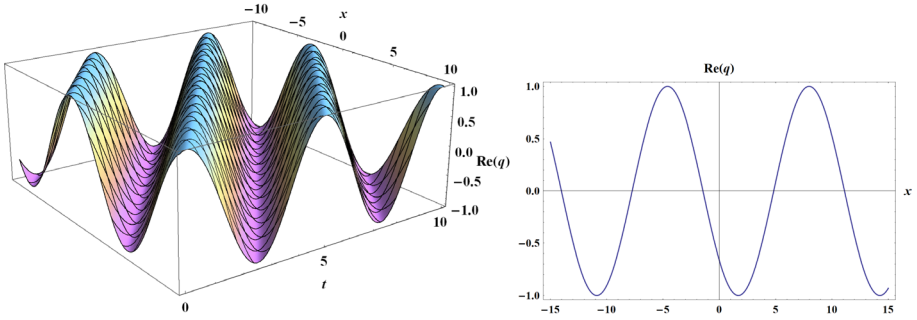
To carry out the similarity reduction, we consider the associated characteristic equation:

$$\frac{dt}{1} = \frac{dx}{\mu} = \frac{du}{\gamma v} = \frac{dv}{-\gamma u},$$

Solving this system yields a new invariant variable:

$$\zeta = x - \mu t,$$

and the following transformation for the complex function:



(a) (b)

Fig. 5 Visualizations (3D and 2D) corresponding to the solution in Eq. (12) for $a = b = \sigma = c = \gamma = 0.5$

$$q(x,t) = F(\zeta)\, e^{-i\left(\frac{\gamma}{\mu}x + G(\zeta)\right)}, \tag{13}$$

where F and G are unknown functions of ζ. Substituting Eq. (13) into the governing Eq. (1) and splitting into real and imaginary parts, we derive a system of reduced ordinary differential equations:

$$\begin{aligned}
&-2a^4\gamma^4\sigma F^3 + 2\mu^4\sigma F^3 - 2a^4\gamma^2\mu^2\sigma F(F')^2 - 2a^2\mu^4\sigma F(F')^2 - 8a^4\gamma^3\mu\sigma F^3 G\\
&-2\gamma\mu^3 FG - \gamma^2\mu^2 F + a^2\gamma^2\mu^3 FG - 4a^4\gamma\mu^3\sigma F(F')^2 G - \mu^4 F(G')^2\\
&-12a^4\gamma^2\mu^2\sigma F^3(G')^2 - 2a^4\mu^4\sigma F(F')^2(G')^2 + a^2\mu^5 F(G')^3 - 8a^4\gamma\mu^3\sigma F^3(G')^3\\
&-2a^4\mu^4\sigma F^3(G')^4 + \mu^4 F' - 2a^2\gamma\mu^4 F' + 2a^4\gamma^2\mu^2\sigma F^2 F' - 2a^2\mu^4\sigma F^2 F'\\
&+2a^4\mu^4\sigma(F')^2 F' - 3a^2\mu^5 GF'' + 4a^4\gamma\mu^3\sigma F^2 GF'' + 2a^4\mu^4\sigma F^2(G')^2 F'\\
&-3a^2\mu^5 FG'' - a^2\mu^5 FG^{(3)} + \mu^5 FG = 0,
\end{aligned}$$

$$\begin{aligned}
&2\gamma\mu^3 F - a^2\gamma^2\mu^3 F - \mu^5 F + 4a^4\gamma^3\mu\sigma F^2 F - 4a^2\gamma\mu^3\sigma F^2 F + 4a^4\gamma\mu^3\sigma(F')^3\\
&+2\mu^4 FG' - 4a^2\gamma\mu^4 FG' + 12a^4\gamma^2\mu^2\sigma F^2 FG' - 4a^2\mu^4\sigma F^2 FG'\\
&+4a^4\mu^4\sigma(F')^3 G - 3a^2\mu^5 F(G')^2 + 12a^4\gamma\mu^3\sigma F^2 F(G')^2 - 2a^2\mu^4\sigma F^3 G'\\
&+\mu^4 FG' - 2a^2\gamma\mu^4 FG' + 2a^4\gamma^2\mu^2\sigma F^3 G' + 4a^4\mu^4\sigma F^2 F(G')^3\\
&+2a^4\mu^4\sigma F(F')^2 G' - 3a^2\mu^5 FGG'' + 4a^4\gamma\mu^3\sigma F^3 GG'' + 2a^4\mu^4\sigma F^3(G')^2 G'\\
&+a^2\mu^5 F^{(3)} = 0.
\end{aligned}$$

Now, assuming the amplitude function to be a constant, i.e., $F(\zeta) = b$, we solve the above equations for $G(\zeta)$ to get:

$$G(\zeta) = \pm\frac{\zeta\left(-4a^2b^2\gamma\sigma + \mu\left(3\mu + \sqrt{9\mu^2 + 8b^2\sigma\left(-1 - a^2(\gamma - 2b^2\sigma)\right)}\right)\right)}{4a^2b^2\mu\sigma} + c,$$

where c denotes a constant of integration. Substituting the expressions of $F(\zeta)$ and $G(\zeta)$ back into Eq. (13), we derive the corresponding invariant solution of Eq. (1):

$$q(x,t) = b\,\exp\left[i\left(\frac{x\gamma}{\mu} \pm \frac{(x-\mu t)\left(-4a^2b^2\gamma\sigma + \mu\left(3\mu + \sqrt{9\mu^2 + 8b^2\sigma\left(-1 - a^2(\gamma - 2b^2\sigma)\right)}\right)\right)}{4a^2b^2\mu\sigma} + c\right)\right]. \tag{14}$$

Here, the parameter μ is constrained by:

$$\mu = \frac{1}{3}\left[\frac{-1 + \sqrt{\beta} - a^6(\gamma - 4b^2\sigma)^2(\gamma - 2b^2\sigma) - a^4(3\gamma - 8b^2\sigma)(\gamma - b^2\sigma) + a^2(-3\gamma + b^2\sigma)}{a^2(-1 + 2a^2(\gamma - 2b^2\sigma))}\right]^{1/2},$$

and

$$\beta = \left(1 + a^2\left(2\gamma - 3b^2\sigma\right) + a^4\left(\gamma - 4b^2\sigma\right)\left(\gamma - 2b^2\sigma\right)\right)^2\left(1 + a^4\left(\gamma - 4b^2\sigma\right)^2 + 2a^2\left(\gamma + 2b^2\sigma\right)\right).$$

The dynamical behaviour of the plane wave solution given in Eq. (14) is illustrated in Fig. 6, where the real part of $q(x, t)$ is plotted.

Having established a family of analytical solutions, it is essential to examine the conserved quantities that govern their evolution. The next section focuses on constructing the conservation laws associated with the equation, which play a crucial role in confirming the integrability and physical consistency of the model.

4 Conservation laws for the NLS–CH equation

This section presents the derivation of conservation laws for the Eq. (1) using Ibragimov's novel conservation theorem [26–28]. According to this approach, we start by formulating the formal Lagrangian associated with the Eq. (1), given as:

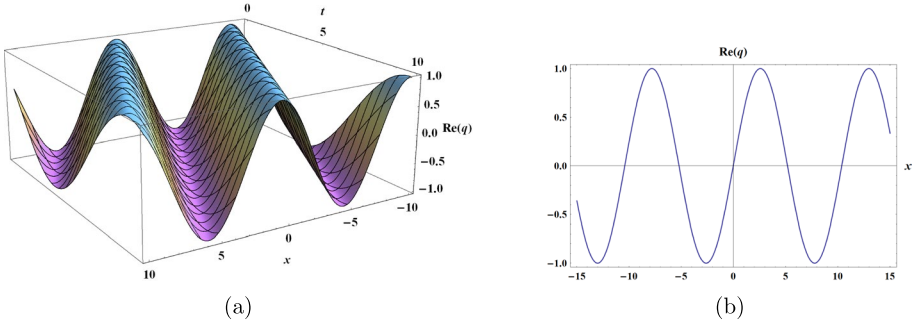
$$\begin{aligned} L = p\left(u_{xx} - v_t + a^2 v_{xxt} + 2\sigma(u - a^2 u_{xx})(u^2 + v^2 - a^2(u_x^2 + v_x^2))\right) \\ + q\left(v_{xx} + u_t - a^2 u_{xxt} + 2\sigma(u^2 + v^2 - a^2(u_x^2 + v_x^2))(v - a^2 v_{xx})\right), \end{aligned} \tag{15}$$

where $p(t, x)$ and $q(t, x)$ are auxiliary dependent variables. The corresponding adjoint equations are obtained through variational derivatives:

$$F^* = \frac{\delta L}{\delta u}, \quad G^* = \frac{\delta L}{\delta v}. \tag{16}$$

Using (16) and the definition of the Lagrangian in (15), we compute:

$$\begin{aligned} F^* = {} & 6pu^2\sigma + p_{xx} - q_t + a^2 q_{txx} \\ & + 2\sigma\left[v(2qu + pv) - a^2\left((u^2 + v^2)p_{xx}\right.\right. \\ & -2vq_x u_x + pu_x^2 + 2puu_{xx} - 2qvu_{xx} - 2qu_x v_x + 3pv_x^2 + 2p_x(uu_x + 2vv_x) + 2(qu + pv)v_{xx}\Big) \\ & + a^4\left(p_{xx}(u_x^2 + v_x^2) + 2\left(v_{xx}(-q_x u_x - qu_{xx} + pv_{xx})\right.\right. \\ & \left.\left.\left.+p_x(u_x u_{xx} + 2v_x v_{xx}) + (-qu_x + pv_x)v_{xxx}\right)\right)\right]. \end{aligned}$$



(a) (b)

Fig. 6 Visualizations (3D and 2D) corresponding to the solution in Eq. (14) for parameters $a = 2$, $b = 1$, $\sigma = -1$, $c = 1$, and $\gamma = 0.5$

$$
\begin{aligned}
G^* = & 2qu^2\sigma + 4puv\sigma + 6qv^2\sigma + p_t + q_{xx} \\
& - a^2\left[p_{txx} + 2\sigma\left((u^2+v^2)q_{xx} + 3qu_x^2 + 2quu_{xx} + 2pvu_{xx}\right.\right. \\
& \left.\left. -2up_xv_x - 2pu_xv_x + qv_x^2 + 2q_x(2uu_x + vv_x) - 2puv_{xx} + 2qvv_{xx}\right)\right] \\
& + 2a^4\sigma\left[q_{xx}(u_x^2+v_x^2) + 2\left(qu_{xx}^2 + u_{xxx}(qu_x - pv_x)\right.\right. \\
& \left.\left. -u_{xx}(p_xv_x + pv_{xx}) + q_x(2u_xu_{xx} + v_xv_{xx})\right)\right].
\end{aligned}
$$

The variational (Euler–Lagrange) operators used are:

$$
\begin{aligned}
\frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u_x} - D_t\frac{\partial}{\partial u_t} + D_x^2\frac{\partial}{\partial u_{xx}} + D_xD_t\frac{\partial}{\partial u_{xt}} + \cdots, \\
\frac{\delta}{\delta v} &= \frac{\partial}{\partial v} - D_x\frac{\partial}{\partial v_x} - D_t\frac{\partial}{\partial v_t} + D_x^2\frac{\partial}{\partial v_{xx}} + D_xD_t\frac{\partial}{\partial v_{xt}} + \cdots,
\end{aligned}
$$

where D_x and D_t are total derivative operators. The Eq. (1) admits three independent one-dimensional symmetry subalgebras. From these, conserved vectors are derived via the conservation theorem:

$$
\begin{aligned}
T^r = & \xi^r L + W_\phi\left[\frac{\partial L}{\partial u^r} - D_s\left(\frac{\partial L}{\partial u_s^r}\right) + D_sD_m\left(\frac{\partial L}{\partial u_{sm}^r}\right) - \cdots\right] \\
& + D_s(W_\phi)\left[\frac{\partial L}{\partial u_s^r} - D_k\left(\frac{\partial L}{\partial u_{sm}^r}\right) + \cdots\right] + \cdots
\end{aligned}
$$

The Lie characteristic function is:

$$
W_\phi = \eta - \xi^x u_x - \xi^t u_t
$$

These conserved vectors satisfy the continuity equation:

$$
D_x(T_j^x) + D_t(T_j^t) = 0, \quad j = 1, 2, 3
$$

We now present the conserved quantities corresponding to each symmetry:

Case 1: For the generator $V_1 = \partial_t$, we obtain:

$$
W_u = -u_t, \quad W_v = -v_t.
$$

Substituting into the conserved vector formula yields:

$$
\begin{aligned}
T_1^t = & a^2q_{xx}u_t - a^2q_xu_{tx} \\
& + pu_{xx} - a^2p_{xx}v_t + a^2p_xv_{tx} + qv_{xx} \\
& + 2\sigma\left(u^2 + v^2 - a^2\left(u_x^2 + v_x^2\right)\right)\left(pu + qv - a^2\left(pu_{xx} + qv_{xx}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
T_1^x = & -pu_{tx} + q_x v_t - qv_{tx} + a^2 \left(q_{tx}u_t - q_x u_{tt} + qu_{ttx} - q_t u_{tx} - p_{tx}v_t - pv_{ttx} + p_t v_{tx}\right. \\
& + 2\sigma \left(2qvu_t u_x - \left(\left(u^2+v^2\right)q_x + 2quu_x\right)v_t + q\left(u^2+v^2\right)v_{tx}\right. \\
& \left. +p\left(\left(u^2+v^2\right)u_{tx} + 2\left(-vu_t + uv_t\right)v_x\right)\right) \\
& + p_x\left(a^2 v_{tt} + u_t\left(1 - 2a^2\left(u^2+v^2\right)\sigma + 2a^4\sigma\left(u_x^2+v_x^2\right)\right)\right) \\
& \left. - 2a^4\sigma\left(-2qu_x u_{xx}v_t + qu_x^2 v_{tx} + 2pu_{xx}v_t v_x + qv_{tx}v_x^2 + pu_{tx}\left(u_x^2+v_x^2\right) - q_x v_t\left(u_x^2+v_x^2\right) + 2u_t\left(qu_x - pv_x\right)v_{xx}\right)\right.
\end{aligned}
$$

Case 2: For the generator $V_2 = \partial_x$, we obtain:

$$W_u = -u_x, \quad W_v = -v_x.$$

Substituting into the formula, we get:

$$T_1^t = \left(-q + a^2 q_{xx}\right)u_x + pv_x - a^2\left(q_x u_{xx} - qu_{xxx} + p_{xx}v_x - p_x v_{xx} + pv_{xxx}\right)$$

$$
\begin{aligned}
T_1^x = & 2pu^3\sigma + 2puv^2\sigma + p_x u_x - pv_t + q_x v_x \\
& + a^2\left(-q_x u_{tx} + q_{tx}u_x - q_t u_{xx} + p_x v_{tx} - p_{tx}v_x - 2\sigma\left(\left(u^2+v^2\right)p_x u_x\right.\right. \\
& \left. +puu_x^2 + \left(\left(u^2+v^2\right)q_x + 2pvu_x\right)v_x - puv_x^2\right) + p_t v_{xx}\left.\right) + 2a^4\sigma\left(p_x u_x\left(u_x^2+v_x^2\right)\right. \\
& \left. +v_x\left(-2pu_{xx}v_x + q_x\left(u_x^2+v_x^2\right) + 2pu_x v_{xx}\right)\right) \\
& \left. + q\left(u_t + 2\sigma\left(v\left(u^2+v^2\right) + a^2\left(-2u_x\left(u - a^2 u_{xx}\right)v_x - vv_x^2 + u_x^2\left(v - 2a^2 v_{xx}\right)\right)\right)\right)\right)
\end{aligned}
$$

Case 3: For the generator $V_3 = v\partial_u - u\partial_v$, the characteristic functions become:

$$W_u = v, \quad W_v = -u.$$

Substitution yields the corresponding conserved vector:

$$T_1^t = pu + qv - a^2\left(up_{xx} + vq_{xx} - p_x u_x + pu_{xx} - q_x v_x + qv_{xx}\right)$$

$$
\begin{aligned}
T_1^x = & -vp_x + uq_x - qu_x + pv_x \\
& + a^2\left(-up_{tx} - vq_{tx} + p_x u_t - pu_{tx} + p_t u_x + q_x v_t - qv_{tx} + q_t v_x + 2\left(u^2+v^2\right)\sigma\left(vp_x - uq_x - qu_x + pv_x\right)\right) \\
& + 2a^4\sigma\left(-vp_x\left(u_x^2+v_x^2\right) + uq_x\left(u_x^2+v_x^2\right) - \left(qu_x - pv_x\right)\left(u_x^2 - 2uu_{xx} + v_x^2 - 2vv_{xx}\right)\right)
\end{aligned}
$$

The conservation laws confirm the integrability and energy balance of the system. To further assess the persistence and physical relevance of the derived soliton solutions, we now perform a detailed stability analysis with respect to small perturbations.

5 Linear stability investigation

To analyze the modulational stability, we employ a standard linear stability approach [29–31]. Let us consider a continuous wave solution to Eq. (1) in the form:

$$q = q_0 \exp\left[i(\kappa x + \omega t)\right], \tag{17}$$

where q_0 is the amplitude of the unperturbed field, κ is the spatial wave number, and ω denotes the temporal frequency. Substituting Eq. (17) into Eq. (1) yields the dispersion relation

$$\omega = \frac{\kappa^2 - 2\sigma q_0^2 + 2a^4\kappa^4\sigma q_0^2}{1 + a^2\kappa^2}.$$

Next, we examine a slightly perturbed solution of the form:

$$q = [q_0 + \theta\phi(x,t)] \exp[i(\kappa x + \omega t)], \tag{18}$$

where $\theta \ll 1$ is a small parameter. Inserting Eq. (18) into Eq. (1) and neglecting higher-order terms in θ, the linearized perturbation equation becomes:

$$\begin{aligned} &- ib^2\phi_t - 2i\kappa\phi_x + 2b^2\sigma q_0^2\left[c(\phi + \phi^*) - ia^2\kappa(\phi_x - \phi_x^*)\right] \\ &- 2a^2b\kappa\phi_{xt} - \phi_{xx} + ia^2b\phi_{xxt} = 0, \end{aligned} \tag{19}$$

with $b = 1 + a^2\kappa^2$ and $c = -1 + a^2\kappa^2$.

We now assume the perturbation $\phi(x,t)$ can be expressed as a combination of Fourier modes:

$$\phi(x,t) = q_1 \exp[i(\hat{\kappa}x - \hat{\omega}t)] + q_2 \exp[-i(\hat{\kappa}x - \hat{\omega}t)], \tag{20}$$

where q_1 and q_2 are complex amplitudes, and $\hat{\kappa}$, $\hat{\omega}$ represent the perturbation wave number and frequency, respectively. Substituting Eq. (20) into Eq. (19) leads to the following system:

$$\begin{aligned} &\left[-b^2\hat{\omega} + \hat{\kappa}(2\kappa + \hat{\kappa})(1 - a^2b\hat{\omega})\right] q_1 + 2b^2\sigma q_0^2\left[(c + a^2\kappa\hat{\kappa})q_1 + (c - a^2\kappa\hat{\kappa})q_2\right] = 0, \\ &\left[b^2\hat{\omega} + \hat{\kappa}(-2\kappa + \hat{\kappa})(1 + a^2b\hat{\omega})\right] q_2 + 2b^2\sigma q_0^2\left[(c + a^2\kappa\hat{\kappa})q_1 + (c - a^2\kappa\hat{\kappa})q_2\right] = 0. \end{aligned}$$

This homogeneous linear system has nontrivial solutions if and only if its determinant vanishes:

$$\begin{vmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{vmatrix} = 0,$$

where

$$\begin{aligned} \psi_{11} &= -b^2\hat{\omega} - \hat{\kappa}(2\kappa + \hat{\kappa})(-1 + a^2b\hat{\omega}) + 2b^2\sigma(c + a^2\kappa\hat{\kappa})q_0^2, \\ \psi_{12} &= 2b^2\sigma(c - a^2\kappa\hat{\kappa})q_0^2, \\ \psi_{21} &= 2b^2\sigma(c + a^2\kappa\hat{\kappa})q_0^2, \\ \psi_{22} &= b^2\hat{\omega} - \hat{\kappa}(2\kappa - \hat{\kappa})(1 + a^2b\hat{\omega}) + 2b^2\sigma(c - a^2\kappa\hat{\kappa})q_0^2. \end{aligned}$$

Solving the determinant leads to the following dispersion relation:

$$\hat{\omega} = \frac{2b^2\kappa\hat{\kappa}\left(1 + a^2b\sigma(b - 2c + a^2\hat{\kappa}^2)q_0^2\right) \pm \sqrt{\Delta}}{b^2\left(b^2 + 2a^2(b - 2a^2\kappa^2)\hat{\kappa}^2 + a^4\hat{\kappa}^4\right)},$$

where the discriminant Δ is given by

$$\Delta = b^2\hat{\kappa}^2\left[\hat{\kappa}^2(b - 4a^2\kappa^2 + a^2\hat{\kappa}^2)^2 + 4b^2\sigma(b - 4a^2\kappa^2 + a^2\hat{\kappa}^2)\right.$$
$$\left.\times(bc + a^2(c - 2a^2\kappa^2)\hat{\kappa}^2)q_0^2\right] + 4a^4b^6\hat{\kappa}^2\kappa^2\sigma^2(b - 2c + a^2\hat{\kappa}^2)^2q_0^4.$$

If $\Delta \geq 0$, then $\hat{\omega}$ is real and the CW solution remains modulationally stable. On the other hand, if $\Delta < 0$, $\hat{\omega}$ becomes complex and the solution is unstable. The modulation instability (MI) gain is defined by:

$$G(\kappa) = 2\,\mathrm{Im}(\hat{\omega}) = \mathrm{Im}\left[\frac{2b^2\kappa\hat{\kappa}\left(1 + a^2b\sigma(b - 2c + a^2\hat{\kappa}^2)q_0^2\right) + \sqrt{\Delta}}{b^2\left(b^2 + 2a^2(b - 2a^2\kappa^2)\hat{\kappa}^2 + a^4\hat{\kappa}^4\right)}\right].$$

Figures 7 and 8 illustrate the impact of various parameters such as input power q_0, wave number κ, and constants a and σ on the MI gain spectrum.

6 Conclusions

This work investigates the recently introduced integrable model known as the Camassa-Holm-Nonlinear Schrödinger (CH-NLS) equation. For the first time, a comprehensive analysis involving Lie symmetries, invariant solutions, conservation laws, and linear stability has been carried out for this equation. Using Lie symmetry techniques, we determined the infinitesimal generators and carried out symmetry reductions, which allowed the CH-NLS equation to be transformed into a reduced nonlinear ordinary differential equation. Group-invariant solutions corresponding to these symmetries were also constructed. Moreover, the conservation laws were derived by employing Ibragimov's general conservation theorem in conjunction with the classical symmetries of the system. Linear stability was examined through modulation instability (MI) analysis applied to the exact solutions, and

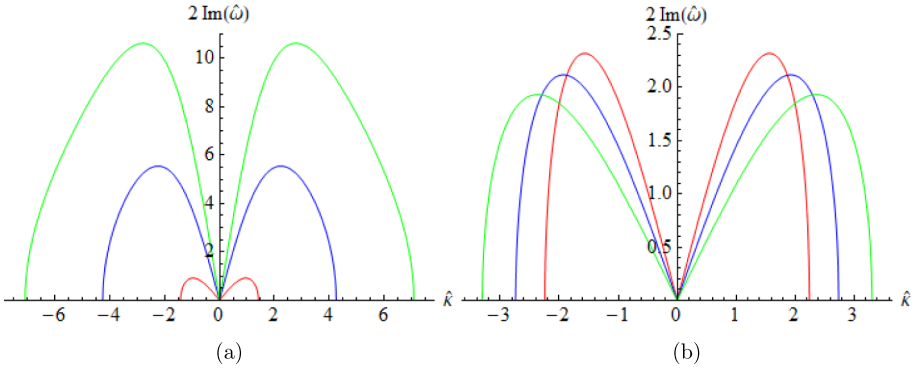


Fig. 7 Modulation instability gain for **a** $\kappa = 0.1,\ a = 0.3,\ \sigma = 0.5$ with $q_0 = 1,\ 3,\ 5$ (bottom to top) and **b** $q_0 = 2,\ a = 0.1,\ \sigma = 0.3$ with $\kappa = 1,\ 3,\ 4$ (top to bottom)

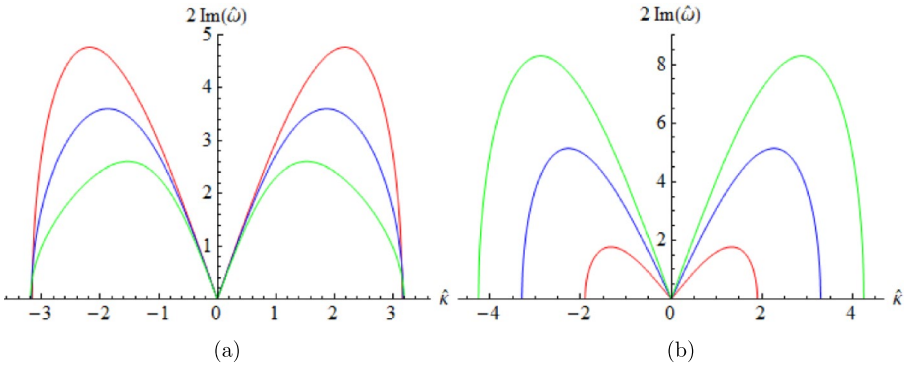


Fig. 8 Gain profile for **a** $q_0 = 5,\ \kappa = 0.2,\ \sigma = 0.1$ and $a = 0.1,\ 0.3,\ 0.5$ (top to bottom); **b** $q_0 = 3,\ \kappa = 0.2,\ a = 0.1$ with $\sigma = 0.1,\ 0.3,\ 0.5$ (bottom to top)

the corresponding MI gain spectrum was calculated. These findings enrich the theoretical understanding of the CH-NLS equation and may serve as a foundation for future studies in nonlinear wave dynamics and soliton theory.

The analytical and symmetry based approaches presented in this work have significant potential for broader applications. The derived invariant solutions and conservation laws can be employed to model wave propagation phenomena in shallow water dynamics, nonlinear optics, and plasma physics. Furthermore, the stability results provide valuable insights into the persistence and robustness of nonlinear wave structures in realistic physical environments. Future research may focus on extending the present methods to fractional-order generalizations of CH–NLS systems. In addition, incorporating numerical simulations to validate the analytical predictions, and exploring chaotic and bifurcation behaviors under varying physical parameters, would further enhance the understanding of such nonlinear dispersive systems.

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Data availability Not applicable.

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