MODERN TRENDS IN SPECIAL FUNCTIONS, FOURIER ANALYSIS AND APPLICATIONS

Editors

Prof. Dr. Clemente CESARANO

Prof. Dr. William RAMIREZ

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PREFACE

This book brings together a series of recent developments in nonlinear differential equations, variational analysis, and the theory of special polynomials, forming a coherent contribution to fields that, despite their distinct origins, intersect through operator methods, generating functions, and functional analytic techniques. We begin with the study of Kirchhoff-type problems and elliptic equations involving critical exponents, employing the Nehari manifold and Lusternik–Schnirelmann category theory to establish existence, multiplicity, and concentration results for positive solutions. These initial chapters, grounded in Sobolev theory and variational methods, lay the foundation for understanding the analytical complexity of models with potential wells, weighted operators, and asymptotic behaviors in critical regimes.

The focus then shifts toward special polynomial families, particularly those arising from the monomiality principle and operational calculus. We investigate matrix extensions of Hermite-type polynomials, along with new U-Bernoulli, U-Euler, and U-Genocchi families. Their periodic functions and Fourier expansions are studied in depth, revealing connections with the Riemann zeta function and with significant combinatorial structures. Here, operators, algebraic frameworks, and generating functions play a central role in deriving identities, recurrence relations, and integral representations with wide applicability.

The final chapters introduce innovative discrete polynomial families—such as the new U–Charlier–Poisson and generalized U–Bernoulli–Korobov types—where properties of orthogonality, three-term recurrence relations, and differential/difference structures are established. Moreover, Szász-type and Brenke-type operators associated with these families are constructed to obtain approximation and convergence results, bridging polynomial theory with numerical analysis and classical positive operator theory.

Overall, this book offers a unified journey through several areas of contemporary mathematics, motivated by the interplay between analysis, combinatorics, special functions, and operator theory. Its aim is to provide the reader with a modern and coherent perspective on these topics, while presenting new theoretical contributions that we hope will serve researchers and advanced students engaged in these fields.

ABOUT THE AUTHORS

Clemente Cesarano is a Professor of Numerical Analysis at the International Telematic University UNINETTUNO, where he coordinates the local Mathematics Section and the PhD Program in Engineering of Technological Innovation.

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Professor Cesarano is the author of numerous scientific publications, textbooks, and monographs in the fields of approximation theory and mathematical analysis. He has participated—also as principal investigator or coordinator—in various nationally and internationally funded research projects.

At UNINETTUNO University, he also oversees the academic coordination of interdisciplinary programs for both undergraduate and postgraduate education and serves as Director of the Master's Program in Project Management.

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Multiplicity results for a problem of $2-q$ Laplacian operator

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Abstract

This work investigates the existence and multiplicity of positive solutions for a class of Kirchhoff–type Laplacian problems. By applying the Nehari manifold approach together with the Lusternik–Schnirelmann category theory, we establish that the problem possesses at least $cat(\Omega)$ distinct positive solutions.

Keywords: positive solutions; Nehari Manifold; critical points; Ljusternik Schnirelmann category.
2020 MSC: 35J60, 58E05, 35J47, 35B09.

1. Introduction

Here, we would like to consider the multiplicity of positive solutions for the Kirchhoff Laplacian type problem:

$$-\Delta u - \Delta_q u + \lambda V(x)u = \mu|u|^{r-2}u + |u|^{2^*-2}u, \qquad x \in \mathbb{R}^N, \tag{1}$$

where $N \geq 4$, $1 < q < \frac{N}{N-1}$, $max\{2, 2^* - q\} < r < 2^*$, $V : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous function, there is some $M_0 > 0$ in which

$$meas\{x \in \mathbb{R}^N \quad V(x) \leq M_0\} < +\infty \tag{2}$$

and

$$\Omega := int(V^{-1}(\{0\}) \tag{3}$$

which is a non-empty bounded open set with smooth boundery.

For litrary, let us give a brief story: In special case, if $q = 2$ the problem (1) is reduces to a simpler case that recently studied by Alves and Barros [4]

$$-\Delta u + \lambda V(x)u = \mu u^{r-1} + u^{2^*-1}, \qquad x \in \mathbb{R}^n. \tag{4}$$

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In fact, using variational method and the Lusternik-Schnirelman category they showed existence of λ^* and μ^* positive such that for $\lambda \geq \lambda^*$ and $\mu \leq \mu^*$ problem (4) has at least $cat(\Omega)$ positive solutions for $4 < r < 6$ when $N = 3$ or $2 < r < 2^* = \frac{2N}{N-2}$ and $N \geq 4$: Rey in [10] proved that if $N \geq 5$; $\lambda = 0$ and $r = 2$; for μ small enough ,the number of solutions of problem (4) is at least $cat(\Omega)$: Here, cat stands for the Lusternik Schnirelmann category of Ω(see [8; 9]).

In [6], Yin and Yang have established the existence and multiplicity of positive solutions for the problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \mu|u|^{r-2}u + |u|^{2^*-2}u, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{5}$$

This problem plays an important role in the limit problem. Our main result is the following.

Theorem 1.1. *Assuming* (2) *and* (3)*. There are* $\mu^*, \lambda^* > 0$ *such that problem* (1) *has at least* $cat(\Omega)$ *positive solutions for* $\mu \in (0, \mu^*)$ *and* $\lambda \in (\lambda^*, \infty)$.

In this paper, we fix used the following notations:

* The usual norms in $H^1(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ will be denoted by $\|\|$ and $|\ |_p$ respectively

** If f is a mensurable function, $\int f(x)dx$ will stand for $\int_{\mathbb{R}^N} f(x)dx$.

This paper is organized as follows: In Section 2, we will recall some required important points in the limit problem. In Section 3, we prove some technical results which are crucial in the proof of Theorem 1. In Section 4, we use the Lusternik-Schnirelmann category theory to prove of the main theorem. The letter C will be repeatedly used to show various positive constants whose exact values are not important.

2. The limit problem

Problem (5) can be reduced to a simple problem

$$\begin{cases} -\Delta u - \Delta_q u = \mu|u|^{r-2}u + |u|^{2^*-2}u, & in \ \ \Omega, \\ u = 0, & on \ \ \partial\Omega \end{cases} \tag{6}$$

Let $I_\mu : H_0^1(\Omega) \to \mathbb{R}$ be the energy functional for problem (6) which is

$$I_\mu(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{q}\int_\Omega |\nabla u|^q dx - \frac{\mu}{r}\int_\Omega |u|^r dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx.$$

Let S be the best Sobolev constant of the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ given by

$$S := \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |\nabla u|^{2^*} dx\right)^{\frac{2}{2^*}}} \tag{7}$$

Yin and Yang [6] proved that:

$$0 < c_\mu < \frac{1}{N} S^{\frac{N}{2}}, \qquad \forall \mu > 0 \tag{8}$$

where

$$c_\mu := \inf_{u \in N_\mu} I_\mu(u)$$

where

$$N_\mu := \{x \in H_0^1(\Omega) : u \neq 0 \;\; and \;\; I'_\mu(u)u = 0\},$$

is the Nehari manifold I_μ, Ω is a smooth bounded domain of $\mathbb{R}^N$. It is proved [6] that there is $r > 0$ small enough so that

$$\Omega_r^+ := \{x \in \mathbb{R}^N : \quad dist(x, \Omega) < r\}$$
$$\Omega_r^- := \{x \in \Omega : \quad dist(x, \partial\Omega) > r\}$$

are homotopically equivalent to Ω. Without loss of generality. We can assume that $0 \in \Omega$ and $B_r(0) \subset \Omega$. Set

$$I_{\mu,r}(u) := \frac{1}{2}\int_{B_r(0)} |\nabla u|^2 dx + \frac{1}{q}\int_{B_r(0)} |\nabla u|^q dx - \frac{\mu}{p}\int_{B_r(0)} |u|^r dx - \frac{1}{2^*}\int_{B_r(0)} |u|^{2^*} dx.$$

then

$$0 < m(\mu) < \frac{1}{N} S^{\frac{N}{2}}.$$

Define

$$\begin{aligned} m(\mu) &:= \inf_{u \in N_{\mu,r}} I_{\mu,r}(u), \\ N_{\mu,r} &:= \{x \in H_0^1(\Omega) : u \neq 0 \;\; and \;\; I'_{\mu,r}(u)u = 0\}, \\ \beta_0 &: H_0^1(\Omega) \setminus \{0\} \to \mathbb{R}^N \\ \beta_0(u) &:= \frac{\int_\Omega |u|^{2^*} x dx}{\int_\Omega |u|^{2^*} dx}. \end{aligned} \tag{9}$$

We will recall and prove some lemmas which are crucial in the proof of the main theorem.

Lemma 2.1. $\lim_{\mu \to 0} c_\mu = \lim_{\mu \to 0} m(\mu) = \frac{1}{N} S^{\frac{N}{2}}$.

Lemma 2.2. *[6, lemma 3.3] There is $\mu^* > 0$ such that if $\mu \in (0, \mu^*)$ and $u \in N_\mu$ with $I_\mu(u) \leq m(\mu)$ then $\beta_0(u) \in \Omega_r^+$.*

Lemma 2.3. *[6, lemma 1.2] There exists a $\mu^* > 0$ such that for any $\mu \in (0, \mu^*)$ problem* (6) *possesses at least $cat(\Omega)$ positive solutions in $H_0^1(\Omega)$.*

3. Preliminary results

From now on, we fix the space $E \subset H^1(\mathbb{R}^N)$ given by

$$E = \{u \in H^1(\mathbb{R}^N) : \int V|u|^2 dx < +\infty\}$$

with inner product

$$\langle u, v\rangle_\lambda := \int (\nabla u \nabla v + \lambda V(x)uv)dx$$

and norm

$$\|u\|_\lambda := \left(\int (|\nabla u|^2 + \lambda V(x)|u|^2)dx\right)^{\frac{1}{2}}$$

Suppose that E is endowed with the norm $\|.\|_\lambda$ and we denote it by E_λ. It is well known that E_λ is a Hilbert space, moreover

$$\|u\|_\lambda \geq \Upsilon\|u\|, \qquad \forall u \in E_\lambda, \qquad \forall \lambda \geq 1 \tag{10}$$

which it shows that the embedding $E_\lambda \hookrightarrow H^1(\mathbb{R}^N)$ is continuous for $\lambda \geq 1$ and embedding

$$E_\lambda \hookrightarrow L^s(\mathbb{R}^N), \qquad \forall s \in [2, 2^*]$$

is also continuous for $\lambda \geq 1$.
Define $I_\eta : E_\lambda \to \mathbb{R}$ by

$$I_\eta(u) := \frac{1}{2}\|u\|_\lambda^2 + \frac{1}{q}\|u\|_q^q - \frac{\mu}{r}\int |u|^r dx - \frac{1}{2^*}\int |u|^{2^*} dx, \tag{11}$$

which is the energy functional respect to (1), is in $C^1(E_\lambda, \mathbb{R})$. Moreover

$$I'_\eta(u)v := \int (\nabla u\nabla v + \lambda V(x)uv)dx + \int (|\nabla u|^{q-2}\nabla u\nabla v dx - \mu\int |u|^{r-2}uv dx - \int |u|^{2^*-2}uv dx, \qquad u, v \in E \tag{12}$$

so

$$J_\eta(u) := I'_\eta(u)u = \|u\|_\lambda^2 + \|u\|_q^q - \mu\int |u|^r dx - \int |u|^{2^*} dx. \tag{13}$$

It is direct to see that critical points of I_η are weak solutions of (4). Set the Nehari manifold of $I_{\lambda,\mu}$ by

$$N_{\lambda,\mu} = \{u \in H \setminus \{0\} | I'_{\lambda,\mu}(u)u = 0\}.$$

For any $u \in N_{\lambda,\mu}$

$$I_\eta(u) = (\frac{1}{2} - \frac{1}{r})\|u\|_\lambda^2 + (\frac{1}{q} - \frac{1}{r})\|u\|_q^q + (\frac{1}{r} - \frac{1}{2^*})\int |u|^{2^*} dx > 0, \tag{14}$$

Thus, $I_{\lambda,\mu}$ is bounded from below on $N_{\lambda,\mu}$ so

$$c_{\lambda,\mu} := \inf_{u\in N_{\lambda,\mu}} I_{\lambda,\mu}(u)$$

.

exists. For any $u \in N_{\lambda,\mu}$

$$J'_\eta(u) = 2\|u\|_\lambda^2 + q\|u\|_q^q - r\mu \int |u|^r dx - 2^* \int |u|^{2^*} dx$$
$$= (2-r)\|u\|_\lambda^2 + (q-r)\|u\|_q^q + (r-2^*) \int |u|^{2^*} < 0. \quad (15)$$

We are going to show that the Mountain Pass Theorem is applicable.

Lemma 3.1. *Suppose that* $1 < q < 2 < r < 2^*$, $\lambda > 0$, $\mu > 0$. *Then*

i) there exist positive numbers ρ *and* d *such that* $I_{\lambda,\mu}(u) \geq \rho$ *for* $\|u\|_\lambda = d$,

ii) there exists $e \in C_0^\infty(\Omega)$ *such that* $\|e\|_\lambda > d$ *and* $I_{\lambda,\mu}(e) < 0$.

PROOF. i) Using the Sobolev embedding

$$H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3), \qquad for \;\; 2 \leq s \leq 2^*,$$
$$I_\mu(u) = \frac{1}{2}\|u\|_\lambda^2 + \frac{1}{q}\|u\|_q^q - \frac{\mu}{r}\int_\Omega |u|^r - \frac{1}{2^*}\int_\Omega |u|^{2^*} \quad (16)$$
$$\geq \frac{1}{2}\|u\|_\lambda^2 + \frac{1}{q}\|u\|_q^q - C\|u\|_\lambda^r - C\|u\|_\lambda^{2^*}. \quad (17)$$

Hence, there exist positive numbers $\rho > 0$ and $d > 0$ such that $I_{\lambda,\mu}(u) \geq \rho$ for $\|u\|_\lambda = d$.
ii) Fix $\phi \in C_0^\infty(\Omega)$ with $supp\phi \subset \Omega$. So

$$g(t) = I_\eta(t\phi) = \frac{t^2}{2}\|\phi\|_\lambda^2 + \frac{t^q}{q}\|\phi\|_q^q - \frac{t^r\mu}{r}\int_\Omega |\phi|^p - \frac{t^{2^*}}{2^*}\int_\Omega |\phi|^{2^*}.$$

Then $\lim_{t\to\infty} I_{\lambda,\mu}(t\phi) = -\infty$ Hance, there exists t_u positive such that $\|t_u\phi\| > d$ and $I_{\lambda,\mu}(t_u\phi) < 0$.

By Lemma 3.1, $I_{\lambda,\mu}$ possesses the mountain pass theorem in Willem [8]. We will denote by $m_{\lambda,\mu}$ the Mountain -Pass level, there is a $(PS)_{m_{\lambda,\mu}}$ sequence (u_n) for $I_{\lambda,\mu}$:

$$I_{\lambda,\mu}(u_n) \to m_{\lambda,\mu} := \inf_{\gamma\in\Gamma} max_{t\in[0,1]} I_{\lambda,\mu}(\gamma(t)) \qquad and \qquad I'_{\lambda,\mu}(u_n) \to 0,$$

where

$$\Gamma = \{\gamma \in C([0,1]), E_\lambda) : \gamma(0) = 0 \quad and \quad \gamma(1) = e\}.$$

Investigating of the following two assertions similar to [3; 4].

1) $c_\eta = m_\eta$.

2) There is $\sigma > 0$, which is independent of μ such that k$\|u\|_\lambda \geq \sigma$ for all $u \in N_\eta$.

Lemma 3.2. *There is* $\tau = \tau(\mu) > 0$ *such that the mountain pass level* c_η *verifies the following inequality*

$$0 < c_\eta < \frac{1}{N} S^{\frac{N}{2}} - \tau, \qquad \forall \lambda > 0.$$

PROOF. From (3)

$$I_{\lambda,\mu}(u) = I_\mu(u) \qquad \forall u \in H_0^1(\Omega),$$

so by definition of c_η and $c_\lambda, c_{\lambda,\mu} \le c_\mu$ for all $\eta > 0$. Then, it is enough to apply (2) to get the desired result.

Lemma 3.3. *Any* $(PS)_d$ *sequence* (w_n) *for* I_η *is bounded in* E_λ. *Moreover,*

$$\limsup_{n\to+\infty} \|w_n\|^2 \le \frac{2pd}{p-2}. \tag{18}$$

PROOF. Let (w_n) be $(PS)_d$ sequence for $I_{\lambda,\mu}$ in E_λ such that $I_{\lambda,\mu}(w_n) = d + o_n(1)$ and $I'_{\lambda,\mu}(w_n) = o_n(1)$.

$$\begin{aligned} I_{\lambda,\mu}(w_n) - \frac{1}{r} I'_{\lambda,\mu}(w_n)(w_n) &\le |I_{\lambda,\mu}(w_n)| + \frac{1}{p}\|I'_{\lambda,\mu}(w_n)(w_n)\|.\|w_n\|_\lambda \\ &\le d + o_n(1) + o_n(1)\|w_n\|_\lambda. \end{aligned} \tag{19}$$

On the other hand, for $n \in \mathbb{N}$

$$\begin{aligned} I_{\lambda,\mu}(w_n) - \frac{1}{r} I'_{\lambda,\mu}(w_n)(w_n) &= (\frac{1}{2} - \frac{1}{r})\|w_n\|_\lambda^2 + (\frac{1}{q} - \frac{1}{r})\|w_n\|^q + (\frac{1}{r} - \frac{1}{2^*}) \int |w_n|^{2^*} dx \\ &\ge (\frac{1}{2} - \frac{1}{r})\|w_n\|_\lambda^2 \end{aligned} \tag{20}$$

Combining the above inequalities, then for $n \in \mathbb{N}$ large enough

$$(\frac{1}{2} - \frac{1}{r})\|w_n\|_\lambda^2 \le d + o_n(1) + o_n(1)\|w_n\|_\lambda. \tag{21}$$

This proves boundedness. Doing lim sup of (21) then (18) follows.

Lemma 3.4. *Let* $\Theta > 0$. *If* $(w_n) \subset E_\lambda$ *is a* $(PS)_d$ *for* I_η *with* $0 \le d \le \Theta$, *then given* $\delta > 0$ *there are* $\lambda_* = \lambda_*(\delta, \Theta)$ *such that*

$$\limsup_{n\to+\infty} \int_{B_R^c} |w_n|^p dx < \delta, \qquad \forall \lambda > \lambda_*.$$

PROOF. See Lemma 3.6 of [4].

Corollary 3.1. *Let* $(v_n) \subset E_\lambda$ *be a sequence such that* $(\|v_n\|_{\lambda_n})$ *is bounded, where* $\lambda_n \to +\infty$ *If* $v_n \rightharpoonup 0$ *in* $H^1(\mathbb{R}^N)$, *then* $v_n \to 0$ *in* $L^p(\mathbb{R}^N)$.

Proof. From Corollary 1 in [4].

Proposition 3.1. *There is $\hat{\lambda} = \hat{\lambda}(\tau) > 0$ such that I_η verifies the $(PS)_{d_\lambda}$ condition for any $d_\lambda \in \left(0, \frac{1}{N} S^{\frac{N}{2}} - \tau\right)$ for all $\lambda \geq \hat{\lambda}$ where τ is as in Lemma 3.2.*

Proof. Let
$$I_\eta(w_n) \to d_\lambda \quad \textit{and} \quad I'_\eta(w_n) \to 0.$$

By Lemma 3.3, there is $w \in E_\lambda$ such that

$$\begin{aligned} w_n \rightharpoonup w \qquad &\text{in} \qquad E_\lambda \\ w_n(x) \to w(x) \qquad &a.e \;\; \text{in} \qquad \mathbb{R}^N \\ w_n \to w \qquad \text{in} \qquad L^s_{Loc}(\mathbb{R}^N), \qquad &0 \leq s < 2^* \end{aligned}$$

Set $v_n := w_n - w$.
$$\int |w_n|^q dx = \int |v_n|^q dx + \int |w|^q dx + o_n(1)$$
and
$$\int |w_n|^{2^*} dx = \int |v_n|^{2^*} dx + \int |w|^{2^*} dx + o_n(1).$$
Then
$$\frac{1}{2}\|v_n\|^2_\lambda + \frac{1}{q}\|v_n\|^q_q - \frac{\mu}{r}\int |v_n|^r dx - \frac{1}{2^*}\int |v_n|^{2^*} dx = d_\lambda - I_{\lambda,\mu}(w) + o_n(1). \tag{22}$$
Since $I'_{\lambda,\mu}(w_n)(w_n) = 0_n(I)$ and $I'_{\lambda,\mu}(w)(w) = 0$, it follows that
$$\|v_n\|^2_\lambda + \|v_n\|^q_q - \mu\int |v_n|^r dx - \int |v_n|^{2^*} dx = o(1). \tag{23}$$
$$\frac{1}{2}\|v_n\|^2_\lambda + \frac{1}{q}\|v_n\|^q_q - \frac{1}{2^*}\int |v_n|^{2^*} dx = d_\lambda - I_{\lambda,\mu}(w) + o_n(1). \tag{24}$$

Since $I'_{\lambda,\mu}(w_n)w_n = o_n(1)$ and $I'_{\lambda,\mu}(w)w = 0$, it follows that
$$\|v_n\|^2_\lambda + \|v_n\|^q_q - \int |v_n|^{2^*} dx = o(1) \tag{25}$$

Assume that for a fix λ, $\|v_n\|^2_\lambda \to h_1 \geq 0$ and $\int |v_n|^{2^*} dx \to h_2 \geq 0$.

If $h_1 = 0$, we deduce that $v_n \to 0$ in E_λ, equivalently, $w_n \to w$ in E_λ, which it completes the proof.

Now assume that h_1 and h_2 are positive By Sobolev embedding
$$\|v_n\|^2_\lambda \leq C(\|v_n\|^r_\lambda + \|v_n\|^{2^*}_\lambda) + o_n(1). \tag{26}$$
Recalling that there is $C > 0$ verifying
$$|t|^r \leq \frac{1}{2C}|t|^2 + C|t|^{2^*}, \qquad \forall t \in \mathbb{R}$$

The last inequality ensures that

$$0 < C_1 := (\frac{1}{2C(C+1)})^{\frac{2}{2^*-2}} \leq \lim_{n\to+\infty} \|v_n\|_\lambda^2.$$

Then there is $C_2 > 0$ in which

$$h_1, h_2 \geq C_2 > 0. \tag{27}$$

On the other hand

$$S \leq \frac{\|v_n\|_\lambda^2}{\left(\int |v_n|^{2^*} dx\right)^{\frac{2}{2^*}}} \leq \frac{h_1}{(h_2 + o_\lambda(1))^{\frac{2}{2^*}}}.$$

Then

$$S^{\frac{N}{2}} \leq \liminf_{\lambda\to+\infty} h_1.$$

Using (24) and since $w \in N_{\lambda,\mu}$, $I_{\lambda,\mu} > 0$ (by (14)), so

$$d_\lambda \geq \frac{1}{q}(\|v_n\|_\lambda^2 + \|v_n\|_q^q) - \frac{1}{2^*}\int_\Omega |v_n|^{2^*} dx$$
$$\liminf_{\lambda\to+\infty} d_\lambda \geq (\frac{1}{q} - \frac{1}{2^*}) \liminf_{\lambda\to+\infty} h_1 \geq \frac{1}{N} S^{\frac{N}{2}}.$$

But this is impossible, since

$$\limsup_{\lambda\to+\infty} d_\lambda \leq \frac{1}{N} S^{\frac{N}{2}} - \tau < \frac{1}{N} S^{\frac{N}{2}}.$$

There is $\hat{\lambda} > 0$ such that $h_1 = 0$ for all $\lambda > \hat{\lambda}$.

This reasult has a direct following corollary.

Corollary 3.2. *There is $\hat{\lambda} > 0$ such that $I_{\lambda,\mu}$ verifies the $(PS)_{d_\lambda}$ condition on $N_{\lambda,\mu}$ for any $d_\lambda \in \left(0, \frac{1}{N} S^{\frac{N}{2}} - \tau\right)$ and $\lambda > \hat{\lambda}$ where τ is as in Lemma 3.2.*

Theorem 3.5. *There is λ^* such that the mountain pass level $c_{\lambda,\mu}$ is a critical level of $I_{\lambda,\mu}$ for all $\lambda \geq \lambda^*$, that is $u_{\lambda,\mu} \in E_{\lambda,\mu}$ verifying*

$$I_{\lambda,\mu}(u_{\lambda,\mu}) = c_{\lambda,\mu} \qquad \text{and} \qquad I'_{\lambda,\mu}(u_{\lambda,\mu}) = 0$$

Proof. From Lemma 3.2, there is $\lambda^* = \lambda^*(\tau)$, $\forall \lambda \geq \lambda^*$, $c_{\lambda,\mu} < \frac{1}{N} S^{\frac{N}{2}} - \tau$. Proposition 3.1 implies that $I_{\lambda,\mu}$ satisfies in $(PS)_{c_{\lambda,\mu}}$ Thus, by mountain pass theorem due to Ambrosetti-Rabinowitz [1], $c_{\lambda,\mu}$ is a critical level of $I_{\lambda,\mu}$ for all $\lambda \geq \lambda^*$.

Definition 3.1. *$(u_n) \subset H^1(\mathbb{R}^N)$ is called a $(PS)_{c,\infty}$ if:*

$$\begin{aligned} &u_n \in E_{\lambda_n} \qquad \text{and} \qquad \lambda_n \to +\infty, \\ &I_{\lambda_n,\mu}(u_n) \to c, \qquad \text{for some} \qquad c \in R, \\ &\|I'_{\lambda_n,\mu}(u_n)\|_{E'_{\lambda_n}} \to 0. \end{aligned} \tag{28}$$

Theorem 3.6. *Let (u_n) be a $(PS)_{c,\infty}$ sequence for $c \in \left(0, \frac{1}{N}S^{\frac{N}{2}}\right)$. Then, there is a subsequence of (u_n) and $u \in H^1(\mathbb{R}^N)$ such that*

$$u_n \rightharpoonup u \qquad in \qquad H^1(\mathbb{R}^N).$$

i) $u \equiv 0$ *in* $\mathbb{R}^N \setminus \Omega$.

ii) $\|u_n - u\|^2_{\lambda_n} \to 0$.

iii) *Moreover,*

$$\begin{aligned} u_n \to u \qquad in \qquad H^1(\mathbb{R}^N), \\ \lambda_n \int V(x)|u_n|^2 dx \to 0, \\ \int_{\mathbb{R}\setminus\Omega} (|\nabla u_n|^2 + \lambda_n V(x)|u_n|^2) dx \to 0, \\ \|u_n\|^2_{\lambda_n} \to \int_\Omega |\nabla u|^2 dx = \|u\|^2_{H^1_0(\Omega)}. \end{aligned} \tag{29}$$

iv) *u is a weak solution of the problem* (6).

Proof. As Lemma 3.3 implying that $(\|u_n\|_{\lambda_n})$ is bounded in R and so (10) implying that (u_n) is bounded in $H^1(\mathbb{R}^N)$ thus there exists a subsequence of (u_n) such that

$$u_n \rightharpoonup u \qquad in \qquad H^1(\mathbb{R}^N).$$

For i) set

$$C_m := \{x \in \mathbb{R}^3 : V(x) > \frac{1}{m}\}.$$

Hence,

$$\bigcup_{m=1}^{+\infty} C_m = \mathbb{R}^N \setminus \bar{\Omega}.$$

Note that,

$$\int_{C_m} |u_n|^2 dx \le \frac{m}{\lambda_n}\|u_n\|^2_{\lambda_n},$$

Fatou's Lemma implies that

$$\int_{C_m} |u|^2 dx \le \liminf_{n\to+\infty} \int_{C_m} |u_n|^2 dx \le \liminf_{n\to+\infty} \frac{m}{\lambda_n}\|u_n\|^2_{\lambda_n} = 0.$$

This implies that $u = 0$ almost everywhere in

$$\mathbb{R}^N \setminus \bar{\Omega}.$$

ii) From i) $\|u\|^2_{H^1_0(\Omega)} = \|u\|^2_{\lambda_n}$ and

$$\|u_n - u\|^2_{\lambda_n} = \|u_n\|^2_{\lambda_n} - \|u\|^2_{H^1_0(\Omega)} + o_n(1) \tag{30}$$

and since $(u_n - u)$ is bounded so $\|u_n\|_{\lambda_n}$ would be a bounded sequence.

$$\begin{aligned}\|u_n\|^2_{\lambda_n} - \|u_n\|^q_q &= I'_{\lambda,\mu}(u_n)u_n + \int (\mu|u_n|^r + |u_n|^{2^*})dx \\ &= \int (\mu|u_n|^r + |u_n|^{2^*})dx + o_n(1).\end{aligned} \tag{31}$$

On the other hand, since $I'_{\lambda,\mu}(u_n)u \to 0$ so

$$\int_\Omega \nabla u_n \nabla u dx + \int_\Omega |\nabla u_n|^{q-2}\nabla u_n \nabla u - \int_\Omega (\mu|u_n|^{r-2}u_n + |u_n|^{2^*}u_n)u dx = o_n(1).$$

It follows that

$$\int |\nabla u|^2 dx + \int |\nabla u|^q dx - \int (\mu|u_n|^r + |u_n|^{2^*})dx = o_n(1), \tag{32}$$

Combining (30), (31) and (32)

$$\begin{aligned}\|u_n - u\|^2_{\lambda_n} &= \|u_n\|^2_{\lambda_n} + \|u_n\|^q_q - (\|u\|^2_{H^1_0(\Omega)} - \|u\|^q_q) - (\|u_n\|^q_q - \|u\|^q_q) \\ &= \int (\mu|u_n|^r + |u_n|^{2^*})dx - \int (\mu|u|^r + |u|^{2^*})dx + (\|u\|^q_q - \|u_n\|^q_q) + o_n(1)\end{aligned} \tag{33}$$

that is

$$\|v_n\|^2_{\lambda_n} = \mu|v_n|^r + |v_n|^{2^*} + (\|u\|^q_q - \|u_n\|^q_q) + o_n(1),$$

where $v_n = u_n - u$.
Corollary 3.1 implies that, $v_n \to 0$ in $L^r(\mathbb{R}^N)$ and from Brézis-Lieb lemma

$$\|v_n\|^2_{\lambda_n} + \|v_n\|^q_q = |v_n|^{2^*}_6 + o_n(1).$$

Now, the same arguments used in the proof of Proposition 3.1 shows that

$$\|v_n\|^2_{\lambda_n} \to 0.$$

iii) It comes from the following inequality and i) that $u \equiv 0$ on Ω^c:

$$0 \le \lambda_n \int V(x)|u_n|^2 dx = \lambda_n(\int_\Omega V(x)|u_n|^2 dx + \int_{\Omega^c} V(x)|u_n|^2 dx) = \lambda_n \int V(x)|u_n - u|^2 dx \le \|v_n\|^2_{\lambda_n}.$$

and

$$\|v_n\|^2_{\lambda_n} = \int (|\nabla v_n|^2 + \lambda_n V(x)|v_n|^2)dx \ge \int_{\Omega^c} (|\nabla v_n|^2 + \lambda_n V(x)|v_n|^2)dx \ge 0.$$

Finally

$$\|v_n\|^2_{\lambda_n} = \int(|\nabla v_n|^2+\lambda_n V(x)|v_n|^2)dx = \int_\Omega(|\nabla v_n|^2+\lambda_n V(x)|v_n|^2)dx+o_n(1) = \int_\Omega |\nabla v_n|^2+o_n(1).$$

iv) For $\varphi \in C_0^\infty(\Omega)$ we have

$$I'_{\lambda,\mu}(u_n)\varphi := \int_\Omega \nabla u_n\nabla\varphi dx + \int_\Omega |\nabla u_n|^{q-2}\nabla u_n\nabla\varphi dx - \mu\int |u_n|^{r-2}u_n\varphi dx - \int |u_n|^4 u_n\varphi dx, \tag{34}$$

(u_n) is a $(PS)_{c,\infty}$ sequence, so

$$I'_{\lambda,\mu}(u_n)\varphi \to 0. \tag{35}$$

Since $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$

$$\int_\Omega \nabla u_n\nabla\varphi dx \to \int_\Omega \nabla u\nabla\varphi dx, \tag{36}$$

and

$$\int_\Omega (\mu|u_n|^{r-2}u_n + |u_n|^{2^*-2}u_n)\varphi dx \to \int_\Omega (\mu|u|^{r-2}u + |u|^{2^*-2}u)\varphi dx \tag{37}$$

and similar [2, Theorem 5.9]

$$\int_\Omega |\nabla u_n|^{q-2}\nabla u_n\nabla\varphi dx \to \int_\Omega |\nabla u|^{q-2}\nabla u\nabla\varphi dx. \tag{38}$$

Therefore,

$$\int_\Omega \nabla u\nabla\varphi dx + \int_\Omega |\nabla u|^{q-2}\nabla u\nabla\varphi dx = \mu\int |u|^{r-2}u\varphi dx + \int_\Omega |u|^{2^*-2}u\varphi dx, \qquad \forall\varphi \in C_0^\infty(\Omega). \tag{39}$$

$C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ so

$$\int_\Omega \nabla uvdx + \int_\Omega |\nabla u|^{q-2}\nabla uvdx = \mu\int_\Omega |u|^{r-2}uvdx + \int_\Omega |u|^{2^*-2}uvdx, \qquad \forall v \in H_0^1(\Omega).$$

Lemma 3.7. *If $\lambda_n \to +\infty$, then $c_{\lambda_n,\mu} \to c_\mu$.*

Proof. See Lemma 3.13 of [4].

The proof of corollaries 3, 4 and 4 are similar to those one in [4; 7].

Corollary 3.3. *Let $\lambda_n \in \mathbb{R}^+$ be a sequence verifying $\lambda_n \to +\infty$ and $u_{\lambda_n,\mu}$ the ground state solution obtained in Theorem 3.5. Then, there is a subsequence of $(u_{\lambda_n,\mu})$ still denoted by itself, and $u \in H_0^1(\Omega)$ such that $u_{\lambda_n,\mu} \to u$ in $H_0^1(\Omega)$ and u is a ground state solution of the limit problem* (6).

Corollary 3.4. *There are $\lambda^* > 0$ large and $\mu^* > 0$ small such that*

$$m(\mu) < 2c_{\lambda,\mu}, \quad \lambda \geq \lambda^* \quad \text{and} \quad \forall\mu \in (0,\mu^*).$$

Corollary 3.5. *If $u \in E_\lambda$ is a nontrivial critical point of $I_{\lambda,\mu}$ such that $I_{\lambda,\mu} \leq m(\mu)$ then u is positive or u is negative.*

corollary 3.5 implies that the nontrivial critical points of $I_{\lambda,\mu}$ are positive solutions of problem (1).

4. Proof of main theorem

Choose $R>0$ such that $\bar{\Omega}\subset B_R=\{x\in\mathbb{R}^N:|x|<R\}$ and set

$$\xi(t)=\begin{cases}1, & 0\le t\le R\\ \frac{R}{t}, & r\ge R.\end{cases} \tag{40}$$

Moreover, we define

$$\beta(u):=\frac{\int|u|^{2^*}\xi(|x|)xdx}{\int|u|^{2^*}dx}\qquad for\quad u\in N_{\lambda,\mu}.$$

Lemma 4.1. *There is $\lambda^*>0$ such that if $u\in N_{\lambda,\mu}$ and $I_{\lambda,\mu}(u)\le m(\mu)$ then $\beta(u)\in\Omega_r^+$ for all $\lambda\ge\lambda^*$.*

Proof. If the conclusion is not true, then there would exist sequences $\lambda_n\to+\infty$ and $u_n\in N_{\lambda_n,\mu}$ in which $I_{\lambda_n,\mu}(u_n)\le m(\mu)$ and

$$\beta(u_n)\notin\Omega_r^+,\qquad\forall n\in\mathbb{N}.$$

Form (14), clearly the sequence $(\|u_n\|_{\lambda_n})$ is bounded in $\mathbb{R}$; (up to subsequence). There is, $u\in H_0^1(\Omega)$ such that

$$u_n\rightharpoonup u\qquad in\quad H^1(\mathbb{R}^N),$$
$$u_n(x)\to u(x)\qquad a.e.\quad in\quad\mathbb{R}^N, \tag{41}$$
$$u_n\to u\qquad in\quad L^t_{Loc}(\mathbb{R}^N)\qquad for\qquad t\in[1,2^*). \tag{42}$$

Moreover,

$$\|v_n\|^2_{\lambda_n}+\|v_n\|^q_q=\mu|v_n|^r_r+|v_n|^{2^*}_{2^*}+o_n(1),$$

where $v_n=u_n-u$. By Corollary 3.1 $|v_n|^r_r\to0$, and so

$$\|v_n\|^2_{\lambda_n}+\|v_n\|^q_q=|v_n|^{2^*}_{2^*}+o_n(1).$$

Arguing as in the proof of Proposition 3.1,

$$\|v_n\|_{\lambda_n}\to0$$

This limit combined with $\|v\|_\lambda\ge\sigma$ implies that

$$u_n\to u\qquad in\qquad H^1(\mathbb{R}^N).\quad u\ne0, I'_\mu(u)u=0\qquad\text{and}\qquad I_{\lambda,\mu}(u_n)\to I_\mu(u).$$

Thus $u\in N_\mu$ and $I_\mu(u)\le m(\mu)$ Applying the Lemma 2.2, so $\beta_0(u)\in\Omega^+_{r/2}$ then

$$\beta_0(u)=\lim_{n\to\infty}\beta(u_n)\in\Omega^+_{r/2}.$$

which is a contradiction.

Lemma 4.2. *If u is a critical point of $I_{\lambda,\mu}$ on $N_{\lambda,\mu}$ then it is a critical point of $I_{\lambda,\mu}$ in $H_0^1(\Omega)$.*

Proof. Let $u \in N_{\lambda,\mu}$ then $I'_{\lambda,\mu}(u)u = 0$.
On the other hand, by the theory of Lagrange multipliers, there exists $\theta \in \mathbb{R}$ such that $I'_{\lambda,\mu}(u) = \theta J'_{\lambda,\mu}(u)$. Thus,

$$0 = I'_{\lambda,\mu}(u)u = \theta J'_{\lambda,\mu}(u)u.$$

Using (15), so $\theta = 0$, Thus u is a critical point of $I_{\lambda,\mu}$ in $H_0^1(\Omega)$.
Let $u_r \in H_0^1(B_r(0))$ is a positive radial ground state solution for $I_{\mu,r}$, that is

$$I_{\mu,r}(u_r) = m(\mu) = \inf_{u \in N_{\mu,r}} I_{\mu,r}(u) \qquad and \qquad I'_{\mu,r}(u_r) = 0.$$

Define $\phi : \Omega^- \to I_{\mu,r}^{m(\mu)}$, where $I_{\mu,r}^{m(\mu)} = \{u \in N_{\mu,r} : I_{\mu,r}(u) \le m(\mu)\}$

$$\begin{aligned}
\phi_r(y)x &= u_r(|x-y|), \qquad x \in B_r(y) \\
\beta(\phi_r(y)) &= \frac{\int |\phi_r(y)(x)|^{2^*}\xi(|x|)x dx}{\int |\phi_r(y)(x)|^{2^*} dx} \\
&= \frac{\int_{B_r(y)} |u_r(|x-y|)|^{2^*}\xi(|x|)x dx}{\int_{B_r(y)} |u_r(|x-y|)|^{2^*} dx} \\
&= \frac{\int_{B_r(0)} |u_r(|z|)|^{2^*}(y+z) dz}{\int_{B_r(0)} |u_r(|z|)|^{2^*} d} \\
&= y, \qquad \forall y \in \Omega_r^-.
\end{aligned} \tag{43}$$

Lemma 4.3. *If $\mu \in (0, \mu^*)$, μ^* is given in Lemma 2.2, then*

$$cat(I_{\lambda,\mu}^{m(\mu)}) \ge cat(\Omega).$$

Proof. Suppose that

$$I_{\lambda,\mu}^{m(\mu)} = A_1 \cup A_2 \cup \ldots \cup A_n.$$

where A_j, $j = 1, 2, ..., n$, is closed and contractible in $I_{\mu,r}^{m(\mu)}$, that is, there exists $h_j \in C([0,1] \times A_j, I_{\lambda,\mu}^{m(\mu)})$ such that

$$h_j(0,u) = u \qquad and \qquad h_j(1,u) = w_j \qquad \forall u \in A_j,$$

where $w_j \in A_j$ is fixed. Any $\phi_r^{-1}(A_j) = B_j$, is closed for $1 \le j \le n$,

$$\Omega_r^- = B_1 \cup B_2 \cup \ldots \cup B_n.$$

Consider the deformation map $g_j : [0,1] \times B_j \to \Omega_r^+$ given by

$$g_j(t,y) = \beta(h_j(t, \phi_r(y))).$$

Then

$$\begin{aligned}
g_j(0,y) &= \beta(h_j(0, \phi_r(y))) = \beta(\phi_r(y)) = y \\
g_j(1,y) &= \beta(h_j(1, \phi_r(y))) = \beta(w_j).
\end{aligned}$$

Thus any β_j is contractible in Ω_r^+. Lemma 4.1 implies that

$$cat(\Omega) = cat_{\Omega_r^+}(\Omega_r^-) \le n.$$

To prove Theorem 1.1 we need the following results.

Proof. of Theorem 1.1. For $0<\mu<\mu^*$ and $\lambda>\lambda^*$,

$$\Omega_r^- \xrightarrow{\phi_r} I_{\lambda,\mu}^{m(\mu)} \xrightarrow{\beta} \Omega_r^+$$

Let $u_r \in H_0^1(B_r) \subset E$ be a minimizer of $I_{\mu,r}$ on $N_{\mu,r}$ with $u_r > 0$ and $\varphi(x) =_r (|. - x|)$, so $\varphi(x) = 0$ in $\mathbb{R}^3 \setminus \Omega$ for every $x \in \Omega_r^-$.

$$\varphi(x) \in N_{\lambda,\mu} \qquad and \qquad I_{\lambda,\mu}(\varphi(x)) = I_{\lambda,r}(\varphi(x)) = m(\mu).$$

Clearly, I and β are even and $\beta \circ \varphi$ is a homotopy equivalence.
Thus Lemma 4.3 implies that $cat(\Omega) \leq cat(I_{\lambda,\mu}^{m(\mu)})$. Since $I_{\lambda,\mu}$ satisfies the $(PS)_c$ condition on $N_{\lambda,\mu}$ for $c \leq m(\mu)$ An standard Lusternik-Schnirelmann theory and Lemma 4.3 yields at least $cat(\Omega)$ of critical points in $N_{\lambda,\mu}$ and consequently, critical points in E_λ. Corollary 3.2 conclude that $I_{\lambda,\mu}$ has at least $cat(\Omega)$ positive solutions.

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Multiplicity and Concentration of Positive Solutions for Weighted Schrödinger Equations with Critical Nonlinearities

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Abstract

This work investigates the nonlinear equation of Schrödinger

$$-\Delta_b w + \beta a(x) w = \eta w + w^{2^*-1}, \quad w \in \mathbb{R}^N,$$

where $2^* = \frac{2N}{N-2}$ denotes the critical Sobolev exponent with $N \geq 4$. Here, $a(x) \geq 0$ is a given potential function. Under the assumption that the parameter $\eta > 0$ is sufficiently small and $\beta > 0$ is large, we establish the existence and multiplicity of positive solutions that exhibit concentration phenomena around the potential well.

Keywords: Schrödinger equations, Weighted Potential, Positive Solutions, Concentration Phenomena, Potential Well, Critical Exponent, Multiplicity of Solutions, Variational Methods.
2020 MSC: 35Q55, 35J10, 35A01, 35B38.

1. Introduction

The nonlinear Schrödinger equation has garnered significant interest in recent years due to its broad range of applications in quantum mechanics, nonlinear optics, and mathematical physics. The study of solutions involving critical Sobolev exponents is particularly challenging due to the loss of compactness and the delicate variational structure.

Several researchers have explored the existence and multiplicity of positive solutions to nonlinear Schrödinger-type equations. For instance, in [28], the authors examined normalized solutions under critical growth conditions and discussed the impact of the potential term. Similarly, [29] addressed magnetic Schrödinger equations and provided new insights into concentration phenomena under Sobolev critical exponents. Recent contributions, such as

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[30], extend these ideas by considering steep potential wells and proving the existence of multi-peak solutions in the presence of critical nonlinearity.

Motivated by these results, in this paper we investigate the following problem:

$$\begin{cases} -\Delta_b w + \beta a(x) w = \eta w + w^{2^*-1} & \text{in } \mathbb{R}^N, \\ w > 0, \quad w \in \mathscr{H}_b^1(\mathbb{R}^N), \end{cases} \tag{$PS^b_{\beta,\eta}$}$$

where $N \geq 4$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, $\beta > 0$, and $\eta \in \mathbb{R}$. The potential function $a(x)$ is assumed to satisfy the following assumptions:

(B_1) $a \in C(\mathbb{R}^N, \mathbb{R})$, $a(x) \geq 0$, and $\Omega := \mathrm{int}(a^{-1}(0))$ is a nonempty bounded set with smooth boundary such that $\bar{\Omega} = a^{-1}(0)$.

(B_2) There exists $M_0 > 0$ such that

$$\mathscr{L}\left\{x \in \mathbb{R}^N : a(x) \leq M_0\right\} < \infty,$$

where $\mathscr{L}$ denotes the Lebesgue measure in $\mathbb{R}^N$.

Under these hypotheses, we aim to establish the existence and multiplicity of positive solutions, particularly in the regime of small η and large β, and investigate their concentration behavior around the potential well.

Definition 1.1. *Let $b \in L^p(\Omega)$ for some $1 < p < \infty$. The* weighted Sobolev space $W^{1,p}(\Omega, b)$ *is defined as the set of all real-valued, Lebesgue measurable functions w defined almost everywhere in Ω such that*

$$\|w\|_{1,p,b} := \left(\int_\Omega |w(x)|^p \, dx + \int_\Omega b(x) |\nabla w(x)|^p \, dx \right)^{1/p} < +\infty.$$

In the special case $p = 2$, we denote $W^{1,2}(\Omega, b)$ by $\mathscr{H}_b^1(\Omega)$ and define the corresponding norm as

$$\|w\|_{\mathscr{H}_b^1} := \left(\int_\Omega |w(x)|^2 \, dx + \int_\Omega b(x) |\nabla w(x)|^2 \, dx \right)^{1/2}. \tag{1}$$

Furthermore, the inner product in $\mathscr{H}_b^1$ is defined by

$$(w, v) := \int_\Omega w(x) v(x) \, dx + \int_\Omega b(x) \nabla w(x) \cdot \nabla v(x) \, dx, \quad \forall w, v \in \mathscr{H}_b^1. \tag{2}$$

A solution w_β of $(PS^b_{\beta,\eta})$ is said to be a *least energy solution* if the corresponding energy functional

$$J^b_{\beta,\eta}(w) := \int_{\mathbb{R}^N} \left(\frac{1}{2} \left(b(x) |\nabla w|^2 + (\beta a(x) - \eta) w^2 \right) - \frac{1}{2^*} |w|^{2^*} \right) dx$$

attains its minimum at $w = w_\beta$ among all nontrivial solutions of $(PS^b_{\beta,\eta})$.

A sequence of solutions (w_n) to $(PS^b_{\beta_n,\eta})$ is said to *concentrate* at a solution w of the following limit problem:

$$\begin{cases} -\Delta_b w = \eta w + w^{2^*-1}, & \text{in } \Omega, \\ w > 0, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

if a subsequence converges strongly to w in the space $\mathscr{H}^1_b(\mathbb{R}^N)$ as $\beta_n \to \infty$.

Let

$$S := \inf_{w \in \mathscr{H}^1_b(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} b(x)|\nabla w|^2\,dx}{\left(\int_{\mathbb{R}^N} |w|^{2^*}\,dx\right)^{2/2^*}}$$

denote the best constant for the Sobolev embedding in the weighted space. We aim to prove the following results.

Theorem 1.1. *Suppose that conditions (B_1) and (B_2) are satisfied, and let $N \geq 4$. Then, for every $0 < \eta < \eta(\Omega)$, there exists a constant $\beta(\eta) > 0$ such that problem $(PS^b_{\beta,\eta})$ admits a least-energy solution w_β for all $\beta \geq \beta(\eta)$.*

Theorem 1.2. *Assume that conditions (B_1) and (B_2) hold, and that $N \geq 4$. Then there exists a constant $0 < \eta^* < \eta_1(\Omega)$ such that, for each $0 < \eta \leq \eta^*$, there exist two constants $\Lambda(\eta) > 0$ and $0 < c(\eta) < \frac{1}{N}S^{N/2}$ with the following property: for all $\beta \geq \Lambda(\eta)$, the problem $(PS^b_{\beta,\eta})$ admits at least $\operatorname{cat}(\Omega)$ positive solutions satisfying*

$$J^b_{\beta,\eta}(w) \leq c(\eta).$$

Theorem 1.3. *Let (w_n) be a sequence of solutions to $(PS^b_{\beta_n,\eta})$, where $\eta \in (0, \eta_1(\Omega))$ and $\beta_n \to \infty$. Suppose that the corresponding energy levels satisfy*

$$J^b_{\beta_n,\eta}(w_n) \to c < \frac{1}{N}S^{N/2} \quad \text{as } n \to \infty.$$

Then, up to a subsequence, (w_n) concentrates at a solution of problem (D^b_η).

2. Variational Compactness and Functional Setting

In this study, we consistently assume that conditions (B_1) and (B_2) hold, with $N \geq 4$. Let $\eta_1(\Omega)$ denote the smallest eigenvalue of the operator $-\Delta$ on the domain Ω subject to the Dirichlet boundary condition $w = 0$. Additionally, we use $|\cdot|_q$ to denote the L^q-norm for $q \in [1, \infty]$.

Define the Hilbert space E_b as

$$E_b = \left\{ w \in \mathscr{H}^1_b(\mathbb{R}^N) : \int_{\mathbb{R}^N} a(x) w^2\,dx < \infty \right\}$$

equipped with the norm

$$\|w\|_b = \left(\|w\|^2_{\mathscr{H}^1_b} + \int_{\mathbb{R}^N} a(x) w^2 \, dx \right)^{1/2}$$

This norm is clearly equivalent to the following family of norms:

$$\|w\|_{b,\beta} = \left(\|w\|^2_{\mathscr{H}^1_b} + \beta \int_{\mathbb{R}^N} a(x) w^2 \, dx \right)^{1/2}, \quad \text{for } \beta > 0.$$

Lemma 2.1. *Let $\beta_n \geq 1$ and $w_n \in E_b$ be a sequence such that $\beta_n \to \infty$ and $\|w_n\|^2_{b,\beta_n} < K$ for some constant $K > 0$. Then there exists a function $w \in \mathscr{H}^1_{0,b}(\Omega)$ such that, up to a subsequence, $w_n \rightharpoonup w$ weakly in E_b and $w_n \to w$ strongly in $L^2(\mathbb{R}^N)$.*

Proof. Since $\|w_n\|^2_b \leq \|w_n\|^2_{b,\beta_n} < K$, the sequence (w_n) is bounded in E_b. Hence, by reflexivity and standard compactness arguments, there exists $w \in E_b$ such that (up to a subsequence) $w_n \rightharpoonup w$ weakly in E_b and $w_n \to w$ in $L^2_{\text{loc}}(\mathbb{R}^N)$.

Define the sets

$$D_m := \left\{ x \in \mathbb{R}^N : |x| \leq m, \ a(x) \geq \frac{1}{m} \right\}, \quad m \in \mathbb{N}.$$

Then, for each m,

$$\int_{D_m} |w_n|^2 \, dx \leq m \int_{D_m} a(x) w_n^2 \, dx \leq \frac{mK}{\beta_n} \to 0 \quad \text{as } n \to \infty.$$

This implies that $w(x) = 0$ almost everywhere in $\mathbb{R}^N \setminus \Omega$, so $w \in \mathscr{H}^1_{0,b}(\Omega)$ due to the smoothness of $\partial\Omega$.

To show strong convergence in $L^2(\mathbb{R}^N)$, let $F := \{x \in \mathbb{R}^N : a(x) \leq M_0\}$, where M_0 is as in assumption (B_2), and denote $F^c := \mathbb{R}^N \setminus F$. Then

$$\int_{F^c} w_n^2 \, dx \leq \frac{1}{\beta_n M_0} \int_{F^c} \beta_n a(x) w_n^2 \, dx \leq \frac{K}{\beta_n M_0} \to 0 \quad \text{as } n \to \infty.$$

Now, consider $B_R := \{x \in \mathbb{R}^N : |x| \leq R\}$ and $B_R^c := \mathbb{R}^N \setminus B_R$. Take any $r \in (1, N/(N-2))$ with Hölder conjugate $r' := r/(r-1)$. Then

$$\int_{B_R^c \cap F} |w_n - w|^2 \, dx \leq \|w_n - w\|^2_{L^{2r}} \mathscr{L}(B_R^c \cap F)^{1/r'} \leq C \|w_n - w\|^2_{E_b} \mathscr{L}(B_R^c \cap F)^{1/r'} \to 0$$

as $R \to \infty$ due to assumption (B_2). Since $w_n \to w$ in L^2_{loc}, we also have

$$\int_{B_R} |w_n - w|^2 \, dx \to 0 \quad \text{as } n \to \infty.$$

Combining these results shows $w_n \to w$ in $L^2(\mathbb{R}^N)$.

We define the self-adjoint operator $\mathscr{B}^b_\beta := -\Delta_b + \beta a$ on the Hilbert space $L^2(\mathbb{R}^N)$, where the associated form domain is E_b. The inner product in $L^2(\mathbb{R}^N)$ is denoted by $(\cdot,\cdot)$. The bilinear form corresponding to $\mathscr{B}^b_\beta$ is given by

$$(\mathscr{B}^b_\beta w, v) := \int_{\mathbb{R}^N} (b(x)\nabla w \cdot \nabla v + \beta a(x) wv)\, dx, \quad \text{for all } w, v \in E_b.$$

Let $\kappa^b_\beta := \inf \sigma(\mathscr{B}^b_\beta)$ denote the lowest point in the spectrum of $\mathscr{B}^b_\beta$. It is straightforward to verify that

$$\kappa^b_\beta = \inf \left\{ (\mathscr{B}^b_\beta w, w) : w \in E_b,\ \|w\|_{L^2} = 1 \right\} \geq 0.$$

Moreover, the mapping $\beta \mapsto \kappa^b_\beta$ is non-decreasing due to the monotonicity of the potential term $\beta a(x)$ in β.

Lemma 2.2. *Let $\eta \in (0, \eta_1(\Omega))$. Then there exists a constant $\beta(\eta) > 0$ such that for all $\beta \geq \beta(\eta)$, the spectral bound satisfies*

$$\kappa^b_\beta \geq \frac{\eta + \eta_1(\Omega)}{2}.$$

As a consequence, for all $w \in E_b$ and $\beta \geq \beta(\eta)$, we have

$$\alpha_\eta \|w\|^2_{b,\beta} \leq \left((\mathscr{B}^b_\beta - \eta) w, w \right),$$

where the constant α_η is defined as

$$\alpha_\eta := \frac{\eta + \eta_1(\Omega)}{\eta_1(\Omega) + 2 + 3\eta}.$$

Proof. Suppose by contradiction that there exists a sequence $\beta_n \to \infty$ such that $\kappa^b_{\beta_n} < \frac{\eta+\eta_1(\Omega)}{2}$ for all n, and assume $\kappa^b_{\beta_n} \to \tau \leq \frac{\eta+\eta_1(\Omega)}{2}$.

Let $w_n \in E_b$ be a sequence satisfying $\|w_n\|_{L^2} = 1$ and

$$\left((\mathscr{B}^b_{\beta_n} - \kappa^b_{\beta_n}) w_n, w_n \right) \to 0.$$

Then, we estimate:

$$\begin{aligned}
\|w_n\|^2_{b,\beta_n} &= \int_{\mathbb{R}^N} \left(b(x)|\nabla w_n|^2 + (1 + \beta_n a(x)) w_n^2 \right) \\
&= \left((\mathscr{B}^b_{\beta_n} - \kappa^b_{\beta_n}) w_n, w_n \right) + (1 + \kappa^b_{\beta_n}) \|w_n\|^2_{L^2} \\
&\leq 2(1 + \eta_1(\Omega)),
\end{aligned}$$

for all sufficiently large n. Hence, by Lemma 2.1, we may assume (up to subsequence) that $w_n \rightharpoonup w$ weakly in E_b and strongly in $L^2(\mathbb{R}^N)$, with $\|w\|_{L^2} = 1$.

Due to the support properties induced by $a(x)$, it follows that $w \in \mathscr{H}^1_{0,b}(\Omega)$. Moreover, using weak lower semicontinuity and convergence, we obtain

$$\int_\Omega b(x)|\nabla w|^2 - \tau |w|^2 \leq \liminf_{n\to\infty} \left((\mathscr{B}^b_{\beta_n} - \kappa^b_{\beta_n}) w_n, w_n \right) = 0.$$

Thus,

$$\int_{\Omega} b(x)|\nabla w|^2 \leq \tau < \eta_1(\Omega),$$

which contradicts the variational characterization of $\eta_1(\Omega)$ as the lowest eigenvalue of $-\Delta_b$ on $\mathscr{H}^1_{0,b}(\Omega)$ with $\|w\|_{L^2} = 1$. This contradiction completes the proof.

Let us consider the energy functional associated with problem $(PS^b_{\beta,\eta})$, defined by

$$\begin{aligned} J^b_{\beta,\eta}(w) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(b(x)|\nabla w|^2 + \beta a(x) w^2 - \eta w^2\right) dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |w|^{2^*} dx \\ &= \frac{1}{2}\left((\mathscr{B}^b_\beta - \eta)w, w\right) - \frac{1}{2^*}\|w\|^{2^*}_{L^{2^*}}. \end{aligned}$$

It is straightforward to verify that $J^b_{\beta,\eta} \in C^1(E_b, \mathbb{R})$, and that its critical points correspond to weak solutions of the equation

$$-\Delta_b w + \beta a(x) w = \eta w + |w|^{2^*-2} w, \quad w \in \mathscr{H}^1_b(\mathbb{R}^N).$$

A sequence $\{w_n\} \subset E_b$ is said to be a *Palais-Smale sequence at level* c (briefly, a $(PS)_c$ sequence) for $J^b_{\beta,\eta}$ if

$$J^b_{\beta,\eta}(w_n) \to c \quad \text{and} \quad \|(J^b_{\beta,\eta})'(w_n)\|_{E_b^*} \to 0 \quad \text{as } n \to \infty.$$

We say that the functional $J^b_{\beta,\eta}$ satisfies the $(PS)_c$ condition if every $(PS)_c$ sequence admits a strongly convergent subsequence in E_b.

Lemma 2.3. *Let $\eta \in (0, \eta_1(\Omega))$ and $\beta \geq \beta(\eta)$. Then every $(PS)_c$ sequence $\{w_n\} \subset E_b$ associated with the functional $J^b_{\beta,\eta}$ is bounded in E_b, and moreover satisfies:*

$$\lim_{n\to\infty} \left((\mathscr{B}^b_\beta - \eta)(w_n, w_n)\right) = \lim_{n\to\infty} \|w_n\|^{2^*}_{L^{2^*}} = Nc. \tag{4}$$

Proof. By assumption, $\{w_n\}$ is a $(PS)_c$ sequence, i.e.,

$$J^b_{\beta,\eta}(w_n) \to c, \quad \text{and} \quad \|(J^b_{\beta,\eta})'(w_n)\|_{E_b^*} \to 0.$$

Using the standard Pohozaev-type identities associated with critical exponents, we compute:

$$J^b_{\beta,\eta}(w_n) - \frac{1}{2^*}\langle (J^b_{\beta,\eta})'(w_n), w_n\rangle = \frac{1}{N}(\mathscr{B}^b_\beta - \eta)(w_n, w_n), \tag{5}$$

$$J^b_{\beta,\eta}(w_n) - \frac{1}{2}\langle (J^b_{\beta,\eta})'(w_n), w_n\rangle = \frac{1}{2}\|w_n\|^{2^*}_{L^{2^*}}. \tag{6}$$

Since $(J^b_{\beta,\eta})'(w_n) \to 0$ in E_b^*, it follows from (5) that $(\mathscr{B}^b_\beta - \eta)(w_n, w_n)$ remains bounded. Lemma 2.2 then guarantees that $\{w_n\}$ is bounded in E_b.

Passing to the limit in (5) and (6) yields the desired identity (4).

Let

$$S = \inf_{w \in \mathscr{H}^1_{0,b}(\Omega)} \frac{\int b(x)|\nabla w|^2 \, dx}{\|w\|^2_{L^{2^*}(\Omega)}} \tag{7}$$

denote the best Sobolev constant corresponding to the weighted embedding $\mathscr{H}^1_{0,b}(\Omega) \hookrightarrow L^{2^*}(\Omega)$.

In what follows, and without loss of generality, we assume that the threshold $\beta(\eta)$ is chosen large enough so that

$$\beta(\eta) \geq \frac{\eta}{M_0},$$

ensuring that $\beta M_0 - \eta \geq 0$ holds for all $\beta \geq \beta(\eta)$.

Proposition 2.1. *Let $\eta \in (0, \eta_1(\Omega))$ and $\beta \geq \beta(\eta)$. Then the functional $J^b_{\beta,\eta}$ satisfies the Palais-Smale condition at any level $c < \frac{1}{N} S^{N/2}$. In other words, every sequence $\{w_n\} \subset E_b$ such that*

$$J^b_{\beta,\eta}(w_n) \to c \quad \textit{and} \quad \|J^{b\prime}_{\beta,\eta}(w_n)\|_{E_b} \to 0 \quad \textit{as } n \to \infty,$$

admits a strongly convergent subsequence in E_b.

Proof. By Lemma 2.3, the sequence $\{w_n\}$ is bounded in E_b. Thus, up to a subsequence, we may assume that $w_n \rightharpoonup w$ weakly in E_b, $w_n \to w$ in $L^2_{\text{loc}}(\mathbb{R}^N)$, and $w_n(x) \to w(x)$ almost everywhere in $\mathbb{R}^N$.

Standard variational arguments imply that the weak limit w satisfies the limiting equation

$$-\Delta_b w + \beta a(x) w = \eta w + |w|^{2^*-2} w \quad \text{in } \mathbb{R}^N.$$

To analyze the convergence behavior of the sequence, we define the remainder sequence by

$$z_n := w_n - w.$$

By the Brezis-Lieb lemma [7; 27], we have

$$|w_n|^{2^*}_{2^*} = |w|^{2^*}_{2^*} + |z_n|^{2^*}_{2^*} + o(1). \tag{8}$$

Moreover, since $J^{b\prime}_{\beta,\eta}(w_n)(w_n) \to 0$, it follows that

$$(\mathscr{B}^b_\beta - \eta)(z_n, z_n) - |z_n|^{2^*}_{2^*} \to 0. \tag{9}$$

Using Lemma 2.3 and equations (8)–(9), we conclude

$$(\mathscr{B}^b_\beta - \eta)(z_n, z_n) \to b \quad \text{and} \quad |z_n|^{2^*}_{2^*} \to b \leq NC < S^{N/2}.$$

As in the proof of Lemma 2.1, one can show that

$$\int_F |z_n|^2 \to 0 \quad \text{as } n \to \infty,$$

where $F := \{x \in \mathbb{R}^N : a(x) \leq M_0\}$, and $F^c = \mathbb{R}^N \setminus F$.

Now, by the definition of the best Sobolev constant S in (7), we estimate:

$$\begin{aligned} S|z_n|_{2^*}^2 &\leq \int_{\mathbb{R}^N} b(x)|\nabla z_n|^2 \\ &\leq \int_{\mathbb{R}^N} b(x)|\nabla z_n|^2 + \int_{F^c} (\beta a(x) - \eta) z_n^2 \\ &\leq (\mathscr{B}_\beta^b - \eta)(z_n, z_n) + \eta \int_F z_n^2 \\ &= (\mathscr{B}_\beta^b - \eta)(z_n, z_n) + o(1). \end{aligned}$$

Taking limits yields:

$$S b^{2/2^*} \leq b.$$

Since $b < S^{N/2}$, it follows that $b = 0$, and therefore $z_n \to 0$ strongly in E_b. Hence, $w_n \to w$ in E_b, completing the proof.

3. Existence via the Nehari Manifold: Proof of Theorems 1.1 and 1.3

We consider the Nehari manifold associated with the functional $J_{\beta,\eta}^b$, defined by

$$\begin{aligned} \mathscr{M}_{\beta,\eta}^b &= \left\{ w \in E_b \setminus \{0\} : J_{\beta,\eta}^{b\prime}(w)(w) = 0 \right\} \\ &= \left\{ w \in E_b \setminus \{0\} : (\mathscr{B}_\beta^b - \eta)(w, w) = \|w\|_{2^*}^{2^*} \right\}. \end{aligned} \tag{10}$$

The Nehari manifold $\mathscr{M}_{\beta,\eta}^b$ is radially diffeomorphic to the unit sphere in E_b with respect to the L^{2^*}-norm, namely,

$$\mathscr{V} := \{v \in E_b : \|v\|_{2^*} = 1\},$$

via the mapping

$$\mathscr{V} \to \mathscr{M}_{\beta,\eta}^b, \qquad v \mapsto \left[(\mathscr{B}_\beta^b - \eta)(v, v)\right]^{\frac{N-2}{4}} v.$$

For each $w \in \mathscr{M}_{\beta,\eta}^b$, the energy functional reduces to

$$J_{\beta,\eta}^b(w) = \frac{1}{N} \left(\mathscr{B}_\beta^b - \eta\right)(w, w).$$

Therefore, the least energy level of $J_{\beta,\eta}^b$ over the Nehari manifold is given by

$$c_{\beta,\eta}^b := \inf_{w \in \mathscr{M}_{\beta,\eta}^b} J_{\beta,\eta}^b(w) = \frac{1}{N} \inf_{v \in \mathscr{V}} \left(\left(\mathscr{B}_\beta^b - \eta\right)(v, v)\right)^{N/2}.$$

Lemma 3.1. *Let $\mathscr{V}_b := \{v \in E_b : \|v\|_{2^*} = 1\}$ and define the Nehari manifold*

$$\mathscr{M}_{\beta,\eta}^b := \left\{ w \in E_b \setminus \{0\} : (\mathscr{B}_\beta^b - \eta)(w, w) = \|w\|_{2^*}^{2^*} \right\}.$$

Then, there exists a homeomorphism $\Phi : \mathscr{V}_b \to \mathscr{M}_{\beta,\eta}^b$, and thus $\mathscr{V}_b \simeq \mathscr{M}_{\beta,\eta}^b$.

Proof. Define the mapping $\Phi : \mathscr{V}_b \to \mathscr{M}^b_{\beta,\eta}$ by

$$\Phi(v) := \left[(\mathscr{B}^b_\beta - \eta)(v,v)\right]^{\frac{N-2}{4}} v.$$

Set $c_0 := \left[(\mathscr{B}^b_\beta - \eta)(v,v)\right]^{\frac{N-2}{4}}$ so that $w = \Phi(v) = c_0 v$. Then:

$$\begin{aligned}(\mathscr{B}^b_\beta - \eta)(w,w) &= (\mathscr{B}^b_\beta - \eta)(c_0 v, c_0 v)\\ &= c_0^2(\mathscr{B}^b_\beta - \eta)(v,v)\\ &= \left[(\mathscr{B}^b_\beta - \eta)(v,v)\right]^{\frac{N}{2}} = \|w\|^{2^*}_{2^*},\end{aligned}$$

since $\|w\|_{2^*} = c_0\|v\|_{2^*} = c_0$.

Hence, $w \in \mathscr{M}^b_{\beta,\eta}$. The mapping Φ is continuous and invertible, with inverse $\Phi^{-1}(w) = w/\|w\|_{2^*} \in \mathscr{V}_b$. Thus, Φ is a homeomorphism.

Lemma 3.2. *Show that $\mathscr{M}^b_{\beta,\eta} \neq \emptyset$.*

Proof. Let $w \in \mathscr{H}^1_{0,b}$ be a nonzero function. Then, there exists some $t > 0$ such that $tw \in \mathscr{M}^b_{\beta,\eta}$, where $\mathscr{M}^b_{\beta,\eta}$ denotes the Nehari manifold associated with the problem.

From the definitions, we compute:

$$(\mathscr{B}^b_\beta - \eta)(tw,tw) = |tw|^{2^*}_{2^*} = t^{2^*}\|w\|^{2^*}_{2^*} = t^{2^*}L_1, \tag{11}$$

$$(\mathscr{B}^b_\beta - \eta)(tw,tw) = t^2\left(\int b(x)|\nabla w|^2 + \beta\int a(x)w^2 - \eta\int w^2\right) = t^2 L_2. \tag{12}$$

Equating (11) and (12), we get:

$$t^2 L_2 = t^{2^*}L_1 \quad \Rightarrow \quad t = \left(\frac{L_2}{L_1}\right)^{\frac{1}{2^*-2}}.$$

Thus, for $t = \left(\frac{L_2}{L_1}\right)^{\frac{1}{2^*-2}}$, we have $tw \in \mathscr{M}^b_{\beta,\eta}$. Consequently, $\mathscr{M}^b_{\beta,\eta} \neq \emptyset$.

If $\mathscr{M}^b_{\beta,\eta} \neq \emptyset$ then $J^b_{\beta,\eta} \neq +\infty$

Proposition 3.1. *If $w \in \mathscr{M}^b_{\beta,\eta}$ is a critical point of $J^b_{\beta,\eta}$ and satisfies $J^b_{\beta,\eta}(w) < 2c^b_{\beta,\eta}$, then w does not change sign. Consequently, $|w|$ is a solution of the problem $(PS^b_{\beta,\eta})$.*

Proof. Since w is a critical point of $J^b_{\beta,\eta}$, we have the relation

$$(\mathscr{B}^b_\beta - \eta)(w,v) = \int_{\mathbb{R}^N} |w|^{2^*-2}wv\,dx$$

for every $v \in E_b$. In particular, this holds for $v = w^\pm$, where $w^\pm = \max\{\pm w, 0\}$.

If both w^+ and w^- are nonzero, then $w^\pm \in \mathscr{M}^b_{\beta,\eta}$, and it follows that

$$J^b_{\beta,\eta}(w) = J^b_{\beta,\eta}(w^+) + J^b_{\beta,\eta}(w^-) \geq 2c^b_{\beta,\eta},$$

which leads to a contradiction.

Similarly, for every domain $D \in \mathbb{R}^N$, we define the functional

$$J^b_{\eta,D}(w) = \frac{1}{2}\int_D (b(x)|\nabla w|^2 - \eta w^2) - \frac{1}{2^*}\int_D |w|^{2^*}$$
$$= \frac{1}{2}(\mathscr{B}^b_0 - \eta)(w)(w) - \frac{1}{2^*}|w|^{2^*}_{2^*} \quad \text{on} \quad \mathscr{H}^1_{0,b}(D),$$

and is associated with the problem (D_η). The corresponding Nehari manifold is defined as

$$\mathscr{M}^b_{\eta,D} = \{w \in \mathscr{H}^1_{0,b}(D) \setminus \{0\} : (\mathscr{B}^b_0 - \eta)(w)(w) = |w|^{2^*}_{2^*}\}.$$

This manifold is radially diffeomorphic to $\mathscr{V}_{b,D} = \{v \in \mathscr{H}^1_{0,b}(D), |v|_{2^*} = 1\}$. Set

$$c^b(\eta, D) := \inf_{w \in \mathscr{M}^b_{\eta,D}} J^b_{\eta,D}(w) = \frac{1}{N} \inf_{v \in \mathscr{V}_D} ((\mathscr{B}^b_0 - \eta)(v)(v))^{N/2}.$$

Lemma 3.3. *Let $\eta \in (0, \eta_1(\Omega))$ and $\beta \geq \beta(\eta)$. Then*

$$\frac{1}{N}(\alpha_\eta S)^{N/2} \leq c^b_{\beta,\eta} < c^b(\eta, \Omega) < \frac{1}{N} S^{N/2}.$$

Proof. By Lemma 2.2, we have the inequality

$$\alpha_\eta \|v\|^2_b \leq \alpha_\eta \|v\|^2_{b,\beta} \leq (\mathscr{B}^b_\beta - \eta)(v, v).$$

Taking the infimum over $v \in \mathscr{V}_b$ yields the first inequality. Since $\mathscr{V}_{b,\Omega} \subset \mathscr{V}_b$ and $(\mathscr{B}^b_\beta)(v, v) = (A^b_0)(v, v)$ for all $v \in \mathscr{V}_{b,\Omega}$, it follows that

$$c^b_{\beta,\eta} \leq c^b(\eta, \Omega).$$

Moreover, Brézis and Nirenberg [8] showed that for every $\eta \in (0, \eta_1(\Omega))$, we have $c^b(\eta, \Omega) < \frac{1}{N} S^{N/2}$, and that $c^b(\eta, \Omega)$ is attained at some function $\tilde{w} > 0$.

If $c^b_{\beta,\eta} = c^b(\eta, \Omega)$, then this minimum would be achieved at $\tilde{w}$. However, since $\tilde{w}$ vanishes outside Ω, this contradicts the strong maximum principle. Therefore, it must hold that $c^b_{\beta,\eta} < c^b(\eta, \Omega)$.

We are now prepared to demonstrate the validity of Theorems 1.1 and 1.3.

Proof (**Proof of Theorem 1.1**). Let $\{w^\beta_n\} \subset \mathscr{M}^b_{\beta,\eta}$ be a minimizing sequence for $J^b_{\beta,\eta}$, i.e.,

$$J^b_{\beta,\eta}(w^\beta_n) \to c^b_{\beta,\eta} \quad \text{as } n \to \infty.$$

By Ekeland's variational principle [15; 27], we may assume that $\{w^\beta_n\}$ is a Palais-Smale sequence at level $c^b_{\beta,\eta}$.

Then, by Proposition 2.1, the $(PS)_c$ condition holds for all $c < \frac{1}{N} S^{N/2}$. Moreover, by Lemma 3.3, we have $c^b_{\beta,\eta} < \frac{1}{N} S^{N/2}$, ensuring compactness.

Therefore, up to a subsequence, $w^\beta_n \to w_\beta$ strongly in E_b, and w_β is a least-energy critical point of $J^b_{\beta,\eta}$. Thus, w_β is a least-energy solution to the problem $(PS^b_{\beta,\eta})$.

Proof (**Proof of Theorem 1.3**). Let $\{w_n\}$ be a sequence of solutions to $(PS^b_{\beta_n,\eta})$ such that $\eta \in (0, \eta_1(\Omega))$, $\beta_n \to \infty$, and

$$J^b_{\beta_n,\eta}(w_n) = \frac{1}{N}(\mathscr{B}^b_{\beta_n} - \eta)(w_n, w_n) \to c.$$

Assume further that the energy satisfies

$$NJ^b_{\beta_n,\eta}(w_n) - (\mathscr{B}^b_{\beta_n} - \eta)(w_n, w_n) \to Nc < S^{N/2}.$$

Then, by Lemmas 2.1 and 2.2, there exists $w \in \mathscr{H}^1_{0,b}(\Omega)$ such that, up to a subsequence, $w_n \rightharpoonup w$ weakly in E_b and $w_n \to w$ strongly in $L^2(\mathbb{R}^N)$. Since each w_n satisfies the Euler-Lagrange equation,

$$\int_{\mathbb{R}^N} (b(x)\nabla w_n \cdot \nabla v + \beta_n a w_n v - \eta w_n v) = \int_{\mathbb{R}^N} |w_n|^{2^*-2} w_n v, \quad \forall v \in E_b,$$

taking the limit with test functions $v \in \mathscr{H}^1_{0,b}(\Omega)$ yields

$$\int_{\mathbb{R}^N} (b(x)\nabla w \cdot \nabla v + \eta w v) = \int_{\mathbb{R}^N} |w|^{2^*-2} w v.$$

Hence, w solves (D^b_η).

Now let $t_n := w_n - w$. Using Brezis-Lieb lemma [7], and orthogonality of cross-terms, we obtain

$$(\mathscr{B}^b_{\beta_n} - \eta)(w_n, w_n) = (\mathscr{B}^b_0 - \eta)(w, w) + (\mathscr{B}^b_{\beta_n} - \eta)(t_n, t_n) + o(1),$$

and

$$|w_n|^{2^*}_{2^*} = |w|^{2^*}_{2^*} + |t_n|^{2^*}_{2^*} + o(1).$$

From the Nehari condition and energy identity, we conclude:

$$(\mathscr{B}^b_{\beta_n} - \eta)(t_n, t_n) - |t_n|^{2^*}_{2^*} \to 0.$$

Assume by contradiction that $|t_n|^{2^*}_{2^*} \to b > 0$. Then Sobolev inequality implies

$$S|t_n|^2_{2^*} \le \|\nabla t_n\|^2_2 \le (\mathscr{B}^b_{\beta_n} - \eta)(t_n, t_n) + o(1) = |t_n|^{2^*}_{2^*} + o(1),$$

which leads to $S^{N/2} \le c < S^{N/2}$, a contradiction. Therefore, $|t_n|_{2^*} \to 0$ and

$$(\mathscr{B}^b_{\beta_n} - \eta)(t_n, t_n) \to 0.$$

Consequently,

$$(\mathscr{B}^b_0 - \eta)(w, w) = \lim_{n\to\infty} (\mathscr{B}^b_{\beta_n} - \eta)(w_n, w_n),$$

and since $a(x) = 0$ in Ω, it follows that $\int a w_n^2 \to 0$. Thus, $w_n \to w$ in E_b, and the sequence concentrates at a solution of (D^b_η).

As a consequence of theorems 1.1 and 1.3, we obtain:

Corollary 3.1. *For each* $\eta \in (0, \eta_1(\Omega))$, *we have*

$$\lim_{\beta\to\infty} c^b_{\beta,\eta} = c^b(\eta, \Omega).$$

4. Proof of Theorem 1.2

To prove Theorem 1.2, we adopt the topological method introduced by Benci and Cerami [6]. Since $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain, there exists a small radius $r > 0$ such that the following inclusions hold:

$$\Omega_{2r}^{+} := \left\{x \in \mathbb{R}^N : \operatorname{dist}(x, \Omega) < 2r\right\},$$
$$\Omega_{r}^{-} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r\},$$

and both sets Ω_{2r}^{+} and Ω_{r}^{-} are homotopically equivalent to Ω. Moreover, we can assume that the open ball $B_r := \{x \in \mathbb{R}^N : |x| < r\}$ is contained in Ω.

Using the argument developed in the proof of Lemma 3.3, it follows that

$$c^b(\eta, \Omega) < c^b(\eta, B_r) < \frac{1}{N} S^{N/2},$$

for every $\eta \in (0, \eta_1(\Omega))$.

For any nonzero function $w \in L^{2^*}(\Omega)$, we define its *center of mass* by

$$\beta(w) := \frac{\int_\Omega |w|^{2^*} x\, dx}{\int_\Omega |w|^{2^*}\, dx}.$$

Recalling a result of Lazzo [19], we state the following lemma:

Lemma 4.1. *There exists a constant* $\eta^\sharp = \eta^\sharp(r) \in (0, \eta_1(\Omega))$ *such that for all* $\eta \in (0, \eta^\sharp]$, *the following statements hold:*

(i) $c^b(\eta, B_r) < 2c^b(\eta, \Omega)$,

(ii) For every $w \in \mathscr{M}^b_{\eta,\Omega}$ *satisfying* $J^b_{\eta,\Omega}(w) \le c^b(\eta, B_r)$, *the center of mass* $\beta(w)$ *lies in* Ω_r^+.

As in [4], we choose $R > 0$ with $\bar{\Omega} \subset B_R$ and set

$$\xi(t) = \begin{cases} 1, & 0 \le t \le R, \\ R/t, & R \le t. \end{cases}$$

Define

$$\beta_0(w) := \frac{\int_{\mathbb{R}^N} \xi(|w|) x dx}{\int_{\mathbb{R}^N} |w|^{2^*} dx} \quad \text{for} \quad w \in L^{2^*}(\mathbb{R}^N \setminus \{0\})$$

Lemma 4.2. *There exists a constant* $\eta^* = \eta^*(r) \in (0, \eta_1(\Omega))$ *and, for each* $\eta \in (0, \eta^*]$, *a number* $\Lambda(\eta) \ge \beta(\eta)$ *such that:*

(i) $c^b(\eta, B_r) < 2c^b_{\beta,\eta}$ *for all* $\beta \ge \Lambda(\eta)$,

(ii) For all $\beta \ge \Lambda(\eta)$ *and for every* $w \in \mathscr{M}^b_{\beta,\eta}$ *with* $J^b_{\beta,\eta}(w) \le c^b(\eta, B_r)$, *we have* $\beta_0(w) \in \Omega^+_{2r}$.

Proof. Assertion (i) follows directly from Lemma 4.1 and Corollary 3.1, which imply:

$$c^b(\eta, B_r) < 2c^b(\eta, \Omega) = 2 \lim_{\beta\to\infty} c^b_{\beta,\eta} \le 2c^b_{\beta,\eta},$$

for all $\beta \ge \Lambda(\eta)$.

We now prove (ii) by contradiction. Suppose that, for arbitrarily small η, there exists a sequence $\{w_n\}$ with $w_n \in \mathscr{M}^b_{\beta_n,\eta}$, $\beta_n \to \infty$, $J^b_{\beta_n,\eta}(w_n) \le c^b(\eta, B_r)$, but $\beta_0(w_n) \notin \Omega^+_{2r}$.

By Lemma 2.1, up to a subsequence, $w_n \rightharpoonup w_\eta$ weakly in E_b, and $w_n \to w_\eta$ in $L^2(\mathbb{R}^N)$.

We consider two cases:

Case 1: $|w_\eta|^{2^*}_{2^*} \le (\mathscr{B}^b_0 - \eta)(w_\eta, w_\eta)$.

Define $t_n := w_n - w_\eta$. Since $a(x) = 0$ in Ω, we have:

$$(\mathscr{B}^b_{\beta_n} - \eta)(w_n, w_n) = (\mathscr{B}^b_0 - \eta)(w_\eta, w_\eta) + (\mathscr{B}^b_{\beta_n} - \eta)(t_n, t_n) + o(1),$$

and by the Brézis–Lieb Lemma [7]:

$$|w_n|^{2^*}_{2^*} = |w_\eta|^{2^*}_{2^*} + |t_n|^{2^*}_{2^*} + o(1).$$

Since $w_n \in \mathscr{M}^b_{\beta_n,\eta}$, we obtain:

$$(\mathscr{B}^b_{\beta_n} - \eta)(t_n, t_n) \le |t_n|^{2^*}_{2^*} + o(1).$$

Assume by contradiction that $|t_n|^{2^*}_{2^*} \to b > 0$. Then:

$$S|t_n|^2_{2^*} \le \int b(x)|\nabla t_n|^2 \le (\mathscr{B}^b_{\beta_n} - \eta)(t_n, t_n) + o(1) \le |t_n|^{2^*}_{2^*} + o(1),$$

which implies $S^{N/2} \le \lim_{n\to\infty} |w_n|^{2^*}_{2^*} < S^{N/2}$, a contradiction.

Therefore, $|t_n|_{2^*} \to 0$, and hence $w_n \to w_\eta$ in $L^{2^*}(\mathbb{R}^N)$. It follows that $\beta_0(w_n) \to \beta(w_\eta)$. Since $J^b_{\eta,\Omega}(w_\eta) \le \lim J^b_{\beta_n,\eta}(w_n) \le c^b(\eta, B_r)$, Lemma 4.1 implies $\beta(w_\eta) \in \Omega^+_r$. This contradicts the assumption $\beta_0(w_n) \notin \Omega^+_{2r}$.

Case 2: $|w_\eta|^{2^*}_{2^*} > (\mathscr{B}^b_0 - \eta)(w_\eta, w_\eta)$.

Then there exists $t \in (0, 1)$ such that $tw_\eta \in \mathscr{M}^b_{\eta,\Omega}$, and hence

$$\begin{aligned} c^b(\eta, \Omega) \le J^b_{\eta,\Omega}(tw_\eta) &= \frac{t^2}{N}(\mathscr{B}^b_0 - \eta)(w_\eta, w_\eta) \\ &< \lim_{n\to\infty} J^b_{\beta_n,\eta}(tw_n) \le c^b(\eta, B_r). \end{aligned}$$

Thus, for n sufficiently large,

$$\big||w_n|^{2^*}_{2^*} - |tw_\eta|^{2^*}_{2^*}\big| < N(c^b(\eta, B_r) - c^b(\eta, \Omega)),$$

and since $c^b(\eta, B_r) - c^b(\eta, \Omega) \to 0$ as $\eta \to 0$, it follows that:

$$|\beta_0(w_n) - \beta(tw_\eta)| < r.$$

However, by Lemma 4.1, $\beta(tw_\eta) \in \Omega^+_r$, contradicting $\beta_0(w_n) \notin \Omega^+_{2r}$. This contradiction completes the proof of (ii).

For a given function $I : M \to \mathbb{R}$, the set $I^{\leq t}$ is defined as:

$$I^{\leq t} = \{z \in M : I(z) \leq t\}.$$

which represents the level set of all points $z \in M$ where $I(z)$ does not exceed t.

The following is an easy consequence of Lusternik-Schnirelmann theory:

Proposition 4.1. *Let $J : M \to \mathbb{R}$ be an even C^1-functional defined on a complete, symmetric, $C^{1,1}$ submanifold $M \subset V \setminus \{0\}$ of a Banach space V. Assume that J is bounded below and satisfies the Palais-Smale condition $(PS)_c$ for all $c \leq t$. Further, suppose there exist continuous maps*

$$X \xrightarrow{L} J^{\leq t} \xrightarrow{\beta} Y$$

such that the composition $\beta \circ L$ is a homotopy equivalence and $\beta(z) = \beta(-z)$ for all $z \in M \cap J^{\leq t}$. Then J admits at least $Cat(X)$ pairs of distinct critical points $\{z, -z\}$ with $J(z) = J(-z) \leq t$.

Proof (**Proof of Theorem 1.2**). Let $0 < \eta < \eta^*$ and $\beta \geq \Lambda(\eta)$. We define the maps

$$\Omega_r^- \xrightarrow{L} \mathscr{M}^b_{\beta,\eta} \cap J^{\leq c^b(\eta,B_r)}_{\beta,\eta} \xrightarrow{\beta_0} \Omega^+_{2r},$$

where β_0 denotes the barycenter map introduced earlier. Lemma 4.2 ensures that β_0 is well defined on this subset of the Nehari manifold.

Let $w_r \in \mathscr{H}^1_{0,b}(B_r) \subset E_b$ be a positive minimizer of J^b_{η,B_r} over $\mathscr{M}^b_{\eta,B_r}$. For each $x \in \Omega_r^-$, define the translated function $l(x) := w_r(\cdot - x)$. Since $B_r(x) \subset \Omega$, we have $l(x) \in \mathscr{M}^b_{\beta,\eta}$ and

$$J^b_{\beta,\eta}(l(x)) = J^b_{\eta,B_r}(w_r) = c^b(\eta, B_r).$$

Moreover, the radial symmetry of w_r yields $\beta_0(l(x)) = x$ for all $x \in \Omega_r^-$. Thus, the composition $\beta_0 \circ L$ is the identity map on Ω_r^- and therefore a homotopy equivalence.

Additionally, since $J^b_{\beta,\eta}(w) = J^b_{\beta,\eta}(-w)$ and $\beta_0(w) = \beta_0(-w)$ for all $w \in E_b \setminus \{0\}$, the symmetry conditions of Proposition 4.1 are satisfied.

By the inequality $c^b(\eta, B_r) < \frac{1}{N} S^{N/2}$ from [8], Proposition 2.1 guarantees that $J^b_{\beta,\eta}$ satisfies the $(PS)_c$ condition for all $c \leq c^b(\eta, B_r)$. Consequently, Proposition 4.1 applies. Invoking also Proposition 3.1 (positivity of critical points) and Lemma 4.2, we deduce that the problem $(NS^b_{\beta,\eta})$ possesses at least $Cat(\Omega)$ positive solutions.

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Some Properties of Hermite-General Matrix Polynomials

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Abstract

In this paper, the Hermite-general matrix polynomials are introduced by using certain operational methods. Also, the differential equation, recurrence relations and other properties for the Hermite-general matrix polynomials are obtained within the context of the monomiality principle. Further, it is shown that these matrix polynomials include many other new special matrix polynomial as particular cases.

Keywords: Hermite-general matrix polynomials, Monomiality principle, Operational

2020 MSC: 33E20; 32W99

1. Introduction

Hermite polynomials are frequently used in many branches of pure and applied mathematics and physics. The importance of multi-variable Hermite polynomials has been recognized in [9] and these polynomials have been used to deal with quantum mechanical and optical beam transport problems. The Hermite matrix polynomials and their extensions and generalizations have been introduced and studied in [2,15,16,18-20, 22-26] for matrices in $\mathbb{C}^{n\times n}$ $(n \in \mathbb{N})$ whose eigenvalues are all situated in the right open half-plane.

We review the definitions and the concepts related to the Hermite matrix polynomials.

Let A be a matrix in $\mathbb{C}^{n\times n}$ such that

$$Re(\mu) > 0, \quad \text{for all } \mu \in \sigma(A), \tag{1.1}$$

where $\sigma(A)$ denotes the set of all the eigenvalues of A. If D_0 is the complex plane cut along the negative real axis and $log(z)$ denotes the principal logarithm of z, then $z^{1/2}$ represents $\exp(\frac{1}{2}log(z))$. If the matrix $A \in \mathbb{C}^{n\times n}$ with $\sigma(A) \subset D_0$, then $A^{1/2} = \sqrt{A}$ denotes the image by $z^{1/2}$ of the matrix functional calculus acting on the matrix A.

We recall that the 2-variable Hermite matrix polynomials (2VHMP) $H_n(x, y, A)$

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are defined by the series [2; p.84]

$$H_n(x,y,A) = n!\sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k y^k (x\sqrt{2A})^{n-2k}}{(n-2k)!k!} \qquad (n \geq 0) \tag{1.2}$$

and specified by the generating function

$$\exp(xt\sqrt{2A} - yt^2 I) = \sum_{n=0}^{\infty} H_n(x,y,A)\frac{t^n}{n!}. \tag{1.3}$$

Also, the 2-variable Hermite matrix polynomials (2VHMP) of the second form $\mathcal{H}_n(x,y;A)$ are defined by the series [23; p.162]

$$\mathcal{H}_n(x,y;A) = n!\sum_{k=0}^{[\frac{n}{2}]} \frac{y^k \left(x\sqrt{\frac{A}{2}}\right)^{n-2k}}{(n-2k)!k!} \qquad (n \geq 0) \tag{1.4}$$

and specified by the generating function

$$\exp\left(xt\sqrt{\frac{A}{2}} + yt^2 I\right) = \sum_{n=0}^{\infty} \mathcal{H}_n(x,y;A)\frac{t^n}{n!}. \tag{1.5}$$

The 2-index 2-variable Hermite matrix polynomials (2I2VHMP) $H_{n,m}(x,y,A)$ are defined by the series [25; p.689]

$$H_{n,m}(x,y,A) = n!\sum_{k=0}^{[\frac{n}{m}]} \frac{(-1)^k y^k (x\sqrt{mA})^{n-mk}}{(n-mk)!k!} \qquad (n \geq 0) \tag{1.6}$$

and specified by the generating function

$$\exp(xt\sqrt{mA} - yt^m I) = \sum_{n=0}^{\infty} H_{n,m}(x,y,A)\frac{t^n}{n!}. \tag{1.7}$$

We note that the 2I2VHMP $H_{n,m}(x,y,A)$ are also defined through the operational rule [25; p.699]

$$H_{n,m}(x,y,A) = \exp\left(-y(\sqrt{mA})^{-m}\frac{\partial^m}{\partial x^m}\right)\left\{(x\sqrt{mA})^n\right\}. \tag{1.8}$$

Very recently, the 3-index 3-variable Hermite matrix polynomials (3I3VHMP) $H_n^{(m,s)}(x,y,z;A)$ are introduced, which are defined by the series [20]

$$H_n^{(m,s)}(x,y,z;A) = n!\sum_{k=0}^{[\frac{n}{s}]} \frac{z^k H_{n-sk,m}(x,y;A)}{k!(n-sk)!} \tag{1.9}$$

and specified by the generating function

$$\exp(xt\sqrt{mA} - yt^m I + zt^s I) = \sum_{n=0}^{\infty} H_n^{(m,s)}(x,y,z;A)\frac{t^n}{n!}. \tag{1.10}$$

The 3I3VHMP $H_n^{(m,s)}(x,y,z;A)$ are also defined through the operational rule [20]

$$H_n^{(m,s)}(x,y,z;A) = \exp\left(z(\sqrt{mA})^{-s}\frac{\partial^s}{\partial x^s}\right)\Big\{H_{n,m}(x,y,A)\Big\}. \tag{1.11}$$

We recall that according to the monomiality principle [4,28], a polynomial set $p_n(x)$ $(n \in \mathbb{N}, x \in \mathbb{C})$ is "quasi-monomial", provided there exist two operators $\hat{M}$ and $\hat{P}$ playing respectively, the role of multiplicative and derivative operators, for the family of polynomials. These operators satisfy the following identities for all $n \in \mathbb{N}$:

$$\hat{M}\{p_n(x)\} = p_{n+1}(x), \tag{1.12a}$$

$$\hat{P}\{p_n(x)\} = np_{n-1}(x). \tag{1.12b}$$

The operators $\hat{M}$ and $\hat{P}$ also satisfy the commutation relation

$$[\hat{P}, \hat{M}] = \hat{1} \tag{1.13}$$

and thus display a Weyl group structure. If $\hat{M}$ and $\hat{P}$ have differential realization, then the differential equation satisfied by $p_n(x)$ is

$$\hat{M}\hat{P}\{p_n(x)\} = np_n(x). \tag{1.14}$$

Assuming here and in the sequel $p_0(x) = 1$, then $p_n(x)$ can be explicity constructed as:

$$p_n(x) = \hat{M}^n\{1\} \tag{1.15}$$

and consequently the generating function of $p_n(x)$ can be cast in the form

$$G(x,t) = \exp(t\hat{M})\{1\} = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!} \qquad (|t| < \infty). \tag{1.16}$$

We note that the 2I2VHMP $H_{n,m}(x,y,A)$ are quasi-monomial under the action of the operators [24; p.43]

$$\hat{M} := x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}, \tag{1.17}$$

$$\hat{P} := \frac{1}{\sqrt{mA}}\frac{\partial}{\partial x}. \tag{1.18}$$

Also, the 3I3VHMP $H_n^{(m,s)}(x,y,z;A)$ are quasi-monomial under the action of the operators [20]

$$\hat{M} := x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + sz(\sqrt{mA})^{-(s-1)}\frac{\partial^{s-1}}{\partial x^{s-1}}, \tag{1.19}$$

$$\hat{P} := (\sqrt{mA})^{-1}\frac{\partial}{\partial x}. \tag{1.20}$$

The special polynomials of two variables are important from the point of view of applications and also these polynomials are helpful in introducing new families of special polynomials. We consider a general family of the polynomials of two variables, namely the 2-variable general polynomials (2VGP) denoted by $p_n(x,y)$ and defined by the generating function

$$e^{xt}\phi(y,t) = \sum_{n=0}^{\infty} p_n(x,y)\frac{t^n}{n!} \qquad (p_0(x,y)=1), \tag{1.21}$$

where $\phi(y,t)$ has (at least the formal) series expansion

$$\phi(y,t) = \sum_{n=0}^{\infty} \phi_n(y)\frac{t^n}{n!} \qquad (\phi_0(y)\neq 0). \tag{1.22}$$

Now, we recall that the 2-variable Appell polynomials (2VAP) $P_n(x,y)$ [7], the Gould-Hopper polynomials (GHP) $H_n^{(s)}(x,y)$ [17], the 2D Appell polynomials (2DAP) $R_n^{(s)}(x,y)$ [3] and the 2-variable generalized Laguerre polynomials (2VGLP) ${}_mL_n(x,y)$ [10] are defined by the generating functions

$$A(yt)\ e^{xt} = \sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{n!}, \tag{1.23}$$

$$e^{xt+yt^s} = \sum_{n=0}^{\infty} H_n^{(s)}(x,y)\frac{t^n}{n!}, \tag{1.24}$$

$$A(t)\ e^{xt+yt^s} = \sum_{n=0}^{\infty} R_n^{(s)}(x,y)\frac{t^n}{n!} \tag{1.25}$$

and

$$e^{yt}\ C_0(-xt^m) = \sum_{n=0}^{\infty} {}_mL_n(x,y)\frac{t^n}{n!}, \tag{1.26}$$

respectively, where $C_0(x)$ denotes the 0^{th} order Tricomi function. The n^{th} order Tricomi functions $C_n(x)$ are defined as [27]:

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!(n+r)!}. \tag{1.27}$$

The 0^{th} order Tricomi function $C_0(\alpha x)$ is also defined by the operational rule

$$C_0(\alpha x) = \exp(-\alpha \hat{D}_x^{-1})\{1\}, \tag{1.28}$$

where $\hat{D}_x^{-1}$ denotes the inverse of the derivative operator $\hat{D}_x := \frac{\partial}{\partial x}$ and is given as:

$$\hat{D}_x^{-n}\{1\} = \frac{x^n}{n!} \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.29}$$

In view of generating functions (1.21), (1.23)-(1.25) and (1.26), we note that the 2VAP $P_n(x,y)$, the GHP $H_n^{(m)}(x,y)$, the 2DAP $R_n^{(s)}(x,y)$ and the 2VGLP ${}_mL_n(y,x)$ belong to family of 2VGP $p_n(x,y)$.

It is worth to mention that for $y=1$ and $y=0$, the 2VAP $P_n(x,y)$ and the 2DAP $R_n^{(s)}(x,y)$ respectively, reduce to the Appell polynomials $A_n(x)$ [1], i.e., we have

$$P_n(x,1) = A_n(x), \tag{1.30}$$

$$R_n^{(s)}(x,0) = A_n(x). \tag{1.31}$$

The Appell polynomials $A_n(x)$ are defined by the generating function

$$A(t)\ e^{xt} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}, \tag{1.32}$$

where $A(t)$ has (at least the formal) expansion

$$A(t) = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!} \qquad (A_0 \neq 0). \tag{1.33}$$

The use of operational identities [4-6, 8, 10, 11, 12-14], currently exploited in the theory of algebraic decomposition of exponential operators, may significantly simplify the study of Hermite matrix generating functions and the discovery of new relations. Recently, Metwally [24] introduced generalized forms of operational rules associated with operators corresponding to the 2I2VHMP $H_{n,m}(x,y,A)$ expansions. The associated generating function is reformulated within the framework of an operational formalism and the theory of exponential operators.

In this paper, the Hermite-general matrix polynomials are introduced by making use of operational identities for decoupling of the exponential operators. The concepts associated with monomiality principle are used to establish their properties. Examples of some members of the Hermite-general matrix polynomials family are considered and certain results for these polynomials are derived.

2. Hermite-general matrix polynomials

To generate the Hermite-general matrix polynomials (HGMP), we take the 2I2VHMP $H_{n,m}(x,y,A)$ as base in the generating function (1.21) of the 2VGP $p_n(x,z)$. Denoting the HGMP by ${}_Hp_{n,m}(x,y,z;A)$, we consider the generating function

$$\exp(\hat{M}t)\ \phi(z,t) = \sum_{n=0}^{\infty} {}_Hp_{n,m}(x,y,z;A)\frac{t^n}{n!}, \tag{2.1}$$

which is the result of replacement of x in equation (1.21) by the multiplicative operator $\hat{M}$ of the 2I2VHMP $H_{n,m}(x,y,A)$.

Using the expression of $\hat{M}$ given in equation (1.17), we find

$$\exp\left(\left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}\right)t\right)\phi(z,t) = \sum_{n=0}^{\infty} {}_Hp_{n,m}(x,y,z;A)\frac{t^n}{n!}, \tag{2.2}$$

which on using the Crofton-type identity [11]

$$f\left(x + m\lambda\frac{d^{m-1}}{dx^{m-1}}\right)\{1\} = \exp\left(\lambda\frac{d^m}{dx^m}\right)\Big\{f(x)\Big\}, \tag{2.3}$$

to decouple the exponential operator in the l.h.s. gives the generating function for the HGMP ${}_Hp_{n,m}(x,y,z;A)$ in the following form:

$$\exp(xt\sqrt{mA} - yt^m I)\ \phi(z,t) = \sum_{n=0}^{\infty} {}_Hp_{n,m}(x,y,z;A)\frac{t^n}{n!}, \tag{2.4}$$

where m, s are both positive integers and A is a matrix in $\mathbb{C}^{n\times n}$ satisfying condition (1.1).

Next, we proceed to find the series definition of the HGMP ${}_Hp_{n,m}(x,y,z;A)$. Breaking the exponential in the l.h.s. of equation (2.4) and then using definition (1.21) and expanding the exponential term in the resultant equation, we find

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} p_n(x\sqrt{mA},z)(-y)^k\frac{t^{n+mk}}{n!k!} = \sum_{n=0}^{\infty} {}_Hp_{n,m}(x,y,z;A)\frac{t^n}{n!},$$

which on replacing n by $n-mk$ in the l.h.s. and then using the lemma [27]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{r}]} A(k,n-rk), \tag{2.5}$$

gives the following series definition of the HGMP ${}_Hp_{n,m}(x,y,z;A)$:

$${}_Hp_{n,m}(x,y,z;A) = n!\sum_{n=0}^{[\frac{n}{m}]}\frac{(-1)^k y^k\ p_{n-mk}(x\sqrt{mA},z)}{k!(n-mk)!}. \tag{2.6}$$

Expanding $\phi(z,t)$ in equation (2.4) by using equation (1.22) and then using equation (1.7) in the l.h.s. of the resultant equation, we find (after equating the coefficients of like powers of t)

$$ {}_Hp_{n,m}(x,y,z;A) = \sum_{k=0}^{n} \binom{n}{k} \phi_{n-k}(z)\; H_{k,m}(x,y,A), \tag{2.7}$$

which in view of equation (1.6) gives the following alternate series definition of the HGMP ${}_Hp_{n,m}(x,y,z;A)$:

$$ {}_Hp_{n,m}(x,y,z;A) = n! \sum_{k=0}^{n} \sum_{r=0}^{[\frac{k}{m}]} \frac{\phi_{n-k}(z)(-1)^r y^r (x\sqrt{mA})^{k-mr}}{r!(k-mr)!(n-k)!}. \tag{2.8}$$

Differentiating equation (2.4) partially with respect to x and y, we get the following matrix differential recurrence relations satisfied by the HGMP ${}_Hp_{n,m}(x,y,z;A)$:

$$\frac{\partial}{\partial x} {}_Hp_{n,m}(x,y,z;A) = n\sqrt{mA}\; {}_Hp_{n-1,m}(x,y,z;A) \quad (n \geq 1)\;, \tag{2.9}$$

$$\frac{\partial}{\partial y} {}_Hp_{n,m}(x,y,z;A) = -\frac{n!}{(n-m)!}\; {}_Hp_{n-m,m}(x,y,z;A) \quad (n \geq m). \tag{2.10}$$

From equation (2.9), we have

$$\frac{\partial^m}{\partial x^m} {}_Hp_{n,m}(x,y,z;A) = (\sqrt{mA})^m \frac{n!}{(n-m)!}\; {}_Hp_{n-m,m}(x,y,z;A) \quad (n \geq m). \tag{2.11}$$

Consequently, from equations (2.10) and (2.11), we have

$$\frac{\partial^m}{\partial x^m} {}_Hp_{n,m}(x,y,z;A) = -(\sqrt{mA})^m \frac{\partial}{\partial y} {}_Hp_{n,m}(x,y,z;A). \tag{2.12}$$

Taking $y = 0$ in equation (2.4) and using equation (1.21) in the l.h.s. of the resultant equation, we find (after equating the coefficients of like powers of t)

$$ {}_Hp_{n,m}(x,0,z;A) = p_n(x\sqrt{mA}, z). \tag{2.13}$$

Now, solving equation (2.12) with the initial condition (2.13), we get the following operational representation:

$$ {}_Hp_{n,m}(x,y,z;A) = \exp\left(-y(\sqrt{mA})^{-m} \frac{\partial^m}{\partial x^m}\right) \Big\{ p_n(x\sqrt{mA}, z) \Big\}. \tag{2.14}$$

Further, replacing y by z, t by $(\sqrt{mA})^{-1}\hat{D}_x$ in expansion (1.22) and multiplying both sides of the resultant equation by $\exp(xt\sqrt{mA} - yt^m I)$ we find

$$\phi(z, (\sqrt{mA})^{-1}\hat{D}_x)\exp(xt\sqrt{mA} - yt^m I) = \sum_{n=0}^{\infty} \phi_n(z)(\sqrt{mA})^{-n}\frac{\hat{D}_x^n}{n!}\exp(xt\sqrt{mA} - yt^m I), \tag{2.15}$$

which on using identity

$$(\sqrt{mA})^{-n}\hat{D}_x^n\Big\{\exp(xt\sqrt{mA} - yt^m I)\Big\} = t^n\,\Big\{\exp(xt\sqrt{mA} - yt^m I)\Big\} \tag{2.16}$$

in the r.h.s. and then using equation (1.22) again in the r.h.s. of the resultant equation gives

$$\phi(z, \sqrt{mA})^{-1}\hat{D}_x)\exp(xt\sqrt{mA} - yt^m I) = \phi(z,t)\exp(xt\sqrt{mA} - yt^m I). \tag{2.17}$$

Using equations (1.7) and (2.4) in the l.h.s. and r.h.s. respectively of the above equation and equating the coefficients of like powers of t in the resultant equation, we get the following operational representation for the HGMP ${}_Hp_{n,m}(x,y,z;A)$:

$${}_Hp_{n,m}(x,y,z;A) = \phi(z, (\sqrt{mA})^{-1}\hat{D}_x)\Big\{H_{n,m}(x,y,A)\Big\}. \tag{2.18}$$

In order to frame the HGMP ${}_Hp_{n,m}(x,y,z;A)$ within the context of monomiality principle formalism, we prove the following results:

Theorem 2.1. *The HGMP ${}_Hp_{n,m}(x,y,z;A)$ are quasi-monomial with respect to the following multiplicative and derivative operators:*

$$\hat{M}_{Hp} := x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + \frac{\phi'\left(z, \hat{D}_x/\sqrt{mA}\right)}{\phi\left(z, \hat{D}_x/\sqrt{mA}\right)} \tag{2.19a}$$

and

$$\hat{P}_{Hp} := (\sqrt{mA})^{-1}\frac{\partial}{\partial x}, \tag{2.19b}$$

respectively.

Proof. Consider the identity

$$(\hat{D}_x/\sqrt{mA})\Big\{\exp(xt\sqrt{mA} - yt^m I)\phi(z,t)\Big\} = t\,\Big\{\exp(xt\sqrt{mA} - yt^m I)\phi(z,t)\Big\}. \tag{2.20}$$

Since, $\phi(z,t)$ is an invertible series and $\frac{\phi'(z,t)}{\phi(z,t)}$ has Taylor's series expansion in power of t, therefore, we have

$$\frac{\phi'(z, \hat{D}_x/\sqrt{mA})}{\phi(z, \hat{D}_x/\sqrt{mA})}\Big\{\exp(xt\sqrt{mA} - yt^m I)\phi(z,t)\Big\}$$

$$= \frac{\phi'(z,t)}{\phi(z,t)}\Big\{ \exp(xt\sqrt{mA} - yt^m I)\phi(z,t)\Big\}, \tag{2.21}$$

where the prime denotes the derivative of the function $\phi(z,t)$ with respect to t.

Now, differentiating equation (2.4) partially with respect to t, we have

$$\left(x\sqrt{mA} - myt^{m-1}I + \frac{\phi'(z,t)}{\phi(z,t)}\right)\Big\{\exp(xt\sqrt{mA} - yt^m I)\phi(z,t)\Big\} = \sum_{n=0}^{\infty} {}_Hp_{n+1,m}(x,y,z;A)\frac{t^n}{n!}. \tag{2.22}$$

Using equations (2.4) and (2.21) in the l.h.s. of equation (2.22), we find

$$x\sqrt{mA}\sum_{n=0}^{\infty} {}_Hp_{n,m}(x,y,z;A)\frac{t^n}{n!} - my\sum_{n=0}^{\infty} {}_Hp_{n,m}(x,y,z;A)\frac{t^{n+m-1}}{n!}$$
$$+\frac{\phi'(z,\hat{D}_x/\sqrt{mA})}{\phi(z,\hat{D}_x/\sqrt{mA})}\sum_{n=0}^{\infty} {}_Hp_{n,m}(x,y,z;A)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_Hp_{n+1,m}(x,y,z;A)\frac{t^n}{n!}, \tag{2.23}$$

which on using equation (2.11) in the l.h.s. gives

$$\left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + \frac{\phi'(z,\hat{D}_x/\sqrt{mA})}{\phi(z,\hat{D}_x/\sqrt{mA})}\right)\sum_{n=0}^{\infty} {}_Hp_{n,m}(x,y,z;A)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} {}_Hp_{n+1,m}(x,y,z;A)\frac{t^n}{n!}. \tag{2.24}$$

Equating the coefficients of like powers of t in both sides of equation (2.24), we get

$$\left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + \frac{\phi'(z,\hat{D}_x/\sqrt{mA})}{\phi(z,\hat{D}_x/\sqrt{mA})}\right) {}_Hp_{n,m}(x,y,z;A)$$
$$= {}_Hp_{n+1,m}(x,y,z;A), \tag{2.25}$$

which in view of the monomiality principle equation (1.12a) yields assertion (2.19a) of Theorem 2.1.

Also, from recurrence relation (2.9) and in view of equation (1.12b), we get assertion (2.19b) of Theorem 2.1.

Corollary 2.1. *The HGMP ${}_Hp_{n,m}(x,y,z;A)$ satisfy the following matrix differential equation:*

$$\left(my(\sqrt{mA})^{-m}\frac{\partial^m}{\partial x^m} - x\frac{\partial}{\partial x} - (\sqrt{mA})^{-1}\frac{\phi'(z,\hat{D}_x/\sqrt{mA})}{\phi(z,\hat{D}_x/\sqrt{mA})}\frac{\partial}{\partial x} + n\right) {}_Hp_{n,m}(x,y,z;A) = 0. \tag{2.26}$$

Proof. Using expressions of $\hat{M}_{HP}$ and $\hat{P}_{HP}$ given in equations (2.19a) and (2.19b) in monomiality principle equation (1.14), we get assertion (2.26) of Corollary 2.1.

Remark 2.1. Since ${}_Hp_{0,m}(x,y,z;A)=I$, therefore in view of monomiality principle equation (1.15), we have

$$ {}_Hp_{n,m}(x,y,z;A)=\left(x\sqrt{mA}-my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}+\frac{\phi'(z,\hat{D}_x/\sqrt{mA})}{\phi(z,\hat{D}_x/\sqrt{mA})}\right)^n\{I\}. \tag{2.27}$$

Remark 2.2. Using relation (2.11) in equation (2.25), we get the following matrix differential recurrence relation satisfied by the HGMP ${}_Hp_{n,m}(x,y,z;A)$:

$$\begin{aligned}x\sqrt{mA}\;{}_Hp_{n,m}(x,y,z;A)&-my\frac{n!}{(n-m+1)!}{}_Hp_{n-m+1,m}(x,y,z;A)\\&+\frac{\phi'(z,\hat{D}_x/\sqrt{mA})}{\phi(z,\hat{D}_x/\sqrt{mA})}{}_Hp_{n,m}(x,y,z;A)={}_Hp_{n+1,m}(x,y,z;A).\end{aligned} \tag{2.28}$$

3. Examples

By making suitable choice for the function $\phi(z,t)$ in equation (2.4), we get the generating functions for some members belonging to the HGMP family ${}_Hp_{n,m}(x,y,z;A)$. The properties of these special matrix polynomials can be obtained from the results derived in previous section. We consider the following examples:

I. Taking $\phi(z,t)=A(zt)$ (that is when the 2VGP $p_n(x,y)$ reduce to the 2VAP $P_n(x,y)$) in generating function (2.4), we find that the Hermite-Appell matrix polynomials (HAMP) ${}_HP_{n,m}(x,y,z;A)$ are defined by the following generating function:

$$\exp(xt\sqrt{mA}-yt^mI)\;A(zt)=\sum_{n=0}^{\infty}{}_HP_{n,m}(x,y,z;A)\frac{t^n}{n!}. \tag{3.1}$$

Since, in view of equations (1.22) and (1.33), we have $\phi_{n-k}(z)=A_{n-k}z^{n-k}$ and therefore from equation (2.8) we get the following series definition for the HAMP ${}_HP_{n,m}(x,y,z;A)$:

$${}_HP_{n,m}(x,y,z;A)=n!\sum_{k=0}^{n}\sum_{r=0}^{[\frac{k}{m}]}\frac{A_{n-k}\;z^{n-k}(-1)^ry^r(x\sqrt{mA})^{k-mr}}{r!(k-mr)!(n-k)!}. \tag{3.2}$$

From equations (2.19a) and (2.19b), we find that the HAMP ${}_HP_{n,m}(x,y,z;A)$ are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\hat{M}_{HP}:=x\sqrt{mA}-my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}+z\frac{A'\left(z\hat{D}_x/\sqrt{mA}\right)}{A\left(z\hat{D}_x/\sqrt{mA}\right)} \tag{3.3a}$$

and

$$\hat{P}_{_HP} := (\sqrt{mA})^{-1}\frac{\partial}{\partial x}, \tag{3.3b}$$

respectively. Also, from equation (2.26), we find that the HAMP ${}_HP_{n,m}(x,y,z;A)$ satisfy the following differential equation:

$$\left(my(\sqrt{mA})^{-m}\frac{\partial^m}{\partial x^m} - x\frac{\partial}{\partial x} - z(\sqrt{mA})^{-1}\frac{A'(z\hat{D}_x/\sqrt{mA})}{A(z\hat{D}_x/\sqrt{mA})}\frac{\partial}{\partial x} + n\right){}_HP_{n,m}(x,y,z;A) = 0. \tag{3.4}$$

Further, from equations (2.14) and (2.18), we get the following operational representations for the HAMP ${}_HP_{n,m}(x,y,z;A)$:

$${}_HP_{n,m}(x,y,z;A) = \exp\left(-y(\sqrt{mA})^{-m}\frac{\partial^m}{\partial x^m}\right)\Big\{P_n(x\sqrt{mA},z)\Big\}, \tag{3.5}$$

$${}_HP_{n,m}(x,y,z;A) = A(z(\sqrt{mA})^{-1}\hat{D}_x)\Big\{H_{n,m}(x,y,A)\Big\}. \tag{3.6}$$

Remark 3.1. In view of relation (1.30), for $z=1$, the 3-variable HAMP ${}_HP_{n,m}(x,y,z;A)$ reduce to the 2-variable Hermite-Appell matrix polynomials ${}_HP_{n,m}(x,y;A)$. Therefore, taking $z=1$ in equations (3.1), (3.2), (3.3a), (3.3b), (3.4), (3.5) and (3.6), we get the corresponding results for the HAMP ${}_HP_{n,m}(x,y;A)$.

II. Taking $\phi(z,t) = C_0(-zt^s)$ (that is when the 2VGP $p_n(x,y)$ reduce to the 2VGLP ${}_sL_n(y,x)$) in generating function (2.4), we find that the Hermite-Laguerre matrix polynomials (HLMP) ${}_HL_n^{(m,s)}(x,y,z;A)$ are defined by the following generating function:

$$\exp(xt\sqrt{mA} - yt^mI)\ C_0(-zt^sI) = \sum_{n=0}^{\infty} {}_HL_n^{(m,s)}(x,y,z;A)\frac{t^n}{n!}. \tag{3.7}$$

Since $p_n(x,y) = {}_sL_n(y,x)$, therefore from equation (2.6), we get the following series definition for the HLMP ${}_HL_n^{(m,s)}(x,y,z;A)$:

$${}_HL_n^{(m,s)}(x,y,z;A) = n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k y^k\ {}_sL_{n-mk}(z,x\sqrt{mA})}{k!(n-mk)!}. \tag{3.8}$$

From equations (2.19a) and (2.19b) and in view of operational definition (1.28), we find that the HLMP ${}_HL_n^{(m,s)}(x,y,z;A)$ are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\hat{M}_{_HL} := x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + s\hat{D}_z^{-1}(\sqrt{mA})^{-(s-1)}\frac{\partial^{s-1}}{\partial x^{s-1}} \tag{3.9a}$$

and

$$\hat{P}_{_HL} := (\sqrt{mA})^{-1}\frac{\partial}{\partial x}, \tag{3.9b}$$

respectively. Also, from equation (2.26), we find that the HLMP ${}_HL_n^{(m,s)}(x,y,z;A)$ satisfy the following differential equation:

$$\left(my(\sqrt{mA})^{-m}\frac{\partial^{m+1}}{\partial z\partial x^m} - x\frac{\partial^2}{\partial z\partial x} - s(\sqrt{mA})^{-(s-1)}\frac{\partial^s}{\partial x^s} + n\frac{\partial}{\partial z}\right){}_HL_n^{(m,s)}(x,y,z;A) = 0. \tag{3.10}$$

Further, from equations (2.14) and (2.18), we get the following operational representations for the HLMP ${}_HL_n^{(m,s)}(x,y,z;A)$:

$${}_HL_n^{(m,s)}(x,y,z;A) = \exp\left(-y(\sqrt{mA})^{-m}\frac{\partial^m}{\partial x^m}\right)\left\{{}_sL_n(z,x\sqrt{mA})\right\}, \tag{3.11}$$

$${}_HL_n^{(m,s)}(x,y,z;A) = \exp\left(\hat{D}_z^{-1}(\sqrt{mA})^{-s}\hat{D}_x^s\right)\ \left\{H_{n,m}(x,y,A)\right\}. \tag{3.12}$$

Remark 3.2. Putting $A = \frac{1}{m} \in \mathbb{C}^{1\times 1}$ and replacing m by s, s by m, x by y, y by $-z$ and z by x in equation (3.7), the HLMP ${}_HL_n^{(m,s)}(x,y,z;A)$ reduce to the recently introduced Laguerre-Gould Hopper polynomials (LGHP) ${}_LH_n^{(m,s)}(x,y,z)$ [21], i.e., we have

$${}_HL_n^{(s,m)}(y,-z,x;1/m) = {}_LH_n^{(m,s)}(x,y,z), \tag{3.13}$$

where the LGHP ${}_LH_n^{(m,s)}(x,y,z)$ are defined by the generating function [21, p. 9933]

$$\exp(yt+zt^s)\ C_0(-xt^m) = \sum_{n=0}^{\infty}{}_LH_n^{(m,s)}(x,y,z)\frac{t^n}{n!}. \tag{3.14}$$

III. Taking $\phi(z,t) = A(t)\ \exp(zt^s)$ (that is when the 2VGP $p_n(x,y)$ reduce to the 2DAP $R_n^{(s)}(x,y)$) in generating function (2.4), we find the following generating function of the Hermite matrix 2D Appell polynomials (HM2DAP) ${}_HR_n^{(m,s)}(x,y,z;A)$ [22]:

$$A(t)\ \exp(xt\sqrt{mA} - yt^mI + zt^sI) = \sum_{n=0}^{\infty}{}_HR_n^{(m,s)}(x,y,z;A)\frac{t^n}{n!}, \tag{3.15}$$

whose properties are discussed in [22]. Also, in view of equation (2.18), we get the following operational representation for the HM2DAP ${}_HR_n^{(m,s)}(x,y,z;A)$:

$${}_HR_n^{(m,s)}(x,y,z;A) = A((\sqrt{mA})^{-1}\hat{D}_x)\exp\left(z(\sqrt{mA})^{-s}\hat{D}_x^s\right)\ \left\{H_{n,m}(x,y,A)\right\}. \tag{3.16}$$

IV. Taking $\phi(z,t) = \exp(zt^s)$ (that is when the 2VGP $p_n(x,y)$ reduce to the GHP $H_n^{(s)}(x,y)$) in generating function (2.4), we find generating function (1.10) of the 3I3VHMP $H_n^{(m,s)}(x,y,z;A)$. The properties of the 3I3VHMP $H_n^{(m,s)}(x,y,z;A)$ are discussed in [20].

4. Summation formulae

We establish some summation formulae connecting the HGMP ${}_Hp_{n,m}(x,y,z;A)$ with certain other special polynomials.

First, we prove the following results:

Theorem 4.1. *For a matrix A in $\mathbb{C}^{n\times n}$ satisfying condition (1.1), the following expansion of the 2VGP $p_n(x,y)$ in a series of the HGMP ${}_Hp_{n,m}(x,y,z;A)$ holds true:*

$$p_n(x\sqrt{mA}, z) = n! \sum_{k=0}^{[\frac{n}{m}]} \frac{y^k \ {}_Hp_{n-mk,m}(x,y,z;A)}{k!(n-mk)!}. \tag{4.1}$$

Proof. From generating function (2.4), we have

$$\exp(xt\sqrt{mA})\ \phi(z,t) = \sum_{n,k=0}^{\infty} \frac{y^k \ {}_Hp_{n,m}(x,y,z;A)}{k!n!} t^{n+mk}. \tag{4.2}$$

Using generating function (1.21) in the l.h.s. of equation (4.2) and replacing n by $n-mk$ in the r.h.s. and then using equation (2.5) in the resultant equation, we find

$$\sum_{n=0}^{\infty} p_n(x\sqrt{mA}, z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]} \frac{y^k \ {}_Hp_{n-mk,m}(x,y,z;A)}{k!(n-mk)!} t^n, \tag{4.3}$$

which on equating the coefficients of like powers of t yields assertion (4.1) of Theorem 4.1.

Theorem 4.2. *For a matrix A in $\mathbb{C}^{n\times n}$ satisfying condition (1.1), the following implicit summation formula for the HGMP ${}_Hp_{n,m}(x,y,z;A)$ holds true:*

$${}_Hp_{n,m}(x+w,y,z;A) = \sum_{k=0}^{n} \binom{n}{k} {}_Hp_{k,m}(x,y,z;A)\ (w\sqrt{mA})^{n-k}. \tag{4.4}$$

Proof. Replacing x by $x+w$ in equation (2.4) and using exponential function property in the resultant equation, we find

$$\exp(xt\sqrt{mA} - yt^m I)\ \phi(z,t)\ \exp(wt\sqrt{mA}) = \sum_{n=0}^{\infty} {}_Hp_{n,m}(x+w,y,z;A)\frac{t^n}{n!}, \tag{4.5}$$

which on using equation (2.4) and expanding the second exponential in the l.h.s., gives

$$\sum_{n,k=0}^{\infty} {}_Hp_{k,m}(x,y,z;A)(w\sqrt{mA})^n \frac{t^{n+k}}{k!n!} = \sum_{n=0}^{\infty} {}_Hp_{n,m}(x+w,y,z;A)\frac{t^n}{n!}. \tag{4.6}$$

Now, using equation (2.5) (for $r = 1$) in the l.h.s. of equation (4.6), we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} {}_Hp_{k,m}(x,y,z;A)(w\sqrt{mA})^{n-k}\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_Hp_{n,m}(x+w,y,z;A)\frac{t^n}{n!}. \tag{4.7}$$

which on equating the coefficients of like powers of t yields assertion (4.4) of Theorem 4.2.

Theorem 4.3. *For a matrix A in $\mathbb{C}^{n\times n}$ satisfying condition (1.1), the following explicit summation formula for the HGMP ${}_Hp_{n,m}(x,y,z;A)$ in terms of the 2I2VHMP $H_{n,m}(x,y,A)$ and the 2VGP $p_n(x,y)$ holds true:*

$${}_Hp_{n,m}(x+w,y,z;A) = \sum_{k=0}^{n} \binom{n}{k} H_{k,m}(x,y,A)\ p_{n-k}(w\sqrt{mA},z) \tag{4.8}$$

Proof. Following the same lines of proof of summation formula (4.4) and using generating functions (1.7), (1.21) and (2.4), we get assertion (4.8) of Theorem 4.3.

Theorem 4.4. *For a matrix A in $\mathbb{C}^{n\times n}$ satisfying condition (1.1), the following implicit summation formula for the HGMP ${}_Hp_{n,m}(x,y,z;A)$ involving the 2I2VHMP $H_{n,m}(x,y,A)$ holds true:*

$${}_Hp_{n,m}(x+w,y+v,z;A) = \sum_{k=0}^{n} \binom{n}{k} {}_Hp_{k,m}(x,y,z;A)\ H_{n-k,m}(w,v,A). \tag{4.9}$$

Proof. Replacing x by $x+w$, y by $y+v$ in equation (2.4) and then following the same lines of proof of summation formula (4.4) and using generating functions (2.4) and (1.7), we get assertion (4.9) of Theorem 4.4.

Theorem 4.5. *The following explicit summation formula for the HGMP ${}_Hp_{n,m}(x,y,z;A)$ in terms of the 3I3VHMP $H_n^{(m,s)}(x,y,z;A)$, 2VGP $p_n(x,y)$ and GHP $H_n^{(s)}(x,y)$ holds true:*

$${}_Hp_{n,m}(x,y,v;A) = \sum_{l=0}^{n}\sum_{k=0}^{n-l} \binom{n}{l}\binom{n-l}{k}\ H_{n-l-k}^{(m,s)}(x,y,z;A)\ p_k(w,v)\ H_l^{(s)}(-w,-z). \tag{4.10}$$

Proof. Consider the product of the generating functions (1.10) and (1.21) of the 3I3VHMP $H_n^{(m,s)}(x,y,z;A)$ and the 2VGP $p_n(x,y)$ respectively, in the following form:

$$\exp(xt\sqrt{mA} - yt^m I + zt^s I)\ \exp(wt)\ \phi(v,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} H_n^{(m,s)}(x,y,z;A)\ p_k(w,v)\frac{t^{n+k}}{n!k!}. \tag{4.11}$$

Breaking the first exponential in the l.h.s. and replacing n by $n-k$ in the r.h.s. of equation (4.11) and then using equation (2.5) (for $r=1$), we have

$$\exp(xt\sqrt{mA} - yt^m I)\ \exp(wt + zt^s)\ \phi(v,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} H_{n-k}^{(m,s)}(x,y,z;A)\ p_k(w,v)\frac{t^n}{n!}. \tag{4.12}$$

Now, shifting the second exponential to the r.h.s. of the above equation and then using generating function (1.24) of the GHP $H_n^{(s)}(x,y)$ in the r.h.s. of the resultant equation, we find

$$\exp(xt\sqrt{mA} - yt^m I)\ \phi(v,t) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\ H_{n-k}^{(m,s)}(x,y,z;A)\ p_k(w,v)\ H_l^{(s)}(-w,-z)\frac{t^{n+l}}{n!l!}, \tag{4.13}$$

which on replacing n by $n-l$ in the r.h.s. becomes

$$\exp(xt\sqrt{mA} - yt^m I)\ \phi(v,t) = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}\ H_{n-l-k}^{(m,s)}(x,y,z;A)$$
$$\times p_k(w,v)\ H_l^{(s)}(-w,-z)\frac{t^n}{n!}. \tag{4.14}$$

Finally, using generating function (2.4) in the l.h.s. of equation (4.14) and then equating the coefficients of like powers of t in the resultant equation, we get assertion (4.10) of Theorem 4.5.

Remark 4.1. Taking $z=0$ in assertion (4.10) of Theorem (4.5) and using the following relations:

$$H_n^{(m,s)}(x,y,0;A) = H_{n,m}(x,y,A), \tag{4.15}$$

$$H_n^{(s)}(x,0) = x^n, \tag{4.16}$$

we deduce the following consequence of Theorem 4.5.

Corollary 4.1. *The following explicit summation formula for the HGMP ${}_Hp_{n,m}(x,y,z;A)$ in terms of the 2I2VHMP $H_{n,m}(x,y,A)$ and 2VGP $p_n(x,y)$ holds true:*

$${}_Hp_{n,m}(x,y,v;A) = \sum_{l=0}^{n}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}\ H_{n-l-k,m}(x,y,A)\ p_k(w,v)\ (-w)^l. \tag{4.17}$$

5. Concluding remarks

In order to further stress the importance of the use of operational methods in introducing new families of special polynomials, here we introduce two more families of special matrix polynomials by using integral representation method. We also establish some integral and operational representations for the HGMP ${}_Hp_{n,m}(x,y,z;A)$.

We recall that the 2-variable generalized truncated matrix polynomials (2VGTMP) $e_n^{(m)}(x,y;A)$ and the 2-variable generalized Chebyshev matrix polynomial (2VGCMP) $U_n^{(m)}(x,y;A)$ are defined by [20]:

$$e_n^{(m)}(x,y;A) = n!\sum_{k=0}^{[\frac{n}{m}]} \frac{(-1)^k y^k (x\sqrt{mA})^{n-mk}}{(n-mk)!} \tag{5.1}$$

and

$$U_n^{(m)}(x,y;A) = \sum_{k=0}^{[\frac{n}{m}]} \frac{(-1)^k (n-k)! y^k (x\sqrt{mA})^{n-mk}}{k!(n-mk)!}\ , \tag{5.2}$$

respectively, where A is a matrix in $\mathbb{C}^{n\times n}$ satisfying condition (1.1).

We note that the 2VGTMP $e_n^{(m)}(x,y;A)$ and the 2VGCMP $U_n^{(m)}(x,y;A)$ are defined in terms of the 2I2VHMP $H_{n,m}(x,y,A)$ by following integral representations:

$$e_n^{(m)}(x,y;A) = \int_0^\infty e^{-t}\ H_{n,m}\,(x,yt,A)\,dt \tag{5.3}$$

and

$$U_n^{(m)}(x,y;A) = \frac{1}{n!}\int_0^\infty e^{-t}\ t^n H_{n,m}\left(x,\frac{y}{t},A\right)dt, \tag{5.4}$$

respectively.

Also, we note the following relations [20]:

$$e_n^{(m)}(x,\hat{D}_y^{-1};A) = H_{n,m}(x,y,A), \tag{5.5}$$

$$U_n^{(m+1)}\left(x\sqrt{\frac{m}{m+1}},y;A\right) = H_{n,m}(x,y,A). \tag{5.6}$$

Now, replacing y by yt in equation (2.7) and multiplying the resultant equation by e^{-t} and then integrating with respect to t between the limits 0 to ∞, we find

$$\int_0^\infty e^{-t}\ {}_Hp_{n,m}\,(x,yt,z;A)\,dt = \sum_{k=0}^{n}\binom{n}{k}\phi_{n-k}(z)\ \int_0^\infty e^{-t}\ H_{k,m}\,(x,yt,A)\,dt, \tag{5.7}$$

which on using equation (5.3) in the r.h.s. becomes

$$\int_0^\infty e^{-t}\ {}_Hp_{n,m}\,(x,yt,z;A)\,dt = \sum_{k=0}^{n}\binom{n}{k}\phi_{n-k}(z)\ e_k^{(m)}(x,y;A). \tag{5.8}$$

In view of equation (2.7), the r.h.s. of equation (5.8) can be viewed as the truncated-general matrix polynomials. Denoting this newly introduced truncated-general matrix polynomials (TGMP) by ${}_e p_{n,m}(x,y,z;A)$, we find

$$\int_0^\infty e^{-t}\ {}_H p_{n,m}\,(x, yt, z; A)\, dt = {}_e p_{n,m}(x,y,z;A), \tag{5.9}$$

where

$${}_e p_{n,m}(x,y,z;A) = \sum_{k=0}^{n} \binom{n}{k} \phi_{n-k}(z)\ e_k^{(m)}(x,y;A). \tag{5.10}$$

Again replacing x by xt, y by yt^{m-1} in equation (2.7) and multiplying the resultant equation by e^{-t} and then integrating with respect to t between the limits 0 to ∞, we find

$$\int_0^\infty e^{-t}\ {}_H p_{n,m}\left(xt, yt^{m-1}, z; A\right) dt = \sum_{k=0}^{n} \binom{n}{k} \phi_{n-k}(z) \int_0^\infty e^{-t}\ H_{k,m}\left(xt, yt^{m-1}, A\right) dt. \tag{5.11}$$

Now, using the relation

$$t^n H_{n,m}(x,y,A) = H_{n,m}(xt, yt^m, A) \tag{5.12}$$

in the r.h.s. of equation (5.11), we have

$$\int_0^\infty e^{-t}\ {}_H p_{n,m}\left(xt, yt^{m-1}, z; A\right) dt = \sum_{k=0}^{n} \frac{n!}{(n-k)!} \phi_{n-k}(z)\, \frac{1}{k!} \int_0^\infty e^{-t}\, t^k\ H_{k,m}\left(x, \frac{y}{t}, A\right) dt, \tag{5.13}$$

which on using equation (5.4) in the r.h.s. becomes

$$\int_0^\infty e^{-t}\ {}_H p_{n,m}\left(xt, yt^{m-1}, z; A\right) dt = \sum_{k=0}^{n} \frac{n!}{(n-k)!} \phi_{n-k}(z)\ U_k^{(m)}(x,y;A). \tag{5.14}$$

In view of equation (2.7), the r.h.s. of equation (5.14) can be viewed as the Chebyshev-general matrix polynomials. Denoting this newly introduced Chebyshev-general matrix polynomials (CGMP) by ${}_U p_{n,m}(x,y,z;A)$, we find

$$\int_0^\infty e^{-t}\ {}_H p_{n,m}\,(x, yt, z; A)\, dt = {}_U p_{n,m}(x,y,z;A), \tag{5.15}$$

where

$${}_U p_{n,m}(x,y,z;A) = \sum_{k=0}^{n} \frac{n!}{(n-k)!} \phi_{n-k}(z)\ U_k^{(m)}(x,y;A). \tag{5.16}$$

Further, we mention the following integral representations of the 2I2VHMP $H_{n,m}(x, y, A)$ in terms of the 2VGTMP $e_n^{(m)}(x, y; A)$ and 2VGCMP $U_n^{(m)}(x, y; A)$:

$$H_{n,m}(x, y, A) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^t \, t^{-1} \, e_n^{(m)} \left(x, \frac{y}{t}; A \right) dt \tag{5.17}$$

and

$$H_{n,m}(x, y, A) = \frac{n!}{2\pi i} \int_{-\infty}^{(0+)} e^t \, t^{-n-1} \, U_n^{(m)}(x, yt; A) dt, \tag{5.18}$$

respectively, which are obtained by using the Hankel formula [27]

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^t \, t^{-z} \, dt \tag{5.19}$$

in series definitions (5.1) and (5.2) of the 2VGTMP $e_n^{(m)}(x, y; A)$ and 2VGCMP $U_n^{(m)}(x, y; A)$, respectively.

Using relations (5.17) and (5.18) respectively in the r.h.s. of equation (2.7), we find

$${}_Hp_{n,m}(x, y, z; A) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^t \, t^{-1} \sum_{k=0}^{n} \binom{n}{k} \phi_{n-k}(z) \, e_k^{(m)} \left(x, \frac{y}{t}; A \right) dt, \tag{5.20}$$

and

$${}_Hp_{n,m}(x, y, z; A) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^t \, t^{-1} \sum_{k=0}^{n} \frac{n!}{(n-k)!} \phi_{n-k}(z) \, U_k^{(m)} \left(\frac{x}{t}, \frac{y}{t^{m-1}}; A \right) dt, \tag{5.21}$$

respectively, which on using equations (5.10) and (5.16) give the following integral representations for the HGMP ${}_Hp_{n,m}(x, y, z; A)$:

$${}_Hp_{n,m}(x, y, z; A) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^t \, t^{-1} \, {}_ep_{n,m} \left(x, \frac{y}{t}, z; A \right) dt \tag{5.22}$$

and

$${}_Hp_{n,m}(x, y, z; A) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^t \, t^{-1} \, {}_Up_{n,m} \left(\frac{x}{t}, \frac{y}{t^{m-1}}, z; A \right) dt, \tag{5.23}$$

respectively.

Finally, in view of equations (2.18), (5.5) and (5.6), we get the following operational representations for the HGMP ${}_Hp_{n,m}(x, y, z; A)$:

$${}_Hp_{n,m}(x, y, z; A) = \phi(z, (\sqrt{mA})^{-1} \hat{D}_x) \left\{ e_n^{(m)}(x, \hat{D}_y^{-1}; A) \right\}, \tag{5.24}$$

$$_{H}p_{n,m}(x,y,z;A) = \phi(z,(\sqrt{mA})^{-1}\hat{D}_x)\Big\{U_n^{(m+1)}\left(x\sqrt{\frac{m}{m+1}},y;A\right)\Big\}. \qquad (5.25)$$

By making suitable choice for the $\phi(z,t)$ as in Section 3, in the results derived above, we can obtain the corresponding results for the members belonging to HGMP family.

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A Note on Fourier Expansions of Periodic U–Bernoulli, U–Euler and U–Genocchi Functions

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Abstract

In this paper, we investigate several analytic properties of the U-Bernoulli, U-Euler, and U-Genocchi polynomials, building on previously studied families of U-type special polynomials. We then derive the Fourier series expansions of the corresponding periodic functions and establish their connections with the Riemann zeta function. In addition, we provide new bounds for the associated U-Bernoulli, U-Euler, and U-Genocchi numbers.

Keywords: New U-Bernoulli polynomials, New U-Euler polynomials, New U-Genocchi polynomials, Fourier expansion, Integral representation, Riemann zeta function.
2010 MSC: 11B68, 11B83, 11B39, 05A19.

1. Introduction

Fourier series and generating function theory is an active branch of modern analysis that has gained importance due to its applications in methods of analysis for mathematical solutions to boundary value problems, engineering, and signal processing in communications. On the other hand, the Riemann zeta function and its generalizations are useful in the investigation of analytic number theory and allied disciplines, especially in the role played by their special values in integral arguments (see [4; 14]).

It is well known that the classical Bernoulli, Euler, and Genocchi polynomials, with $x \in \mathbb{R}$, are respectively defined by the following generating functions (see [7; 16]):

$$\frac{ze^{zx}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}, \quad (|z| < 2\pi), \tag{1}$$

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$$\frac{2e^{zx}}{e^z+1}=\sum_{n=0}^{\infty}E_n(x)\frac{z^n}{n!},\quad (|z|<\pi), \tag{2}$$

$$\frac{2ze^{zx}}{e^z+1}=\sum_{n=0}^{\infty}G_n(x)\frac{z^n}{n!},\quad (|z|<\pi). \tag{3}$$

For $x=0$, the numbers $B_n(0)=B_n$, $E_n(0)=E_n$ and $G_n(0)=G_n$ appear.

If $n>2$, the Fourier series for periodic Bernoulli functions of period 1 for even index is given by (see [10])

$$\tilde{B}_{2n}(x)=(-1)^{n+1}\frac{2(2n)!}{(2\pi)^{2n}}\sum_{m=1}^{\infty}\frac{\cos(2m\pi x)}{m^{2n}};\quad x\in\mathbb{R},\quad (n\in\mathbb{N}), \tag{4}$$

and for odd index

$$\tilde{B}_{2n+1}(x)=(-1)^{n+1}\frac{2(2n+1)!}{(2\pi)^{2n+1}}\sum_{m=1}^{\infty}\frac{\sin(2m\pi x)}{m^{2n+1}};\quad x\in\mathbb{R},\quad (n\in\mathbb{N}). \tag{5}$$

By (5) and integration of the series for $\frac{\tilde{B}_{2n+1}(x)}{x}$ leads to the series representation of the Riemann zeta function $\zeta(2n+1)$, $n\in\mathbb{N}$ (cf. [10]):

$$\zeta(2n+1)=(-1)^{n+1}\frac{(2\pi)^{2n+1}}{2(2n+1)!}\int_0^1 B_{2n+1}(x)\cot\left(\frac{\pi x}{2}\right)\,dx. \tag{6}$$

Given that, $\tilde{E}_n(x+2)=-\tilde{E}_n(x+1)=\tilde{E}_n(x)$, $x\in\mathbb{R}$, if $n\in\mathbb{N}$, the Fourier series for the periodic Euler functions of period 2 is given by (see [11]):

$$\tilde{E}_{2n}(x)=4(-1)^n(2n)!\sum_{m=0}^{\infty}\frac{\sin((2m+1)\pi x)}{[(2m+1)\pi]^{2n+1}};\quad (x\in\mathbb{R}). \tag{7}$$

Using (7) and integration of the series for $\frac{\tilde{E}_{2n}(x)}{x}$ leads to the integral representation of $\zeta(2n+1)$, $n\in\mathbb{N}$ (cf. [11]).

Given that, $\tilde{G}_n(x+2)=-\tilde{G}_n(x+1)=\tilde{G}_n(x)$, for $x\in\mathbb{R}$, the Fourier series for the periodic Genocchi functions of period 2, is given by the relation (see [12]):

$$\tilde{G}_{2n+1}(x)=4(-1)^n(2n+1)!\sum_{m=0}^{\infty}\frac{\sin((2m+1)\pi x)}{[(2m+1)\pi]^{2n+1}};\quad (x\in\mathbb{R}). \tag{8}$$

By using (8) and integration of the series for $\frac{\tilde{G}_{2n+1}(x)}{x}$ leads to the integral representation of $\zeta(2n+1)$, $n\in\mathbb{N}$ (cf. [12]).

In the present work, considering [9], we investigate Fourier expansions for the new periodic U-Bernoulli, U-Euler, and U-Genocchi functions for finding new relationships with the Riemann zeta function, respectively. Moreover, we study special bounds for the respective numbers. In the next section, some formulas for the U-Bernoulli, U-Euler, and U-Genocchi polynomials will be derived, and we will recall some known results necessary for the present investigation. In Section 3, we give the Fourier series of the functions f_n. Also, we give a connection with U-Bernoulli numbers and the Riemann zeta function, and we find a bound for these numbers. We do the same job in sections 4 and 5 for the two families, $g_n(x)$ and $h_n(x)$.

2. Preliminary and basic results

Let $s = \sigma + i\rho$, be a complex number with $\sigma,\ \rho \in \mathbb{R}$. The Riemann zeta function is defined by (see [1, p. 249])

$$\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s}, \qquad \sigma > 0. \tag{9}$$

The Euler formula is given by

$$e^{ix} = \cos(x) + i\sin(x). \tag{10}$$

We can see from (10) that (cf. [5])

$$i\tan x = 1 - \frac{2}{e^{2ix}+1}. \tag{11}$$

Moreover, let us notice that:

$$i\tan x = \frac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}} = 1 - \frac{2}{e^{2ix}-1} + \frac{4}{e^{4ix}-1}. \tag{12}$$

From (12), it follows that

$$\tan(-x) = -i + \frac{2i}{e^{-2ix}-1} - \frac{4i}{e^{-4ix}-1},$$

therefore

$$x\tan(-x) = -ix + \frac{2ix}{e^{-2ix}-1} - \frac{4ix}{e^{-4ix}-1}.$$

It is known that the integer part function $\lfloor x \rfloor$ is defined by

$$\lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \leq x\}.$$

The new family of U-Bernoulli polynomials $M_n(x)$ of degree n in variable x is defined by the following generating function (see [9])

$$F_M(x,z) = \left(\frac{z}{e^{-z}-1}\right) e^{-xz} = \sum_{n=0}^{\infty} M_n(x)\frac{z^n}{n!}, \quad (|z| < 2\pi). \tag{13}$$

Some U-Bernoulli polynomials are:

$M_0(x) = -1,$ $\quad M_1(x) = x - \frac{1}{2},$
$M_1(x) = x - \frac{1}{2},$ $\quad M_4(x) = -x^4 + 2x^3 - x^2 + \frac{1}{30},$
$M_2(x) = -x^2 + x - \frac{1}{6},$ $\quad M_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x.$

If $x = 0$ in (13), $M_n = M_n(0)$ will be called the U-Bernoulli numbers, hence, we have

$$\frac{z}{e^{-z}-1} = \sum_{n=0}^{\infty} M_n\frac{z^n}{n!}, \quad (|z| < 2\pi). \tag{14}$$

Some of these numbers are:

$$M_0 = -1; \quad M_1 = -\frac{1}{2}; \quad M_2 = -\frac{1}{6}; \quad M_3 = 0.$$

We can easily prove that for $n \in \mathbb{N}$ odd,

$$M_n = 0, \qquad \forall\, n \geq 3. \tag{15}$$

The new family of U-Euler polynomials $A_n(x)$ of degree n in variable x is defined by the following generating function (see [9])

$$E_A(x;z) = \left(\frac{2}{e^{-\frac{z}{2}}+1}\right) e^{-\frac{xz}{2}} = \sum_{n=0}^{\infty} A_n(x)\frac{z^n}{n!}, \quad (|z| < 2\pi). \tag{16}$$

Some U-Euler polynomials are:

$A_0(x) = 1,$ $\quad A_3(x) = -\frac{1}{8}x^3 + \frac{3}{16}x^2 - \frac{1}{32},$
$A_1(x) = \frac{1}{4} - \frac{x}{2},$ $\quad A_4(x) = \frac{1}{16}x^4 - \frac{1}{8}x^3 + \frac{1}{16}x,$
$A_2(x) = \frac{1}{4}x^2 - \frac{1}{4}x,$ $\quad A_5(x) = -\frac{1}{32}x^5 + \frac{5}{64}x^4 - \frac{5}{64}x^2 + \frac{1}{64}.$

When $x = 0$ in (16), $A_n = A_n(0)$ we will call them the U-Euler numbers. We have therefore

$$\frac{2}{e^{-\frac{z}{2}}+1} = \sum_{n=0}^{\infty} A_n\frac{z^n}{n!}, \quad (|z| < 2\pi). \tag{17}$$

Some of these numbers are:

$$A_0 = 1; \quad A_1 = \frac{1}{4}; \quad A_2 = 0; \quad A_3 = -\frac{1}{32}.$$

.

In (17), we can prove that if $n \in \mathbb{N}$ is an even number,

$$A_n = 0, \qquad \forall n \geq 2. \tag{18}$$

The new family of U-Genocchi polynomials $V_n(x)$ of degree n in variable x is defined by (see [9])

$$G_V(x,z) = \left(\frac{2z}{e^{-\frac{z}{2}}+1}\right) e^{-\frac{xz}{2}} = \sum_{n=0}^{\infty} V_n(x)\frac{z^n}{n!}, \quad (|z| < 2\pi). \tag{19}$$

Some U-Genocchi polynomials are:

$V_0(x) = 0,$
$V_1(x) = 1,$
$V_2(x) = -x + \frac{1}{2},$
$V_3(x) = \frac{3}{4}x^2 - \frac{3}{4}x,$
$V_4(x) = -\frac{1}{2}x^3 + \frac{3}{4}x^2 - \frac{1}{8},$
$V_5(x) = \frac{5}{16}x^4 - \frac{5}{8}x^3 + \frac{5}{16}x.$

For $x = 0$ in (19), $V_n = V_n(0)$, we will call them the U-Genocchi numbers. So

$$\frac{2z}{e^{-\frac{z}{2}}+1} = \sum_{n=0}^{\infty} V_n\frac{z^n}{n!}, \qquad (|z| < 2\pi). \tag{20}$$

Some of these numbers are:

$$V_0 = 0; \quad V_1 = 1; \quad V_4 = -\frac{1}{8}.$$

Starting from (19), it is possible to establish, for $n \geq 1$ a relationship with the U-Euler polynomials employing

$$V_n(x) = nA_{n-1}(x). \tag{21}$$

Furthermore, by setting $x = 0$ in (21), it follows

$$V_n = nA_{n-1}, \qquad \forall\, n \geq 1. \tag{22}$$

Proposition 2.1. *The polynomials $M_n(x)$ satisfy*

$$M_n'(x) = -nM_{n-1}(x), \qquad \forall n \in \mathbb{N}. \tag{23}$$

$$M_n(x+1) - M_n(x) = n(-1)^{n-1}x^{n-1}, \qquad \forall n \in \mathbb{N}. \tag{24}$$

Proof. In view of (13), we have

$$\frac{\partial}{\partial x}\left(\frac{ze^{-xz}}{e^{-z}-1}\right) = -n\sum_{n=1}^{\infty} M_{n-1}(x)\frac{z^n}{n!}.$$

Besides,

$$\frac{\partial}{\partial x}\left(\frac{ze^{-xz}}{e^{-z}-1}\right) = \sum_{n=0}^{\infty} M_n'(x)\frac{z^n}{n!}.$$

So, the uniqueness of the power series leads us to

$$\frac{M_n'(x)}{n!} = \frac{-nM_{n-1}(x)}{n!}.$$

Hence, assertion (23) follows.

We will now provide (24). We start from (13)

$$\frac{z}{e^{-z}-1}e^{-(x+1)z} - \frac{z}{e^{-z}-1}e^{-xz} = \frac{ze^{-xz}(e^{-z}-1)}{e^{-z}-1} = ze^{-xz}.$$

Turning to the power series, we have that

$$\sum_{n=0}^{\infty}[M_{n+1}(x+1) - M_{n+1}(x)]\frac{z^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty}(-1)^n x^n (n+1)\frac{z^{n+1}}{(n+1)!}.$$

So, assertion (24) holds. Proposition 2.1 is proved.

Taking $x = 0$ in (24) for $n \geq 2$, we get

$$M_n(1) \quad = \quad M_n. \tag{25}$$

Proposition 2.2. *Let $n \in \mathbb{N}$, then the polynomials $A_n(x)$ satisfy*

$$A_n'(x) = -\frac{n}{2}A_{n-1}(x), \tag{26}$$

$$A_n(x+1) + A_n(x) = \frac{(-1)^n}{2^{n-1}}x^n. \tag{27}$$

Proof. By differentiating (16), with respect to x, the power series' uniqueness leads us to (26).

We will show the assertion (27). Starting from (16) we have

$$\frac{2e^{-\frac{(x+1)z}{2}}}{e^{-\frac{z}{2}}+1}+\frac{2e^{-\frac{xz}{2}}}{e^{-\frac{z}{2}}+1}=2e^{-\frac{xz}{2}}.$$

Turning to the power series we have that

$$\sum_{n=0}^{\infty}A_n(x+1)\frac{z^n}{n!}+\sum_{n=0}^{\infty}A_n(x)\frac{z^n}{n!}=2\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\frac{x^nz^n}{n!}, \tag{28}$$

from 28 we obtain (27). Proposition 2.2 is proved.

Let's notice that, making $x=0$ in (27), we obtain:

$$\begin{aligned} A_n(1)+A_n(0) &= 0,\\ A_n(1) &= -A_n. \end{aligned} \tag{29}$$

Theorem 2.1. *For every $n\in\mathbb{N}$, the polynomials $V_n(x)$ satisfy the properties:*

$$V_n'(x)=-\frac{n}{2}V_{n-1}(x), \tag{30}$$

$$V_n(x+1)+V_n(x)=\frac{n(-1)^{n-1}}{2^{n-2}}x^{n-1}. \tag{31}$$

Proof. From (26), we have

$$A_n'(x)=-\frac{n}{2}A_{n-1}(x).$$

By using (21), we obtain

$$A_n'(x)=-\frac{1}{2}V_n(x),$$

then

$$V_n(x)=-2A_n'(x).$$

Whence it follows that

$$\begin{aligned} V_n'(x) &= -\frac{n(n-1)}{2}\frac{V_{n-1}(x)}{n-1}.[-2A_n'(x)]'\\ &= -2\left[-\frac{n}{2}A_{n-1}(x)\right]'\\ &= n\left[-\frac{n-1}{2}A_{n-2}(x)\right]'=-\frac{n(n-1)}{2}A_{n-2}(x)\\ &= -\frac{n(n-1)}{2}\frac{V_{n-1}(x)}{n-1}. \end{aligned}$$

Then, the U-Genocchi polynomials satisfy (30).

We will now prove the assertion (31). Starting from (27) and using (21), we have that

$$\begin{aligned} A_n(x+1)+A_n(x) &= \frac{(-1)^n}{2^{n-1}}x^n \\ \Leftrightarrow \frac{V_{n+1}(x+1)}{n+1}+\frac{V_{n+1}(x)}{n+1} &= \frac{(-1)^n}{2^{n-1}}x^n. \end{aligned}$$

So, (31) follows. Thus, Theorem 2.1 is completely proved.

For $x=0$ in (31), we get

$$V_n(1) = -V_n. \tag{32}$$

3. Fourier series of the periodic U-Bernoulli functions

In this section, we give the Fourier series of a periodic function involving $U-$Bernoulli polynomials and introduce the relationship of the U-Bernoulli numbers with the Riemann zeta function. Furthermore, we give a bound for U-Bernoulli numbers.

Theorem 3.1. *Let $n \in \mathbb{N}$,*

$$f_n(x) := M_n(x-\lfloor x \rfloor), \quad (x \in \mathbb{R}). \tag{33}$$

Then, the Fourier series for $f_n(x)$ is

$$f_n(x) = M_n(x-\lfloor x \rfloor) = \frac{(-1)^n n!}{(2\pi i)^n} \sum{}' \frac{e^{2\pi ikx}}{k^n}, \tag{34}$$

where $\sum'$ denotes a sum over the integers not including 0.

Proof. The function f_n is a periodic function with a period $T=1$.

$$f_n(x+1) = M_n(x+1-\lfloor x+1 \rfloor) = M_n(x-\lfloor x \rfloor) = f_n(x).$$

So, $f_n(x)$ has a representation in Fourier series, which is given by (see [4]):

$$f_n(x) = \sum_{k\in\mathbb{Z}} \hat{f}_n(k) e^{2\pi inx}.$$

Firstly, note that for every $x \in [0,1)$,

$$f_n(x) = M_n(x-\lfloor x \rfloor) = M_n(x-0) = M_n(x).$$

Now, let's calculate the coefficients $\hat{f}_n(k)$. For $k=0$, applying (23) we have

$$\hat{f}_n(0) = \int_0^1 f_n(x)e^{-2\pi i0x}dx = \frac{-1}{n+1}[M_{n+1}(0) - M_{n+1}(1)] = 0, \qquad \forall n \geq 1.$$

Now, for $k > 0$, $n = 1$, we obtain

$$\hat{f}_1(k) = \int_0^1 M_1(x - \lfloor x \rfloor)e^{-2\pi ikx}dx = \int_0^1 \left(x - \frac{1}{2}\right) e^{-2\pi ikx}dx.$$

Integrating by parts, it follows

$$\hat{f}_1(k) = \left[-\left(x - \frac{1}{2}\right)\left(\frac{e^{-2\pi ikx}}{2\pi ikx}\right)\right]_0^1 + \frac{1}{2\pi ik}\int_0^1 e^{-2\pi ikx}.$$

If $u = 2\pi ikx$, we see that

$$\int_0^1 e^{-2\pi ikx} = \int_0^1 [\cos(2\pi kx) - i\sin(2\pi kx)]dx = 0.$$

This implies

$$\hat{f}_1(k) = \frac{-1}{2\pi ik}. \tag{35}$$

Let's see now the case $k > 0$, $n > 1$.

$$\hat{f}_n(k) = \int_0^1 M_n(x - \lfloor x \rfloor)e^{-2\pi ikx}dx = \int_0^1 M_n(x)e^{-2\pi ikx}dx.$$

Integrating by parts, with $u = M_n(x)$ and $dv = e^{-2\pi ikx}dx$, and using (25) we get

$$\begin{aligned}\hat{f}_n(k) &= \frac{-1}{2\pi ik}[M_n(1) - M_n(0)] - \frac{n}{2\pi ik}\int_0^1 M_{n-1}(x)e^{-2\pi ikx}dx, \\ &= -\frac{n}{2\pi ik}\int_0^1 M_{n-1}(x)e^{-2\pi ikx}dx.\end{aligned}$$

Integrating by parts again, with $u = M_{n-1}(x)$ and $dv = e^{-2\pi ikx}dx$, we can apply (25), to get

$$\begin{aligned}\hat{f}_n(k) &= \frac{-n}{2\pi ik}\left\{\left[-\frac{M_{n-1}(x)}{2\pi ik}e^{-2\pi ikx}\right]_0^1 - \frac{n-1}{2\pi ik}\int_0^1 M_{n-2}(x)e^{-2\pi ikx}dx\right\}, \\ &= \frac{(-1)^2 n(n-1)}{(2\pi ik)^2}\int_0^1 M_{n-2}(x)e^{-2\pi ikx}dx.\end{aligned}$$

So, integrating by parts $(n-1)$ times we get

$$\hat{f}_n(k) = \frac{(-1)^{n-1}n!}{(2\pi ik)^{n-1}} \int_0^1 M_1(x)e^{-2\pi ikx}dx.$$

Now, it follow from (35) that

$$\hat{f}_n(k) = \frac{(-1)^n n!}{(2\pi ik)^n}.$$

This finishes the proof of Theorem 3.1.

Theorem 3.2. *For $n \geq 1$, the U-Bernoulli numbers satisfy the following relationship with the Riemann zeta function:*

$$M_{2n} = \frac{2(2n)!(-1)^n}{(2\pi)^{2n}}\zeta(2n). \tag{36}$$

Proof. We employ the Theorem 3.1. By making $x = 0$ in (33), we have

$$\begin{aligned} f_n(0) &= \frac{(-1)^n n!}{(2\pi i)^n} \sum_{k\in\mathbb{Z}}{}' \frac{e^{2\pi ik(0)}}{k^n} \\ &= M_n(0) = M_n = \frac{(-1)^n n!}{(2\pi i)^n} \sum_{k\in\mathbb{Z}}{}' \frac{1}{k^n}. \end{aligned}$$

Therefore,

$$M_{2n} = \frac{(2n)!}{(2\pi i)^{2n}} \sum_{k\in\mathbb{Z}}{}' \frac{1}{k^{2n}}. \tag{37}$$

Notice that, from (9)

$$\begin{aligned} \sum_{k\in\mathbb{Z}}{}' \frac{1}{k^{2n}} &= \cdots + \frac{1}{(-2)^{2k}} + \frac{1}{(-1)^{2k}} + \frac{1}{(1)^{2k}} + \frac{1}{(2)^{2k}} + \cdots \\ &= 2\sum_{k\in\mathbb{Z}}{}' \frac{1}{k^{2n}} = 2\zeta(2n). \end{aligned} \tag{38}$$

Consequently, (37) becomes

$$M_{2n} = \frac{(2n)!}{(2\pi)^{2n}(i^2)^n}2\zeta(2n) = \frac{2(2n)!}{(2\pi)^{2n}(-1)^n}\zeta(2n).$$

So, (36) follows and Theorem 3.2 is proved.

Theorem 3.3. *For all $n \geq 1$, the U-Bernoulli numbers M_n satisfy the inequality,*

$$\frac{(2n)!}{(2\pi)^{2n}} \leq |M_{2n}| \leq \frac{(2n)!}{(2\pi)^{2n}} \frac{\pi^2}{3}. \tag{39}$$

Proof. From (36), gives

$$|M_{2n}| = \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k\geq 1} \frac{1}{k^{2n}}. \tag{40}$$

Now, let's consider the following chain of inequalities:

$$\begin{aligned} & k^{2n} \geq k^2 \\ \Leftrightarrow \quad & \sum_{k\geq 1} \frac{1}{k^{2n}} \leq \sum_{k\geq 1} \frac{1}{k^2} \\ \Leftrightarrow \quad & \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k\geq 1} \frac{1}{k^{2n}} \leq \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k\geq 1} \frac{1}{k^2}, \end{aligned}$$

hence

$$|M_{2n}| \leq \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k\geq 1} \frac{1}{k^2}.$$

Now, by Theorem 3.2 with $n = 1$, we have

$$\sum_{k\geq 1} \frac{1}{k^2} = \frac{(2\pi)^2}{2(-1)2} M_2 = \frac{4\pi^2}{-4}\left(-\frac{1}{6}\right) = \frac{\pi^2}{6},$$

therefore

$$\frac{2(2n)!}{(2\pi)^{2n}} \sum_{k\geq 1} \frac{1}{k^2} = \frac{(2n)!}{(2\pi)^{2n}} \frac{\pi^2}{3}.$$

So that,

$$|M_{2n}| \leq \frac{(2n)!}{(2\pi)^{2n}} \frac{\pi^2}{3}, \qquad n \geq 1. \tag{41}$$

On the other hand, notice that

$$\frac{2(2n)!}{(2\pi)^{2n}} \sum_{k\geq 1} \frac{1}{k^2} \geq \frac{(2n)!}{(2\pi)^{2n}},$$

then

$$|M_{2n}| \geq \frac{(2n)!}{(2\pi)^{2n}}, \qquad k \geq 1. \tag{42}$$

Therefore, we derive (39) from (41) and (42). Theorem 3.3 is proved.

4. Fourier series of the periodic U-Euler functions

In this section, we give the Fourier series of a periodic function involving $U-$Euler polynomials. Furthermore, we give a bound for U-Euler numbers.

Theorem 4.1. *Let $n \in \mathbb{N}$,*

$$g_n(x) := A_n(x - \lfloor x \rfloor), \qquad (x \in \mathbb{R}). \tag{43}$$

Then, the Fourier series for $g_n(x)$ is

$$g_n(x) = \sum_{k\in\mathbb{Z}} \left[2\sum_{j=1}^{n-1} \left(-\frac{1}{2}\right)^{j-1} \binom{n}{j-1} \frac{A_{n-j+1}(j-1)!}{(2k\pi i)^j} + \left(-\frac{1}{2}\right)^n \frac{n!}{(2k\pi i)^n} \right] e^{2k\pi i x}. \tag{44}$$

Proof. In analogy to (33), we find that g_n is also a periodic function with period $T = 1$, hence, it admits a Fourier series expansion given by

$$g_n(x) = \sum_{k\in\mathbb{Z}} \hat{g}_n(k) e^{2\pi i k x},$$

where

$$\hat{g}_n(k) = \int_0^1 g_n(x) e^{-2\pi i k x} dx.$$

Now, let's calculate the coefficients $\hat{g}_n(k)$.

For $k = 0$, $n \in \mathbb{N}$ and using (26) and (29), we get

$$\begin{aligned} \hat{g}_n(0) &= \int_0^1 A_n(x - \lfloor x \rfloor) e^{-2\pi i 0 x} dx \\ &= -\frac{2}{n+1}[-2A_{n+1}] = \frac{(-2)^2 A_{n+1}}{n+1}, \end{aligned}$$

this is true for all $n \geq 1$. For $k > 0$ and $n = 1$, from (26) and (29), follows

$$\hat{g}_1(k) = \int_0^1 A_1(x - \lfloor x \rfloor) e^{-2\pi i k x} dx = \int_0^1 \left(\frac{1}{4} - \frac{x}{2}\right) e^{-2\pi i k x} dx.$$

Integrating by parts, with $u = \frac{1}{4} - \frac{x}{2}$, $dv = e^{-2\pi i k x} dx$, using (26) and (29) gives

$$\begin{aligned} \hat{g}_1(k) &= \left[-\frac{1}{2\pi i k} \left(\frac{1}{4} - \frac{x}{2}\right) e^{-2\pi i k x} \right]_0^1 - \int_0^1 -\frac{1}{2\pi i k} e^{-2\pi i k x} \left(-\frac{1}{2}\right) dx \\ &= -\frac{1}{2\pi i k} \left[\left(\frac{1}{4} - \frac{x}{2}\right) e^{-2\pi i k x} \right]_0^1 - \frac{1}{2\pi i k} \frac{1}{2} \int_0^1 e^{-2\pi i k x} dx. \end{aligned}$$

Therefore,

$$\hat{g}_1(k) = -\frac{1}{2k\pi i}\left(-\frac{1}{2}\right). \tag{45}$$

Let's consider now, $n > 1$ and $k > 0$. We have in this case,

$$\hat{g}_n(k) = \int_0^1 A_n(x - \lfloor x \rfloor)e^{-2\pi ikx}dx = \int_0^1 A_n(x)e^{-2\pi ikx}dx,$$

integrating by parts, with $u = A_n(x)$, $dv = e^{-2\pi ikx}dx$, and using (26), (29) we get

$$\hat{g}_n(k) = -\frac{1}{2\pi ik}\left[A_n(1)e^{-2\pi ik} - A_n(0)\right] - \frac{n}{2}\frac{1}{2\pi ik}\int_0^1 A_{n-1}(x)e^{-2\pi ikx}dx.$$

So that,

$$\hat{g}_n(k) = -\frac{1}{2\pi ik}\left[-2A_n\right] - \frac{n}{2}\frac{1}{2\pi ik}\int_0^1 A_{n-1}(x)e^{-2\pi ikx}dx.$$

Integrating by parts the last integral, with $u = A_{n-1}(x)$, $du = -\frac{(n-1)}{2}A_{n-2}(x)$ and using (26) and (29) it follows that

$$\begin{aligned}
\hat{g}_n(k) &= -\frac{[-2A_n]}{2k\pi i} - \frac{n}{2}\frac{1}{2\pi ik}\left\{-\frac{1}{2\pi ik}\left[A_{n-1}(x)e^{-2k\pi ix}\right]_0^1 - \frac{(n-1)}{2}\frac{1}{2\pi ik}\int_0^1 A_{n-2}(x)e^{-2k\pi ix}dx\right\} \\
&= \frac{2A_n}{2k\pi i} + \left(-\frac{1}{2}\right)\frac{2nA_{n-1}}{(2\pi ik)^2} + \left(-\frac{1}{2}\right)^2\frac{n(n-1)}{(2k\pi i)^2}\int_0^1 A_{n-2}(x)e^{-2k\pi ix}dx.
\end{aligned}$$

Integrating by parts the last integral, with $u = A_{n-2}(x)$, $du = -\frac{(n-2)}{2}A_{n-3}(x)dx$ by (26), (29) and using the notation

$\Theta = \frac{2A_n}{2k\pi i} + \left(-\frac{1}{2}\right)\frac{2nA_{n-1}}{(2\pi ik)^2}$, we obtain

$$\begin{aligned}
\hat{g}_n(k) &= \Theta + \left(-\frac{1}{2}\right)^2\frac{n(n-1)}{(2k\pi i)^2}\left\{-\frac{1}{2k\pi i}[A_{n-2}(x)e^{-2k\pi i}]_0^1 - \frac{(n-2)}{2}\frac{1}{2k\pi i}\int_0^1 A_{n-3}(x)e^{-2k\pi ix}dx\right\} \\
&= \Theta + \left(-\frac{1}{2}\right)^2\frac{n(n-1)}{(2k\pi i)^2}\frac{2A_{n-2}}{2k\pi i} + \left(-\frac{1}{2}\right)^2\frac{n(n-1)}{(2k\pi i)^2}\frac{-(n-2)}{2}\frac{1}{2k\pi i}\int_0^1 A_{n-3}(x)e^{-2k\pi ix}dx \\
&= \Theta + \left(-\frac{1}{2}\right)^2\frac{2n(n-1)A_{n-2}}{(2k\pi i)^3} + \left(-\frac{1}{2}\right)^3\frac{n(n-1)(n-2)}{(2k\pi i)^3}\int_0^1 A_{n-3}(x)e^{-2k\pi ix}dx.
\end{aligned}$$

Integrating by parts the last integral $(n-1)$ times, we can say

$$\begin{aligned}\hat{g}_n(k) &= \sum_{j=0}^{n-2}\left(-\frac{1}{2}\right)^j \frac{2A_{n-j}}{(2k\pi i)^{j+1}}\frac{n!}{(n-j)!} + \left(-\frac{1}{2}\right)^{n-1}\frac{n(n-1)\cdots(n-(n-1))}{(2k\pi i)^{n-1}}\int_0^1 A_1(x)e^{-2k\pi i x}dx\\ &= \sum_{j=1}^{n-1}\left(-\frac{1}{2}\right)^{j-1}\frac{2A_{n-j+1}(j-1)!}{(2k\pi i)^j}\frac{n!}{(j-1)!(n-j+1)!} + \left(-\frac{1}{2}\right)^n \frac{n!}{(2k\pi i)^n}.\end{aligned}$$

Then,

$$\hat{g}_n(k) = \left(-\frac{1}{2}\right)^n \frac{n!}{(2k\pi i)^n} + \sum_{j=1}^{n-1}\left(-\frac{1}{2}\right)^{j-1}\binom{n}{j-1}\frac{2A_{n-j+1}(j-1)!}{(2k\pi i)^j}. \tag{46}$$

Therefore, from (7) and (46) follows (44). Theorem 4.1 is completely the proved.

Proposition 4.1. *The $U-$Bernoulli numbers defined in (14) and the tangent function are related by*

$$\tan(-x) = \sum_{n=0}^{\infty}\frac{M_{2n+2}}{(2n+2)!}4^{n+1}[1-4^{n+1}](-1)^{n+1}x^{2n+1}. \tag{47}$$

Proof. By (14) and (2) with $z = 2ix$ and $z = 4ix$ and taking into account (15), we have the following

$$\begin{aligned}x\tan(-x) &= -ix + \sum_{n=0}^{\infty}\frac{M_n}{n!}[2^n - 4^n]i^n x^n\\ &= -ix + \left(-\frac{1}{2}\right)[-2]ix + \sum_{n=2}^{\infty}\frac{M_n}{n!}[2^n-4^n]i^n x^n\\ &= \sum_{n=1}^{\infty}\frac{M_{2n}}{(2n)!}[4^n - 4^n 4^n](-1)^n x^{2n}.\end{aligned}$$

Hence,

$$x\tan(-x) = \sum_{n=1}^{\infty}\frac{M_{2n}}{(2n)!}4^n[1-4^n](-1)^n x^{2n}.$$

So (47) follows. Proposition 4.1 is demonstrated.

Proposition 4.2. *The U-Euler numbers defined in (17) and the tangent function are related by*

$$\tan(-x) = \sum_{n=0}^{\infty} A_{2n+1} \frac{4^{2n+1}(-1)^{n+1}x^{2n+1}}{(2n+1)!}. \quad (48)$$

PROOF. Using together (16) and (11) with $z = 4ix$ and taking into account (18), it follows that

$$\begin{aligned} i\tan(-x) &= 1 - \sum_{n=0}^{\infty} A_n \frac{(4ix)^n}{n!} = 1 - \sum_{n=0}^{\infty} A_n \frac{4^n i^n x^n}{n!} \\ &= -\sum_{n=0}^{\infty} A_{n+1} \frac{4^{n+1} i^{n+1} x^{n+1}}{(n+1)!} = -i\sum_{n=0}^{\infty} A_{n+1} \frac{4^{n+1} i^n x^{n+1}}{(n+1)!} \\ &= -i\sum_{n=0}^{\infty} A_{2n+1} \frac{4^{2n+1} i^{2n} x^{2n+1}}{(2n+1)!}. \end{aligned}$$

So,

$$i\tan(-x) = i\sum_{n=0}^{\infty} A_{2n+1} \frac{4^{2n+1}(-1)^{n+1}x^{2n+1}}{(2n+1)!}. \quad (49)$$

Thus, by (49), the desire result follows. Thus, we complete the proof of the Proposition 4.2.

Proposition 4.3. *For $n \geq 1$, the numbers M_n and A_n are related by means of the following formula*

$$M_{2n} = \frac{n2^{2n-1}}{(1-4^n)} A_{2n-1}. \quad (50)$$

PROOF. We have from (47) and (48) that

$$\begin{aligned} \frac{M_{2n+2}}{(2n+2)!} 4^{n+1}[1-4^{n+1}](-1)^{n+1} &= \frac{A_{2n+1}}{(2n+1)!} 4^{2n+1}(-1)^{n+1} \\ &= \frac{A_{2n+1} 4^{2n+1}(-1)^{n+1}(2n+2)!}{4^{n+1}[1-4^{n+1}](-1)^{n+1}(2n+1)!} \\ &= \frac{A_{2n+1} 4^n 4^{n+1}(2n+1)!(2n+2)}{4^{n+1}[1-4^{n+1}](2n+1)!} \\ &= \frac{A_{2n+1} 4^n (2n+2)}{[1-4^{n+1}]}. \end{aligned}$$

Then,

$$M_{2(n+1)} = \frac{4^n(2n+2)}{[1-4^{n+1}]} A_{2n+1}.$$

From the previous equality, it follows that

$$\begin{aligned} M_{2n} &= \frac{A_{2(n-1)+1}4^{n-1}(2(n-1)+2)}{[1-4^{(n-1)+1}]} \\ &= \frac{A_{2n-1}4^{n-1}(2n)}{1-4^n} \\ &= \frac{A_{2n-1}4^n n}{2(1-4^n)} = \frac{n2^{2n-1}}{1-4^n}A_{2n-1}. \end{aligned}$$

Proposition 4.3 is demonstrated.

Proposition 4.4. *For $n \geq 1$, the U-Euler numbers defined in* (17) *are related to the Riemann zeta function as follows:*

$$\zeta(2n) = \frac{4^{n-1}(2\pi)^{2n}(-1)^n}{2(2n-1)![1-4^n]}A_{2n-1}. \tag{51}$$

Furthermore,

$$\frac{(4^n-1)(2n)!}{n2^{2n-1}(2\pi)^{2n}} \leq |A_{2n-1}| \leq \frac{2(4^n-1)(2n)!\pi^2}{3n(4\pi)^{2n}}. \tag{52}$$

PROOF. The representation (51) follows from (36) and (50)

$$\frac{2(2n)!(-1)^n}{(2\pi)^{2n}}\zeta(2n) = \frac{nA_{2n-1}2^{2n-1}}{1-4^n}.$$

So, we obtain

$$\zeta(2n) = \frac{A_{2n-1}4^{n-1}(2\pi)^{2n}(-1)^n}{2(2n-1)![1-4^n]}.$$

From (50) it follows that

$$A_{2n-1} = \frac{(1-4^n)}{n2^{2n-1}}M_{2n}.$$

This implies

$$|A_{2n-1}| = \left|\frac{(1-4^n)}{n2^{2n-1}}M_{2n}\right| = \frac{(4^n-1)}{n2^{2n-1}}|M_{2n}|. \tag{53}$$

Now, from (41) and (53) we have

$$|A_{2n-1}| \leq \frac{2(4^n-1)(2n)!\pi^2}{3n(4\pi)^{2n}}. \tag{54}$$

On the other hand, from (42) and (53), we get

$$|A_{2n-1}| \geq \frac{4^n - 1}{n2^{2n-1}} \frac{(2n)!}{(2\pi)^{2n}}. \tag{55}$$

Then, from (54) and (55), (52) follows. Proposition 4.4 is completely proved.

5. Fourier series of the periodic *U*-Genocchi functions

In this section, we give the Fourier series of a periodic function involving U-Genocchi polynomials. Furthermore, we give a bound for U-Genocchi numbers.

Theorem 5.1. *Let $n \geq 1$, $x \in \mathbb{R}$ and let's consider the function*

$$h_n(x) := V_n(x - \lfloor x \rfloor), \tag{56}$$

where $\lfloor x \rfloor$ is the integer part function. Then, the Fourier series for $h_n(x)$ is given by

$$h_n(k) = n \sum_{k \in \mathbb{Z}} \left[2 \sum_{j=1}^{n-2} \left(-\frac{1}{2}\right)^{j-1} \binom{n-1}{j-1} \frac{V_{n-j+2}}{n-j+2} \frac{(j-1)!}{(2k\pi i)^j} + \left(-\frac{1}{2}\right)^{n-1} \frac{(n-1)!}{(2k\pi i)^n} \right] e^{2k\pi i x}. \tag{57}$$

Proof. Analogously to (33), the function h_n is also a periodic function with period $T = 1$, hence, it admits a Fourier series expansion given by

$$h_n(x) = \sum_{k \in \mathbb{Z}} \hat{h}_n(k) e^{2\pi i k x},$$

where

$$\hat{h}_n(k) = \int_0^1 h_n(x) e^{-2\pi i k x} dx.$$

Using the relationship between U-Euler and U-Genocchi polynomials given by (21), we can get a Fourier expansion for the U-Genocchi polynomial as follows: Firstly, let's see that

$$h_n(x) = V_n(x - \lfloor x \rfloor) = nA_{n-1}(x - \lfloor x \rfloor) = ng_{n-1}(x).$$

Hence, applying (21) to the Fourier series for $g_n(x)$ given by (44) yields (57). This completes the proof.

On the other hand, the relationship between U-Euler and U-Genocchi numbers given by (22) allows us to find directly, the relationship between the U-Genocchi numbers and the Riemann zeta function. We can now state the following result.

Proposition 5.1. *For $n \geq 1$, numbers V_n are related to the Riemann zeta function as follows*

$$\zeta(2n) = \frac{4^{n-2}(2\pi)^{2n}(-1)^n}{n(2n-1)![1-4^n]}V_{2n}. \tag{58}$$

Furthermore,

$$\frac{4(4^n-1)(2n)!}{2^{2n}(4\pi)^{2n}} \leq |V_{2n}| \leq \frac{4(4^n-1)(2n)!\pi^2}{3(4\pi)^{2n}}. \tag{59}$$

Proof. From (21) and (51), (58) follows.

$$\begin{aligned}\zeta(2n) &= \frac{4^{n-1}(2\pi)^{2n}(-1)^n}{2(2n-1)![1-4^n]}A_{2n-1} = \frac{4^{n-1}(2\pi)^{2n}(-1)^n}{2(2n-1)![1-4^n]}\frac{V_{2n}}{2n} \\ &= \frac{4^{n-1}(2\pi)^{2n}(-1)^n}{4n(2n-1)![1-4^n]}V_{2n} = \frac{4^{n-2}(2\pi)^{2n}(-1)^n}{n(2n-1)![1-4^n]}V_{2n}.\end{aligned}$$

Using (52), we get a bound for the U-Genocchi numbers as follows:

$$\begin{aligned} \frac{(4^n-1)(2n)!}{n2^{2n-1}(2\pi)^{2n}} &\leq |A_{2n-1}| \leq \frac{2(4^n-1)(2n)!\pi^2}{3n(4\pi)^{2n}} \\ \Leftrightarrow \frac{(4^n-1)(2n)!}{n2^{2n-1}(2\pi)^{2n}} &\leq \left|\frac{V_{2n}}{2n}\right| \leq \frac{2(4^n-1)(2n)!\pi^2}{3n(4\pi)^{2n}} \\ \Leftrightarrow \frac{2(4^n-1)(2n)!}{2^{2n}2^{-1}(2\pi)^{2n}} &\leq |V_{2n}| \leq \frac{4(4^n-1)(2n)!\pi^2}{3(4\pi)^{2n}} \\ \Leftrightarrow \frac{4(4^n-1)(2n)!}{2^{2n}(4\pi)^{2n}} &\leq |V_{2n}| \leq \frac{4(4^n-1)(2n)!\pi^2}{3(4\pi)^{2n}}. \end{aligned}$$

Proposition 5.1 is completely proved.

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A Research Announcement on New Parametric U–Charlier–Poisson Polynomials and Their Szász–Type Operators

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Abstract

In this work, we present a new family of parametric U–Charlier–Poisson–type polynomials, denoted by $G_n^{[2+J]}(x;\alpha,\beta,\lambda)$. We establish several foundational properties of this family, including explicit representations, an associated orthogonality structure, and a link with derivatives of harmonic functions. Furthermore, we introduce Szász–type operators constructed from these polynomials and investigate their approximation behavior by means of Korovkin's theorem, thereby obtaining convergence results for the proposed operators.

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1. Introduction

Throughout this article, $\mathbb{N}$ will mean the set of natural numbers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$,likewise $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$ will denote the set of real numbers, positive real numbers, and the set of complex numbers. As usual, will denote by $C[0,\infty)$ the set of all functions f continuous in the interval $[0,\infty)$. The notation $UC[0,\infty)$ will denote the space of functions uniformly continuous on $[0,\infty)$. The space of all polynomials in one variable with real coefficients is denoted by $\mathbb{P}$, and $\log(z)$ denotes the principal value of the multi-valued logarithm function. In [5], a famous theorem about linear operators is published, known as the Korovkin Theorem, which states that a sequence of linear operators under certain conditions converges uniformly in each subset of the locally compact domain. Korovkin Theorem, in its many applications, was also used to demonstrate the convergence of Szász operators, which are defined by (see [9, p. 239, Eq. (2)]):

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \tag{1}$$

where $f \in C[0,\infty)$, $n \in \mathbb{N}$, and $x \geq 0$. The generalizations of Szász operators by using polynomials, especially defined via generating functions, have been frequently studied lately. These kinds

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of generalizations provide a range of new sequences of operators to approximation theory highly advantageous when interpolating positive continuous functions [9]. A known generalization of (1) can be obtained using the Appell polynomials given by (cf. [12]):

$$P_n(f;x) := \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{2}$$

considering that $p_k(x) \geq 0$, for $x \in [0,\infty)$ and $g(1) \neq 0$.

Some time later, Serhan Varma, et al., in ([12, p.122 Eq. (1.7)]), generalized (1) in the following way: first, they use the Brenke-type polynomials, which are defined by the following generating function:

$$\zeta(z)\xi(xz) = \sum_{k=0}^{\infty} p_k(x) \frac{z^k}{k!}, \tag{3}$$

where ζ and ξ are analytical functions. Second, they introduce the linear positive operators including the Brenke-type, polynomials which are given by (see [12, p. 121, Eq. (1.12)]):

$$L_n(f;x) := \frac{1}{\zeta(1)\xi(nx)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{4}$$

where $x \geq 0$ and $n \in \mathbb{N}$. It is observed that if $\xi(z) = e^z$ in (3), then the operators (4) concerning (3) lead to (2) with respect to the Appell polynomials, and if $\xi(z) = e^z$ and $\zeta(z) = 1$ in (4), we have (1).

On the other hand, when using the Brenke-type polynomials given in (3), with $\xi(z) = e^z$ and $\zeta(t) = \left(1 - \frac{z}{a}\right)^u$, we have the classic Charlier-Poisson polynomials, which are defined by (see [10, p. 458, Eq. (1.2)]):

$$e^z \left(1 - \frac{z}{a}\right)^u = \sum_{k=0}^{\infty} C_k(a,u) \frac{z^k}{k!}, \qquad a \neq 0. \tag{5}$$

Then, Serhan Varma, et al., introduce the positive linear operators involving Charlier-Poisson polynomials (see [11, p. 119, Eq. (1.6)]) by replacing $\xi(z) = e^z$ and $\zeta(z) = \left(1 - \frac{z}{a}\right)^u$ in (4), as follows:

$$L_n(f;x,a) = e^{-1}\left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k(a, -(a-1)nx)}{k!} f\left(\frac{k}{n}\right), \tag{6}$$

where $a > 1$ and $x \geq 0$. We see that if in (6) we take on both sides $a \to \infty$ and $x \to x - \frac{1}{n}$, then we get the Szász operators given in (1). The convergence and bounding properties of these operators were also investigated [11]. Furthermore, in [1], a study of Charlier-Poisson polynomials is presented, in particular, their explicit representation given by

$$C_n(x,\alpha) = \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} k! (-\alpha)^{n-k}. \tag{7}$$

The Charlier-Poisson polynomials $C_n(x,\alpha)$, $x \in \mathbb{N}_0$, and, $\alpha > 0$, are orthogonal with respect to the Poisson distribution with mean α, that is,

$$\sum_{x=0}^{\infty} C_m(x,\alpha)C_n(x,\alpha)\frac{e^{-\alpha}\beta^x}{x!} = \alpha^n n!\delta_{mn}, \quad m,n \in \mathbb{N}_0, \tag{8}$$

where δ_{mn} is the Kronecker delta.

Our contribution aims to introduce a new family of discrete polynomials, called new parametric U-Charlier-Poisson type polynomials, and we investigate some of their properties. Thus, the operators obtained from Brenke-type polynomials are applied to the said polynomials. The outline of this work is as follows: In Section 2, we study some basic results of operators obtained from Brenke-type polynomials applied to Charlier-Poisson polynomials and other results necessary for developing this work. In Section 3, we introduce the new parametric U-Charlier-Poisson type polynomials and explore some of their properties. In Section 4, we investigate the orthogonality relation. Finally, in Section 5, we apply the Szász-type operators (4), obtained from Brenke-type polynomials to the new family of polynomials to study the convergence and bounding properties.

2. Background and previous results

Let f be some function of a real variable x. The backward and forward difference operators Δ and ∇ respectively, are defined as (see [6, p. 19–20]):

$$\nabla f(x) := f(x) - f(x-1), \tag{9}$$
$$\Delta f(x) := f(x+1) - f(x). \tag{10}$$

Given two real sequences $\{a_k\}$ and $\{b_k\}$, if $b_{-1} = 0$, then (see [6, p. 20])

$$\sum_{k=0}^{\infty}(\Delta a_k)b_k = -\sum_{k=0}^{\infty} a_k \nabla b_k. \tag{11}$$

Furthermore, for $f_1(x)$ and $f_2(x)$ with real values, the following property is satisfied (cf. [3]):

$$\nabla(f_1(x)f_2(x)) = f_1(x)\nabla f_2(x) + f_2(x-1)\nabla f_1(x). \tag{12}$$

The falling factorial x of order n is (see [4])

$$\langle x\rangle_n := x(x-1)\cdots(x-n+1), \text{ with } \langle x\rangle_0 = 1, \tag{13}$$

and the rising factorial x of order n is (see [4])

$$(x)_n := x(x+1)\cdots(x+n-1), \quad (x)_0 := 1. \tag{14}$$

The rising factorial and the falling factorial fulfill the following relationship (see [3]):

$$(x)_n = \frac{\Gamma(n+x)}{\Gamma(x)}, \tag{15}$$
$$\langle x\rangle_n = \frac{x!}{(x-n)!}, \tag{16}$$

where $\Gamma(x)$ is the Gamma function.

On the other hand, the digamma function $\psi(x)$ is defined as the logarithmic derivative of the gamma function $\Gamma(x)$ (see [7])

$$\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{17}$$

The generalized harmonic numbers function is given by (see [7])

$$H_n^m(x) = \sum_{k=0}^{n-1} \frac{1}{(k+x)^m}, \quad n, m \in \mathbb{N}. \tag{18}$$

If $m = 1$ in (18), then

$$H_n^{(1)}(x) = \sum_{k=0}^{n-1} \frac{1}{k+x}. \tag{19}$$

If $x = 0$ in (18), we have

$$H_n^m(0) = H_n^m = \sum_{k=1}^{n} \frac{1}{k^m}, \tag{20}$$

where H_n^m denotes the so-called n-th harmonic numbers of order m.

Notice that from (15) and (17) follows

$$\begin{aligned} \frac{d}{dx}(x)_n &= \frac{\Gamma(n+x)}{\Gamma(x)} \left(\frac{d}{dx} \ln(\Gamma(n+x) - \frac{d}{dx} \ln(\Gamma(x) \right) \\ &= \frac{\Gamma(n+x)}{\Gamma(x)} \left(\psi(n+x) - \psi(x) \right). \end{aligned} \tag{21}$$

By (15), (19), and (21), we obtain

$$(x)_n = \frac{1}{H_n^{(1)}(x)} \frac{d}{dx}(x)_n. \tag{22}$$

The Stirling numbers of the first kind, $s(n,k)$, appear as the coefficients in the following generating function (see [8]):

$$\frac{(\log(1+z))^k}{k!} = \sum_{n=k}^{\infty} s(n,k) \frac{z^n}{n!}. \tag{23}$$

In addition, they also satisfy

$$\langle x \rangle_n = \sum_{k=0}^{n} s(n,k) x^k. \tag{24}$$

Note that from (24), we can write

$$(1+z)^x = \sum_{n=0}^{\infty} \binom{x}{n} z^n = \sum_{n=0}^{\infty} \langle x \rangle_n \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} s(n,k) \frac{z^n}{n!} \right) x^k. \tag{25}$$

Now, (cf. [1, p. 170 Eq. (1.1)]) it is possible to represent the Charlier-Poisson polynomials given in (5) as follows:

$$e^{-\alpha w}(1+w)^x = \sum_{n=0}^{\infty} C_n(x,\alpha)\frac{w^n}{n!}, \tag{26}$$

with $\alpha \neq 0$. Note that by taking $\alpha = a$, $w = -\dfrac{z}{a}$, $x = u$ in (26) we have (5).

It is worth noting that the classical orthogonal polynomials possess a weight function that conforms to the Pearson equation of the form

$$\nabla\left[\sigma(x)\omega(x)\right] = \tau(x)\omega(x), \tag{27}$$

whit σ a polynomial of degree ≤ 2 and τ a polynomial of degree ≤ 1. We note that in (27) the backward difference operator ∇, given in (9), is used for orthogonal polynomials on the lattice and it is replaced by differentiation in the case of orthogonal polynomials on an interval of the real line. The Pearson equation is an important part of the theory of classical orthogonal polynomials because it lets us find many useful properties of these polynomials.

Let $f \in UC[0,\infty)$, If $\delta > 0$, the modulus of continuity of the function f, denoted by $\omega(f;\delta)$ is defined by (cf. [11])

$$\omega(f;\delta) := \sup_{x,y\in[0,\infty)} |f(x)-f(y)|, \;\; where \;\; |x-y| < \delta. \tag{28}$$

Additionally, it is well known that,

$$|f(x)-f(y)| \leq \omega(f,\delta)\left(\frac{|x-y|}{\delta}+1\right). \tag{29}$$

Also, we have if f is uniformly continuous, then

$$|f(x)-f(y)| \leq \omega(f,\delta). \tag{30}$$

The following Proposition summarizes some properties of the operators defined in (6).

Proposition 2.1. *For $n \in \mathbb{N}$, let $L_n(f;x,a)$ the positive linear operators involving Charlier-Poisson polynomials in the variable $x \geq 0$. Then, the following statements hold.*

1. *[11, Lemma 1] The operators defined in (6) satisfy the following identities:*
   (i) $L_n(1;x,a) = 1$.
   (ii) $L_n(s;x,a) = x + \dfrac{1}{n}$.
   (iii) $L_n(s^2;x,a) = x^2 + \dfrac{x}{n}\left(3+\dfrac{1}{a-1}\right) + \dfrac{2}{n^2}$.
2. *[11, Theorem 1] Let E be the set given by*

$$S := \left\{f : [0,\infty) \to \mathbb{R} : |f(x)| \leq Me^{Ax}, \;\; A \in \mathbb{R} \; and \; M \in \mathbb{R}^+\right\}.$$

If $f \in C[0,\infty) \cap S$, then

$$\lim_{n\to\infty} L_n(f;x,a) = f(x). \tag{31}$$

That is, the operators defined in (6) converge uniformly on every compact subset of $[0,\infty)$.

3. *[11, Theorem 2] Let $f \in UC[0,\infty) \cap S$. Then the operators L_n given in (6) satisfy*

$$|L_n(f;x,a) - f(x)| \leq \left\{1 + \sqrt{x\left(1 + \frac{1}{a-1}\right) + \frac{2}{n}}\right\} \omega\left(f; \frac{1}{\sqrt{n}}\right), \tag{32}$$

with ω given by (28).

3. New parametric U-Charlier-Poisson type polynomials and some of their properties

In this section, we shall introduce a new class of discrete polynomials, which we denote by $G_n^{[2+J]}(x;\alpha,\beta,\lambda)$ and will we call new parametric U-Charlier-Poisson type polynomials. Furthermore, we obtain some of their properties.

Definition 3.1. *For a fixed $J \in \mathbb{N}$, $\beta, \lambda \in \mathbb{R}$ and $\alpha \neq 0$, the new family of parametric U-Charlier-Poisson type polynomials $G_n^{[2+J]}(x;\alpha,\beta,\lambda)$ in the variable $x \in \mathbb{N}_0$ are defined by the means of the power series expansion at 0 of the following generating function:*

$$u(x;z;\alpha,\beta,\lambda) = \left[\beta e^{-\alpha z} + \lambda(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}\right](1+z)^x = \sum_{n=0}^{\infty} G_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!}. \tag{33}$$

From (33) and taking $A_j(\lambda,\alpha) = \lambda(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}$, the first parametric U-Charlier-Poisson type polynomials are obtained, which are:

$$\begin{aligned}
G_0^{[2+J]}(x;\alpha,\beta,\lambda) &= \beta + A_j(\lambda,\alpha),\\
G_1^{[2+J]}(x;\alpha,\beta,\lambda) &= -\alpha\beta + A_j(\lambda,\alpha) + x(\beta + A_j(\lambda,\alpha)),\\
G_2^{[2+J]}(x;\alpha,\beta,\lambda) &= \alpha^2\beta - 2\alpha\beta x + x(x-1)(\beta + A_j(\lambda,\alpha)),\\
G_3^{[2+J]}(x;\alpha,\beta,\lambda) &= -\alpha^3\beta + \alpha^2\beta x - 2\alpha\beta x(x-1) + x(x-1)(x-2)(-\alpha\beta + \beta + A_j(\lambda,\alpha)).
\end{aligned}$$

Note that if $\beta = 1$ and $\lambda = 0$, $z = w$ in (33), we have the classic Charlier-Poisson polynomials given in (26). Therefore, the generating function of $G_n^{[2+J]}(x;\alpha,\beta,\lambda)$ in (33) includes, as its special cases, the generating function of the Charlier-Poisson polynomials, i.e., $C_n(x,\alpha) = G_n^{[2+J]}(x;\alpha,1,0)$.

Substituting $x = 0$ in (33), we have

$$u(0;z;\alpha,\beta,\lambda) = \left[\beta e^{-\alpha z} + \lambda(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}\right] = \sum_{n=0}^{\infty} G_n^{[2+J]}(0,\alpha,\beta,\lambda)\frac{z^n}{n!}, \tag{34}$$

where $G_n^{[2+J]}(0,\alpha,\beta,\lambda) = G_n^{[2+J]}(\alpha,\beta,\lambda)$ denotes the parametric U-Charlier-Poisson type numbers. In view of (34), we can compute a few values of the numbers $G_n^{[2+J]}(\alpha,\beta,\lambda)$ as follows:

$$G_0^{[2+J]}(\alpha,\beta,\lambda) = \beta + A_j(\lambda,\alpha), \qquad G_3^{[2+J]}(\alpha,\beta,\lambda) = -\beta\alpha^3,$$
$$G_1^{[2+J]}(\alpha,\beta,\lambda) = -\beta\alpha, \qquad G_4^{[2+J]}(\alpha,\beta,\lambda) = \beta\alpha^4,$$
$$G_2^{[2+J]}(\alpha,\beta,\lambda) = \alpha^2\beta, \qquad G_5^{[2+J]}(\alpha,\beta,\lambda) = -\beta\alpha^5.$$

One can use $A_j(\lambda,\alpha)$ in the following manner:

$$A_j(\lambda,\alpha) = \lambda(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2} = \sum_{n=0}^{\infty} U_n^{[2+J]}(\alpha)\frac{\lambda^n}{n!}. \tag{35}$$

Whereby some $U_n^{[2+J]}(\alpha)$ are

$$U_o^{[2+J]}(\alpha) = (-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}, \qquad U_2^{[2+J]}(\alpha) = 2(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2},$$
$$U_1^{[2+J]}(\alpha) = (-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}, \qquad U_3^{[2+J]}(\alpha) = 6(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}.$$

Proposition 3.1. *Let $\beta \in \mathbb{R} - \{0\}$, $J \in \mathbb{N}$ fixed, and $\left\{G_k^{[2+J]}(\alpha,\beta,\lambda)\right\}_{k=0}^{\infty}$ be a parametric U-Charlier-Poisson type sequence of numbers defined as in* (34). *Then, the following relationship is fulfilled:*

$$G_n^{[2+J]}(\alpha,\beta,\lambda) = (-1)^n\beta\alpha^n, \tag{36}$$
$$\text{with} \quad G_0^{[2+J]}(\alpha,\beta,\lambda) = \beta + A_j(\lambda,\alpha). \tag{37}$$

Proof. By using (34) follows

$$\sum_{n=0}^{\infty} G_n^{[2+J]}(\alpha,\beta,\lambda)\frac{z^n}{n!} = A_j(\lambda,\alpha) + \beta\sum_{n=0}^{\infty}(-1)^n\alpha^n\frac{z^n}{n!}$$
$$\Leftrightarrow \quad G_0^{[2+J]}(\alpha,\beta,\lambda) + \sum_{n=1}^{\infty} G_n^{[2+J]}(\alpha,\beta,\lambda)\frac{z^n}{n!} = \beta\sum_{n=0}^{\infty}(-1)^n\alpha^n\frac{z^n}{n!} + A_j(\lambda,\alpha).$$

With what we have,

$$G_0^{[2+J]}(\alpha,\beta,\lambda) = \sum_{n=1}^{\infty}\left[(-1)^n\alpha^n\beta - G_n^{[2+J]}(\alpha,\beta,\lambda)\right]\frac{z^n}{n!} + (\beta + A_j(\lambda,\alpha)). \tag{38}$$

From (38) follows (36) and (37). Proposition 3.1 is proved.

With its proof, the following proposition provides a concise formula for the parametric U-Charlier-Poisson type polynomials $G_n^{[2+J]}(x;\alpha,\beta,\lambda)$.

Proposition 3.2. *Given a fixed $J \in \mathbb{N}$, let $\left\{G_n^{[2+J]}(x;\alpha,\beta,\lambda)\right\}_{n=0}^{\infty}$ be a parametric U-Charlier-Poisson type sequence of polynomials, defined as in (33). Then, the following explicit representation hold:*

$$G_n^{[2+J]}(x;\alpha,\beta,\lambda) = \beta C_n(x,\alpha) + \lambda(-\alpha)^{-J}\left[\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}\right]\langle x\rangle_n\,, \tag{39}$$

where $\langle x\rangle_n$, is the falling factorial defined in (13).

Proof. Using the generating function of the parametric U-Charlier-Poisson type polynomials given in (33), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!} &= \left[\beta e^{-\alpha z} + \lambda(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}\right](1+z)^x \\
&= \beta\sum_{n=0}^{\infty} C_n(x,\alpha)\frac{z^n}{n!} + \lambda(-\alpha)^{-J}\left[\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}\right]\sum_{n=0}^{\infty}\binom{x}{n}z^n \\
&= \beta\sum_{n=0}^{\infty} C_n(x,\alpha)\frac{z^n}{n!} + \lambda(-\alpha)^{-J}\left[\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}\right]\sum_{n=0}^{\infty}\langle x\rangle_n\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\left[\beta C_n(x,\alpha) + \lambda(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}\langle x\rangle_n\right]\frac{z^n}{n!}.
\end{aligned}$$

Considering the above expression, we thus have (39). Proposition 3.2 is demonstrated.

Proposition 3.3. *For a fixed $J \in \mathbb{N}$, let $\left\{G_k^{[2+J]}(x;\alpha,\beta,\lambda)\right\}_{k=0}^{\infty}$ be a parametric U-Charlier-Poisson type sequence of polynomials defined by (33). If $\beta \to 0$, and $\lambda \to 1$, then the following identity holds:*

$$\sum_{k=0}^{n}\binom{n}{k}G_k^{[2+J]}(x;\alpha,0,1)C_{n-k}(-\alpha,-x) = \alpha^n\sum_{n=0}^{\infty} n!\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}. \tag{40}$$

Proof. Let us write (33) as

$$\begin{aligned}
\left[\beta e^{-\alpha z}(1+z)^x + A_j(\lambda,\alpha)(1+z)^x\right]e^{\alpha z}(1+z)^{-x} &= \left(\sum_{n=0}^{\infty} G_n^{[2+J]}(x;\beta,\alpha,\lambda)\frac{z^n}{n!}\right)e^{\alpha z}(1+z)^{-x} \\
&= \left(\sum_{n=0}^{\infty} G_n^{[2+J]}(x;\beta,\alpha,\lambda)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} C_n(-\alpha,-x)\frac{z^n}{n!}\right).
\end{aligned}$$

From the above expression and (35), we have

$$\beta + e^{\alpha z}A_j(\lambda,\alpha) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}G_k^{[2+J]}(x;\beta,\alpha,\lambda)C_{n-k}(-x,-\alpha)\frac{z^n}{n!}.$$

$$\Leftrightarrow \quad \beta + \sum_{n=0}^{\infty} \alpha^n \left(\sum_{n=0}^{\infty} U_n^{[2+J]}(\lambda;\alpha) \frac{\lambda^n}{n!} \right) \frac{z^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} G_k^{[2+J]}(x;\beta,\alpha,\lambda) C_{n-k}(-x,-\alpha) \frac{z^n}{n!}.$$

Then, taking $\beta \to 0$, and $\lambda \to 1$, follows

$$\sum_{n=0}^{\infty} \alpha^n \left(\sum_{n=0}^{\infty} n! \prod_{m=2}^{2+J} \frac{2^m-1}{2^m-2} \right) \frac{z^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} G_k^{[2+J]}(x;\beta,\alpha,\lambda) C_{n-k}(-\alpha,-x) \frac{z^n}{n!},$$

from which (40) follows. Proposition 3.3 is demonstrated.

Proposition 3.4. *For a fixed $J \in \mathbb{N}$, $\beta \in \mathbb{R} - \{0\}$, let $\left\{ G_k^{[2+J]}(x;\alpha,\beta,\lambda) \right\}_{k=0}^{\infty}$ be a parametric U-Charlier-Poisson type sequence of polynomials defined by (33). Then, we have the following relationship:*

$$A_j(\lambda,\alpha) C_n(x,-\alpha) + \sum_{k=0}^{n} \beta s(n,k) x^k = \sum_{l=0}^{n} \binom{n}{l} G_l^{[2+j]}(x;\alpha,\beta,\lambda) \alpha^{n-l}, \tag{41}$$

where $s(n,k)$ is defined by (24).

Proof. From (33), implies that

$$\begin{aligned} \beta(1+z)^x &= \left[\sum_{n=0}^{\infty} G_n^{[2+J]}(x;\alpha,\beta,\lambda) \frac{z^n}{n!} - A_j(\lambda,\alpha)(1+z)^x \right] \frac{1}{e^{-\alpha z}} \\ &= \left(\sum_{n=0}^{\infty} G_n^{[2+J]}(x;\alpha,\beta,\lambda) \frac{z^n}{n!} \right) \left(\sum_{n=0}^{\infty} \alpha^n \frac{z^n}{n!} \right) - A_j(\lambda,\alpha) \sum_{n=0}^{\infty} C_n(x,-\alpha) \frac{z^n}{n!}. \end{aligned}$$

Now, using (25) follows:

$$\begin{aligned} \beta \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} s(n,k) x^k \right) \frac{z^n}{n!} &= \sum_{n=0}^{\infty}\sum_{l=0}^{n} \binom{n}{l} G_l^{[2+J]}(x;\alpha,\beta,\lambda) \alpha^{n-l} \frac{z^n}{n!} - A_j(\lambda,\alpha) \sum_{n=0}^{\infty} C_n(x,-\alpha) \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} G_l^{[2+J]}(x;\alpha,\beta,\lambda) \alpha^{n-l} - A_j(\lambda,\alpha) C_n(x,-\alpha) \right) \frac{z^n}{n!}, \end{aligned}$$

which yields (41). Our Proposition 3.4 is proven.

Proposition 3.5. *For a fixed $J \in \mathbb{N}$, $\beta \in \mathbb{R} - \{0\}$ the following relations hold for the parametric U-Charlier-Poisson type polynomials defined by (33):*

$$n \frac{\partial}{\partial x} G_{n-1}^{[2+J]}(x;\alpha,\beta,\lambda) = \sum_{k=1}^{n} (-1)^k (n-k) \langle n \rangle_k G_{n-k-1}^{[2+J]}(x;\alpha,\beta,\lambda), \quad (G_{-n}^{[2+J]} \equiv 0), \tag{42}$$

$$\frac{1}{\alpha} G_{n+1}^{[2+J]}(x;\alpha,\beta,\lambda) - \aleph(x;z;\alpha) \frac{\partial}{\partial x} G_n^{[2+J]}(x;\alpha,\beta,\lambda) + G_n^{[2+J]}(x;\alpha,\beta,\lambda) - A_j(\lambda,\alpha)\langle x \rangle_n = 0, \tag{43}$$

where $\alpha \in \mathbb{R} - \{0\}$, $z \in \mathbb{C} - \{0,-1\}$, $n \in \mathbb{N}$ with

$$\aleph(x;z;\alpha) = \frac{x}{\alpha} \left[\frac{1}{(1+z)\log(1+z)} \right], \tag{44}$$

and $A_j(\lambda,\alpha)$ given in (35).

Proof. To prove (42), we note that by differentiating (33) with respect to x, we can write

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}G_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!} = \left(\sum_{n=1}^{\infty}(-1)^n\frac{z^n}{n}\right)\left(\sum_{n=0}^{\infty}G_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!}\right)$$

$$\Leftrightarrow \sum_{n=1}^{\infty}\frac{\partial}{\partial x}G_{n-1}^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^{n-1}}{(n-1)!} = \left(\sum_{n=1}^{\infty}(-1)^n(n-1)!\frac{z^n}{n}\right)\left(\sum_{n=0}^{\infty}G_{n-1}^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^{n-1}}{(n-1)!}\right)$$

$$\Leftrightarrow \sum_{n=1}^{\infty}n\,\frac{\partial}{\partial x}G_{n-1}^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!} = \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^k(k-1)!\binom{n}{k}(n-k)G_{n-1-k}^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!}$$

$$\Leftrightarrow n\,\frac{\partial}{\partial x}G_{n-1}^{[2+J]}(x;\alpha,\beta,\lambda) = \sum_{k=1}^{n}(-1)^k(k-1)!\binom{n}{k}(n-k)G_{n-1-k}^{[2+J]}(x;\alpha,\beta,\lambda).$$

Of the above expression and applying (16) follows (42).

Now to prove (43), we derive (33) with respect to z as follows:

$$\frac{\partial}{\partial z}u(x;z;\alpha,\beta,\lambda) = \sum_{n=0}^{\infty}G_{n+1}^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!}, \tag{45}$$

and

$$\frac{\partial}{\partial z}u(x;z;\alpha,\beta,\lambda) = \frac{x}{(1+z)}\left[(1+z)^x(\beta\,e^{-\alpha z}+A_j(\lambda,\alpha))\right]-\alpha\,\beta\,e^{-\alpha z}(1+z)^x. \tag{46}$$

Likewise, if we derive (33) with respect to x, we have the following:

$$\frac{\partial}{\partial x}u(x;z;\alpha,\beta,\lambda) = \sum_{n=0}^{\infty}\frac{\partial}{\partial x}G_n^{[2+J]}(x;\alpha,\beta,\lambda)\,\frac{z^n}{n!}, \tag{47}$$

$$\frac{\partial}{\partial x}u(x;z;\alpha,\beta,\lambda) = (1+z)\log(1+z)(\beta\,e^{-\alpha z}+A_j(\lambda,\alpha)). \tag{48}$$

By using (45), (46), (47), and (48), we obtain

$$\frac{1}{\alpha}\frac{\partial}{\partial z}u(x;z;\alpha,\beta,\lambda)-\frac{1}{\alpha}\left[\frac{x}{(1+z)\log(1+z)}\right]\frac{\partial}{\partial x}u(x;z;\alpha,\beta,\lambda)$$
$$+\,u(x;z;\alpha,\beta,\lambda)-(1+z)^x\,A_j(\lambda,\alpha)=0$$

$$\Leftrightarrow \frac{1}{\alpha}\sum_{n=0}^{\infty}G_{n+1}^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!}-\frac{1}{\alpha}\sum_{n=0}^{\infty}\left[\frac{x}{(1+z)\log(1+z)}\right]\frac{\partial}{\partial x}G_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!}$$
$$+\sum_{n=0}^{\infty}G_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!}-A_j(\lambda,\alpha)\sum_{n=0}^{\infty}\binom{x}{n}z^n=0$$

$$\Leftrightarrow \sum_{n=0}^{\infty}\frac{1}{\alpha}G_{n+1}^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!}-\sum_{n=0}^{\infty}\frac{1}{\alpha}\left[\frac{x}{(1+z)\log(1+z)}\right]\frac{\partial}{\partial x}G_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!}$$
$$+\sum_{n=0}^{\infty}G_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{z^n}{n!}-\sum_{n=0}^{\infty}A_j(\lambda,\alpha)\langle x\rangle_n\frac{z^n}{n!}=0.$$

Of the previous equation taking $\aleph(x;z;\alpha)$ as in (44), (43) follows. Proposition 3.5 is proved.

Proposition 3.6. *For a fixed $J \in \mathbb{N}$, $\beta \in \mathbb{R} - \{0\}$, let $G_n^{[2+J]}(x;\alpha,\beta,\lambda)$ be the parametric U-Charlier-Poisson type polynomials. Then the following statement holds:*

$$\frac{d}{dx}(x)_n = \frac{H_n(x)}{\beta}\left[-\lambda(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2} + \sum_{k=0}^{n}(-1)^n\alpha^k G_{n-k}^{[2+J]}(-x;\alpha,\beta,\lambda)\right], \tag{49}$$

using $(x)_n$ given by (14), and $H_n(x) = H_n^{(1)}(x)$ given in (19).

Proof. Taking $z \to -z$, and $x \to -x$ in (33), we have

$$\begin{aligned}
\beta(1-z)^{-x} &= e^{-\alpha z}\sum_{n=0}^{\infty}(-1)^n G_n^{[2+J]}(-x;\alpha,\beta,\lambda)\frac{z^n}{n!} - A_j e^{-\alpha z}(1-z)^{-x}\\
&= \sum_{n=0}^{\infty}(-1)^n\alpha^n\frac{z^n}{n!}\sum_{n=0}^{\infty}(-1)^n G_n^{[2+J]}(-x;\alpha,\beta,\lambda)\frac{z^n}{n!} - A_j(\lambda,\alpha)\sum_{n=0}^{\infty}(-1)^n C_n(-x,\alpha)\frac{z^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(-1)^n\alpha^k G_{n-k}^{[2+J]}(-x;\alpha,\beta,\lambda)\frac{z^n}{n!} - A_j(\lambda,\alpha)\sum_{n=0}^{\infty}(-1)^n C_n(-x,\alpha)\frac{z^n}{n!}.
\end{aligned}$$

Then, for $|z| < 1$, using the Binomial Theorem, we have

$$\begin{aligned}
\frac{1}{(1-z)^x} &= \sum_{n=0}^{\infty}\beta^{-1}\left[\sum_{k=0}^{n}\binom{n}{k}(-1)^n\alpha^k G_{n-k}^{[2+J]}(-x;\alpha,\beta,\lambda) - A_j(\lambda,\alpha)(-1)^n C_n(-x,\alpha)\right]\frac{z^n}{n!}\\
\sum_{n=0}^{\infty}(x)_n\frac{z^n}{n!} &= \sum_{n=0}^{\infty}\beta^{-1}\left[\sum_{k=0}^{n}\binom{n}{k}(-1)^n\alpha^k G_{n-k}^{[2+J]}(-x;\alpha,\beta,\lambda) - A_j(\lambda,\alpha)(-1)^n C_n(-x,\alpha)\right]\frac{z^n}{n!}\\
(x)_n &= \sum_{k=0}^{n}(-1)^n\beta^{-1}\alpha^k G_{n-k}^{[2+J]}(-x;\alpha,\beta,\lambda) - \beta^{-1}A_j(-1)^n C_n(-x,\alpha).
\end{aligned}$$

This way, using (22) follows (49). This completes the demonstration of Proposition 3.6.

4. Orthogonality relationship of the polynomials $G_n^{[2+J]}(x;\alpha,\beta,\lambda)$

The main aim of this section is to obtain the relation of orthogonality satisfied by the polynomials $G_n^{[2+J]}(x;\alpha,\beta,\lambda)$, and apply it to study a relationship between these polynomials and the operator ∇ given in (9).

For a fixed $J \in \mathbb{N}$, we define the parametric U-Charlier-Poisson discrete weight function ω^α as

$$\omega^\alpha(x;\beta,\lambda) = \frac{e^{-\alpha}\alpha^x}{x!}\left|\mathscr{M}(\beta,\lambda,\alpha) + i\Theta(\beta,\lambda,\alpha)\right|^{-2}, \tag{50}$$

where $\alpha \in \mathbb{R}$, $\alpha < 0$; β, $\lambda \in \mathbb{R} - \{0\}$, on the lattice $\mathbb{N}$; $z, w \in \mathbb{C}$; $z = a_1 + ia_2$, $w = c_1 + ic_2$ in the circle $C(0,|\eta|)$ and $|\eta| = \min\{|z|,|w|\}$. While $\mathscr{M}(\beta,\lambda,\alpha)$ and $\Theta(\beta,\lambda,\alpha)$ are given by

$$\begin{aligned}
\mathscr{M}(\beta,\lambda,\alpha) &= \beta\left(\beta + A_j(\lambda,\alpha)(\varepsilon_2\cos(c_2\alpha) + \varepsilon_1\cos(a_2\alpha))\right)\\
&\quad + \left[A_j(\lambda,\alpha)\right]^2\varepsilon_1\varepsilon_2\cos(\alpha(a_2+c_2)), && (51)\\
\Theta(\beta,\lambda,\alpha) &= \beta A_j(\lambda,\alpha)\left(\varepsilon_2\sin(c_2\alpha) + \varepsilon_1\sin(a_2\alpha)\right)\\
&\quad + \left[A_j(\lambda,\alpha)\right]^2\varepsilon_1\varepsilon_2\sin(\alpha(a_2+c_2)), && (52)
\end{aligned}$$

where $A_j(\lambda,\alpha)$ given in (35), $\varepsilon_1 = e^{a_1\alpha}$, and $\varepsilon_2 = e^{c_1\alpha}$.

With the weight $\omega^\alpha(x;\beta,\lambda)$ given in (50), we can introduce on $\mathbb{P}$ the inner product as follows:

$$\langle f,g\rangle_{\omega^\alpha} = \sum_{x=0}^{\infty} f(x)g(x)\omega^\alpha(x;\beta,\lambda), \tag{53}$$

where $f,g\in\mathbb{P}$. Which has positive weights for every $\alpha<0$

The Pearson equation concerning (27) for weight (50) is now of the form

$$\nabla\omega^\beta(x;\alpha,\beta,\lambda) = \left(\frac{\alpha-x}{\alpha}\right)\omega^\beta(x;\alpha,\beta,\lambda). \tag{54}$$

Theorem 4.1. *For a fixed $J\in\mathbb{N}$, if $\alpha\in\mathbb{R}$, $\alpha<0$, β, $\lambda\in\mathbb{R}-\{0\}$, and $m,n\in\mathbb{N}_{\not{F}}$. Then, the parametric U-Charlier-Poisson type polynomials for the weight (50) satisfy the following orthogonality relation:*

$$\sum_{x=0}^{\infty} G_m^{[2+J]}(x;\alpha,\beta,\lambda)G_n^{[2+J]}(x;\alpha,\beta,\lambda)\frac{e^{-\alpha}\alpha^x}{x!}\left|\Omega(\beta,\lambda,\alpha)\right|^{-2} = \frac{\Gamma(n+1)\alpha^n}{\Omega(\beta,\lambda,\alpha)}\delta_{m,n}. \tag{55}$$

Whit $\Omega(\beta,\lambda,\alpha) = \mathscr{M}(\beta,\lambda,\alpha)+i\Theta(\beta,\lambda,\alpha)$.

Proof. One can see that from (33) follows:

$$\begin{aligned} L_G(x;z,\alpha,\beta,\lambda) &= \beta\left(\sum_{n=0}^{\infty}(-\alpha)^n\frac{z^n}{n!}\right)\sum_{n=0}^{\infty}\binom{x}{n}z^n + A_j(\lambda,\alpha)\sum_{n=0}^{\infty}\binom{x}{n}z^n \\ &= \beta\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(-\alpha)^{n-k}\langle x\rangle_k\frac{z^n}{n!} + A_j(\lambda,\alpha)\sum_{n=0}^{\infty}\langle x\rangle_n\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\left[\beta\sum_{k=0}^{n}\binom{n}{k}(-\alpha)^{n-k}\langle x\rangle_k + A_j(\lambda,\alpha)\langle x\rangle_n\right]\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\beta\binom{n}{k}(-\alpha)^{n-k}\frac{\langle x\rangle_k}{n!} + A_j(\lambda,\alpha)\frac{\langle x\rangle_n}{n!}\right]z^n. \end{aligned}$$

This implies that

$$L_G(x;z,\alpha,\beta,\lambda) = \sum_{n=0}^{\infty} D_n^{[2+J]}(x;\alpha,\beta,\lambda)z^n, \tag{56}$$

where

$$D_n^{[2+J]}(x;\alpha,\beta,\lambda) = \sum_{k=0}^{n}\beta\binom{n}{k}(-\alpha)^{n-k}\frac{\langle x\rangle_k}{n!} + A_j(\lambda,\alpha)\frac{\langle x\rangle_n}{n!}. \tag{57}$$

Similarly, we obtain

$$L_G(x;w,\alpha,\beta,\lambda) = \sum_{n=0}^{\infty} D_m^{[2+J]}(x;\alpha,\beta,\lambda)w^n, \tag{58}$$

with

$$D_n^{[2+J]}(x;\alpha,\beta,\lambda) = \sum_{k=0}^{m} \beta \binom{m}{k} (-\alpha)^{m-k} \frac{\langle x \rangle_k}{m!} + A_j(\lambda,\alpha) \frac{\langle x \rangle_m}{m!}. \quad (59)$$

On the other hand, we have

$$\begin{aligned} L_G(x;z,\alpha,\beta,\lambda) L_G(x;w,\alpha,\beta,\lambda) &= \left[\beta e^{-\alpha z} + A_j(\lambda,\alpha)\right]\left[\beta e^{-\alpha w} + A_j(\lambda,\alpha)\right](1+z)^x (1+w)^x \\ &= e^{-\alpha z} e^{-\alpha w} (\beta + A_j(\lambda,\alpha) e^{\alpha z})(\beta + A_j(\lambda,\alpha) e^{\alpha w})(1+z)^x (1+w)^x, \end{aligned}$$

and so, we have that

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\alpha^k L_G(x;z,\alpha,\beta,\lambda) L_G(x;w,\alpha,\beta,\lambda)}{k!} &= (\beta + A_j(\lambda,\alpha)e^{\alpha z})(\beta + A_j(\lambda,\alpha)e^{\alpha w}) e^{-\alpha z} e^{-\alpha w} e^{\alpha(1+z)(1+w)} \\ &= (\beta + A_j(\lambda,\alpha)e^{\alpha z})(\beta + A_j(\lambda,\alpha)e^{\alpha w}) e^{\alpha} e^{\alpha z w}. \end{aligned}$$

So,

$$\sum_{k=0}^{\infty} L_G(x;z,\alpha,\beta,\lambda) L_G(x;w,\alpha,\beta,\lambda) \frac{e^{-\alpha}\alpha^k}{k!} = (\beta + A_j(\lambda,\alpha)e^{\alpha z})(\beta + A_j(\lambda,\alpha)e^{\alpha w}) \sum_{n=0}^{\infty} \alpha^n \frac{z^n w^n}{n!}. \quad (60)$$

By using (56) and (58) on the left side of (60), we found

$$\begin{aligned} \sum_{k=0}^{\infty} L_G(x;z,\alpha,\beta,\lambda) L_G(x;w,\alpha,\beta,\lambda) \frac{e^{-\alpha}\alpha^k}{k!} &= \sum_{k=0}^{\infty} \frac{e^{-\alpha}\alpha^k}{k!} \sum_{n=0}^{\infty} D_n^{[2+J]}(x;\alpha,\beta,\lambda) z^n \sum_{m=0}^{\infty} D_m^{[2+J]}(x;\alpha,\beta,\lambda) w^n \\ &= \sum_{m,n=0}^{\infty} \sum_{k=0}^{\infty} D_m^{[2+J]}(x;\alpha,\beta,\lambda) D_n^{[2+J]}(x;\alpha,\beta,\lambda) \frac{e^{-\alpha}\alpha^k}{k!} z^n w^n. \quad (61) \end{aligned}$$

By combining Equation (60) with Equation (61), we have that.

$$\sum_{n=0}^{\infty} (\beta + e^{\alpha z} b)(\beta + e^{\alpha w} b) \frac{\alpha^n z^n w^n}{n!} = \sum_{m,n=0}^{\infty} \sum_{k=0}^{\infty} D_m^{[2+J]}(x;\alpha,\beta,\lambda) D_n^{[2+J]}(x;\alpha,\beta,\lambda) \frac{e^{-\alpha}\alpha^k}{k!} z^n w^n.$$

Which results in

$$\sum_{k=0}^{\infty} D_m^{[2+J]}(k;\alpha,\beta,\lambda) D_n^{[2+J]}(k;\alpha,\beta,\lambda) \frac{e^{-\alpha}\alpha^k}{k!} = \begin{cases} \left[\dfrac{\alpha^n(\beta + A_j(\lambda,\alpha)e^{\alpha z})(\beta + A_j(\lambda,\alpha)e^{\alpha w})}{n!}\right], & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases}$$

$$\Leftrightarrow \sum_{k=0}^{\infty} D_m^{[2+J]}(k;\alpha,\beta,\lambda) D_n^{[2+J]}(k;\alpha,\beta,\lambda) \frac{e^{-\alpha}\alpha^k}{k!} = \left[\frac{\alpha^n(\beta + A_j(\lambda,\alpha)e^{\alpha z})(\beta + A_j(\lambda,\alpha)e^{\alpha w})}{n!}\right] \delta_{m,n}.$$

And so we can see that

$$\sum_{k=0}^{\infty} G_m^{[2+J]}(x,\alpha,\beta,\lambda)G_n^{[2+J]}(x,\alpha,\beta,\lambda)\frac{e^{-\alpha}\alpha^k}{k!} = n!\alpha^n(\beta+A_j(\lambda,\alpha)e^{\alpha z})(\beta+A_j(\lambda,\alpha)e^{\alpha w})\delta_{m,n}. \tag{62}$$

Now, from Equation (62), we consider the following product:

$$(\beta+A_j(\lambda,\alpha)e^{\alpha z})(\beta+A_j(\lambda,\alpha)e^{\alpha w}) = \beta^2+\beta\varepsilon_2 A_j(\lambda,\alpha)e^{ic_2\alpha}+\beta\varepsilon_1 A_j(\lambda,\alpha)e^{ia_2\alpha} + [A_j(\lambda,\alpha)]^2\varepsilon_1\varepsilon_2 e^{ia_2\alpha}e^{ic_2\alpha}. \tag{63}$$

Finally, we take into consideration the following: we develop the calculations in (63) and substitute Equations (51) and (52) into the result, then organizing (62) with these calculations we get (55), which completes the proof of Theorem 4.1.

Again employing (63) with certain conditions provides

Corollary 4.1. *For a fixed $J\in\mathbb{N}$, if $\alpha\in\mathbb{R}$, $\alpha<0$, β, $\lambda\in\mathbb{R}-\{0\}$, and $m,n\in\mathbb{N}_{\not{v}}$. Assume that $z_1=a_1+ia_2$, $z_2=c_1+ic_2$, with $a_1,c_1\to 0$ and $a_2\to c_1=\zeta$ in the circle $C(0,|\eta|)$. Then, the parametric U-Charlier-Poisson type polynomials satisfy the following orthogonality relation*

$$\sum_{x=0}^{\infty} G_m^{[2+J]}(x,\alpha,\beta,\lambda)G_n^{[2+J]}(x,\alpha,\beta,\lambda)\frac{e^{-\alpha}\alpha^x}{x!}\left|\Omega_1(\beta,\lambda,\alpha)\right|^{-2} = \frac{\Gamma(n+1)\alpha^n}{\Omega_1(\beta,\lambda,\alpha)}\delta_{m,n}.$$

Whit $\Omega_1(\beta,\lambda,\alpha)=\mathscr{M}_1(\beta,\lambda,\alpha)+i\Theta_1(\beta,\lambda,\alpha)$. Also $\mathscr{M}_1(\beta,\lambda,\alpha)$ and $\Theta_1(\beta,\lambda,\alpha)$ are given by

$$\mathscr{M}_1(\beta,\lambda,\alpha) = \beta\Big(\beta+2A_j(\lambda,\alpha)\cos(\zeta\alpha)+[A_j(\lambda,\alpha)]^2\cos(2\alpha\zeta)\Big), \tag{64}$$

$$\Theta_1(\beta,\lambda,\alpha) = 2\beta A_j(\lambda,\alpha)\sin(\zeta\alpha)+[A_j(\lambda,\alpha)]^2\sin(2\zeta\alpha). \tag{65}$$

Using the orthogonality property of the polynomials $G_n^{[2+J]}(x,\alpha,\beta,\lambda)$, the summation by parts given in (11), and the Pearson equation given in (54), we can see the following structure relation:

Proposition 4.1. *The parametric U-Charlier-Poisson type polynomials given in (33), satisfy*

$$\Delta G_n^{[2+J]}(x;\alpha,\beta,\lambda) = a_{n-1,n}^{\alpha}G_{n-1}^{[2+J]}(x;\alpha,\beta,\lambda), \tag{66}$$

where $a_{n-1,n}^{\alpha}$ are the Fourier coefficients.

Proof. Writing the polynomials $\Delta G_n^{[2+J]}(x;\alpha,\beta,\lambda) = G_n^{[2+J]}(x+1;\alpha,\beta,\lambda)-G_n^{[2+J]}(x;\alpha,\beta,\lambda)$ in terms of $\left\{G_n^{[2+J]}(x,\alpha,\beta,\lambda)\right\}_{n\geq 0}$, we have

$$G_n^{[2+J]}(x+1;\alpha,\beta,\lambda)-G_n^{[2+J]}(x;\alpha,\beta,\lambda) = \sum_{k=0}^{n-1}a_{k,n}^{\alpha}G_k^{[2+J]}(x;\alpha,\beta,\lambda), \tag{67}$$

where

$$a_{k,n}^{\alpha} = \frac{\left\langle \Delta G_n^{[2+J]}(x;\alpha,\beta,\lambda), G_k^{[2+J]}(x;\alpha,\beta,\lambda)\right\rangle_{\omega^{\alpha}}}{\left\langle G_k^{[2+J]}(x;\alpha,\beta,\lambda), G_k^{[2+J]}(x;\alpha,\beta,\lambda)\right\rangle_{\omega^{\alpha}}}, \qquad k=0,1,\ldots,n-1.$$

This way, applying (11) and (12) follows

$$\begin{aligned}\left\langle G_k^{[2+J]}, G_k^{[2+J]} \right\rangle_{\omega^\alpha} a_{k,n}^\alpha &= \sum_{L=0}^{\infty} \left(\Delta G_n^{[2+J]}(L;\alpha,\beta,\lambda) G_k^{[2+J]}(L;\alpha,\beta,\lambda) \right) \omega^\alpha(L,\beta,\lambda) \\ &= -\sum_{L=0}^{\infty} G_n^{[2+J]}(L;\alpha,\beta,\lambda) \nabla \left(\omega^\alpha(L,\beta,\lambda) G_k^{[2+J]}(L;\alpha,\beta,\lambda) \right) \\ &= -\sum_{L=0}^{\infty} G_n^{[2+J]}(L;\alpha,\beta,\lambda) \left[\omega^\alpha(L,\beta,\lambda) \nabla G_k^{[2+J]}(L;\alpha,\beta,\lambda) + G_k^{[2+J]}(L-1;\alpha,\beta,\lambda) \nabla \omega^\alpha(L) \right] \\ &= -\sum_{L=0}^{\infty} G_n^{[2+J]}(L;\alpha,\beta,\lambda) \omega^\alpha(L,\beta,\lambda) \nabla G_k^{[2+J]}(L;\alpha,\beta,\lambda) \\ &\quad -\sum_{L=0}^{\infty} G_n^{[2+J]}(L-1;\alpha,\beta,\lambda) G_k^{[2+J]}(L-1;\alpha,\beta,\lambda) \nabla \omega^\alpha(L;\beta,\lambda).\end{aligned}$$

Now, due to orthogonality of $G_n^{[2+J]}(L-1;\alpha,\beta,\lambda)$, since ∇G_k^α has degree $k-1$, we have the first sum is zero. For the second sum, substituting (54) yields

$$\begin{aligned}\left\langle G_k^{[2+J]}, G_k^{[2+J]} \right\rangle_{\omega^\alpha} a_{k,n}^\alpha &= -\sum_{L=0}^{\infty} G_n^{[2+J]}(L;\alpha,\beta,\lambda) G_k^{[2+J]}(L-1;\alpha,\beta,\lambda) \omega^\alpha(L,\beta,\lambda) \frac{\alpha - L}{\alpha} \\ &= -\frac{1}{\alpha} \sum_{L=0}^{\infty} G_n^{[2+J]}(L;\alpha,\beta,\lambda) G_k^{[2+J]}(L-1;\alpha,\beta,\lambda) \omega^\alpha(L,\beta,\lambda). \qquad (68)\end{aligned}$$

This sum is zero for $k+1 < n$, so only $a_{n-1,n}^\alpha$ can be non-zero. Therefore, from (68) follows (66). Proposition 4.1 is proved.

5. Szász-type operators including the parametric U-Charlier-Poisson type polynomials

In this section, we present a linear positive Szász-type operator given by (4) involving the U-Charlier-Poisson type polynomials. With the help of the Korovkin Theorem, we study the convergence and some properties.

We define the Szász-type operators, including the generating function of the parametric U-Charlier-Poisson type polynomials given in (33), with $\alpha = a$, $z = -\frac{1}{a}$, and $x = -(a-1)nx$ as follows:

$$J_n(f,x) = (\beta e + A_j(\lambda,\alpha))^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{G_k^{[2+J]}(-(a-1)nx, a, \beta, \lambda)}{k!} f\left(\frac{k}{n}\right), \qquad (69)$$

where $f \in C[0,\infty)$, $n \in \mathbb{N}$, $\beta e \neq A_j(\lambda,\alpha)$, and $x \geq 0$.

Lemma 5.1. *For $n \in \mathbb{N}$ and $x \geq 0$, the operators J_n defined by (69), satisfy the following identities:*

1. $J_n(1,x) = 1$,
2. $J_n(s,x) = x + \dfrac{\beta e}{n(\beta e + A_j(\lambda,\alpha))}$,

3. $J_n(s^2,x) = x^2 + x\left(\dfrac{1}{n(a-1)+\dfrac{2\beta e}{n(\beta e + A_j(\lambda,\alpha))}+\dfrac{1}{n}}\right) + \dfrac{2\beta e}{n^2(\beta e + A_j(\lambda,\alpha))},$

with $\beta e \neq A_j(\lambda,\alpha)$.

PROOF. Using the generating function of the parametric U-Charlier-Poisson type polynomials, given by (33), we can see that

$$\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx;a,\beta,\lambda)}{k!} = \left(1-\frac{1}{a}\right)^{-(a-1)nx}(\beta e + A_j(\lambda,\alpha)), \tag{70}$$

$$\sum_{k=0}^{\infty}\frac{kG_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!} = \left(1-\frac{1}{a}\right)^{-(a-1)nx}[\beta e + nx(\beta e + A_j(\lambda,\alpha))]. \tag{71}$$

$$\sum_{k=0}^{\infty}\frac{k^2G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!} = \left(1-\frac{1}{a}\right)^{-(a-1)nx} \times \left[n^2x^2(\beta e + A_j(\lambda,\alpha)) + n^2x(\beta e + A_j(\lambda,\alpha))\Phi + 2\beta e\right] \tag{72}$$

where

$$\Phi = \left(\frac{1}{n(a-1)+\dfrac{2\beta e}{n(\beta e + A_j(\lambda,\alpha))}+\dfrac{1}{n}}\right).$$

Then, multiplying in each of the equations (70)-(72) by the right multiplicative inverses and using the Definition of the operators (69) the assertions of the lemma are obtained.

Theorem 5.2. *Let* $S := \{f : [0,\infty) \to \mathbb{R} : |f(x)| \le Me^{Ax}\}$*, where* $A \in \mathbb{R}$*. If* $f \in C[0,\infty) \cap S$*, then*

$$\lim_{n\to\infty} J_n(f,x) = f. \tag{73}$$

That is, the operators defined in (69) converge uniformly on every compact subset of $[0,\infty)$*.*

PROOF. By using the Lemma 5.1, we have

$$\lim_{n\to\infty} J_n(s^i;x,a) = x^i, \quad i = 0,1,2.$$

In this way, using Korovkin's Theorem [2], convergence is guaranteed in each compact subset of $[0,\infty)$.

The next result gives the rate of convergence of the sequence J_n to f by means of the modulus of continuity.

Theorem 5.3. *Let* $f \in UC[0,\infty) \cap S$*. Then the operators* J_n *satisfy the inequality that follows:*

$$|J_n(f,x) - f(x)| \le \left\{1 + \sqrt{\Upsilon_n(x;\beta,\lambda)}\right\}\omega\left(f;\frac{1}{\sqrt{n}}\right), \tag{74}$$

with

$$\Upsilon_n(x;\beta,\lambda)=\left[\frac{x}{n}\left[\frac{n^2(\beta e+H)^2-2\beta e}{n^3(a-1)(\beta e+H)^2+3\beta e+H}\right]+\frac{2\beta e}{n^2(\beta e+H)}\right],\quad a\neq 1$$

where $H=\left(\beta e+\lambda(-\alpha)^{-J}\prod_{m=2}^{2+J}\frac{2^m-1}{2^m-2}\right)^{-1}$.

Proof. By using (28), (30), the Definition of the new operators given in (69), and identity 1 of the Lemma 5.1, we can write

$$|J_n(f,x)-f(x)|=\left|H\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}f\left(\frac{k}{n}\right)-1\cdot f(x)\right|.$$

Thereupon

$$\begin{aligned}|J_n(f,x)-f(x)| &= \left|H\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left(f\left(\frac{k}{n}\right)-f(x)\right)\right|\\ &\leq H\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left|f\left(\frac{k}{n}\right)-f(x)\right|.\end{aligned}$$

This way of (29) follows:

$$\begin{aligned}|J_n(f,x)-f(x)| &\leq \left\{H\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left(\frac{1}{\delta}\left|\frac{k}{n}-x\right|+1\right)\omega(f,\delta)\right\}\\ &\leq \left\{1+\frac{1}{\delta}H\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left|\frac{k}{n}-x\right|\right\}\omega(f,\delta)\end{aligned}\quad(75)$$

On the other hand, it holds by Cauchy-Schwarz inequality for series, and Lemma 5.1 the following:

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left|\frac{k}{n}-x\right| &\leq \left(H^{-1}\left(1-\frac{1}{a}\right)^{-(a-1)nx}\right)^{1/2}\\ &\times\left(\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left(\frac{k}{n}-x\right)^2\right)^{1/2}.\end{aligned}$$

Then, taking

$$\phi=\left(H^{-1}\left(1-\frac{1}{a}\right)^{-(a-1)nx}\right)^{1/2}\left(\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left(\frac{k}{n}-x\right)^2\right)^{1/2},$$

is fulfilled

$$\begin{aligned}
\phi &= \left(H^{-1}\left(1-\frac{1}{a}\right)^{-(a-1)nx}\right)^{1/2} \\
&\times \left(H^{-1}\left(1-\frac{1}{a}\right)^{-(a-1)nx} H\left(1-\frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{G_k^{[2+J]}(-(a-1)nx, a, \beta, \lambda)}{k!} \left(\frac{k}{n}-x\right)^2\right)^{1/2} \\
&= H^{-1}\left(1-\frac{1}{a}\right)^{-(a-1)nx} \left[H\left(1-\frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{G_k^{[2+J]}(-(a-1)nx, a, \beta, \lambda)}{k!} \left(\frac{k^2}{n^2}-2\frac{kx}{n}+x^2\right)\right]^{1/2} \\
&= H^{-1}\left(1-\frac{1}{a}\right)^{-(a-1)nx} \left[J_n(s^2,x)-2xJ_n(s,x)+x^2J_n(1,x)\right]^{1/2}.
\end{aligned}$$

So, of (75) and the above expression, we get

$$|J_n(f,x)-f(x)| \leq \left\{1+\frac{1}{\delta}\sqrt{\Upsilon_n(x;\beta,\lambda)}\right\}\omega\left(f,\delta\right).$$

By choosing $\delta := \delta_n = \dfrac{1}{\sqrt{n}}$, we arrive at the desired result. Theorem 5.3 is proved.

Lemma 5.4. *For $x \in [0,\infty)$, the sequence of operators J_n given in (69), satisfy the following property*

$$J_n((s-x)^2,x) = x\left(\frac{1}{n(a-1)+\dfrac{2\beta e}{n(\beta e+A_j(\lambda,\alpha))}+\dfrac{1}{n}} - \frac{2\beta e}{n(\beta e+A_j(\lambda,\alpha))}\right) + \frac{2\beta e}{n^2(\beta e+A_j(\lambda,\alpha))},$$

with $\beta e \neq A_j(\lambda,\alpha)$.

Proof. By taking advantage of the linearity property of J_n operators, we have

$$J_n((s-x)^2,x) = J_n(s^2,x)-2xJ_n(s,x)-x^2J_n(1,x).$$

Next, we apply Lemma 5.1, we obtain the desired outcome.

Theorem 5.5. *Let $f \in C[0,\infty)\cap S$ and $x \in [0,\infty)$. The operators J_n satisfy the inequality that follows:*

$$|J_n(f,x)-f(x)| \leq 2\omega(f;\sqrt{\tau_n}), \tag{76}$$

where

$$\tau_n = x\left(\frac{1}{n(a-1)+\dfrac{2\beta e}{n(\beta e+A_j(\lambda,\alpha))}+\dfrac{1}{n}} - \frac{2\beta e}{n(\beta e+A_j(\lambda,\alpha))}\right) + \frac{2\beta e}{n^2(\beta e+A_j(\lambda,\alpha))}.$$

Proof. By the identity 1 of the Lemma 5.1, and using the modulus of continuity property, it is fulfilled

$$\begin{aligned}|J_n(f,x)-f(x)| &\le H\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left|f\left(\frac{k}{n}\right)-f(x)\right| \\ &\le \left\{1+H\left(1-\frac{1}{a}\right)^{(a-1)nx}\frac{1}{\delta}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left|\frac{k}{n}-x\right|\right\}\omega(f,\delta).\end{aligned}$$

On the other hand, by the Lemma 5.4, and the Cauchy-Schwarz inequality, it holds

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left|\frac{k}{n}-x\right| &\le \sqrt{(H)^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}}\times \\ &\quad \left\{\sum_{k=0}^{\infty}\frac{G_k^{[2+J]}(-(a-1)nx,a,\beta,\lambda)}{k!}\left|\frac{k}{n}-x\right|\right\}^{1/2} \\ &\le H\left(1-\frac{1}{a}\right)^{-^{-1}(a-1)nx}\{\tau_n\}^{1/2}.\end{aligned}$$

This way, $|J_n(f,x)-f(x)|\le\left\{1+\frac{1}{\delta}\sqrt{\tau_n}\right\}^{1/2}$. Thus, by taking $\delta=\sqrt{\tau_n}$, we have the desired result.

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A Research Announcement on Generalized Discrete U–Bernoulli–Korobov Polynomials

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Abstract

The aim of this work is to introduce a new family of generalized discrete U–Bernoulli–Korobov–type polynomials. We provide several explicit representations of this class, together with connections to other well–known families of special polynomials. In addition, we establish properties involving the forward and backward difference operators Δ and ∇. Finally, we examine the orthogonality structure satisfied by these polynomials and derive the corresponding three-term recurrence relation.

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1. Introduction

The study of generating functions and their various extensions leads to polynomials and numbers known for their exceptional and valuable properties, which have applications in some branches of mathematics, probability, engineering, and other scientific disciplines. Many mathematical physics issues can be solved analytically thanks to the recent developments in generating functions theory [12; 13; 20**?**]. We can find certain results related to the generating functions for the Bernoulli polynomials and degenerate Bernoulli polynomials. Also, in the literature, we find various kinds of versions of the Euler, Genocchi, and Dahee degenerate polynomials, see for example [2; 3; 8; 10; 11; 17–19]. On the other hand, in recent years the investigations of discrete orthogonal polynomials have gained high attention for their applications on functional equations and differential and their use to establish various analytic number theory properties (cf. [4–6; 9]).

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The motivation of this work is to introduce a new family of generalized discrete U–Bernoulli–Korobov–kind polynomials equipped with a parameter that outlines the advantages of techniques associated with the generating functions we have in mind to give some representative properties, and we show that these polynomials are orthogonal to $\mathbb{N}$ with respect to the inner product that will be studied. Here $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{N}$ will be denoted the sets of the numbers complex, real, positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{P}$ is the space of all polynomials in one variable with real coefficients, and $\log(z)$ denotes the principal value of the multi-valued logarithm function.

The outline of this paper is as follows: In Section 2, we provide well-known basic formulas and definitions that we shall need to use for the rest of the work. In Section 3, a new class of discrete polynomials is introduced using their generating function. We derive certain properties and explicit formulas for these polynomials. Also, we study relations with the Korobov polynomials, the Stirling numbers of the first kind, and the Dahee and Cauchy numbers. Moreover, in section 4, we establish that these new polynomials satisfy an orthogonality relationship. Finally, we study that they satisfy to three-term recurrence relation.

2. Background and preliminary results

The classical Bernoulli polynomials $B_n(x)$, are defined by employing the following generating function (see [16; 17]):

$$\left(\frac{z}{e^z - 1}\right) e^{zx} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}, \quad (|z| < 2\pi). \tag{1}$$

For $x = 0$, we find from (1) the classical Bernoulli numbers B_n given by $B_n := B_n(0) = B_n^{(0)}$, $(n \in \mathbb{N}_0)$.

The Bernoulli polynomials of the second kind $b_n(x)$, are defined as below (see [3, p. 167, Eq(1.2)]):

$$\frac{z}{\log(1+z)}(1+z)^x = \sum_{n=0}^{\infty} b_n(x) \frac{z^n}{n!}, \quad (|z| < 1). \tag{2}$$

For $x = 0$ in (2), $b_n := b_n(0)$, $(n \in \mathbb{N}_0)$ is called Bernoulli numbers of the second kind. (cf. [8; 9]). The Bernoulli polynomials of the second kind are called also, Korobov polynomials of the first kind.

The Daehee polynomials $D_n(x)$ are defined by employing the generating function (see [7; 8; 10]):

$$\frac{\log(1+z)}{z}(1+z)^x = \sum_{n=0}^{\infty} D_n(x) \frac{z^n}{n!}, \quad (|z| < 1). \tag{3}$$

If $x = 0$, in (3) $D_n := D_n(0)$ denotes the so called Daehee numbers.

The falling factorial x of order n; $\langle x \rangle$, is (see [10]):

$$\langle x \rangle_n = x(x-1)\cdots(x-n+1), \quad n \geq 1; \langle x \rangle_0 = 1. \tag{4}$$

The Cauchy numbers of the first kind C_n, are given by (see [8]):

$$\frac{z}{\log(1+z)} = \sum_{n=0}^{\infty} C_n \frac{z^n}{n!}, \quad (|z| < 1). \tag{5}$$

The Stirling numbers of the first kind $s(n,k)$, appear as the coefficients in the following generating function (see [17]):

$$\frac{(\log(1+z))^k}{k!} = \sum_{n=k}^{\infty} s(n,k) \frac{z^n}{n!}, \quad (|z| < 1). \ (k \in \mathbb{N}_0). \tag{6}$$

These numbers can also be given as (see [10; 17]).

$$\langle x \rangle_n = \sum_{k=0}^{n} s(n,k) x^n. \tag{7}$$

Of the classical exponential function, is received

$$e^{-\alpha z} - 1 = \sum_{m=0}^{\infty} \frac{(-\alpha)^{m+1} z^{m+1}}{(m+1)!}. \tag{8}$$

Let f be some function of real variable x, the backward and forward difference operators Δ and ∇ respectively are defined as (see [14]):

$$\nabla f(x) := f(x) - f(x-1), \tag{9}$$
$$\Delta f(x) := f(x+1) - f(x). \tag{10}$$

Further, for any real number a we have

$$\Delta_a f(x) := f(x+a) - f(x). \tag{11}$$

If $a = 1$ in (11), we obtain (10).

It is also satisfied (see [14]).

$$\nabla f(x) = \Delta f(x) - \Delta \nabla f(x). \tag{12}$$
$$\nabla(f(x)g(x)) = f(x)\nabla g(x) + g(x-1)\nabla f(x). \tag{13}$$

For two arbitrary sequences $\{c_k\}_{k\geq 0}$ and $\{d_k\}_{k\geq 0}$, if $d_{-1} = 0$ then applying summation by parts there holds (see [14]):

$$\sum_{k=0}^{\infty} (\Delta c_k) d_k = -\sum_{k=0}^{\infty} c_k \nabla d_k \tag{14}$$

3. New Family of generalized discrete U−Bernoulli−Korobov−kind polynomials

In this section, a new class of discrete polynomials is introduced which we denote by $\mathscr{P}_n(x;\alpha)$ and will we call generalized discrete U−Bernoulli−Korobov−kind polynomials, and study certain properties and explicit formulas that satisfy these new polynomials.

Definition 3.1. *The new family of generalized discrete U−Bernoulli−Korobov−kind polynomials $\mathscr{P}_n(x;\alpha)$ of degree n in the variable x and parameter $\alpha \in \mathbb{R}-\{0\}$ are defined through the generating function*

$$L(x,z;\alpha) = \left(\frac{z}{e^{-z\alpha}-1}\right)(1+z)^x = \sum_{n=0}^{\infty} \mathscr{P}_n(x;\alpha)\frac{z^n}{n!}, \quad \left(|z| < \frac{2\pi}{|\alpha|}\right). \tag{15}$$

By using (15) we can compute the first generalized discrete U−Bernoulli−Korobov−kind polynomials $\mathscr{P}_n(x;\alpha)$, as follows:

$$\begin{aligned}
\mathscr{P}_0(x;\alpha) &= -\frac{1}{\alpha},\\
\mathscr{P}_1(x;\alpha) &= -\frac{x}{\alpha} - \frac{1}{2},\\
\mathscr{P}_2(x;\alpha) &= -\frac{x^2}{\alpha} + \left(\frac{1-\alpha}{\alpha}\right)x - \frac{\alpha}{6},\\
\mathscr{P}_3(x;\alpha) &= \left(\frac{-1}{\alpha}\right)x^3 + \frac{3(2-\alpha)}{2\alpha}x^2 + \frac{(3\alpha-\alpha^2-4)}{2\alpha}x,\\
\mathscr{P}_4(x;\alpha) &= \left(\frac{-1}{\alpha}\right)x^4 + \left(\frac{6-2\alpha}{\alpha}\right)x^3 + \left(\frac{-\alpha^2+6\alpha-11}{\alpha}\right)x^2 + \left(\frac{\alpha^2-4\alpha+6}{\alpha}\right)x + \frac{\alpha^3}{30},\\
\mathscr{P}_5(x;\alpha) &= \left(\frac{-1}{\alpha}\right)x^5 + \left(\frac{20-5\alpha}{2\alpha}\right)x^4 + \left(\frac{45\alpha-105-5\alpha^2}{3\alpha}\right)x^3 + \left(\frac{10\alpha^2-55\alpha+100}{2\alpha}\right)x^2\\
&\quad + \left(\frac{\alpha^4+20\alpha^2+90\alpha-144}{6\alpha}\right)x.
\end{aligned}$$

For $x=0$, in (15) corresponds to the generating function of the generalized U−Bernoulli−Korobov−kind numbers, $\mathscr{P}_n(\alpha) = \mathscr{P}_n := \mathscr{P}(0;\alpha)$ given by

$$\frac{z}{e^{-z\alpha}-1} = \sum_{n=0}^{\infty} \mathscr{P}_n(\alpha)\frac{z^n}{n!}, \quad \left(|z| < \frac{2\pi}{|\alpha|}\right). \tag{16}$$

From (16), we get some of these numbers as below:

$$\mathscr{P}_0(\alpha) = -\frac{1}{\alpha}; \quad \mathscr{P}_1(\alpha) = -\frac{1}{2}; \quad \mathscr{P}_2(\alpha) = -\frac{\alpha}{6}; \quad \mathscr{P}_3(\alpha) = 0; \quad \mathscr{P}_4(\alpha) = \frac{\alpha^3}{30}; \quad \mathscr{P}_5(\alpha) = 0.$$

Proposition 3.1. *Let $\alpha \in \mathbb{R}-\{0\}$, and $\{\mathscr{P}_n(\alpha)\}_{n\geq 0}$ be a sequence of generalized $U-$Bernoulli–Korobov–kind numbers. Then, the following relationship is fulfilled.*

$$\sum_{k=0}^{n} \frac{(-\alpha)^{k+1}}{(k+1)} \binom{n}{k} \frac{\mathscr{P}_{n-k}(\alpha)}{n!} = \begin{cases} 1, & si\ n = 0, \\ 0, & si\ n \neq 0. \end{cases}$$

Proof. By using (16) we have

$$\begin{aligned} \frac{z}{e^{-\alpha z}-1} &= \sum_{n=0}^{\infty} \mathscr{P}_n(\alpha)\frac{z^n}{n!}, \qquad |z| < \frac{2\pi}{|\alpha|}. \\ z &= (e^{-\alpha z}-1)\sum_{n=0}^{\infty} \mathscr{P}_n(\alpha)\frac{z^n}{n!}. \end{aligned} \tag{17}$$

From (8) in (17) it follows that

$$z = \alpha z \left(\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,\alpha^n}{(n+1)!}\, z^n \right) \left(\sum_{n=0}^{\infty} \mathscr{P}_n(\alpha)\frac{z^n}{n!} \right). \tag{18}$$

In (18), we obtain

$$1 = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-\alpha)^{k+1}}{(k+1)} \binom{n}{k} \mathscr{P}_{n-k}(\alpha)\,\frac{z^n}{n!}. \tag{19}$$

Comparing the coefficients in (19) completes the proof.

Proposition 3.2. *Let $\alpha \in \mathbb{R}-\{0\}$, and $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$ be a sequence of generalized discrete $U-$Bernoulli–Korobov–kind polynomials. Then, the following relations hold:*

$$\mathscr{P}_n(x;\alpha) = \sum_{k=0}^{\infty} \langle x\rangle_k \binom{n}{k} \mathscr{P}_{n-k}(\alpha), \tag{20}$$

$$\mathscr{P}_n(x;\alpha) - \mathscr{P}_n(\alpha) = \sum_{k=0}^{n-1} \frac{n}{k+1} \binom{n-1}{k} \langle x\rangle_{k+1}\, \mathscr{P}_{n-1-k}(\alpha), \tag{21}$$

with $\langle x\rangle_k$ given in (4).

Proof. From (15) and (16), we can write

$$\begin{aligned} \sum_{k=0}^{n} \mathscr{P}_n(x;\alpha)\frac{z^n}{n!} &= \left(\sum_{n=0}^{\infty} \mathscr{P}_n(\alpha)\frac{z^n}{n!} \right) \left(\sum_{n=0}^{\infty} \binom{x}{n} z^n \right) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{x}{k}\binom{n}{k} k!\mathscr{P}_{n-k}(\alpha)\frac{z^n}{n!}. \end{aligned} \tag{22}$$

As a result of (22), we obtain (20).

The assertion (21) follows by utilizing (15), (16), and the Cauchy product rule, which finally yields

$$\sum_{n=0}^{\infty}\left[\mathscr{P}_n(x;\alpha)-\mathscr{P}_n(\alpha)\right]\frac{z^n}{n!} = \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}\binom{x}{k+1}\binom{n-1}{k}nk!\mathscr{P}_{n-1-k}(\alpha)\right)\frac{z^n}{n!}, \quad (23)$$

by comparing coefficients in (23), we obtain (21).

Proposition 3.3. *The following summation formulae for the generalized discrete U−Bernoulli–Korobov−kind polynomials $\mathscr{P}_n(x;\alpha)$ and $\mathscr{P}_n(y;\beta)$ with $\alpha,\beta\in\mathbb{R}-\{0\}$ and $n\in\mathbb{N}$ hold true:*

$$\mathscr{P}_n(x+y;\alpha) = \sum_{k=0}^{n}\binom{n}{k}\langle x+y\rangle_k\,\mathscr{P}_{n-k}(\alpha), \quad (24)$$

$$\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_n(x+y;\alpha)\mathscr{P}_{n-k}(\beta) = \sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}(x;\alpha)\mathscr{P}_k(y;\beta), \quad (25)$$

$$\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_n(x+y;\alpha)\mathscr{P}_{n-k}(\alpha) = \sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}(x;\alpha)\mathscr{P}_k(y;\alpha), \quad (26)$$

$$\mathscr{P}_n(x;\alpha) = \sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_k(x;\alpha)\alpha^{n-k}+\sum_{k=0}^{n-1}n\,\langle x\rangle_k\binom{n-1}{k}\alpha^{n-k-1}. \quad (27)$$

Proof. The representation (24), follows from (4) and (15). On the other hand, because of (15) for α, β, and $x,y\in\mathbb{Z}^+$, we have

$$\left(\frac{z}{e^{-\alpha z}-1}\right)(1+z)^x = \sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}, \quad (28)$$

$$\left(\frac{z}{e^{-\beta z}-1}\right)(1+z)^y = \sum_{n=0}^{\infty}\mathscr{P}_n(y;\beta)\frac{z^n}{n!}. \quad (29)$$

Multiplying member by member to (28) and (29), we deduce

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}(\beta)\mathscr{P}_k(x+y;\alpha)\frac{z^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}(x;\alpha)\mathscr{P}_k(y;\beta)\frac{z^n}{n!}. \quad (30)$$

Therefore, of (30) we derive (25). Similarly, we can obtain (26).

To prove of (27), multiplying (15) by $e^{\alpha z}$ leads to

$$z\left(\sum_{n=0}^{\infty}\frac{\alpha^n z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\binom{x}{n}z^n\right) = \sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}-\left(\sum_{n=0}^{\infty}\frac{\alpha^n z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}\right).$$

Hence,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x!}{(x-k)!k!}\frac{\alpha^{n-k}}{(n-k)!}\frac{n!z^{n+1}}{n!}=\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}-\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\mathscr{P}_k(x;\alpha)}{k!}\frac{\alpha^{n-k}}{(n-k)!}\frac{n!z^n}{n!}. \tag{31}$$

From (4) and (31), we derive

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\langle x\rangle_k\binom{n-1}{k}\alpha^{n-k-1}n\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left[\mathscr{P}_n(x;\alpha)-\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_k(x;\alpha)\alpha^{n-k}\right]\frac{z^n}{n!}, \tag{32}$$

whence the formula (27) follows of (32).

Theorem 3.1. *For every $n\in\mathbb{N}$ and $\alpha\in\mathbb{R}-\{0\}$, the generalized discrete U–Bernoulli–Korobov–kind polynomials satisfy.*

$$\begin{aligned}(n-1)\mathscr{P}_n(x;\alpha)&-n\,x\,\mathscr{P}_{n-1}(x-1;\alpha)\\ &=\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l(\alpha)^{l+1}\mathscr{P}_{n-j-l}(\alpha)\mathscr{P}_j(x;\alpha).\end{aligned} \tag{33}$$

PROOF. By differentiating both sides of (15) with respect to z, we get

$$\frac{(1+z)^x}{(e^{-\alpha z}-1)}+\frac{x\,z\,(z+1)^{x-1}}{e^{-\alpha z}-1}+\frac{\alpha z e^{-\alpha z}(1+z)^x}{(e^{-\alpha z}-1)^2} = \sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)n\frac{z^{n-1}}{n!}.$$

So,

$$\begin{aligned}\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)n\frac{z^n}{n!}&-\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}-\sum_{n=0}^{\infty}nx\mathscr{P}_{n-1}(x-1;\alpha)\frac{z^n}{n!}\\ &=\alpha\left(\sum_{n=0}^{\infty}(-\alpha)^n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(\alpha)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}\right),\end{aligned}$$

therefore

$$\begin{aligned}\sum_{n=0}^{\infty}[n\mathscr{P}_n(x;\alpha)-\mathscr{P}_n(x;\alpha)-nx\mathscr{P}_{n-1}(x-1;\alpha)]\frac{z^n}{n!}&=\alpha\left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}(-\alpha)^l\mathscr{P}_{n-l}(\alpha)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}\right)\\ &=\alpha\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-\alpha)^l\mathscr{P}_{n-j-l}(\alpha)\mathscr{P}_j(x;\alpha)\right)\frac{z^n}{n!}.\end{aligned}$$

As a result of the above expression (33), follows.

Theorem 3.2. *The following relations hold for the generalized discrete U−Bernoulli−Korobov−kind polynomials defined in* (15).

$$\frac{\partial \mathscr{P}_n(x;\alpha)}{\partial x} = \sum_{k=0}^{n-1}(-1)^k\, n\binom{n-1}{k}\frac{k!}{k+1}\mathscr{P}_{n-k-1}(x;\alpha), \quad (n\in\mathbb{N}), \tag{34}$$

$$(n-1)\mathscr{P}_n(x;\alpha) - n\,\gamma(x,z)\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x;\alpha) - n\,\psi(z;\alpha)\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x;\alpha) = 0, \tag{35}$$

where $\alpha\in\mathbb{R}-\{0\}$, $z\in\mathbb{C}-\{0,-1\}$ *and* $n\in\mathbb{N}$, *with*

$$\gamma(x,z) = \frac{x}{(1+z)\log(1+z)}, \quad \textit{and } \psi(z;\alpha) = \frac{\alpha\, e^{-\alpha z}}{(e^{-\alpha z}-1)\log(1+z)}.$$

Proof. By differentiating (15) with respect to x, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\partial \mathscr{P}_n(x;\alpha)}{\partial x}\frac{z^n}{n!} &= \left(\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}z^{n+1}\right)\\ &= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\mathscr{P}_{n-1-k}(x;\alpha)(-1)^k\binom{n-1}{k}\frac{k!}{(k+1)(n-1)!}z^n.\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty}\frac{\partial \mathscr{P}_n(x;\alpha)}{\partial x}\frac{z^n}{n!} = \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{n\,k!}{(k+1)}\mathscr{P}_{n-1-k}(x;\alpha)\frac{z^n}{n!}.$$

As a result of these computations, we obtain (34).

To prove (35), we differentiate (15) concerning z as follows:

$$\frac{\partial}{\partial z}L(x,z;\alpha) = \sum_{n=1}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^{n-1}}{(n-1)!}, \tag{36}$$

and

$$\frac{\partial}{\partial z}L(x,z;\alpha) = \frac{(1+z)^x}{(e^{-\alpha z}-1)} + \left[\frac{z\,(1+z)^x}{(e^{-\alpha z}-1)}\right]\left[\frac{x}{(1+z)}\right] + \left[\frac{z\,(1+z)^x}{(e^{-\alpha z}-1)}\right]\left[\frac{\alpha\, e^{-\alpha z}}{(e^{-\alpha z}-1)}\right]. \tag{37}$$

Furthermore, differentiating (15) concerning x, we have

$$\frac{\partial}{\partial x}L(x,z;\alpha) = \sum_{n=0}^{\infty}\frac{\partial}{\partial x}\mathscr{P}_n(x,\alpha)\frac{z^n}{n!}, \tag{38}$$

$$\frac{\partial}{\partial x}L(x,z;\alpha) = \frac{z\,(1+z)^x\log(1+z)}{(e^{-\alpha z}-1)}. \tag{39}$$

Combining (37) with (38) and (39), we can be written

$$\frac{\partial}{\partial z}L(x,z;\alpha)-\left[\frac{x}{(1+z)\log(1+z)}+\frac{\alpha\, e^{-\alpha z}}{(e^{-\alpha z}-1)\log(1+z)}\right]\frac{\partial}{\partial x}L(x,z;\alpha)-\frac{(1+z)^{x}}{(e^{-\alpha z}-1)}=0. \quad (40)$$

Thus, from (40) we have

$$z\frac{\partial}{\partial z}L(x,z;\alpha)-\left[\frac{zx}{(1+z)\log(1+z)}+\frac{z\alpha e^{-\alpha z}}{(e^{-\alpha z}-1)\log(1+z)}\right]\frac{\partial}{\partial x}L(x,z;\alpha)-L(x,z;\alpha)=0. \quad (41)$$

Hence, from (38) and (41), and after simplifying, we can get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\, n\,\frac{z^n}{n!}-\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\,\frac{z^n}{n!}-\\ \sum_{n=0}^{\infty}\left[\frac{x}{(1+z)\log(1+z)}+\frac{\alpha\, e^{-\alpha z}}{(e^{-\alpha z}-1)\log(1+z)}\right]\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x;\alpha)\frac{nz^n}{n!} &= 0,\end{aligned}$$

and consequently

$$\begin{aligned}n\mathscr{P}_n(x;\alpha)-\mathscr{P}_n(x;\alpha)-\left[\frac{n\,x}{(1+z)\log(1+z)}\right]\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x;\alpha)\\ -\left[\frac{n\,\alpha\, e^{-\alpha z}}{(e^{-\alpha z}-1)\log(1+z)}\right]\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x;\alpha)=0. \quad (42)\end{aligned}$$

In (42) doing $\gamma(x,z)=\dfrac{x}{(1+z)\log(1+z)}$, $\psi(z;\alpha)=\dfrac{\alpha\, e^{-\alpha z}}{(e^{-\alpha z}-1)(\log(1+z))}$ follows (35).
Theorem 3.2 is proved.

Theorem 3.3. *Given $\alpha\in\mathbb{R}-\{0\}$, and let $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$ be a sequence of generalized discrete U–Bernoulli–Korobov–kind polynomials. Then, the following assertions hold:*

$$\mathscr{P}_n(x;\alpha)=\sum_{k=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}x^k\mathscr{P}_{n-j}(\alpha)s(j,k),\quad \textit{with } s(n,k) \textit{ given in } (6). \quad (43)$$

$$\mathscr{P}_n(x;\alpha)=\sum_{q=0}^{n}\sum_{l=0}^{q}\sum_{j=0}^{n-q}\sum_{k=0}^{\infty}\binom{n}{q}\binom{q}{l}\binom{n-q}{l}x^k\mathscr{P}_{q-l}(\alpha)s(l,k)b_{n-q-j}D_j, \quad (44)$$

where b_n and D_n are given in (2) and (3), respectively.

$$\mathscr{P}_n(x;\alpha)=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\binom{n}{k}b_{n-k}(x)\mathscr{P}_{k-j}(\alpha)D_j, \quad (45)$$

where $b_n(x)$ is defined in (2).

$$\mathscr{P}_n(x;\alpha)=\sum_{l=0}^{n}\sum_{j=0}^{l}\sum_{q=0}^{n-l}\sum_{k=0}^{\infty}\binom{n}{l}\binom{l}{j}\binom{n-l}{q}C_jD_{l-j}s(q,k)\mathscr{P}_{n-l-q}(\alpha)x^k, \tag{46}$$

where C_n is defined in (5).

PROOF. The statement (43) follows from (6) and (15).
By using (2), (3), (6), and (15), we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}&=\left(\sum_{n=0}^{\infty}\mathscr{P}_n(\alpha)\frac{z^n}{n!}\right)\left(e^{x\log(1+z)}\right)\\&=\left(z\sum_{n=0}^{\infty}\mathscr{P}_n(\alpha)\frac{z^n}{n!}\right)\left(\sum_{k=0}^{\infty}\frac{x^k}{z}\frac{[\log(1+z)]^k}{k!}\right)\\&=\left(\sum_{n=0}^{\infty}\mathscr{P}_n(\alpha)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}s(n,k)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}b_n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}D_n\frac{z^n}{n!}\right)\sum_{k=0}^{\infty}x^k\\&=\sum_{n=0}^{\infty}\left[\sum_{q=0}^{n}\binom{n}{q}\binom{q}{l}\binom{n-q}{j}\sum_{l=0}^{q}\sum_{j=0}^{n-q}\sum_{k=0}^{\infty}x^k\mathscr{P}_{q-l}(\alpha)s(l,k)b_{n-q-j}D_j\right]\frac{z^n}{n!},\end{aligned}$$

from which (44) follow. Taking (2), (3) into account, and, by (15) we can find (45).

To prove (46), we use (5) as well as (6), and (15). Thus, we deduce

$$\begin{aligned}\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}&=\left(\sum_{n=0}^{\infty}C_n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}D_n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(\alpha)\frac{z^n}{n!}\right)\left(\sum_{k=0}^{\infty}x^k\frac{[\log(1+z)]^k}{k!}\right)\\&=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{j=0}^{l}\sum_{q=0}^{n-l}\sum_{k=0}^{\infty}\binom{n}{q}\binom{l}{j}\binom{n-q}{q}C_jD_{l-j}s(q,k)\mathscr{P}_{n-l-q}(\alpha)x^k\frac{z^n}{n!},\end{aligned}$$

from which assertion (46) follows. This completes the proof.

Theorem 3.4. *Let $\alpha\in\mathbb{R}-\{0\}$ and $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$ be a sequence of generalized discrete $U-$Bernoulli$-$Korobov$-$kind polynomials. Then, the following relations hold:*

$$\Delta_a\mathscr{P}_n(x;\alpha)=\sum_{k=0}^{n}\binom{n}{k}\langle a\rangle_k\,\mathscr{P}_{n-k}(x;\alpha)-\mathscr{P}_n(x;\alpha), \tag{47}$$

$$\Delta\mathscr{P}_n(x;\alpha)=n\mathscr{P}_{n-1}(x;\alpha), \tag{48}$$

$$\nabla\mathscr{P}_n(x;\alpha)=n\mathscr{P}_{n-1}(x-1;\alpha), \tag{49}$$

$$\Delta\mathscr{P}_n(x;\alpha)+n\Delta\mathscr{P}_{n-1}(x;\alpha)=n\mathscr{P}_{n-1}(x+1;\alpha). \tag{50}$$

with ∇ and Δ_a the operators given in (9) *and* (11)*, respectively.*

Proof. We see that from (11) and (15) it follows

$$\sum_{n=0}^{\infty} \Delta_a \mathscr{P}(x;\alpha)\frac{z^n}{n!} = \frac{z}{e^{-\alpha z}-1}(1+z)^x(1+z)^a - \frac{z}{e^{-\alpha z}-1}(1+z)^x$$
$$= \left(\sum_{n=0}^{\infty} \mathscr{P}_n(x;\alpha)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\binom{a}{n}z^n\right) - \sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}. \tag{51}$$

Hence, in (51) we have

$$\sum_{n=0}^{\infty} \Delta_a \mathscr{P}(x;\alpha)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{a}{k}\frac{n!}{(n-k)!}\mathscr{P}_{n-k}(x;\alpha) - \mathscr{P}_n(x;\alpha)\right)\frac{z^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\langle a\rangle_k\, \mathscr{P}_{n-k}(x;\alpha) - \mathscr{P}_n(x;\alpha)\right)\frac{z^n}{n!},$$

from which, (47) follows. For the case $a=1$, we obtain (48).

To prove (49), we see that of (9) and (15), we get

$$\sum_{n=0}^{\infty}\nabla\mathscr{P}_n(x;\alpha)\frac{z^n}{n!} = \left(\frac{z^2}{e^{-\alpha z}-1}\right)(1+z)^x\left(\frac{1}{1+z}\right),$$

and consequently

$$\sum_{n=1}^{\infty}\nabla\mathscr{P}_n(x;\alpha)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\mathscr{P}_n(x-1;\alpha)\frac{z^{n+1}}{n!}$$
$$= \sum_{n=1}^{\infty}\mathscr{P}_{n-1}(x-1;\alpha)\,n\,\frac{z^n}{n!},$$

from which, (49) follows.

Taking (15) into account, as well as using the operator $\varDelta$, we get the following expression

$$(1+z)\sum_{n=0}^{\infty}\varDelta\mathscr{P}_n(x;\alpha)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\mathscr{P}_n(x+1;\alpha)\frac{z^{n+1}}{n!}$$

thus, we have

$$\sum_{n=1}^{\infty}[\varDelta\mathscr{P}_n(x;\alpha) + n\varDelta\mathscr{P}_{n-1}(x;\alpha) - n\mathscr{P}_{n-1}(x+1;\alpha)]\frac{z^n}{n!} = 0,$$

and, as a consequence, (50) follows. Hence, Theorem 3.4 is proved.

On the other hand, by using (49) and (50), we can see that the polynomials $\mathscr{P}_n(x;\alpha)$ satisfy (12) in such a way that

$$\nabla\mathscr{P}_n(x;\alpha) = \Delta\mathscr{P}_n(x;\alpha) - \Delta\nabla\mathscr{P}_n(x;\alpha). \tag{52}$$

Proposition 3.4. *Let $\alpha\in\mathbb{R}-\{0\}$, $n\in\mathbb{R}$ and $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$ be a sequence of generalized discrete−kind polynomials. Then, the following relations hold:*

$$\Delta(2\mathscr{P}_n(x;\alpha) + n\mathscr{P}_{n-1}(x;\alpha)) - 2\Delta\nabla\mathscr{P}_n(x;\alpha) = 2n\mathscr{P}_{n-1}(x;\alpha) + n(n-1)\mathscr{P}_{n-2}(x;\alpha), \tag{53}$$

$$\Delta\mathscr{P}_n(x;\alpha) - \Delta\nabla\mathscr{P}_n(x;\alpha) = n\mathscr{P}_{n-1}(x;\alpha) - n(n-1)\mathscr{P}_{n-2}(x-1;\alpha). \tag{54}$$

Proof. By using (15) and applying the operator Δ it follows

$$\left(\frac{z}{e^{-\alpha z}-1}\right)(1+z)^{x+1}(1+z) = (1+2z+z^2)\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}$$

$$\sum_{n}^{\infty}\left[\mathscr{P}_n(x+1;\alpha) - \mathscr{P}_n(x;\alpha)\right]\frac{z^n}{n!} = 2\sum_{n=1}^{\infty}n\mathscr{P}_{n-1}(x;\alpha)\frac{z^n}{n!} + \sum_{n=2}^{\infty}n(n-1)\mathscr{P}_{n-2}(x;\alpha)\frac{z^n}{n!}$$
$$-\sum_{n=0}^{\infty}\mathscr{P}_n(x+1;\alpha)\frac{z^{n+1}}{n!}$$

$$\sum_{n=1}^{\infty}\Delta\mathscr{P}_n(x;\alpha)\frac{z^n}{n!} = \sum_{n=1}^{\infty}[2n\mathscr{P}_{n-1}(x;\alpha) + n(n-1)\mathscr{P}_{n-2}(x;\alpha)$$
$$-n\mathscr{P}_{n-1}(x+1;\alpha)]\frac{z^n}{n!},$$

whence

$$n\mathscr{P}_{n-1}(x+1;\alpha) = 2n\mathscr{P}_{n-1}(x;\alpha) + n(n-1)\mathscr{P}_{n-2}(x;\alpha) - \Delta\mathscr{P}_n(x;\alpha). \tag{55}$$

Now, taking (50), (55) into account, and by (52), we get

$$2\Delta\mathscr{P}_n(x;\alpha) + n\Delta\mathscr{P}_{n-1}(x;\alpha) = 2n\mathscr{P}_{n-1}(x;\alpha) + n(n-1)\mathscr{P}_{n-2}(x;\alpha),$$

thus

$$2\Delta\mathscr{P}_n(x;\alpha) - 2\Delta\nabla\mathscr{P}_n(x;\alpha) + n\Delta\mathscr{P}_{n-1}(x;\alpha) = 2n\mathscr{P}_{n-1}(x;\alpha) + n(n-1)\mathscr{P}_{n-2}(x;\alpha),$$

and, as a consequence, we obtain (53).

To prove (54), we use (15) and the operator ∇, then

$$(1+z)^2\sum_{n=0}^{\infty}\mathscr{P}_n(x-1;\alpha)\frac{z^n}{n!} = (1+z)\sum_{n=0}^{\infty}\mathscr{P}_n(x;\alpha)\frac{z^n}{n!}$$

$$2n\mathscr{P}_{n-1}(x-1;\alpha) - n\mathscr{P}_{n-1}(x;\alpha) + n(n-1)\mathscr{P}_{n-2}(x-1;\alpha) = \nabla\mathscr{P}_n(x;\alpha),$$

this implies,

$$2n\mathscr{P}_{n-1}(x-1;\alpha) = n\mathscr{P}_{n-1}(x;\alpha) - n(n-1)\mathscr{P}_{n-2}(x-1;\alpha) + \nabla\mathscr{P}_n(x;\alpha). \tag{56}$$

From (49), (56) and by using (50), we have

$$\nabla\mathscr{P}_n(x;\alpha) + n(n-1)\mathscr{P}_{n-2}(x-1;\alpha) - n\mathscr{P}_{n-1}(x;\alpha) = 0,$$

from which, (54) follows. This completes the proof.

4. Orthogonality of the generalized discrete U−Bernoulli−Korobov−kind polynomials

Let ω^α be the discrete weights function

$$\omega^\alpha(x;\beta) = \frac{(-\alpha)^x e^\alpha (1-e^{\alpha\beta})^2}{x!}, \quad x \in \mathbb{N}, \tag{57}$$

with $\alpha < 0$, $z,\, v \in \mathbb{C}$ and $\lambda_1 \in \mathrm{Re}(z)$, $\sigma_1 \in \mathrm{Re}(v)$, $\beta = \lambda_1 = \sigma_1$.

With this weight, we can consider on $\mathbb{P}$, the inner product $\langle f,\, g\rangle_{\omega^\alpha}$

$$\langle f,\, g\rangle_{\omega^\alpha} = \sum_{x=0}^{\infty} f(x)g(x)\omega^\alpha(x;\beta), \tag{58}$$

which has positive weights for every $\alpha < 0$.

The weight function $\omega^\alpha(x;\beta)$ satisfies the Pearson−type difference equation

$$\begin{aligned}\nabla\omega^\alpha(x;\beta) &= \omega^\alpha(x;\beta) - \omega^\alpha(x-1;\beta)\\ &= \left(1 - \frac{x}{(-\alpha)}\right)\left(\frac{e^\alpha(-\alpha)^x(1-e^{\alpha\beta})^2}{x!}\right)\\ &= \left(\frac{\alpha+x}{\alpha}\right)\omega^\alpha(x;\beta).\end{aligned} \tag{59}$$

Theorem 4.1. *If $\alpha \in \mathbb{R}$, with $\alpha < 0$ and $m,\, n \in \mathbb{N}$. Then, the generalized discrete U−Bernoulli −Korobov−kind polynomials satisfy the following orthogonality relation:*

$$\sum_{x=0}^{\infty} \mathscr{P}_m(x;\alpha)\mathscr{P}_n(x;\alpha)\,\omega^\alpha(x;\beta) = (-\alpha)^{n-1}\, n^2\, \Gamma(n)\delta_{mn}, \tag{60}$$

with $\omega^\alpha(x;\beta)$ given in (57) and $|z|,\, |v| < \dfrac{2\pi}{|\alpha|}$

Proof. Using (16), the Cauchy product property and taking into account the binomial theorem, we can see that

$$L(x,z;\alpha)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{x}{k}\frac{\mathscr{P}_{n-k}(\alpha)}{(n-k)!}\,z^{n}. \tag{61}$$

Hence,

$$L(x,z;\alpha)=\sum_{n=0}^{\infty}L_n(x;\alpha)z^{n}, \tag{62}$$

we note that

$$L_n(x;\alpha)=\sum_{k=0}^{n}\binom{x}{k}\frac{\mathscr{P}_{n-k}(\alpha)}{(n-k)!}. \tag{63}$$

Therefore by using (4), it follows that

$$L_n(x;\alpha)=\sum_{k=0}^{n}\frac{\langle x\rangle_k}{k!}\frac{\mathscr{P}_{n-k}(\alpha)}{(n-k)!}. \tag{64}$$

Likewise, of (15) we see that

$$L(x,v;\alpha)=\sum_{m=0}^{\infty}L_m(x;\alpha)\,v^{m}, \tag{65}$$

hence

$$L_m(x;\alpha)=\sum_{k=0}^{m}\binom{x}{k}\frac{\mathscr{P}_{m-k}(\alpha)}{(m-k)!}=\sum_{k=0}^{m}\frac{\langle x\rangle_k}{k!}\frac{\mathscr{P}_{m-k}(\alpha)}{(m-k)!}. \tag{66}$$

Now, for any k it follows from (61) and (65) that

$$(-\alpha)^{k}L(k,z;\alpha)L(k,v;\alpha)=\left[\frac{zve^{\alpha z+\alpha v}}{(1-e^{\alpha z})(1-e^{\alpha v})}\right][-\alpha(1+z)(1+v)]^{k}, \tag{67}$$

hence

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-\alpha)^{k}L(k,z;\alpha)L(k,v;\alpha)}{k!}&=\frac{zve^{\alpha z+\alpha v}}{(1-e^{\alpha z})(1-e^{\alpha v})}\sum_{k=0}^{\infty}\frac{[-\alpha(1+z)(1+v)]^{k}}{k!}\\&=\left[\frac{zv\,e^{-\alpha}}{(1-e^{\alpha z})(1-e^{\alpha v})}\right]e^{-\alpha zv}\\&=\sum_{n=0}^{\infty}\left[\frac{n\,e^{-\alpha}(-\alpha)^{n-1}}{(1-e^{\alpha z})(1-e^{\alpha v})}\right]\frac{z^{n}\,v^{n}}{n!}.\end{aligned} \tag{68}$$

On the other hand, because of (62) and (65) we also have

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-\alpha)^k L(k,z;\alpha)L(k,v;\alpha)}{k!} &= \sum_{k=0}^{\infty}\frac{(-\alpha)^k}{k!}\sum_{n=0}^{\infty}L_n(k;\alpha)\,z^n\sum_{m=0}^{\infty}L_m(k;\alpha)\,v^m\\ &= \sum_{m,n=0}^{\infty}\sum_{k=0}^{\infty}L_m(k;\alpha),L_n(k;\alpha)\,\frac{(-\alpha)^k}{k!}\,z^n v^m. \qquad (69)\end{aligned}$$

So, from (68) and (69) follows

$$\sum_{m,n=0}^{\infty}\sum_{k=0}^{\infty}L_m(k;\alpha)L_n(k;\alpha)\,\frac{(-\alpha)^k}{k!}\,z^n v^m = \sum_{n=0}^{\infty}\left[\frac{e^{-\alpha}(-\alpha)^{n-1}}{(1-e^{\alpha z})(1-e^{\alpha v})}\right]\frac{nz^n v^n}{n!}. \qquad (70)$$

Comparing the coefficients of $z^n\, v^m$ in (70), we conclude

$$\sum_{k=0}^{\infty}L_m(k;\alpha)L_n(k;\alpha)\,\frac{(-\alpha)^k}{k!} = \begin{cases}\dfrac{(-\alpha)^{n-1}\,n\,e^{-\alpha}}{n!}\left[\dfrac{1}{(1-e^{\alpha z})(1-e^{\alpha v})}\right], & \text{si } m=n,\\[2ex] 0, & \text{si } m\neq n.\end{cases} \qquad (71)$$

If we now consider that $t_1 = e^{\alpha z}$, $t_2 = e^{\alpha v}$ and $z=\lambda_1+i\lambda_2$, $v=\sigma_1+i\sigma_2$ then,

$$t_1 = e^{\alpha z} = e^{\alpha\lambda_1}e^{i\,\alpha\lambda_2} \text{ and } t_2 = e^{\alpha v} = e^{\alpha\sigma_1}e^{i\,\alpha\sigma_2},$$

so $|t_1| = e^{\alpha\lambda_1}$, $|t_2| = e^{\alpha\sigma_1}$ with $|\lambda_1|$, $|\lambda_2|$, $|\sigma_1|$, $|\sigma_2| < \dfrac{2\pi}{|\alpha|}$.
Thus,

$$(1-e^{\alpha z})(1-e^{\alpha v}) = (1-|t_1|e^{i\,\alpha\lambda_2})(1-|t_2|e^{i\,\alpha\sigma_2}). \qquad (72)$$

If now we establish $\lambda_2,\sigma_2 \longrightarrow 0$ and $\beta=\lambda_1=\sigma_1$, then we can be written (72) as

$$(1-e^{\alpha z})(1-e^{\alpha v}) = (1-e^{\alpha\beta})^2.$$

Therefore in (71), we find

$$\sum_{x=0}^{\infty}\frac{\mathscr{P}_m(x;\alpha)}{m!}\frac{\mathscr{P}_n(x;\alpha)}{n!}\omega^{\alpha}(x;\beta) = \frac{n\,(-\alpha)^{n-1}}{n!}\delta_{mn}, \quad \text{where } \delta_{mn} \text{ is the Kronecker delta.}$$

And, as a consequence (60) follows. Theorem 4.1 is proved.

Due to Theorem 4.1, we obtain a three-term recurrence relation that the sequence $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$ satisfies.

Theorem 4.2. *Let $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$ be the sequence generalized discrete $U-$Bernoulli–Korobov −kind polynomials which are orthogonal on $\mathbb{N}$ for the inner product (58). Then, they satisfy the following three-term recurrence relation:*

$$x\mathscr{P}_{n-1}(x;\alpha) = \gamma_n \mathscr{P}_n(x;\alpha) + \xi_n \mathscr{P}_{n-1}(x;\alpha) + \lambda_n \mathscr{P}_{n-2}(x;\alpha), \quad n > 2, \tag{73}$$

with

$$\begin{aligned} \gamma_n &= \frac{n\,\alpha}{2}, \\ \xi_n &= \left[(s(n-1,n-2) - s(n,n-1)) - \frac{\alpha(2n+1)}{6}\right], \\ \lambda_n &= \frac{(-\alpha)(n-1)^3}{n(n-3)^2}\frac{\Gamma(n-1)}{\Gamma(n-2)}, \end{aligned} \tag{74}$$

and $s(n,k)$ given in (6).

Proof. To prove (73), we first expand the polynomial $x\mathscr{P}_{n-1}(x;\alpha)$, which is of degree n in terms of $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$:

$$x\mathscr{P}_{n-1}(x;\alpha) = \sum_{k=0}^{n} \frac{a(k,n-1)}{g_k(\alpha)} \mathscr{P}_n(x;\alpha), \tag{75}$$

with $\alpha < 0$ is a fixed parameter, $n \in \mathbb{N}$ and $x \in \mathbb{N}_0$. From the orthogonality of $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$, we obtain

$$\begin{aligned} \frac{a(k,n-1)}{g_k(\alpha)} &= \frac{\langle x\mathscr{P}_{n-1}(x;\alpha), \mathscr{P}_k(x;\alpha)\rangle_{\omega^\alpha}}{\langle \mathscr{P}_k(x;\alpha), \mathscr{P}_k(x;\alpha)\rangle_{\omega^\alpha}} \\ &= \frac{\langle \mathscr{P}_{n-1}(x;\alpha), x\mathscr{P}_k(x;\alpha)\rangle_{\omega^\alpha}}{\langle \mathscr{P}_k(x;\alpha), \mathscr{P}_k(x;\alpha)\rangle_{\omega^\alpha}}. \end{aligned}$$

As $x\mathscr{P}_k(x;\alpha)$ is a polynomial of degree $k+1$, by orthogonality $a(k,n-1) = 0$ for $k < n-2$ and therefore (75) can be written in the form

$$\begin{aligned} x\mathscr{P}_{n-1}(x;\alpha) = {} & \frac{a(n,n-1)}{g_n(\alpha)}\mathscr{P}_n(x;\alpha) + \frac{a(n-1,n-1)}{g_{n-1}(\alpha)}\mathscr{P}_{n-1}(x;\alpha) \\ & + \frac{a(n-2,n-1)}{g_{n-2}(\alpha)}\mathscr{P}_{n-2}(x;\alpha). \end{aligned} \tag{76}$$

On the other hand, taking (6), (20) into account, and (76), we can obtain:

$$\begin{aligned} \mathscr{P}_n(x;\alpha) = {} & \mathscr{P}_0(\alpha)x^n + \left(\mathscr{P}_0(\alpha)s(n,n-1) + \mathscr{P}_1(\alpha)\binom{n}{n-1}\right)x^{n-1} \\ & + \left(\mathscr{P}_0(\alpha)s(n,n-2) + \mathscr{P}_1(\alpha)\binom{n}{n-1}s(n,n-2) + \mathscr{P}_2(\alpha)\binom{n}{n-2}\right)x^{n-2} + \cdots, \end{aligned} \tag{77}$$

$$\mathscr{P}_{n-1}(x;\alpha) = \mathscr{P}_1(\alpha)\binom{n}{n-1}x^{n-1} + \left(\mathscr{P}_1(\alpha)s(n-1,n-2)\binom{n}{n-1} + \mathscr{P}_2(\alpha)\binom{n}{n-2}\right)x^{n-2} + \cdots, \tag{78}$$

$$\mathscr{P}_{n-2}(x;\alpha) = \mathscr{P}_1(\alpha)\binom{n-1}{n-2}x^{n-2} + \left(\mathscr{P}_1(\alpha)s(n-2,n-3)\binom{n-1}{n-2} + \mathscr{P}_2(\alpha)\binom{n-1}{n-3}\right)x^{n-3} + \cdots, \tag{79}$$

$$x\mathscr{P}_{n-1}(x;\alpha) = \mathscr{P}_1(\alpha)\binom{n}{n-1}x^{n} + \left(\mathscr{P}_1(\alpha)s(n-1,n-2)\binom{n}{n-1} + \mathscr{P}_2(\alpha)\binom{n}{n-2}\right)x^{n-1} + \cdots, \tag{80}$$

moreover

$$x\mathscr{P}_{n-2}(x;\alpha) = \mathscr{P}_1(\alpha)\binom{n-1}{n-2}x^{n-1} + \left(\mathscr{P}_1(\alpha)s(n-2,n-3)\binom{n-1}{n-2} + \mathscr{P}_2(\alpha)\binom{n-1}{n-3}\right)x^{n-2} + \cdots, \tag{81}$$

also, we can write $x\mathscr{P}_{n-2}(x;\alpha)$ in terms of $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$, we have

$$\begin{aligned} x\mathscr{P}_{n-2}(x;\alpha) &= \sum_{k=0}^{n-1}\frac{a(k,n-2)}{g_k(\alpha)}\mathscr{P}_k(x;\alpha) \\ &= \frac{a(n-1,n-2)}{g_{n-1}(\alpha)}\mathscr{P}_{n-1}(x;\alpha) + \frac{a(n-2,n-2)}{g_{n-2}(\alpha)}\mathscr{P}_{n-2}(x;\alpha) \\ &\quad + \frac{a(n-3,n-2)}{g_{n-3}(\alpha)}\mathscr{P}_{n-3}(x;\alpha). \end{aligned} \tag{82}$$

Now, from (79) and (82) we deduce:

$$\frac{a(n-1,n-2)}{g_{n-1}(\alpha)} = \left(\frac{n-1}{n}\right). \tag{83}$$

So,

$$x\mathscr{P}_{n-2}(x;\alpha) = \left(\frac{n-1}{n}\right)\mathscr{P}_{n-1}(x;\alpha) + P(x). \tag{84}$$

From (76) and (84) it is seen that

$$\begin{aligned} \frac{a(n-2,n-1)}{g_{n-2}(\alpha)} &= \frac{\langle \mathscr{P}_{n-1}(x;\alpha), x\mathscr{P}_{n-2}(x;\alpha)\rangle_{\omega^\alpha}}{g_{n-2}(\alpha)} \\ &= \frac{(n-1)g_{n-1}(\alpha)}{n\, g_{n-2}(\alpha)}. \end{aligned} \tag{85}$$

Also, using (76), (77) on (78), we obtain

$$\frac{a(n,n-1)}{g_n(\alpha)} = \frac{\mathscr{P}_1(\alpha)}{\mathscr{P}_0(\alpha)}\binom{n}{n-1}. \tag{86}$$

Substitution of (85) and (86) into (76) gives

$$\begin{aligned} x\mathscr{P}_n(x;\alpha) = \frac{\mathscr{P}_1(\alpha)}{\mathscr{P}_0(\alpha)}\binom{n}{n-1}\mathscr{P}_n(x;\alpha) + \frac{a(n-1,n-2)}{g_{n-1}(\alpha)}\mathscr{P}_{n-1}(x;\alpha) \\ + \frac{(n-1)g_{n-1}(\alpha)}{n\, g_{n-2}(\alpha)}\mathscr{P}_{n-2}(x;\alpha). \end{aligned} \tag{87}$$

Comparing the coefficients of the highest terms on the left-hand and right-hand sides of (87), we have

$$\frac{a(n-1,n-2)}{g_{n-1}(\alpha)} = (s(n-1,n-2) - s(n,n-1)) - \frac{\alpha(2n+1)}{6}. \tag{88}$$

Because of Theorem 4.1, it follows

$$\frac{a(n-2,n-1)}{g_{n-2}(\alpha)} = \frac{(-\alpha)(n-1)^3}{n(n-3)^2}\frac{\Gamma(n-1)}{\Gamma(n-2)} \tag{89}$$

and from (86)

$$\frac{a(n,n-1)}{g_n(\alpha)} = \frac{n\alpha}{2}. \tag{90}$$

Theorem 4.2 is proved.

By using the orthogonality of the polynomials $\mathscr{P}_n(x;\alpha)$, we give the following relation.

Proposition 4.1. *The generalized discrete U–Bernoulli–Korobov–kind polynomials, which are orthogonal with respect to the inner product* (58)*, fulfill the relation*

$$\varDelta\mathscr{P}_n(x;\alpha) = J^{\alpha}_{k,n}\mathscr{P}_{n-1}(x;\alpha), \tag{91}$$

where $J^{\alpha}_{k,n}$ *are the Fourier coefficients.*

Proof. If we write the polynomial $\varDelta\mathscr{P}_n(x;\alpha)$ in terms of $\{\mathscr{P}_n(x;\alpha)\}_{n\geq 0}$, we have

$$\mathscr{P}_n(x+1;\alpha) - \mathscr{P}_n(x;\alpha) = \sum_{k=0}^{n-1} J^{\alpha}_{k,n}\mathscr{P}_k(x;\alpha),$$

besides, for $0 \leq k \leq n-1$

$$J^{\alpha}_{k,n} = \frac{\langle \varDelta\mathscr{P}_n, \mathscr{P}_k\rangle_{\omega^\alpha}}{\langle \mathscr{P}_k, \mathscr{P}_k\rangle_{\omega^\alpha}}.$$

Hence, by (4) and (13), we have

$$\begin{aligned}
\langle \mathscr{P}_k, \mathscr{P}_k \rangle_{\omega^\alpha} J^\alpha_{k,n} &= \langle \Delta \mathscr{P}_n, \mathscr{P}_k \rangle_{\omega^\alpha} \\
&= \sum_{g=0}^{\infty} \Delta \mathscr{P}_n(g;\alpha) \mathscr{P}_k(g;\alpha) \omega^\alpha(g;\beta) \\
&= -\sum_{g=0}^{\infty} \mathscr{P}_n(g;\alpha) \nabla(\omega^\alpha(g;\beta) \mathscr{P}_k(g;\alpha)) \\
&= -\sum_{g=0}^{\infty} \mathscr{P}_n(g;\alpha) \omega^\alpha(g;\beta) \nabla \mathscr{P}_k(g;\alpha) - \sum_{g=0}^{\infty} \mathscr{P}_n(g;\alpha) \mathscr{P}_k(g-1;\alpha) \nabla \omega^\alpha(g;\beta),
\end{aligned}$$

from which, by orthogonality, the first sum is zero since $\nabla \mathscr{P}_k$ is degree $k < n+1$. For the second sum let us consider (59)

$$\langle \mathscr{P}_k, \mathscr{P}_k \rangle_{\omega^\alpha} J^\alpha_{k,n} = -\frac{1}{\alpha} \sum_{g=0}^{\infty} \mathscr{P}_n(g;\alpha) \mathscr{P}_k(g-1;\alpha)(\alpha+g)\omega^\alpha(g;\beta),$$

if we use orthogonality again, only $J^\alpha_{n-1,n}$ can be non zero, and as a consequence (91), follows. Proposition 4.1 is proved.

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