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Advancements in Nonlinear Dynamics: Lie Symmetry Applications in the Jaulent-Miodek Equation

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Abstract

This research presents a detailed analysis of the nonlinear Jaulent-Miodek (J-M) equation through the lens of Lie symmetries. Our primary objective is to comprehensively identify the symmetry group and the optimal systems of Lie sub-algebras pertinent to the J-M equation. We delve into the Lie invariants associated with symmetry generators and demonstrate their contribution to forming similarity-reduced equations that encapsulate the essence of the original equation. Moreover, the study introduces a two-step methodology for establishing the conservation laws relevant to the J-M equation. The initial phase involves identifying suitable multipliers essential for calculating these laws. Subsequently, we utilise symbolic computation to derive these conservation laws formally. This in-depth exploration of the equation's symmetries and conservation laws not only enhances our understanding of the J-M equation's intrinsic properties but also aids in simplifying and solving the equation under various conditions.

Keywords: Lie symmetries, Jaulent-Miodek equation, Symmetry group, Lie invariants, Conservation laws, Wave phenomena, Plasma physics.

1 | Introduction

Partial Differential Equations (PDEs) are fundamental to various domains within the applied sciences and engineering, encapsulating essential principles such as the conservation of mass, momentum, energy, and

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electric charge. These principles, in the form of continuity equations, are evident in a wide array of disciplines, including fluid mechanics, quantum physics, plasma physics, elasticity, gas dynamics, electromagnetism, magnetohydrodynamics, and nonlinear optics. The significance of conservation laws [1], [2] in PDE analysis is profound, as they provide the means for exploring the equations' integrability and applying linearization mappings, which are crucial for establishing the existence and uniqueness of solutions and for assessing the stability and global behavior of such solutions [3–6].

Conservation laws are acknowledged as essential in both the practical application and theoretical investigation of differential equations [7], [8]. While not all conservation laws derived from PDEs may possess a direct physical interpretation, they play a significant role in the study of PDEs' integrability. This study is pivotal for comprehending the behavior and properties of PDEs and their solutions, making the identification and understanding of these laws vital steps in the analytical process. The pertinence of these laws is substantiated by scholarly research, which underscores their indispensable role in solving complex mathematical challenges within the scientific community.

The Jaulent-Miodek (J-M) equation, a notable mathematical expression in the field of differential equations, is presented as follows:

$$\begin{aligned} &f_t + f_{xxx} + \frac{3}{2}gg_{xxx} + \frac{9}{2}g_x g_{xx} - 6ff_x - 6fgg_x - \frac{3}{2}g^2 f_x = 0. \\ &g_t + g_{xxx} - 6gf_x - 6fg_x - \frac{15}{2}g^2 g_x = 0, \end{aligned} \tag{1}$$

where the subscripts denote partial derivatives concerning time t and spatial variables x. This set of *Eq. (1)* forms a coupled system that can be directly linked to the J-M spectral problem, as indicated in [9]. In this context, the variables $f(x,t)$ and $g(x,t)$ typically represent the physical quantities of interest, such as wave amplitudes, electric fields, or fluid densities, depending on the specific application. $f(x,t)$ often represents a primary wave field or potential function, which describes the evolution of a physical quantity over space and time. For example, in plasma physics, f can be associated with the electric field potential, representing how the electric field changes as waves propagate through a plasma medium. $g(x,t)$ represents an auxiliary field or an additional physical quantity linked to the main variable f. In certain contexts, g might correspond to the density of a secondary fluid component, magnetic field intensity, or another wave function that interacts with f.

The J-M equation is an important mathematical model in the study of nonlinear wave phenomena, appearing in various domains such as fluid dynamics, plasma physics, and optical fiber systems. Nonlinear PDEs, like the J-M equation describe complex physical processes, including the evolution of wave propagation and interactions in dispersive media. The seminal connection between the coupled J-M equation system and the Euler-Darboux equation was elucidated in the pioneering work of Matsuno [1], [2], which has been influential in extending our understanding of integrable systems. Historically, the study of nonlinear dynamics has centered on integrable systems that exhibit soliton behavior, a phenomenon in which solitary waves maintain their shape during propagation and interactions. The J-M equation is a member of this class of integrable equations, known for its complex wave structures and interactions. Many studies have employed various analytical and numerical techniques to investigate the J-M equation. For instance, Matsuno [1] demonstrated the connection between the J-M equation and the Euler-Darboux equation, enriching the theoretical foundation of this nonlinear system. More recent studies, such as Fan [10], [11], used advanced mathematical tools to derive exact solutions, while Mohebbi et al. [12], [13] applied numerical methods to explore the properties of the J-M equation. Despite these advancements, many aspects of the J-M equation, particularly its symmetry properties and conservation laws, remain underexplored. This gap underscores the need for a more systematic examination of the J-M equation using Lie symmetry analysis. Among the various approaches applied to the study of differential equations, the method involving symmetry groups stands out prominently.The utilization of this method for analyzing differential equations can be traced back to the groundbreaking efforts of Sophus Lie, who introduced the classical Lie method in the latter part of the 19th century [14–16]. This classical method laid the groundwork for a robust framework for exploring solutions to differential equations through symmetries. In modern research, the theory of Lie group transformations has been profoundly influential in constructing exact solutions for nonlinear PDEs. The adaptability of Lie's approach to a vast array of problems in PDEs renders it a tool of immense value. It is widely considered one of the most vigorously pursued areas of research within the domain of nonlinear PDE theory and its numerous applications. Current investigations extend the utility of Lie group analysis in seeking analytic solutions, exploring symmetries, and understanding the intrinsic properties of nonlinear PDEs. This progress highlights the enduring relevance and continuous evolution of Lie's foundational work in contemporary mathematical research. Obtaining the solutions of fractional-order PDEs [17–19] using the Lie symmetry approach is also very challenging, and we have extended our ideas to these types of systems.

While previous studies have explored the J-M equation through various analytical and numerical techniques [11], this paper adopts a unique approach by systematically applying the Lie symmetry method to derive a comprehensive set of conservation laws. Unlike earlier works, which primarily focused on exact solutions or the connections to other integrable systems, our study combines the multiplier method with symbolic computation, providing deeper insights into the equation's structure, integrability, and potential applications.

The analysis of *Eq. (1)* is comprehensively explained in a manner that guides the reader through each step. Our work begins with Section 1, titled "Introduction to the Equation", which discusses the equation being solved in this study. In Section 2, we find the symmetries of *Eq. (1)* and its Lie symmetry groups. It also discusses some useful applications of *Eq. (1)*. In Section 3, we focus on constructing the optimal system of One-Dimensional (1D) subalgebras of *Eq. (1)*, providing a foundation for further symmetry analysis. Section 4 explores the Lie invariants and formulates similarity-reduced equations [20–22] that emerge from the infinitesimal symmetries of *Eq. (1)*. Section 5 focuses on deriving the conservation laws of *Eq. (1)* by identifying suitable multipliers. This section also unveils new conservation laws, which we elucidate through the symbolic computation of these laws, thereby enriching the current understanding and future study of *Eq. (1)*. In Section 6, we discuss the comparison with existing literature, the importance of our results, applications, and some limitations. And the final section, Section 7, presents the conclusion. The main goal of this study is to present these findings clearly and provide a stepping stone for future research in the field.

2 | Derivation of Lie Symmetries

Lie symmetry analysis is a method used to determine the continuous symmetries of differential equations. A Lie symmetry of a differential equation is a transformation that maps solutions of the equation to other solutions. We focus on the methodology for identifying symmetries in the context of the J-M equation [23–26]. By identifying these symmetries, we gain valuable insights into the structure and potential solutions of the equation.

In this expanded discussion, we start by considering a one-parameter Lie group of infinitesimal transformations acting on the independent variables x and $t$and the dependent variables f and g. These transformations are expressed as

$$\begin{aligned}\bar{x} &= x + \varepsilon\xi(x,t,f,g) + O(\varepsilon^2),\\ \bar{t} &= x + \varepsilon\Phi(x,t,f,g) + O(\varepsilon^2),\\ \bar{f} &= x + \varepsilon\eta_1(x,t,f,g) + O(\varepsilon^2),\\ \bar{g} &= x + \varepsilon\eta_2(x,t,f,g) + O(\varepsilon^2),\end{aligned} \tag{2}$$

where ε represents the group parameter, and ξ, Φ, η_1, and η_2 serve as the infinitesimal transformations corresponding to the independent and dependent variables, respectively. The corresponding symmetry generator, often referred to as a vector field, is given by:

$$\begin{aligned}X = \; &\xi(x,t,f,g)\,\partial_x + \Phi(x,t,f,g)\,\partial_t\\ &+\eta_1(x,t,f,g)\,\partial_f + \eta_2(x,t,f,g)\,\partial_g.\end{aligned} \tag{3}$$

To determine the functions ξ, Φ, η_1, and η_2, we require that this vector field X satisfies the invariance condition of the J-M equation under the Lie group of transformations. The invariance condition implies that if $f(x,t)$ and $g(x,t)$ are solutions of the J-M equation, then $\bar{f}(\bar{x},\bar{t})$ and $\bar{g}(\bar{x},\bar{t})$ must also be solutions. By substituting the infinitesimal transformations into the J-M equation and expanding in terms of ϵ, we obtain an equation in terms of ϵ. Setting the coefficient of ϵ to zero yields a system of determining equations for ξ, Φ, η_1, and η_2. The Lie algebra $\mathcal{G}$ of the J-M equation's symmetries is generated by the vector fields χ_1, χ_2, χ_3, which are

$$\chi_1 = \partial_t, \quad \chi_2 = \partial_x.$$
$$\chi_3 = t\,\partial_t + \frac{1}{3}x\,\partial_x - \frac{2}{3}f\,\partial_f - \frac{1}{3}g\,\partial_g. \tag{4}$$

To understand the algebraic structure, we calculate the commutators $[\chi_i, \chi_j]$ for $i, j = 1,2,3$. For instance:

$$[\chi_1, \chi_3] = -\chi_1, \quad [\chi_2, \chi_3] = -\frac{1}{3}\chi_2, \quad [\chi_1, \chi_2] = 0.$$

The commutator table below summarizes these relationships, providing insight into the algebraic structure of the Lie algebra.

Table 1. Commutator table.

[,]	χ_1	χ_2	χ_3
χ_1	0	0	χ_1
χ_2	0	0	$\frac{1}{3}\chi_2$
χ_3	$-\chi_1$	$-\frac{1}{3}\chi_2$	0

We investigate the effects of a one-parameter Lie group [23-27] of infinitesimal transformations applied to the variables, denoted as $x^1 = x, x^2 = t, f^1 = f, f^2 = g$. This examination focuses on understanding how these transformations influence the behavior and properties of these variables.

Table 1 showcases the commutation relations within the 3- dimensional Lie algebra, denoted as $\mathcal{G}$. This algebra is characterized by the vector fields χ_1, χ_2, χ_3,which collectively define the algebra's structure and interactions.

Theorem 1. It states that for any solution $A(t,x)$ and $B(t,x)$ of the $J-M$ equation, the transformed functions by the group actions $G_1(s), G_2(s)$, and $G_3(s)$ are also solutions, as follows:

$$G_1(\varepsilon)\cdot A(t,x) = A(t-\varepsilon, x). \tag{5}$$
$$G_1(\varepsilon)\cdot B(t,x) = B(t-\varepsilon, x).$$
$$G_2(\varepsilon)\cdot A(t,x) = A(t, x-\varepsilon).$$
$$G_2(\varepsilon)\cdot B(t,x) = B(t, x-\varepsilon).$$
$$G_3(\varepsilon)\cdot A(t,x) = e^{-\left(\frac{2}{3}\right)\varepsilon} A\left(e^{-\varepsilon}t, e^{-\left(\frac{1}{3}\right)\varepsilon}x\right).$$
$$G_3(\varepsilon)\cdot B(t,x) = e^{-\left(\frac{1}{3}\right)\varepsilon} B\left(e^{-\varepsilon}t, e^{-\left(\frac{1}{3}\right)\varepsilon}x\right).$$

3 | Optimizing the Systematic Approach

In this section, we delve into the methodology for deducing the optimal system and reduced forms for *Eq. (1)*, leveraging the symmetrical group characteristics delineated in *Table 2*, which details the adjoint representation for the infinitesimal generators as follows:

Table 2. Infinitesimal generators in adjoining form.

+	χ_1	χ_2	χ_3
χ_1	χ_1	χ_2	$\chi_3 - \epsilon\chi_1$
χ_2	χ_1	χ_2	$\chi_3 - \frac{1}{3}\epsilon\chi_2$
χ_3	$e^{\epsilon}\chi_1$	$e^{\frac{1}{3}\epsilon}\chi_2$	χ_3

In the context of the original PDE, characterized by two independent variables, we employ symmetry reduction to transform it into an ordinary differential equation. This transition is fundamental to simplifying and solving complex differential equations.

A standard and widely cited method, as referenced in [23], provides a systematic classification of 1D subalgebras into conjugacy classes. This classification is pivotal, as it parallels the categorization of orbits within the adjoint representation group, establishing a conjugacy relationship among the 1D subalgebras.

The classification issue mirrors the task of classifying orbits for the adjoint action. A 1D subalgebra within a Lie algebra $\mathcal{G}$ is represented by a non-null vector, and the classification challenge is approached by applying general elements X in $\mathcal{G}$ to adjoint transformations aimed at achieving maximal simplification. To construct the optimal subalgebra system of $\mathcal{g}$, we introduce the Lie series representation as follows:

$$+\left(\exp(\epsilon(\chi_i))\chi_j\right) = \chi_j - \epsilon[\chi_i, \chi_j] + \frac{\epsilon^2}{2}\left[\chi_i, [\chi_i, \chi_j]\right] - \cdots, \tag{6}$$

where $[\chi_i, \chi_j]$ represents the Lie bracket or commutator within the Lie algebra, with ϵ as a varying parameter and subscripts i, j taking the values 1 to 3. This procedure elucidates the adjoint representation, as illustrated in *Table 2*.

The optimal system of 1D subalgebras plays a critical role in reducing the complexity of the J-M equation. By classifying the subalgebras into conjugacy classes, we reduce the infinite-dimensional symmetry group to more manageable 1D systems. This reduction simplifies the problem significantly, allowing us to systematically derive invariant solutions by transforming the original PDE into an ODE under the action of the corresponding symmetry generator.

This optimal system is a powerful tool because it acts as a minimal set of representatives from the conjugacy classes. Each representative leads to a specific symmetry reduction, transforming the PDE into a simpler ODE. These reduced equations are often easier to solve and provide critical insights into the nature of the original solutions.

For example, the subalgebra spanned by χ_3 reduces the J-M equation to a third-order nonlinear ODE, while subalgebras like $\alpha\chi_1 + \chi_2$ lead to further reductions that simplify the analysis of the system. This approach not only aids in solving the equation but also offers a clearer understanding of the equation's structure and symmetry properties.

Thus, constructing the optimal system is fundamental to efficiently exploring the solution space of the J-M equation. It serves as a foundation for further symmetry analysis and reveals invariant properties of the system, contributing to a more comprehensive understanding of its dynamics.

Theorem 2. (1) χ_3, (2) $\alpha\chi_1 + \chi_2$, and (3) χ_1 offer an optimum system of 1D Lie algebras of the $J - M$ *Eq. (1)*, where $\alpha \in R$ and $\alpha \neq 0$.

Proof: consider the symmetry group g of *Eq. (1)*, as outlined in *Table 2*, and let

$$\tilde{X} = \zeta_1\chi_1 + \zeta_2\chi_2 + \zeta_3\chi_3. \quad (7)$$

represent a non-zero vector field in g. Our goal is to reduce as many coefficients of $f_i, i = 1,2,3$, as we can by strategically using adjoint maps on $\tilde{X}$.

Case 1. Initially, assume $\zeta_3 \neq 0$. We can normalize $\tilde{X}$ so that $\zeta_3 = 1$. From *Table 2*, applying $+(\exp(\zeta_1\chi_1))$ and $+(\exp(3\zeta_2\chi_2))$ to $\tilde{X}$ will eliminate the coefficients of χ_1 and χ_2. Hence, any 1D subalgebra formed by a $\tilde{X}$ with $\zeta_3 \neq 0$ is like the subalgebra spanned by χ_3.

Case 2. For other 1D subalgebras spanned by vectors like the above but with $\zeta_3 = 0$, if $\zeta_2 \neq 0$, we scale it to make $\zeta_2 = 1$. Referring to *Table 2* again, no further simplifications are possible in this scenario. Therefore, each 1D subalgebra created by a $\tilde{X}$ with $\zeta_3 = 0$ and $\zeta_2 \neq 0$ is equivalent to the subalgebra spanned by $\alpha\chi_1 + \chi_2$, with α as an arbitrary constant.

Case 3. In situations where $\zeta_3 = 0, \zeta_2 = 0$, and $\zeta_1 \neq 0$, it follows that every 1D subalgebra produced by $\tilde{X}$ is analogous to the subalgebra spanned by χ_1.

4 | Symmetry based Reduction

The calculation of invariants linked to symmetry operators is achieved by integrating their characteristic equations. Take, for instance, the characteristic equation of the operator

$$\chi_3 = t\,\partial_t + \frac{1}{3}x\,\partial_x - \frac{2}{3}f\,\partial_f - \frac{1}{3}g\,\partial_g. \quad (8)$$

and

$$\frac{dt}{t} = \frac{3dx}{x} = -\frac{3df}{2f} = -\frac{3dg}{g}. \quad (9)$$

The derived invariants in this context are $\varrho = xt^{-1/3}, \psi = ft^{2/3}$, and $\Psi = gt^{1/3}$. Hence, the solution to our equation becomes $f = \psi t^{-2/3}, g = \Psi t^{-1/3}$. By replacing the derivatives of f and g with ϱ, ψ, and Ψ in *Eq. (1)*, we obtain a coupled system of Ordinary Differential Equations (ODE):

$$\begin{aligned} &4\psi + 2\varrho\psi_\varrho - 6\psi_{\varrho\varrho\varrho} - 9\Psi\Psi_{\varrho\varrho\varrho} - 27\Psi_\varrho\Psi_\varrho\varrho \\ &\qquad +36\psi\psi_\varrho + 36\psi\Psi\Psi_\varrho + 9\Psi^2\psi_\varrho = 0, \\ &2\Psi + 2\varrho\Psi_\varrho - 6\Psi_{\varrho\varrho\varrho} + 36\Psi\psi_\varrho + 36\psi\Psi_\varrho + 45\Psi^2\Psi_\varrho = 0. \end{aligned} \quad (10)$$

For the operator $\alpha\chi_1 + \chi_2$, the equations are

$$\begin{aligned} &\alpha\psi_\varrho + \psi_{\varrho\varrho} + \frac{3}{2}\Psi\Psi_{\varrho\varrho\varrho} + \frac{9}{2}\Psi_\varrho\Psi_{\varrho\varrho} \\ &\qquad -6\psi\psi_\varrho - 6\psi\Psi\Psi_\varrho - \frac{3}{2}\Psi^2\psi_\varrho = 0. \\ &\alpha\Psi_\varrho + \Psi_{\varrho\varrho\varrho} - 6\Psi\psi_\varrho - 6\psi\Psi_\varrho - \frac{15}{2}\Psi^2\Psi_\varrho = 0. \end{aligned} \quad (11)$$

with the corresponding invariants for the above operator being $\varrho = x - \alpha t, \psi = f$, and $\Psi = g$.

5 | Conservation Laws

To address the conservation laws [2], [18], various approaches are employed, such as those based on Noether's theorem and the multiplier method. These methods establish a connection between the conserved vector of a PDE and its Lie–Bäcklund symmetry generators, along with the direct method and others [3], [4], [23].

Definition 1 ([28]). A local conservation law for a system of PDEs can be defined as follows:

$$\Delta_\tau\left(x, f^{(n)}\right) = 0, \quad \text{for } \tau = 1, \dots, l, \tag{12}$$

where the expression involves a set of independent variables $\bar{x} = x^1, \dots, x^p$ and a set of dependent variables $\bar{f} = f^1, \dots, f^q$.

Here, $f^{(n)}$ denotes the collection of all derivatives of the functions f with respect to the variables x, covering all orders from 0 up to n. This conservation law can be represented by a divergence expression:

$$D_i T_1^i[f] = D_1 T_1^1[f] + \cdots + D_n T_1^n[f] = 0. \tag{13}$$

And is valid for all solutions of system *Eq. (12)*. $T_1^i[f] = T_1^i(x, f, \partial f, \dots, \partial_f^r), i = 1, \dots, n$, are called fluxes of the conservation law, and the highest-order derivative (r) present in the fluxes $T_1^i[f]$ is called the order of a conservation law [4].

Remark 1. When one of the independent variables in system *Eq. (12)* is time t, the conservation law *Eq. (13)* is represented as

$$D_t T_2[f] + \operatorname{div} T_1[f] = 0, \tag{14}$$

where $\operatorname{div} T_1[f] = D_i T_1^i[f] = D_1 T_1^1[f] + \cdots + D_n T_1^n[f]$ is a spatial divergence and $x = (x^1, \dots, x^{n-1})$ are $n - 1$ spatial variables. Here $T_2[f]$ is referred to as a density, and $T_1^i[f]$ as spatial fluxes of the conservation law *Eq. (14)*.

5.1 | Computation of Conservation Laws with Finding Multiplier

In our research, we explore the derivation of the conservation law through the multiplier method [29]. The multiplier method is a technique used to identify conservation laws associated with differential equations. In this approach, multipliers are specific functions that, when multiplied by the original differential equation, transform it into divergence form. This transformation is essential because it enables the identification of conserved quantities.

The process involves finding these multipliers

$$\{\Lambda_\tau\}_{\tau=1}^{l} = \left\{\Lambda_\tau\left(x, \psi, \partial\psi, \dots, \partial_\psi^r\right)\right\}_{\tau=1}^{l}. \tag{15}$$

Such that the resulting expression can be written as a divergence of some vector field, which implies a conservation law:

$$\Lambda_\tau[\psi]\Delta_\tau[\psi] \equiv D_i T_2^i[\psi]. \tag{16}$$

is an identity for any arbitrary function $\psi(x)$. Therefore, for solutions $\psi(x) = f(x)$ to system *Eq. (11)*, where $\Delta_\tau[\psi]$ is not singular, a local conservation law is established as $\Lambda_\tau[f]\Delta_\tau[f] \equiv D_i T_1^i[f] = 0$.The method is systematic and powerful, especially for nonlinear equations, as it enables the discovery of conservation laws that might not be immediately apparent.

In this study, the multiplier method was used as the first step in identifying conservation laws for the J-M equation. This involved determining suitable multipliers that could convert the equation into a form that directly yields conserved quantities.

Definition 2 ([4]). We define the Euler operator in relation to ψ^j as

$$L_{\psi^J} = \frac{\partial}{\partial \psi^j} - D_i \frac{\partial}{\partial \psi_i^j} + \cdots + (-1)^s D_{i_1} \cdots D_{i_i} \frac{\partial}{\partial \psi_{i_1 \cdots i_s}^j} + \cdots. \tag{17}$$

for $j = 1, \ldots, q$.

Theorem 3. A collection of non-singular local multipliers $\left\{\Lambda_\tau\left(x, \psi, \partial\psi, \ldots, \partial_\psi^r\right)\right\}_{\tau=0}^{l}$ establishes a local conservation law for the system $\Delta_\tau\left(x, \psi^{(n)}\right)$ if and only if there exists a set of equalities

$$L_{\psi^j}\left(\Lambda_\tau(x, \psi, \partial\psi, \ldots, \partial^r\psi)\Delta_\tau\left(x, \psi^{(n)}\right)\right) = 0, \quad j = 1, \ldots, q. \tag{18}$$

which are valid for any arbitrary functions $\psi(x)$ [4].

The collection of equations represented by *Eq. (18)* results in a set of linear equations that are essential for identifying all possible sets of local conservation law multipliers for the system detailed in *Eq. (12)*. In this context, we focus on all local conservation law multipliers that take the forms:

$$\Lambda_1 = \alpha(t, x, f, g, f_t, g_t, f_x, g_x, f_{tt}, g_{tt}, f_{xx}, g_{xx}). \tag{19}$$

and

$$\Lambda_2 = \beta(t, x, f, g, f_t, g_t, f_x, g_x, f_{tt}, g_{tt}, f_{xx}, g_{xx}). \tag{20}$$

which are associated with *Eq. (1)*. The determining *Eq. (18)* for the J-M equation is

$$\begin{aligned}
&L_f\left[\Lambda_1\left(f_t + f_{xxx} + \frac{3}{2} g g_{xxx} + \frac{9}{2} g_x g_{xx}\right.\right.\\
&\left.-6 f f_x - 6 f g g_x - \frac{3}{2} g^2 f_x\right)\\
&\left.+\Lambda_2\left(g_t + g_{xxx} - 6 g f_x - 6 f g_x - \frac{15}{2} g^2 g_x\right)\right] \equiv 0\\
&L_g\left[\Lambda_1\left(f_t + f_{xxx} + \frac{3}{2} g g_{xxx} + \frac{9}{2} g_x g_{xx}\right.\right.\\
&\left.-6 f f_x - 6 f g g_x - \frac{3}{2} g^2 f_x\right)\\
&\left.+\Lambda_2\left(g_t + g_{xxx} - 6 g f_x - 6 f g_x - \frac{15}{2} g^2 g_x\right)\right] \equiv 0,
\end{aligned} \tag{21}$$

where $f(x, t)$ and $g(x, t)$ are unspecified functions (arbitrary functions). The separation of *Eq. (21)* based on the third-order derivatives of f results in a system of PDEs. The solutions to this system represent the collection of local multipliers corresponding to all significant local conservation laws of the J-M equation.

The obtained solution for the J-M equation of the determining System *Eq. (21)* can be expressed as follows:

$$\begin{aligned}\alpha = \ & c_1x + 6c_1tf + \frac{9}{2}c_1tg^2 + c_2fg + \frac{1}{6}c_3fc_2g_{xx} \\ & + \frac{5}{12}c_2g^3 + \frac{3}{4}c_3g^2 + c_4g + c_5 \\ \beta = - \ & \frac{1}{6}c_2f_{xx} + \frac{1}{2}c_1xg - \frac{5}{24}c_2g_x^2 - \frac{5}{12}g_{xx} - c_2g - \frac{3}{2}c_1tg_{xx} \\ & - \frac{1}{4}c_3g_{xx} + \frac{15}{4}c_1tg^3 + 9c_1tfg + \frac{1}{2}c_2f^2 + \frac{5}{4}c_2fg^2 \\ & + \frac{3}{2}c_3fg + c_4f + \frac{35}{96}c_2g^4 + \frac{5}{8}c_3g^3 + \frac{3}{4}c_4g^2 + \frac{1}{2}c_5g + c_6.\end{aligned} \tag{22}$$

here, c_1, c_2, c_3, c_4, c_5, and c_6 represent arbitrary constants. So local multipliers are given by

I. $\alpha = 0, \quad \beta = 1.$

II. $\alpha = 1, \quad \beta = \frac{1}{2}g.$

III. $\alpha = g, \beta = f + \frac{3}{4}g^2.$

IV.

$$\Lambda_1 = \alpha(t, x, f, g, f_t, g_t, f_x, g_x, f_{tt}, g_{tt}, f_{xx}, g_{xx}). \tag{23}$$

V. $\alpha = f + \frac{3}{4}g^2, \quad \beta = -\frac{1}{4}g_{xx} + \frac{3}{2}fg + \frac{5}{8}g^3.$

VI. $\alpha = x + 6tg + \frac{9}{2}tg^2, \quad \beta = \frac{1}{2}xg - \frac{3}{2}tg_{xx} + \frac{15}{4}t^3 + 9tfg.$

VII. $\alpha = fg - \frac{1}{6}g_{xx} + \frac{5}{12}g^3, \quad \beta = -\frac{1}{6}f_{xx} - \frac{5}{24}g_x^2 - \frac{5}{12}gg_{xx} + \frac{1}{2}f^2 + \frac{5}{4}fg^2 + \frac{35}{96}g^4.$

The multipliers α and β are key in defining a significant local conservation law represented as $D_xT_1 + D_tT_2 = 0$, which is characterized by

$$\begin{aligned}D_xT_1 + D_tT_2 \equiv \alpha(f_t + \ & f_{xxx} + \frac{3}{2}gg_{xxx} + \frac{9}{2}g_xg_{xx} \\ & -6ff_x - 6fgg_x - \frac{3}{2}g^2f_x\Big) \\ + \ & \beta\Big(g_t + g_{xxx} - 6gf_x - 6fg_x - \frac{15}{2}g^2g_x\Big).\end{aligned} \tag{24}$$

To determine the conserved quantities T_2 and T_1, it's necessary to reverse the total divergence operator. This involves integrating (by parts) certain multidimensional expressions that include arbitrary functions and their derivatives, a complex task. The homotopy operator, as described in [30], offers a potent and effective algorithmic approach (explicit formula) developed from the principles of homological algebra and variational bicomplexes.

Definition 3 ([6]). The two-component vector operator known as the 2-dimensional homotopy operator is represented by $\left(H^{(x)}_{f(x,t)}h, H^{(t)}_{f(x,t)}h\right)$. It is defined as follows:

$$H^{(x)}_{\mathbf{f}(x,t)}h = \int_0^1 \left(\sum_{j=0}^{q} I^{(x)}_{f'}h\right)[\varrho\mathbf{f}]\frac{d\varrho}{\varrho}. \tag{25}$$

$$H^{(t)}_{\mathbf{f}(x,t)}h = \int_0^1 \left(\sum_{j=0}^{q} I^{(t)}_{fj}h\right)[\varrho\mathbf{f}]\frac{d\varrho}{\varrho}.$$

The x-integrand, $I^{(x)}_{f(x,t)^{(x)}}h$, is given by

$$I^{(x)}_{fj}h=\sum_{k_1=1}^{\Gamma^j_1}\sum_{k_2=0}^{\Gamma^j_2}\left(\sum_{i_1=0}^{k_1-1}\sum_{i_2=0}^{k_2}E^{(x)}f^j_{x^{i_1}x^{i_2}}(-D_x)^{k_1-i_1-1}\times(-D_t)^{k_2-i_2}\right)\frac{\partial h}{\partial f^j_{x^{k_1}t^{k_2}}},\tag{26}$$

where Γ^j_1,Γ^j_2 are the order of h in f to x and t, respectively, with combinatorial coefficient $E^{(x)}=E(i_1,i_2,k_1,k_2)$, were

$$E(i_1,i_2,k_1,k_2)=\frac{\binom{i_1+i_2}{i_1}\binom{k_1+k_2-i_1-i_2-1}{k_1-i_1-1}}{\binom{k_1+ki_2}{k_1}}.\tag{27}$$

Similarly, t-integrand, $I^{(j)}_{f(x,t)^j}h$, defined as

$$I^{(t)}_{fj}h=\sum_{k_1=0}^{\Gamma^j_1}\sum_{k_2=1}^{\Gamma^j_2}\left(\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{k_2-1}E^{(t)}f^j_{x^{j_1}x^{j_2}}(-D_x)^{k_1-i_1}\times(-D_t)^{k_2-i_2-1}\right)\frac{\partial h}{\partial f_{x\quad k_1t^{k_2}}},\tag{28}$$

where $E^{(t)}=E(i_2,i_1,k_2,k_1)$.

We apply the homotopy operator to find conserved quantities T_1 and T_2 which yield multipliers $\alpha=0$ and $\beta=1$. We have

$$\alpha\left(f_t+f_{xxx}+\frac{3}{2}gg_{xxx}+\frac{9}{2}g_xg_{xx}-6ff_x-6fgg_x-\frac{3}{2}g^2f_x\right)+\beta\left(g_t+g_{xxx}-6gf_x-6fg_x-\frac{15}{2}g^2g_x\right)=g_t+g_{xxx}-6gf_x-6fg_x-\frac{15}{2}g^2g_x.\tag{29}$$

The integrands *Eq. (26)* and *Eq. (28)* are

$$I^{(t)}_fh=0,\quad I^{(x)}_fh=-6fg,\quad I^{(t)}_gh=g$$
$$I^{(x)}_gh=-6gf-\frac{15}{2}g^3+g_{xx}.\tag{30}$$

Apply *Eq. (25)* to the integrands *Eq. (30)*. Therefore

$$T_2=H^{(x)}_{f(x,t)}h=6fg-g_{xx}+\frac{5}{2}g^3$$
$$T_1=H^{(t)}_{f(x,t)}h=-g.\tag{31}$$

So, we have the conservation law of the J-M equation with respect to multipliers $\alpha = 0$ and $\beta = 1$:

$$D_t(-g) + D_x\left(6fg - g_{xx} + \frac{5}{2}g^3\right) = 0. \tag{32}$$

And similarly, conservation laws with respect to other multipliers are given as follows:

I. $\alpha = 1$ and $\beta = \frac{1}{2}g$:

$$T_1 = -\frac{1}{4}g^2 - f$$
$$T_2 = \frac{15}{16}g^4 + \frac{9}{2}fg^2 - 2gg_{xx} - \frac{5}{4}g_x^2 + 3f^2 - f_{xx}. \tag{33}$$

II. $\alpha = g$ and $\beta = g + \frac{3}{4}g^2$:

$$T_1 = -\frac{1}{4}g^3 - fg$$
$$T_2 = \frac{9}{8}g^3 + 6fg^3 - \frac{9}{4}f^2g_{xx} + 6f^2g - fg_{xx} - gf_{xx} + f_xg_x. \tag{34}$$

III. $\alpha = f + \frac{3}{4}g^2$ and $\beta = -\frac{1}{4}g_{xx} + \frac{3}{2}fg + \frac{5}{8}g^3$:

$$\begin{aligned} T_1 = {} & -\frac{5}{32}g^4 - \frac{3}{4}fg^2 + \frac{1}{8}gg_{xx} - \frac{1}{2}f^2 \\ T_2 = {} & \frac{25}{32}g^6 + \frac{39}{8}fg^4 - \frac{7}{8}g^3g_{xx} + \frac{15}{2}g^2f^2 - \frac{3}{2}fg_x^2 \\ & -3fgg_{xx} - \frac{3}{4}g^2f_{xx} + \frac{3}{2}gf_xg_x + 2f^3 + \frac{1}{2}f_x^2 + \frac{1}{8}g_{xx}^2 \\ & -ff_{xx} - \frac{1}{8}gg_{tx} + \frac{1}{8}g_tg_x. \end{aligned} \tag{35}$$

IV. $\alpha = fg - \frac{1}{6}g_{xx} + \frac{5}{12}g^3$ and $\beta = -\frac{1}{6}f_{xx} - \frac{5}{24}g_x^2 - \frac{5}{12}gg_{xx} + \frac{1}{2}f^2 + \frac{5}{4}fg^2 + \frac{35}{96}g^4$:

$$\begin{aligned} T_1 = {} & -\frac{7}{96}g^5 - \frac{5}{12}fg^3 + \frac{5}{72}gg_x^2 - \frac{1}{2}gf^2 \\ & +\frac{5}{36}g^2g_{xx} = \frac{1}{12}gf_{xx} + \frac{1}{12}fg_{xx} \\ T_2 = {} & \frac{25}{64}g^7 + \frac{45}{16}fg^5 - \frac{25}{48}g^3g_x^2 + \frac{23}{4}g^3f^2 - \frac{95}{96}g^4g_{xx} \\ & -\frac{11}{4}fg^2g_{xx} - \frac{5}{12}g^3f_{xx} + 3gf^3 - \frac{5}{4}fgg_x^2 - \frac{1}{2}f^2g_{xx} \\ & -\frac{5}{36}g^2g_{tx} + \frac{5}{36}gg_tg_x + \frac{1}{3}gg_{xx}^2 + \frac{5}{24}g_x^2g_{xx} - fgf_{xx} \\ & +\frac{1}{12}f_tg_x - \frac{1}{12}gf_{tx} + \frac{1}{12}g_tf_x + \frac{1}{6}f_{xx}g_{xx} - \frac{1}{12}fg_{tx}. \end{aligned} \tag{36}$$

V. $\alpha = x + 6tf + \frac{9}{2}tg^2$ and $\beta = \frac{1}{2}xg - \frac{3}{2}tg_{xx} + \frac{15}{4}tg^3 + 9tfg$:

$$
\begin{aligned}
T_1 = \;& -\frac{15}{16}tg^4 - \frac{9}{2}tfg^2 + \frac{3}{4}tgg_{xx} - \frac{1}{4}xg^2 - 3tf^2 - xf \\
T_2 = \;& \frac{75}{16}tg^6 + \frac{117}{4}tfg^4 + \frac{15}{16}xg^4 + 45tf^2g^2 - \frac{21}{2}tg^3g_{xx} \\
& +9tgf_xg_x - 9tfg_x^2 + \frac{9}{2}xfg^2 - 18tfgg_{xx} - \frac{9}{2}tg^2f_{xx} \\
& +12tf^3 + \frac{5}{4}gg_x + \frac{3}{4}tg_{xx}^2 - 2xgg_{xx} + \frac{3}{4}tg_tg_x - 6tff_{xx} \\
& +3xf^2 - \frac{5}{4}xg_x^2 + 3tf_x^2 - \frac{3}{4}tgg_{tx} - xf_{xx} + f_x.
\end{aligned}
\tag{37}
$$

5.2 | Symbolic Computation to Deriving Conservation Law

Once the multipliers are found, symbolic computation is utilized to compute the conserved quantities. By utilizing algorithms based on the homotopy operator, the symbolic computation method allows the derivation of conserved quantities from the expressions obtained through the multiplier method. This approach is particularly advantageous because it handles the complexity and algebraic manipulation involved in higher-order and nonlinear PDEs. It ensures that the resulting conservation laws are derived efficiently and accurately, which would be highly cumbersome to do manually.

The homotopy operator method is based on homological algebra and the variational bicomplex, providing an algorithmic means to reverse the divergence operator. It transforms the divergence expressions into a manageable form and systematically integrates the terms by parts to isolate the conserved densities and fluxes. This method is well suited for symbolic computation tools, making it highly efficient for deriving conservation laws.

The process begins with the calculation of the density T_2, followed by the computation of the flux T_1.The computation of T_1 necessitates the application of the homotopy operator. In line with the method proposed by Hereman et al. [5], [31], a potential density is formulated as a linear blend of differential terms with unknown coefficients, which remains consistent with the scaling symmetry of the specific PDE. By establishing T_2, we can calculate D_tT_2 and eliminate all temporal derivatives; D_tT_2 should be a divergence. Consequently, based on *Theorem 4* of [13], it is required that

$$
L_{\psi^j}(D_tT_2) = 0, \quad j = 1, \dots, N. \tag{38}
$$

The process results in a linear equation system to determine the unknown coefficients. By integrating the solution of this system into the proposed expression for T_2, the actual density is obtained. Subsequently, the expression for T_1, defined as $T_1 = \mathrm{div}^{-1}(D_tT_2)$, is calculated using the homotopy operator. J-M equation is invariant under the scaling (dilation) symmetry *Eq. (4)*:

$$
(t, x, f, g) \longrightarrow (\varrho^3 t, \varrho x, \varrho^{-2} f, \varrho^{-1} g). \tag{39}
$$

The conservation law, as stated in *Eq. (14)*, is a requisite for solutions derived from *Eq. (1)*. Our objective is to identify polynomial conservation laws that follow the PDE's scaling symmetry. Our task, then, is to pinpoint a polynomial conservation law that diverges from this scaling symmetry. We opt to assign a scaling factor to a specific element within *Eq. (14)*, designating this factor as the rank (R) of that element. Subsequently, we formulate a potential candidate for this element, comprising a linear amalgamation of monomial expressions (each of identical rank R) combined with coefficients yet to be determined. By actively eliminating terms that are equivalent to divergence, this proposed candidate becomes more concise and of a reduced order.

For the J-M equation we will compute the density T_2 of a fixed rank; for example, $R = -3$. We construct a list of differential terms which contains all powers of dependent variables and their derivatives and products of them of rank -3:

$$\begin{aligned}\Theta = \{ \quad & \{f_x^3, g^3, tf^3, xf^2, ff_x, fg, f^2 f_{xxx}, f_x^3 f_{xx}, g^3 f_{xx}, gf_x^2 \\ & g_x f_x^3, g^3 g_x, tf^3 f_{xx}, xf^2 f_{xx}, ff_x f_{xx}, fgf_{xx} \\ & f^2 f_{xx} f_{xxx}, gf^2 f_{xx}, g^2 f_x f_{xx}, tf^3 g_x, xf^2 g_x, fg_x f_x \\ & fgg_x, f^2 g_x f_{xxx}, gg_x f_x^2, g^2 f_x g_x\}.\end{aligned} \tag{40}$$

By removing all terms that are divergences or divergence equivalent to other terms in Θ, we have

$$\begin{aligned}\Theta = \{ \quad & \{f_x^3, g^3, tf^3, xf^2, g^2 f_x, g^3 f_{xx}, gf_x^2, g_x f_x^3 \\ & tf^3 f_{xx}, xf^2 f_{xx}, fgf_{xx}, f^2 f_{xx} f_{xxx}, gf_x^2 f_{xx} \\ & g^2 f_x f_{xx}, tf^3 g_x, xf^2 g_x, ff_x g_x, f^2 g_x f_{xxx} \\ & gg_x f_x^2, g^2 g_x f_x\}.\end{aligned} \tag{41}$$

Now, by forming a candidate density combining the terms in Θ linearly with undetermined coefficients c_i,

$$\begin{aligned}T_2 = \quad & c_1 f_x^3 + c_2 g^3 + c_3 tf^3 + c_4 xf^2 + c_5 g^2 f_x + c_8 g^3 f_{xx} \\ & + c_7 gf_x^2 + c_8 g_x f_x^3 + c_9 tf^3 f_{xx} + c_{10} xf^2 f_{xx} \\ & + c_{11} fgf_{xx} + c_{12} f^2 f_{xx} f_{xxx} + c_{13} gf_x^2 f_{xx} \\ & + c_{14} g^2 f_x f_{xx} + c_{15} tf^3 g_x + c_{16} xf^2 g_x + c_{17} ff_x g_x \\ & + c_{18} f^2 g_x f_{xxx} + c_{19} gg_x f_x^2 + c_{20} g^2 g_x f_x.\end{aligned} \tag{42}$$

Compute the total derivative with respect to t of *Eq. (42)*, and set

$$F = -D_t T_2. \tag{43}$$

After replacing f_t with

$$-f_{xxx} - \frac{3}{2} gg_{xxx} - \frac{9}{2} g_x g_{xx} + 6ff_x + 6fgg_x + \frac{3}{2} g^2 f_x.$$

and g_t by

$$-g_{xxx} + 6gf_x + 6fg_x + \frac{15}{2} g^2 g_x.$$

Eq. (43) must be a divergence, use *Eq. (38)*, and require

$$L_{f(t,x)} F = 0, \quad L_{g(x,t)} F = 0. \tag{44}$$

The solution of system *Eq. (44)* is

$$c_1 = 0, \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = 0. \tag{45}$$

$$c_5 = 0, \quad c_0 = c_{20}, \quad c_7 = c_{11}, \quad c_8 = \frac{1}{3} c_{13}.$$

$$c_9 = 0, \quad c_{10} = 0, \quad c_{12} = 0, \quad c_{14} = c_{19}.$$

$$c_{15} = 0, \quad c_{16} = 0, \quad c_{17} = c_{11}, \quad c_{18} = 0.$$

Where c_{11}, c_{13}, c_{19}, and c_{20} is arbitrary.

Case 4. Substitute *Eq. (45)* and $c_{11} = 1, c_{13} = 0, c_{19} = 0$, and $c_{20} = 0$ into *Eq. (42)* and *Eq. (43)* given $T_2 = gf_x^2 + fgf_{xx} + ff_x g_x$ and

$$\begin{aligned}F = \ & f_xg_xf_{xxx} + 2gf_xf_{xxxx} + \frac{9}{2}f_xg_x^2g_{xx} + 9gf_xg_{xx}^2 \\ & +ff_xg_{xxxx} + 3g^2f_xg_{xxxx} + \frac{9}{2}fg_xg_{xx} + fg_xf_{xxxx} \\ & +6fg_x^2g_{xxx} + \frac{3}{2}fg^2g_{xxxxx} + \frac{3}{2}g^2f_{xx}g_{xxx} + gf_{xx}f_{xxx} \\ & +gff_{xxxxx} + ff_{xx}g_{xxx} - 12f^2g_xf_{xx} - \frac{9}{2}g^3f_xf_{xx} \\ & -\frac{3}{2}fg^3f_{xxx} - 6f^2g^2g_{xxx} - 6gf^2f_{xxx} - 6f^2f_xg_{xx} \\ & -30ff_x^2g_x - 27g^2g_xf_x^2 + f_x^2g_{xxx} - 6f^2g_x^3 - 18gf_x^3 \\ & +\frac{27}{3}gf_xg_xg_{xxx} + \frac{9}{2}gg_xg_{xx}f_{xx} + 9fgg_xg_{xxxx} \\ & +15fgg_{xx}g_{xxx} - \frac{69}{2}fg^2f_xg_{xx} - 27fg^2g_xf_{xx} \\ & -48fgf_xf_{xx} - 57fgf_xg_x^2 - 24gf^2g_xg_{xx}.\end{aligned} \tag{46}$$

As F is the divergence of T_1, denoted by $F = \text{div}\, T_1$, we can determine the flux T_1 utilizing the 1D homotopy operator, which effectively inverts divergences. By applying the formulas of the 1D homotopy operator as delineated in *Eq. (25)*, and subsequently eliminating the curl component from the flux T_1, we can yield the desired computation as fellows:

$$\begin{aligned}T_1 = - \ & \frac{3}{2}g^3f_x^2 - \frac{3}{2}fg^3f_{xx} - 6g^2f^2g_{xx} - \frac{45}{2}fg^2f_xg_x \\ & -18fgf_x^2 - \frac{3}{2}g^2f_xg_{xxx} + \frac{9}{2}gg_xf_xf_{xx} + \frac{3}{2}fg^2g_{xxxx} \\ & -6gf^2f_{xx} + 6fgg_xg_{xxx} + \frac{9}{2}fgg_{xx}^2 - 6f^2f_xg_x \\ & +gf_xf_{xxx} + fgf_{xxxx} + ff_xg_{xxx}.\end{aligned} \tag{47}$$

Case 5. Substitute $c_{11} = 0, c_{13} = 1, c_{19} = 0$, and $c_{20} = 0$ into *Eq. (42)*, given

$$\begin{aligned}T_2 = \ & \frac{1}{3}g_xf_x^3 + gf_x^2f_{xx} \\ T_1 = \ & -\frac{1}{6}f_x^2(36fg^2g_{xx} + 9g^3f_{xx} + 69g^2f_xg_x \\ & +36fgg_x^2 - 36gg_xg_{xxx} + 12ff_xg_x \\ & -27gg_{xx}^2 + 36fgf_{xx} + 48gf_x^2 \\ & -9g^2g_{xxxx} - 6gf_{xxxx} - 2f_xg_{xxx}).\end{aligned} \tag{48}$$

Case 6. Substitute $c_{11} = 0, c_{13} = 0, c_{19} = 1, c_{20} = 0$,

$$\begin{aligned}T_2 = \ & g^2f_xf_{xx} + gg_xf_x^2 \\ T_1 = \ & -\frac{1}{2}gf_x(12fg^2g_{xx} + 3g^3f_{xx} + 33g^2f_xg_x \\ & +12fgg_x^2 - 12gg_xg_{xxx} + 12fgf_{xx} \\ & +12ff_xg_x - 3g^2g_{xxxx} + 24gf_x^2 - 9gg_{xx}^2 \\ & -2gf_{xxxx} - 2f_xg_{xxx}).\end{aligned} \tag{49}$$

Case 7. finally, $c_{11} = 0, c_{13} = 0, c_{19} = 0, c_{20} = 1$,

$$T_2 = \frac{1}{3}g^3f_{xx} + g^2f_xg_x \tag{50}$$

$$T_1 = -\frac{1}{6}g^2(3g^3f_{xx} + 12fg^2g_{xx} + 63g^2g_xf_x$$
$$+12fgg_x^2 + 48gf_x^2 - 9gg_{xx}^2$$
$$-12gg_xg_{xxx} + 36ff_xg_x - 3g^2g_{xxxx}$$
$$+12fgf_{xx} - 2gf_{xxxx} - 6f_xg_{xxx}).$$

6 | Discussion

6.1 | Comparison with Existing Literature:

The results of this study provide a deeper understanding of the J-M equation's symmetry properties and conservation laws, building on the foundational work by Matsuno [1], who initially linked the J-M equation to the Euler-Darboux equation. Unlike previous studies that focused primarily on deriving exact solutions or numerical methods (e.g., Fan [10], Mohebbi et al. [11]), this research has taken a systematic approach using Lie symmetry analysis to derive a broader set of conservation laws. This offers a more comprehensive view of the J-M equation's structure and symmetry properties.

6.2 | Significance of the Findings

Identifying the optimal system of 1D subalgebras and the corresponding similarity reductions provides crucial insights into the underlying dynamics of the J-M equation. This allows for a more efficient reduction of the original PDE into simpler ODEs, making it easier to derive exact solutions. The conservation laws obtained through the multiplier method and symbolic computation add a new layer of understanding to the integrability and solvability of the J-M equation, which are critical for applications in wave phenomena, fluid dynamics, and plasma physics.

6.3 | Implications and Applications

The derived conservation laws and symmetry properties can significantly impact practical applications in physics and engineering, particularly in modeling wave propagation, soliton dynamics, and other nonlinear phenomena in dispersive media. This study’s approach to systematically identifying Lie symmetries and conservation laws can be extended to other nonlinear PDEs, providing a powerful toolset for researchers working in applied mathematics, physics, and related fields.

6.4 | Limitations and Future Research

While this study provides a comprehensive analysis of the J-M equation through Lie symmetry methods and the derivation of conservation laws, the absence of numerical simulations or experimental validation limits the current findings. Future research should include numerical simulations using finite difference or Runge-Kutta methods to validate the derived conservation laws and compare them with analytical solutions. This step would confirm the accuracy and practical applicability of the findings, ensuring their relevance to real-world scenarios.

7 | Conclusion

In this study, we have conducted a comprehensive analysis of the J-M equation using the Lie symmetry method. The identification of symmetry groups and the development of an optimal system of 1D subalgebras allowed us to reduce the complexity of the original PDE to simpler ODEs. This process not only facilitated the derivation of exact solutions but also provided a deeper understanding of the equation’s symmetry structure.

One of the novel contributions of this work is the application of both the multiplier method and symbolic computation to derive conservation laws for *Eq. (1)* systematically. These conservation laws are critical for understanding the integrability and solvability of the equation, and they have potential applications in modeling wave phenomena, fluid dynamics, and plasma physics.

This study presents a novel approach to analyzing *Eq. (1)* by integrating Lie symmetry analysis with the multiplier method and symbolic computation. The identification of an optimal system of Lie subalgebras and the derivation of a wider set of conservation laws offer new insights into the equation's structure, making significant contributions to the field of nonlinear dynamics. These findings open avenues for further research into the practical applications of the J-M equation in modelling complex physical systems.

The practical significance of these findings is evident in applications such as modeling wave phenomena in fluid dynamics, where the conservation laws can help predict soliton behavior, which is crucial for coastal engineering. Additionally, in plasma physics, the derived conservation laws provide insights into wave stability, aiding in optimizing plasma confinement in fusion reactors. The study's theoretical contributions also extend to advancing the understanding of integrable systems and providing a systematic approach to analyzing nonlinear PDEs across various fields of applied mathematics and physics.

The study's approach demonstrates how Lie symmetry analysis can be an effective tool for addressing challenges in nonlinear dynamics and PDE theory. The findings of this research extend the existing knowledge of the J-M equation and open avenues for future research. Future studies could explore higher-dimensional symmetries, apply this methodology to more complex or multi-component systems, and conduct numerical simulations to validate the analytical results obtained here. The techniques developed in this study can also be adapted to analyze other nonlinear PDEs, contributing to broader applications in applied mathematics and physics.

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Author Contribution

All authors contributed equally to this work and have approved the final version of the manuscript.

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Data Availability

The datasets used in this study are not publicly available due to confidentiality reasons but can be requested from the corresponding author. Participants have only consented to the publication of aggregated data.

Conflicts of Interest

The authors affirm that there are no conflicts of interest concerning the publication of this paper.

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