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Research article

Oscillatory behavior of solutions of third order semi-canonical dynamic equations on time scale

Ahmed M. Hassan[1,*], **Clemente Cesarano**[2], **Sameh S. Askar**[3] **and Ahmad M. Alshamrani**[3]

[1] Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt

[2] Section of Mathematics, International Telematic University Uninettuno, CorsoVittorio Emanuele II, 39, 00186 Roma, Italy

[3] Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: ahmed.mohamed@fsc.bu.edu.eg.

Abstract: This paper investigates the oscillatory behavior of nonlinear third-order dynamic equations on time scales. Our main approach is to transform the equation from its semi-canonical form into a more tractable canonical form. This transformation simplifies the analysis of oscillation behavior and allows us to derive new oscillation criteria. These criteria guarantee that all solutions to the equation oscillate. Our results extend and improve upon existing findings in the literature, particularly for the special cases where $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. Additionally, we provide illustrative examples to demonstrate the practical application of the developed criteria.

Keywords: third order; oscillatory; semi-canonical; non-canonical; canonical
Mathematics Subject Classification: 34K11, 34C10, 34N05

1. Introduction

A recently developed theory gaining significant interest is time scale $\mathbb{T}$, introduced by Stefan Hilger to bridge the gap between continuous and discrete analysis [1]. In simpler terms, it aims to unify the study of differential equations (governing continuous change) and difference equations (modeling discrete jumps) [2].

The core concept of time scale $\mathbb{T}$ involves defining a time domain as any non-empty, closed set of real numbers. The familiar differential and difference equations emerge as special cases when the time scale is the set of all real numbers or integers, respectively.

To gain a comprehensive understanding, it is necessary to review some basic concepts of time scale theory. The forward and backward jump operators $\sigma,\ \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(\ell) = \inf\{s \in \mathbb{T} \mid s > \ell\} \quad \text{and} \quad \rho(\ell) = \sup\{s \in \mathbb{T} \mid s < \ell\},$$

(supplemented by $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$). A point $\ell \in \mathbb{T}$ is called right-scattered, right-dense, left-scattered or left-dense if $\sigma(\ell) > \ell$, $\sigma(\ell) = \ell$, $\rho(\ell) < \ell$, $\rho(\ell) = \ell$ holds, respectively. The set $\mathbb{T}^\kappa$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum; otherwise, it is $\mathbb{T}$ without this left-scattered maximum. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(\ell) = \sigma(\ell) - \ell$. Hence, the graininess function is constant 0 if $\mathbb{T} = \mathbb{R}$, while it is constant ℓ for $\mathbb{T} = \mathbb{Z}$. However, a time scale $\mathbb{T}$ could have nonconstant graininess. A function $h : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous and is written $h \in C_{rd}(\mathbb{T}, \mathbb{R})$, provided that h is continuous at right dense points and at left dense points in $\mathbb{T}$, left hand limits exist, and are finite. We say that $h : \mathbb{T} \to \mathbb{R}$ is differentiable at $\ell \in \mathbb{T}$ whenever

$$h^\Delta := \lim_{s \to \ell} \frac{h(\ell) - h(s)}{\ell - s}$$

exists when $\sigma(\ell) = \ell$ (here by $s \to \ell$ it is understood that s approaches ℓ in the time scale), and when h is continuous at ℓ and $\sigma(\ell) > \ell$ it is

$$h^\Delta := \lim_{s \to \ell} \frac{h(\sigma(\ell)) - h(\ell)}{\mu(\ell)}.$$

The product and quotient rules [3, Theorem 1.20] for the derivative of the product hk and the quotient h/k of two differentiable functions h and k are as follows:

$$(hk)^\Delta(\ell) = h^\Delta(\ell)k(\ell) + h(\sigma(\ell))k^\Delta = h(\ell)k^\Delta(\ell) + h^\Delta(\ell)k(\sigma(\ell)), \tag{1.1}$$

$$\left(\frac{h}{k}\right)^\Delta(\ell) = \frac{h^\Delta(\ell)k(\ell) - h(\ell)k^\Delta(\ell)}{k(\ell)k(\sigma(\ell))}. \tag{1.2}$$

The chain rule [3, Theorem 1.90] for the derivative of the composite function $h \circ k$ of a continuously differentiable function $h : \mathbb{R} \to \mathbb{R}$ and a (delta) differentiable function $k : \mathbb{T} \to \mathbb{R}$ results in

$$(h \circ k)^\Delta = \left\{ \int_0^1 h'(k + s\mu k^\Delta) ds \right\} g^\Delta. \tag{1.3}$$

A function $h : \mathbb{T} \to \mathbb{R}$ is called *rd*-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by

$$\mathrm{C_{rd}} = \mathrm{C_{rd}}(\mathbb{T}) = \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}).$$

The set of functions $h : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$\mathrm{C^1_{rd}} = \mathrm{C^1_{rd}}(\mathbb{T}) = \mathrm{C^1_{rd}}(\mathbb{T}, \mathbb{R}).$$

Finally, if $h : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $h^\sigma : \mathbb{T} \to \mathbb{R}$ by $h^\sigma(\ell) = h(\sigma(\ell))$ for all $\ell \in \mathbb{T}$.

Bohner and Peterson's book [3] provides a comprehensive overview and organization of this new calculus. Beyond these basic cases, numerous other time scales can be defined, leading to a wealth of applications. One such application is the study of population dynamic models, as explored in [4]. To delve deeper into the theory, readers can consult the referenced papers [5, 6] and monographs [3, 7].

Recent years have seen a surge in research on the oscillation and non-oscillation of solutions to dynamic equations on time scales. For further exploration, readers can refer to the references provided [8, 9, 13–19].

The present paper investigates the asymptotic behavior of solutions to the semi-canonical third dynamic equation

$$\left(a_2(\ell)(a_1(\ell)x^{\Delta}(\ell))^{\Delta}\right)^{\Delta} + p(\ell)x^{\gamma}(\delta(\ell)) = 0, \quad \ell \in [\ell_0, \infty)_{\mathbb{T}}, \tag{1.4}$$

where γ is the ratio of positive odd integers.

In this paper, we consider the following conditions:

(i) $a_1(\ell) \in C^2_{rd}([\ell_0, \infty)_{\mathbb{T}}, (0, \infty))$, $a_2(\ell) \in C^1_{rd}([\ell_0, \infty)_{\mathbb{T}}, (0, \infty))$, $p(\ell) \in C([\ell_0, \infty)_{\mathbb{T}}, (0, \infty))$ and Eq (1.4) is in semi-canonical form, i.e.,

$$\int_{\ell_0}^{\infty} \frac{\Delta s}{a_2(s)} = \infty \text{ and } \int_{\ell_0}^{\infty} \frac{\Delta s}{a_1(s)} < \infty; \tag{1.5}$$

(ii) $\delta \in C^1_{rd}([\ell_0, \infty)_{\mathbb{T}})$, $\delta^{\Delta}(\ell) \geq 0$, and $\lim_{\ell\to\infty} \delta(\ell) = \infty$.

Let us recall that a solution of Eq (1.4) is a nontrivial real-valued function x satisfying the equation for $\ell \geq \ell_x$ for some $\ell_x \geq \ell_{x_0}$ such that $x \in C^1([\ell_x, \infty)_{\mathbb{T}}, \mathbb{R})$, $a_1 x^{\Delta} \in C^1([\ell_x, \infty)_{\mathbb{T}}, \mathbb{R})$, and $a_2(a_1(x^{\Delta}))^{\Delta} \in C^1([\ell_x, 1)_{\mathbb{T}}, \mathbb{R})$. We exclude solutions that vanish identically in some neighborhood of infinity, assuming that such solutions exist for Eq (1.4). A solution $x(\ell)$ of Eq (1.4) is termed oscillatory if it exhibits arbitrarily large zeros on $[\ell_x, \infty)_{\mathbb{T}}$; otherwise, it is classified as non-oscillatory.

The study of oscillatory behavior in Eq (1.4) often hinges on its form. Equation (1.4) is in canonical form if

$$\int_{\ell_0}^{\infty} \frac{\Delta s}{a_1(s)} = \int_{\ell_0}^{\infty} \frac{\Delta s}{a_2(s)} = \infty,$$

and it is in non-canonical form if

$$\int_{\ell_0}^{\infty} \frac{\Delta s}{a_1(s)} < \infty \text{ and } \int_{\ell_0}^{\infty} \frac{\Delta s}{a_2(s)} < \infty.$$

If either

$$\int_{\ell_0}^{\infty} \frac{\Delta s}{a_1(s)} < \infty \text{ and } \int_{\ell_0}^{\infty} \frac{\Delta s}{a_2(s)} = \infty, \tag{S_1}$$

or

$$\int_{\ell_0}^{\infty} \frac{\Delta s}{a_1(s)} = \infty \text{ and } \int_{\ell_0}^{\infty} \frac{\Delta s}{a_2(s)} < \infty, \tag{S_2}$$

then we will say that (1.4) is in semicanonical form.

The groundwork for studying third-order dynamic equations on general time scales was laid by Erbe et al. [13], who focused on equations of the form

$$(a_2(\ell)((a_1(\ell)x^{\Delta}(\ell))^{\Delta}))^{\Delta} + p(\ell)f(x(\ell)) = 0, \quad \ell \in [\ell_0, \infty)_{\mathbb{T}}, \tag{1.6}$$

where $a_1, a_2, p \in C_{rd}(\ell_0, \infty)_{\mathbb{T}}$, $f \in C(\mathbb{R}, \mathbb{R})$ R is continuous and satisfies $uf(u) > 0$ for $u \neq 0$. Additionally, for each $k > 0$, there exists $M = M_k > 0$ such that $f(u)/u \geq M$, $|u| \geq k$. Using the

Riccati transformation technique, they established sufficient conditions that guarantee every solution to this equation either oscillates or converges to zero.

Building on Erbe et al.'s work [13], Hassan [14] investigated a more general form of the third-order equation

$$(a_2(\ell)((a_1(\ell)x^{\Delta}(\ell))^{\Delta})^{\alpha})^{\Delta} + f(\ell, x(\delta(\ell))) = 0, \quad \ell \in [\ell_0, \infty)_{\mathbb{T}}, \tag{1.7}$$

where $\alpha \geq 1$ and $\delta(\ell) \leq \ell$, in the canonical form.

In the particular case of $\mathbb{T} = \mathbb{R}$ and $\gamma = 1$, Chatzarakis et al. [11] established new oscillation criteria for the differential equation

$$(a_2(\ell)(a_1(\ell)x'(\ell))')' + p(\ell)x(\delta(\ell)) = 0,$$

in the canonical form. Recently, techniques have been developed to study the oscillatory behavior of solutions to third-order equations. Moaaz et al. [21, 22] extended the improved methods used in studying second-order equations [23, 24]. The development of oscillation criteria for delay differential equations of odd orders can also be observed through the works [25, 26].

Our literature review indicates a scarcity of research on the oscillatory behavior of solutions to Eq (1.4) when it takes the semi-canonical form (S_1). This paper tackles Eq (1.4) in its less-studied semi-canonical form. We begin by transforming it into the more common canonical form. This transformation allows us to then establish new criteria for determining when solutions to Eq (1.4) oscillate.

2. Main results

To enhance readability, we'll use the following symbols:

$$A(\ell) := \int_{\ell}^{\infty} \frac{\Delta s}{a_1(s)}, \quad a(\ell) := a_1(\ell)A(\ell)A^{\sigma}(\ell), \quad r(\ell) := \frac{a_2(\ell)}{A^{\sigma}(\ell)},$$

$$P(\ell) := p(\ell)A^{\gamma}(\ell), \quad \phi(\ell) := \int_{\ell_1}^{\ell} \frac{\Delta s}{r(s)}, \quad \psi(\ell) := \int_{\ell_1}^{\ell} \frac{\phi(s)\Delta s}{a(s)}, \text{ and } z(\ell) := \frac{x(\ell)}{A(\ell)}.$$

Lemma 2.1. *[27] Assume that x is an eventually positive solution of* (1.4) *satisfying* (1.5)*. Then there exists $\ell_1 \in [\ell_0, \infty)_{\mathbb{T}}$ such that x satisfies one of the following three cases:*

(I) $x^{\Delta} > 0$, $(a_1(\ell)(x^{\Delta}(\ell)))^{\Delta} > 0$, $(a_2(\ell)((a_1(\ell)(x^{\Delta}(\ell)))^{\Delta})^{\Delta} < 0$*;*
(II) $x^{\Delta} < 0$, $(a_1(\ell)(x^{\Delta}(\ell)))^{\Delta} > 0$, $(a_2(\ell)((a_1(\ell)(x^{\Delta}(\ell)))^{\Delta})^{\Delta} < 0$*;*
(III) $x^{\Delta} < 0$, $(a_1(\ell)(x^{\Delta}(\ell)))^{\Delta} < 0$, $(a_2(\ell)((a_1(\ell)(x^{\Delta}(\ell)))^{\Delta})^{\Delta} < 0$.

Theorem 2.1. *Assume that*

$$\int_{\ell}^{\infty} \frac{\Delta s}{r(s)} = \infty. \tag{2.1}$$

Then the semi-canonical dynamic Eq (1.4) *has a solution $x(\ell)$ if and only if the corresponding canonical equation*

$$(r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta})^{\Delta} + P(\ell)z^{\gamma}(\delta(\ell)) = 0, \tag{2.2}$$

admits the solution $z(\ell) = \frac{x(\ell)}{A(\ell)}$.

Proof. Referring back to $\sigma(\ell)$ as the forward jump operator and performing differentiation yields

$$
\begin{aligned}
\frac{a_2(\ell)}{A^\sigma(\ell)}\left(a_1(\ell)A(\ell)A^\sigma(\ell)\left(\frac{x(\ell)}{A(\ell)}\right)^\Delta\right)^\Delta &= \frac{a_2(\ell)}{A^\sigma(\ell)}\left\{a_1(\ell)A(\ell)A^\sigma(\ell)\left[\frac{x^\Delta(\ell)A(t)-x(t)A^\Delta(t)}{A(t)A^\sigma(\ell)}\right]\right\}^\Delta \\
&= \frac{a_2(\ell)}{A^\sigma(\ell)}\left\{a_1(\ell)x^\Delta(\ell)A(\ell)-a_1(\ell)x(\ell)A^\Delta(\ell)\right\}^\Delta \\
&= \frac{a_2(\ell)}{A^\sigma(\ell)}\left\{\left(a_1(\ell)x^\Delta(\ell)\right)^\Delta A^\sigma(\ell)+a_1(\ell)x^\Delta(\ell)A^\Delta(\ell)+x^\Delta(\ell)\right\} \\
&= a_2(\ell)\left(a_1(\ell)x^\Delta(\ell)\right)^\Delta. \qquad (2.3)
\end{aligned}
$$

From (2.1), we have

$$
\int_\ell^\infty \frac{A^\sigma(s)}{a_2(s)}\Delta s=\infty, \qquad (2.4)
$$

and

$$
\int_\ell^\infty \frac{\Delta s}{a_1(s)A(s)A^\sigma(s)}=\int_\ell^\infty\left(\frac{1}{A(s)}\right)^\Delta \Delta s=\lim_{\ell\to\infty}\left(\frac{1}{A(\ell)}-\frac{1}{A(\ell_0)}\right)=\infty. \qquad (2.5)
$$

Combining (2.3) with (1.4), we obtain

$$
\begin{aligned}
(a_2(\ell)(a_1(\ell)x^\Delta(\ell))^\Delta)^\Delta+p(\ell)x^\gamma(\delta(\ell)) &=0 \\
\left(\frac{a_2(\ell)}{A^\sigma(\ell)}\left(a_1(\ell)A(\ell)A^\sigma(\ell)\left(\frac{x(\ell)}{A(\ell)}\right)^\Delta\right)^\Delta\right)^\Delta+p(\ell)A^\gamma(\ell)\frac{x^\gamma(\delta(\ell))}{A^\gamma(\ell)} &=0 \\
(r(\ell)(a(\ell)z^\Delta(\ell))^\Delta)^\Delta+P(\ell)z^\gamma(\delta(\ell)) &=0. \qquad (2.6)
\end{aligned}
$$

It is clear that $\frac{x(\ell)}{A(\ell)}$ is a solution of (2.6). Moreover, considering (2.4) and (2.5), it is apparent that Eq (2.6) is in canonical form and from [28] this canonical from is unique. □

Theorem 2.1 significantly streamlines the analysis of Eq (1.4) by reducing it to the scope of (2.2), thereby directing our focus towards only two classifications of solutions that ultimately exhibit positivity, i.e., either

$$
z(\ell)>0,\ \ a(\ell)z^\Delta(\ell)<0,\ \ r(\ell)(a(\ell)z^\Delta(\ell)>0,\ (r(\ell)(a(\ell)z^\Delta(\ell))^\Delta)^\Delta<0,
$$

and in this case, we denote $z\in\aleph_0$ or

$$
z(\ell)>0,\ \ a(\ell)z^\Delta(\ell)>0,\ \ r(\ell)(a(\ell)z^\Delta(\ell))^\Delta>0,\ (r(\ell)(a(\ell)z^\Delta(\ell))^\Delta)^\Delta<0,
$$

and for this characteristic, we indicate that $z\in\aleph_2$.

Theorem 2.2. *Let* $\gamma\geq 1$ *and* (2.1) *hold. Suppose that*

$$
\begin{aligned}
\limsup_{\ell\to\infty}\left\{\frac{1}{\phi^\gamma(\delta(\ell))}\int_{\ell_1}^{\delta(\ell)}\phi(\sigma(s))P(s)\psi^\gamma(s)\Delta s+\frac{1}{\phi^{\gamma-1}(\delta(\ell))}\int_{\delta(\ell)}^{\ell}\frac{P(s)\psi^\gamma(s)}{\phi^\gamma(\delta(s))}\Delta s\right. \\
\left.+\phi(\delta(\ell))\int_\ell^\infty P(s)\psi^\gamma(s)\Delta s\right\}=\begin{cases}\infty, & \gamma>1,\\ 1, & \gamma=1,\end{cases} \qquad (2.7)
\end{aligned}
$$

and

$$\int_{\ell_0}^{\infty}\frac{1}{a(u)}\int_{u}^{\infty}\frac{1}{r(v)}\int_{v}^{\infty}P(s)\Delta s\Delta v\Delta u=\infty. \tag{2.8}$$

Then every non-oscillatory solution $z(\ell)$ *of* (1.4) *satisfies* $\lim_{\ell\to\infty}\frac{x(\ell)}{A(\ell)}=0$.

Proof. Let $x(\ell)$ be a non-oscillatory solution of Eq (1.4), where $x(\ell)>0$, and $x(\delta(\ell))>0$ for $\ell\geq\ell_1$ for some $\ell_1\geq\ell_0$. According to Theorem 2.1, the corresponding function $z(\ell)=\frac{x(\ell)}{A(\ell)}$ is a positive solution of (2.2), implying that either $z\in\aleph_0$ or $z\in\aleph_2$ for $\ell\geq\ell_1$.

Let us examine the case where $z\in\aleph_2$. In this case, we observe that

$$\begin{aligned}a(\ell)z^{\Delta}(\ell)&\geq\int_{\ell_1}^{\ell}r^{-1}(s)r(s)(a(s)z^{\Delta}(s))^{\Delta}\Delta s\\&\geq r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta}\int_{\ell_1}^{\ell}\frac{\Delta s}{r(s)}\\&\geq r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta}\phi(\ell).\end{aligned}$$

Hence ,

$$\begin{aligned}\left(\frac{a(\ell)z^{\Delta}(\ell)}{\phi(\ell)}\right)^{\Delta}&=\frac{\phi(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta}-(a(\ell)z^{\Delta}(\ell))\phi^{\Delta}(\ell)}{\phi(\ell)\phi^{\sigma}(\ell)}\\&=\frac{\phi(\ell)r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta}-(a(\ell)z^{\Delta}(\ell))}{r(\ell)\phi(\ell)\phi^{\sigma}(\ell)}\leq 0.\end{aligned} \tag{2.9}$$

Consequently, it can be inferred from (2.9) that

$$z(\ell)\geq\int_{\ell_1}^{\ell}z^{\Delta}(s)\Delta s=\int_{\ell_1}^{\ell}\frac{a(s)z^{\Delta}(s)}{\phi(s)}\frac{\phi(s)}{a(s)}\Delta s\geq\frac{a(\ell)z^{\Delta}(\ell)}{\phi(\ell)}\psi(\ell). \tag{2.10}$$

Combining (2.10) with (2.2), we see that $\frac{a(\ell)z^{\Delta}(\ell)}{\phi(\ell)}\psi(\ell)$ is a positive solution to the dynamic inequality

$$\left(r(\ell)\chi^{\Delta}(\ell)\right)^{\Delta}+\frac{P(\ell)\psi^{\gamma}(\ell)}{\phi^{\gamma}(\delta(\ell))}\chi^{\gamma}(\delta(\ell))\leq 0, \tag{2.11}$$

where $\chi(\ell):=a(\ell)z^{\Delta}(\ell)$. Integration (2.11) from ℓ to ∞ and considering the nonincreasing nature of $\chi(\ell)/\phi(\ell)$, we obtain

$$\chi^{\Delta}(\ell)\geq\frac{1}{r(\ell)}\int_{\ell}^{\infty}\frac{P(s)\psi^{\gamma}(s)}{\phi^{\gamma}(\delta(s))}\chi^{\gamma}(\delta(s))\Delta s.$$

Therefore,

$$\begin{aligned}\chi(\ell)&\geq\int_{\ell_1}^{\ell}\frac{1}{r(s)}\int_{s}^{\infty}\frac{P(u)\psi^{\gamma}(u)}{\phi^{\gamma}(\delta(u))}\chi^{\gamma}(\delta(u))\Delta u\Delta s\\&=\int_{\ell_1}^{\ell}\frac{1}{r(s)}\int_{s}^{\ell}\frac{P(u)\psi^{\gamma}(u)}{\phi^{\gamma}(\delta(u))}\chi^{\gamma}(\delta(u))\Delta u\Delta s+\int_{\ell_1}^{\ell}\frac{1}{r(s)}\int_{\ell}^{\infty}\frac{P(u)\psi^{\gamma}(u)}{\phi^{\gamma}(\delta(u))}\chi^{\gamma}(\delta(u))\Delta u\Delta s.\end{aligned} \tag{2.12}$$

Integrating by parts, we obtain

$$\chi(\ell) \geq \int_{\ell_1}^{\ell} \phi(\sigma(s)) \frac{P(s)\psi^{\gamma}(s)}{\phi^{\gamma}(\delta(s))} \chi^{\gamma}(\delta(s))\Delta s + \phi(\ell) \int_{\ell}^{\infty} \frac{P(s)\psi^{\gamma}(s)}{\phi^{\gamma}(\delta(s))} \chi^{\gamma}(\delta(s))\Delta s. \tag{2.13}$$

It follows that

$$\begin{aligned}
\chi(\delta(\ell)) &\geq \int_{\ell_1}^{\delta(\ell)} \phi(\sigma(s)) \frac{P(s)\psi^{\gamma}(s)}{\phi^{\gamma}(\delta(s))} \chi^{\gamma}(\delta(s))\Delta s + \phi(\delta(\ell)) \int_{\delta(\ell)}^{\infty} \frac{P(s)\psi^{\gamma}(s)}{\phi^{\gamma}(\delta(s))} \chi^{\gamma}(\delta(s))\Delta s \\
&= \int_{\ell_1}^{\delta(\ell)} \phi(\sigma(s)) \frac{P(s)\psi^{\gamma}(s)}{\phi^{\gamma}(\delta(s))} \chi^{\gamma}(\delta(s))\Delta s + \phi(\delta(\ell)) \int_{\delta(\ell)}^{\ell} \frac{P(s)\psi^{\gamma}(s)}{\phi^{\gamma}(\delta(s))} \chi^{\gamma}(\delta(s))\Delta s \\
&\quad + \phi(\delta(\ell)) \int_{\ell}^{\infty} \frac{P(s)\psi^{\gamma}(s)}{\phi^{\gamma}(\delta(s))} \chi^{\gamma}(\delta(s))\Delta s.
\end{aligned} \tag{2.14}$$

Utilizing the monotonicity characteristics of $\chi(\ell)$ and $\chi(\ell)/\phi(\ell)$, we have $\chi(\delta(\ell)) \leq \chi(\delta(s))$ and $\frac{\chi(\delta(s))}{\phi(\delta(s))} \geq \frac{\chi(\delta(\ell))}{\phi(\delta(\ell))}$ for $s \geq \ell$, hence (2.14) takes the form

$$\begin{aligned}
\chi(\delta(\ell)) &\geq \frac{\chi^{\gamma}(\delta(\ell))}{\phi^{\gamma}(\delta(\ell))} \int_{\ell_1}^{\delta(\ell)} \phi(\sigma(s))P(s)\psi^{\gamma}(s)\Delta s + \frac{\chi^{\gamma}(\delta(\ell))}{\phi^{\gamma-1}(\delta(\ell))} \int_{\delta(\ell)}^{\ell} \frac{P(s)\psi^{\gamma}(s)}{\phi^{\gamma}(\delta(s))}\Delta s \\
&\quad + \phi(\delta(\ell))\chi^{\gamma}(\delta(\ell)) \int_{\ell}^{\infty} P(s)\psi^{\gamma}(s)\Delta s,
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
\chi^{1-\gamma}(\delta(\ell)) &\geq \frac{1}{\phi^{\gamma}(\delta(\ell))} \int_{\ell_1}^{\delta(\ell)} \phi(\sigma(s))P(s)\psi^{\gamma}(s)\Delta s + \frac{1}{\phi^{\gamma-1}(\delta(\ell))} \int_{\delta(\ell)}^{\ell} \frac{P(s)\psi^{\gamma}(s)}{\phi^{\gamma}(\delta(s))}\Delta s \\
&\quad + \phi(\delta(\ell)) \int_{\ell}^{\infty} P(s)\psi^{\gamma}(s)\Delta s.
\end{aligned} \tag{2.16}$$

This contradicts (2.7). Subsequently, let us assume that $z \in \aleph_0$. Then $\lim_{\ell\to\infty} z(\ell) = k \geq 0$, and we propose that $k = 0$. If not, it would imply $z(\ell) \geq k > 0$. Integrating (2.2) from ℓ to ∞ yields

$$r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta} \geq \int_{\ell}^{\infty} P(s)z^{\gamma}(\delta(s))\Delta s \geq k^{\gamma} \int_{\ell}^{\infty} P(s)\Delta s.$$

Therefore,

$$-a(\ell)z^{\Delta}(\ell) \geq k^{\gamma} \int_{\ell}^{\infty} \frac{1}{r(u)} \int_{u}^{\infty} P(s)\Delta s\Delta u,$$

and

$$z(\ell_1) \geq k^{\gamma} \int_{\ell_1}^{\infty} \frac{1}{a(u)} \int_{u}^{\infty} \frac{1}{r(v)} \int_{v}^{\infty} P(s)\Delta s\Delta v\Delta u.$$

This leads to a contradiction to (2.8). Thus, we conclude: $\lim_{\ell\to\infty} z(\ell) = \lim_{\ell\to\infty} \frac{x(\ell)}{A(\ell)} = 0$, and, the proof of the theorem is complete. □

Theorem 2.3. *Let* $0 < \gamma < 1$ *and* (2.1) *hold. If* (2.8) *and*

$$\limsup_{\ell\to\infty}\left\{\frac{1}{\phi(\delta(\ell))}\int_{\ell_1}^{\delta(\ell)}\phi(\sigma(s))P(s)\psi^{\gamma}(s)\Delta s+\int_{\delta(\ell)}^{\ell}\frac{P(s)\psi^{\gamma}(s)}{\phi^{\gamma}(\delta(s))}\Delta s\right.$$
$$\left.+\phi^{\gamma}(\delta(\ell))\int_{\ell}^{\infty}P(s)\psi^{\gamma}(s)\Delta s\right\}=\infty \tag{2.17}$$

hold, then every non-oscillatory solutionz(ℓ) *of* (1.4) *satisfies* $\lim_{\ell\to\infty}\frac{x(\ell)}{A(\ell)}=0$.

Proof. Let $x(\ell)$ be a non-oscillatory solution of Eq (1.4), where $x(\ell) > 0$, and $x(\delta(\ell)) > 0$ for $\ell \geq \ell_1$ for some $\ell_1 \geq \ell_0$. According to Theorem 2.1, the corresponding function $z(\ell) = \frac{x(\ell)}{A(\ell)}$ is a positive solution of (2.2), implying that either $z \in \aleph_0$ or $z \in \aleph_2$ for $\ell \geq \ell_1$.

First, let us assume that $z \in \aleph_2$. Proceeding similarly to the proof of Theorem 2.2, we arrive at (2.15). Dividing (2.16) by $\phi^{1-\gamma}(\delta(\ell)$, we obtain

$$\left(\frac{\chi(\delta(\ell))}{\phi(\delta(\ell))}\right)^{1-\gamma}\geq\frac{1}{\phi(\delta(\ell))}\int_{\ell_1}^{\delta(\ell)}\phi(\sigma(s))P(s)\psi^{\gamma}(s)\Delta s+\int_{\delta(\ell)}^{\ell}\frac{P(s)\psi^{\gamma}(s)}{\phi^{\gamma}(\delta(s))}\Delta s$$
$$+\phi^{\gamma}(\delta(\ell))\int_{\ell}^{\infty}P(s)\psi^{\gamma}(s)\Delta s. \tag{2.18}$$

In view of the decreasing nature of $\chi(\delta(\ell))/\phi(\delta(\ell))$ and the fact that $0 < \gamma < 1$, there exists a constant $C > 0$ such that

$$\left(\frac{\chi(\delta(\ell))}{\phi(\delta(\ell))}\right)^{1-\gamma}\leq C.$$

Taking the lim sup as $\ell \to \infty$, we establish a contradiction to (2.18), and consequently, $z \notin \aleph_2$.

Subsequently, let us assume that $z \in \aleph_0$. Proceeding similarly to the proof of Theorem 2.2, it becomes evident that condition (2.8) once more leads to the conclusion that $\lim_{\ell\to\infty}\frac{x(\ell)}{A(\ell)} = 0$. This completes the proof. □

Theorem 2.4. *Suppose that conditions (i), (ii), and* $\delta^{\Delta}(\ell) > 0$ *are satisfied on* $[\ell_0,\infty)_{\mathbb{T}}$, $\gamma \leq 1$, *and there exists a function* $\xi(\ell)$ *such that*

$$\xi^{\Delta}(\ell)\geq 0,\ \ \xi(\ell)>\ell\ ,\text{and}\ \theta(\ell)=\delta(\xi(\xi(\ell)))<\ell. \tag{2.19}$$

If

$$\liminf_{\ell\to\infty}\int_{\delta(\ell)}^{\ell}P(s)\psi^{\gamma}(\delta(s))\Delta s\begin{cases}=\infty, & \gamma<1,\\ >1/e, & \gamma=1,\end{cases} \tag{2.20}$$

and

$$\liminf_{\ell\to\infty}\int_{\theta(\ell)}^{\ell}\left(\frac{1}{a(s)}\int_{s}^{\xi(\ell)}\frac{1}{r(u)}\int_{u}^{\xi(u)}P(v)\Delta v\Delta u\right)\Delta s\begin{cases}=\infty, & \gamma<1,\\ >1/e, & \gamma=1,\end{cases} \tag{2.21}$$

for all $\ell_1 \geq \ell_0$, *then Eq* (1.4) *is oscillatory.*

Proof. Let $x(\ell)$ be a non-oscillatory solution of Eq (1.4), where $x(\ell) > 0$, and $x(\delta(\ell)) > 0$ for $\ell \geq \ell_1$ for some $\ell_1 \geq \ell_0$. According to Theorem 2.1, the corresponding function $z(\ell) = \frac{x(\ell)}{A(\ell)}$ is a positive solution of (2.2), implying that either $z \in \aleph_0$ or $z \in \aleph_2$ for $\ell \geq \ell_1$. Assuming that $z(\ell) \in \aleph_2$, we have

$$a(\ell)z^{\Delta}(\ell) \geq \int_{\ell_1}^{\ell} r^{-1}(s)r(s)(a(s)z^{\Delta}(s))^{\Delta}\Delta s \geq r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta}\phi(\ell).$$

It follows that

$$z^{\Delta}(\ell) \geq \frac{r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta}\phi(\ell)}{a(\ell)}. \tag{2.22}$$

Integrating the above inequality from ℓ_2 to ℓ, we obtain

$$\begin{aligned} z(\ell) &\geq \int_{\ell_2}^{\ell} \frac{r(s)(a(s)z^{\Delta}(s))^{\Delta}\phi(s)}{a(s)}\Delta s \\ &\geq r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta} \int_{\ell_2}^{\ell} \frac{\phi(s)}{a(s)}\Delta s \\ &= r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta}\psi(\ell). \end{aligned} \tag{2.23}$$

There exists $\ell_3 \geq \ell_2$ such that $\delta(\ell) \geq \ell_2$ for all $\ell \geq \ell_3$. Then, we have

$$z(\delta(\ell)) \geq r(\delta(\ell))(a(\delta(\ell))z^{\Delta}(\delta(\ell)))^{\Delta}\psi(\delta(\ell)), \quad \text{for all} \quad \ell \geq \ell_3.$$

Combining this with (2.2) yields

$$Y^{\Delta}(\ell) + P(\ell)\psi^{\gamma}(\delta(\ell))Y^{\gamma}(\delta(\ell)) \leq 0, \quad \text{for} \quad \ell \geq \ell_3, \tag{2.24}$$

where $Y(\ell) := r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta}$. Integrating (2.24) from $\delta(\ell)$ to ℓ, we have

$$\begin{aligned} Y(\delta(\ell)) &\geq Y(\delta(\ell)) - Y(\ell) \\ &\geq Y^{\gamma}(\delta(\ell)) \int_{\delta(\ell)}^{\ell} P(s)\psi^{\gamma}(\delta(s))\Delta s. \end{aligned} \tag{2.25}$$

Hence,

$$Y^{1-\gamma}(\delta(\ell)) \geq \int_{\delta(\ell)}^{\ell} P(s)\psi^{\gamma}(\delta(s))\Delta s \quad \text{for} \quad \ell \geq \ell_3.$$

According to [29, Theorem 1], we reach the intended contradiction.

Now, consider $z \in \aleph_0$. Integrating (2.2) from ℓ to $\xi(\ell)$, we obtain

$$\begin{aligned} r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta} &\geq \int_{\ell}^{\xi(\ell)} P(s)z^{\gamma}(\delta(s))\Delta s \\ &\geq z^{\gamma}(\delta(\xi(\ell))) \int_{\ell}^{\xi(\ell)} P(s)\Delta s, \end{aligned}$$

where $\theta(\ell) := \delta(\xi(\xi(\ell)))$. Consequently,

$$(a(\ell)z^{\Delta}(\ell))^{\Delta} \geq \frac{z^{\gamma}(\delta(\xi(\ell)))}{r(\ell)} \int_{\ell}^{\xi(\ell)} P(s)\Delta s. \tag{2.26}$$

Integrating (2.26) from ℓ to $\xi(\ell)$, we have

$$\begin{aligned}
-a(\ell)z^{\Delta}(\ell)\int_{\ell}^{\xi(\ell)}\frac{z^{\gamma}(\delta(\xi(s)))}{r(s)}\int_{s}^{\xi(s)}P(u)\Delta u\Delta s
&\geq z^{\gamma}(\delta(\xi(\xi(\ell))))\int_{\ell}^{\xi(\ell)}\frac{1}{r(s)}\int_{s}^{\xi(s)}P(u)\Delta u\Delta s\\
&= z^{\gamma}(\theta(\ell))\int_{\ell}^{\xi(\ell)}\frac{1}{r(s)}\int_{s}^{\xi(s)}P(u)\Delta u\Delta s. \qquad (2.27)
\end{aligned}$$

It follows that

$$z^{\Delta}(\ell)+\left(\frac{1}{a(\ell)}\int_{\ell}^{\xi(\ell)}\frac{1}{r(s)}\int_{s}^{\xi(s)}P(u)\Delta u\Delta s\right)z^{\gamma}(\theta(\ell))\leq 0. \qquad (2.28)$$

The remainder of the proof follows a similar pattern to the one described above and is therefore omitted. □

Theorem 2.5. *Let* (2.1) *hold. Assume that there exists a function* $\rho(\ell)\in C^{1}_{rd}(\mathbb{T},\mathbb{R}^{+})$ *, such that*

$$\limsup_{\ell\to\infty}\int_{\ell_0}^{\ell}\left(P(s)\rho(s)\frac{\psi(\delta(s))}{\phi(s)}\lambda^{\gamma-1}-\frac{\rho^{\Delta}(s)r(s)}{4\rho(s)}\right)\Delta s=\infty, \qquad (2.29)$$

and (2.8) *hold. Then every solution* $z(\ell)$ *of* (1.4) *is oscillatory or satisfies* $\lim_{\ell\to\infty}\frac{x(\ell)}{A(\ell)}=0$.

Proof. Let $x(\ell)$ be a non-oscillatory solution of Eq (1.4), where $x(\ell)>0$, and $x(\delta(\ell))>0$ for $\ell\geq\ell_1$ for some $\ell_1\geq\ell_0$. According to Theorem 2.1, the corresponding function $z(\ell)=\frac{x(\ell)}{A(\ell)}$ is a positive solution of (2.2), implying that either $z\in\aleph_0$ or $z\in\aleph_2$ for $\ell\geq\ell_1$.

Firstly, let us consider $z\in\aleph_2$; then we have $r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta}$ is decreasing, and moreover,

$$\begin{aligned}
r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta} &\geq \int_{\ell}^{\infty}P(s)z^{\gamma}(\delta(s))\Delta s\\
&\geq z^{\gamma}(\ell(s))\int_{\ell}^{\infty}P(s)\Delta s. \qquad (2.30)
\end{aligned}$$

Let us define the generalized Riccati substitution

$$\omega(\ell)=\rho(\ell)\frac{r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta}}{a(\ell)z^{\Delta}(\ell)}. \qquad (2.31)$$

Applying both the product rule and the quotient rule, we obtain

$$\begin{aligned}
\omega^{\Delta}(\ell) &= (r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta})^{\Delta}\left(\frac{\rho(\ell)}{a(\ell)z^{\Delta}(\ell)}\right)+(r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta})^{\sigma}\left(\frac{\rho(\ell)}{a(\ell)z^{\Delta}(\ell)}\right)^{\Delta}\\
&= (r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta})^{\Delta}\left(\frac{\rho(\ell)}{a(\ell)z^{\Delta}(\ell)}\right)\\
&\quad + (r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta})^{\sigma}\left(\frac{(a(\ell)z^{\Delta}(\ell))\rho^{\Delta}(\ell)-\rho(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta}}{(a(\ell)z^{\Delta}(\ell))(a(\ell)z^{\Delta}(\ell))^{\sigma}}\right)
\end{aligned}$$

$$\leq -P(\ell)\rho(\ell)\left(\frac{z^{\gamma}(\delta(\ell))}{a(\ell)z^{\Delta}(\ell)}\right)+\frac{\rho_{+}^{\Delta}(\ell)}{\rho(\sigma(\ell))}\omega(\sigma(\ell))$$
$$-\rho(\ell)(r(\ell)(a(\ell)z^{\Delta}(\ell))^{\Delta})^{\sigma}\frac{(a(\ell)z^{\Delta}(\ell))^{\Delta}}{(a(\ell)z^{\Delta}(\ell))(a(\ell)z^{\Delta}(\ell))^{\sigma}}. \tag{2.32}$$

Using the monotonicity of $r(\ell)(a(\ell)r^{\Delta}(\ell))^{\Delta}$ and $(a(\ell)r^{\Delta}(\ell))$, we have

$$\left(a(\ell)z^{\Delta}(\ell)\right)^{\Delta} \geq \frac{r(\sigma(\ell))}{r(\ell)}\left(a(\sigma(\ell))z^{\Delta}(\sigma(\ell))\right)^{\Delta}, \tag{2.33}$$

and

$$\frac{1}{a(\ell)z^{\Delta}(\ell)} \geq \frac{1}{a(\sigma(\ell))z^{\Delta}(\sigma(\ell))}. \tag{2.34}$$

Combining (2.33) and (2.34) with (2.32), we obtain

$$\omega^{\Delta}(\ell) \leq -P(\ell)\rho(\ell)\left(\frac{z^{\gamma}(\delta(\ell))}{a(\ell)z^{\Delta}(\ell)}\right)+\frac{\rho_{+}^{\Delta}(\ell)}{\rho(\sigma(\ell))}\omega(\sigma(\ell))-\frac{\rho(\ell)}{r(\ell)\rho^{2}(\sigma((\ell))}\omega^{2}(\sigma(\ell)). \tag{2.35}$$

From (2.9), (2.10), and the fact that $\delta(\ell) \leq \ell$, we have

$$\frac{z(\delta(\ell))}{a(\ell)z^{\Delta}(\ell)} \geq \frac{\psi(\delta(\ell))}{\phi(\ell)}, \quad \text{for } \ell \geq \ell_{3}. \tag{2.36}$$

This, together with (2.33), leads to

$$\omega^{\Delta}(\ell) \leq -P(\ell)\rho(\ell)\frac{\psi(\delta(\ell))}{\phi(\ell)}z^{\gamma-1}(\delta(\ell))+\frac{\rho_{+}^{\Delta}(\ell)}{\rho(\sigma(\ell))}\omega(\sigma(\ell))-\frac{\rho(\ell)}{r(\ell)\rho^{2}(\sigma((\ell))}\omega^{2}(\sigma(\ell)). \tag{2.37}$$

Since $z^{\Delta}(\ell) > 0$, then there exists a constant $\lambda > 0$ such that $z(\ell) \geq \lambda$ for $\ell \geq \ell_3$. Consequently, (2.37) can be expressed as

$$\omega^{\Delta}(\ell) \leq -P(\ell)\rho(\ell)\frac{\psi(\delta(\ell))}{\phi(\ell)}\lambda^{\gamma-1}+\frac{\rho_{+}^{\Delta}(\ell)}{\rho(\sigma(\ell))}\omega(\sigma(\ell))-\frac{\rho(\ell)}{r(\ell)\rho^{2}(\sigma((\ell))}\omega^{2}(\sigma(\ell))$$
$$\leq -P(\ell)\rho(\ell)\frac{\psi(\delta(\ell))}{\phi(\ell)}\lambda^{\gamma-1}+\frac{\rho^{\Delta}(\ell)r(\ell)}{4\rho(\ell)}. \tag{2.38}$$

Integrating both sides of (2.38) from $\ell_4 > \ell_3$ to ℓ, we obtain

$$\int_{\ell_4}^{\ell}\left(P(s)\rho(s)\frac{\psi(\delta(s))}{\phi(s)}\lambda^{\gamma-1}-\frac{\rho^{\Delta}(s)r(s)}{4\rho(s)}\right)\Delta s \leq \omega(\ell_4), \tag{2.39}$$

which contradicts (2.29). Now, assume that $z \in \aleph_0$. Proceeding similarly to the proof of Theorem 2.2, it becomes evident that condition (2.8) once more leads to the conclusion that $\lim_{\ell\to\infty}\frac{x(\ell)}{A(\ell)} = 0$. This completes the proof. □

Theorem 2.6. *Let* $\gamma = 1$ *,*

$$\limsup_{\ell\to\infty}\int_{\delta(\ell)}^{\ell}\left(\frac{1}{a(v)}\int_{v}^{\ell}\frac{1}{r(u)}\int_{u}^{\ell}P(s)\Delta s\Delta u\right)\Delta v > 1, \tag{2.40}$$

and assume that there exists a function $\rho(\ell) \in C^1_{rd}(\mathbb{T}, \mathbb{R}^+)$, such that

$$\limsup_{\ell\to\infty} \int_{\ell_0}^{\ell} \left(P(s)\rho(s)\frac{\psi(\delta(s))}{\phi(s)} - \frac{\rho^\Delta(s)r(s)}{4\rho(s)} \right) \Delta s = \infty. \tag{2.41}$$

Then every solution of (1.4) *is oscillatory.*

Proof. Let $x(\ell)$ be a non-oscillatory solution of Eq (1.4), where $x(\ell) > 0$, and $x(\delta(\ell)) > 0$ for $\ell \geq \ell_1$ for some $\ell_1 \geq \ell_0$. According to Theorem 2.1, the corresponding function $z(\ell) = \frac{x(\ell)}{A(\ell)}$ is a positive solution of (2.2), implying that either $z \in \aleph_0$ or $z \in \aleph_2$ for $\ell \geq \ell_1$. Assume $z(\ell) \in \aleph_0$. Integrating (2.2) from v to ℓ yields

$$r(v)(a(v)z^\Delta(v))^\Delta \geq \int_v^\ell P(s)z(\delta(s))\Delta s \geq z(\delta(\ell)) \int_v^\ell P(s)\Delta s.$$

Integrating again twice from v to ℓ, we obtain

$$z(v) \geq z(\delta(\ell)) \int_v^\ell \left(\frac{1}{a(v)} \int_v^\ell \frac{1}{r(u)} \int_u^\ell P(s)\Delta s \Delta u \right) \Delta v.$$

Replacing v with $\delta(\ell)$ leads to contradiction to (2.40). Hence, every positive solution $z(\ell)$ does not satisfy $\aleph_0$. Therefore, if (2.40) holds, then $z(\ell) \in \aleph_2$. Proceeding as in Theorem (2.5) with $\gamma = 1$, completes the proof. □

Example 2.1. *Consider the third order linear differential equation*

$$\left(\frac{1}{\ell} \left(\ell^2 \left(x'(\ell) \right) \right)' \right)' + \frac{p_0}{\sqrt{\ell}} x(\alpha\ell) = 0, \quad \ell \geq 1, \tag{2.42}$$

where p_0 is a constant and $\alpha \in (0,1)$. Here $a_2(\ell) = \frac{1}{\ell}$, $a_1(\ell) = \ell^2$, $p(\ell) = \frac{p_0}{\sqrt{\ell}}$ and $\delta(\ell) = \alpha\ell$. It is clear that (2.42) *is semi-canonical. Since $A(\ell) = \frac{1}{\ell}$, $a(\ell) = r(\ell) = 1$, and $P(\ell) = \frac{p_0}{\ell^{3/2}}$, the corresponding canonical equation is*

$$z'''(\ell) + \frac{p_0}{\ell^{3/2}} x(\alpha\ell) = 0. \tag{2.43}$$

It is clear that (2.1) *holds. Applying Theorem 2.6, we have*

$$\begin{aligned} \limsup_{\ell\to\infty} \int_{\delta(\ell)}^{\ell} \left(\frac{1}{a(v)} \int_v^\ell \frac{1}{r(u)} \int_u^\ell P(s)\Delta s \Delta u \right) \Delta v &= \limsup_{\ell\to\infty} \int_{\alpha\ell}^{\ell} \left(\int_v^\ell \int_u^\ell \frac{p_0}{s^{3/2}} ds du \right) dv \\ &= \lim_{\ell\to\infty} (-\alpha^2 + (1-2\alpha)\ \sqrt{\alpha} + 4)\ell^{3/2} > 1 \end{aligned}$$

and by choosing $\rho(\ell) = \ell$

$$\limsup_{\ell\to\infty} \int_{\ell_0}^{\ell} \left(P(s)\rho(s)\frac{\psi(\delta(s))}{\phi(s)} - \frac{\rho^\Delta(s)r(s)}{4\rho(s)} \right) \Delta s = \limsup_{\ell\to\infty} \int_{\ell_0}^{\ell} \left(\frac{p_0}{s^{3/2}} \frac{\alpha^2 s^3}{s} - \frac{1}{4s} \right) ds = \infty.$$

It follows that (2.42) *is oscillatory. Also, by Theorems 2.2 and 2.4, Eq* (2.42) *is oscillatory or* $\lim_{\ell\to\infty} \frac{x(\ell)}{A(\ell)} = 0$.

Remark 2.1. *It is worth noting that the existing results in [10, 30–32] cannot be directly applied to Eq* (2.42) *due to the fact that $a_1(\ell) \neq 1$.*

Example 2.2. *Consider the second order difference equation*

$$\Delta\left(\frac{1}{\ell+1}\Delta(\ell(\ell+1))\Delta x(\ell)\right) + p_0 x^{1/2}(\ell-2) = 0, \quad \ell \geq 1, \tag{2.44}$$

where p_0 *is a constant. Here* $a_2(\ell) = \frac{1}{\ell+1}$, $a_1(\ell) = \ell(\ell+1)$, $p(\ell) = p_0$ *and* $\delta(\ell) = \ell - 2$. *It is clear that* (2.44) *is semi-canonical. Since* $A(\ell) = \frac{1}{\ell}$, $a(\ell) = r(\ell) = 1$, *and* $P(\ell) = \frac{p_0}{\ell^{1/2}}$, *the corresponding canonical equation is*

$$\Delta(\Delta(\Delta(z(\ell)))) + \frac{p_0}{\ell^{1/2}} x(\ell-2) = 0. \tag{2.45}$$

It is clear that (2.1) *and* (2.8) *hold. Further,* (2.17) *becomes*

$$\limsup_{\ell\to\infty}\left\{\frac{1}{\ell-3}\sum_{1}^{\ell-2}(s+1)\frac{p_0}{s^{1/2}}\frac{s}{2} + \sum_{\ell-2}^{\ell}\frac{sp_0}{\sqrt{2s(s-2)}} + \sqrt{(\ell-2)}\sum_{\ell}^{\infty}\frac{p_0 s}{\sqrt{2s}}\right\} = \infty.$$

Hence, by Theorem 2.3, every solution is oscillatory or $\lim_{\ell\to\infty}\frac{x(\ell)}{A(\ell)} = 0$.

3. Conclusions

The results of this study are presented in a novel and generalizable framework, highlighting their broad applicability. Our approach involves a unique transformation that converts the equation from the semi-canonical form to the more tractable canonical form. This transformation facilitates the derivation of new oscillation criteria with fewer restrictions compared to the existing literature. Theorems 2.4 and 2.6 illustrate our criteria, ensuring that all solutions oscillate. The results obtained are consistent with the results in [11, 13, 14] and can be extended to non linear difference equations. Our approach has the potential to be extended to both non-canonical and semi-canonical forms (as defined in (S_2)), potentially leading to new oscillation conditions.

Author contributions

Ahmed M. Hassan: Writing-original draft, Writing-review and editing, Making major revisions; Clemente Cesarano: Supervision, Writing-review and editing; Sameh S. Askar: Formal analysis, Writing-original draft; Ahmad M. Alshamrani: Writing-original draft, Making major revisions. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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