CHAPTER

11

Solvability of the Cauchy problem for a fractionally loaded equation with variable coefficients

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11.1 Introduction and problem statement

At present, special attention is being paid to the most rapidly developing direction of mathematical physics on a global scale – methods of studying direct and inverse problems [1], [2], [3], [4], [5]. These areas have become one of the most important problems in mathematical physics and engineering. Due to the wide application of this problem and the novelty and complexity of its theory, it has attracted the attention of many scientists. In recent years, the control of heat generation processes has been rapidly developing, since the thermal conductivity and relaxation function of each medium are different, and these quantities are closely related to the initial state and properties of the medium. For this reason, the study of initial, initial-boundary value problems, and the problems posed for loaded integro-differential equations of fractional heat transfer is a targeted scientific study.

We should note that the scientific studies [6], [7], and [8] were dedicated to the Cauchy problem for loaded equations, while works [9] and [10] focused on the heat equation without loading. These studies formulated the problem and proposed methods that were utilized in our research.

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In this chapter, we investigate the Cauchy problem for a heat equation involving fractional derivatives. We establish the unique solvability of the direct problem for a multidimensional loaded heat equation with variable coefficients in Hölder spaces. The loaded term is formulated using the fractional Riemann–Liouville derivative. The findings of this study are applicable to the analysis of heat equations with fractional loads, particularly in cases where the order is higher and the loaded term is represented as a fractional function.

Before proceeding to the formulation of the problem, we give some definitions and propositions in Holder spaces [11].

We introduce the following notations:

Let R^n denote the n-dimensional Euclidean space, where $x = (x^{(1)}, ..., x^{(n)}) \in R^n$;

Let R_T^n represent the $(n+1)$-dimensional Euclidean space, with points denoted as (x, y), where $x \in R^n$ and $y \in (0, T]$, and

$$R_T^{n-1} = \{(x', y) | x' \in R^{n-1}, 0 < y < T\},$$
$$\left(\overline{R}_T^{n-1}\right) = \left\{\left(x', y\right) x' \in R^{n-1}, 0 \leq y \leq T\right\}.$$

Let $f(x)$ be a function defined on R^n

Definition 11.1.1. If for any two points $x^{(1)}, x^{(2)} \in R^n$:

$$|f(x^{(1)}) - f(x^{(2)})| \leq A|x^{(1)} - x^{(2)}|^l, A = const > 0, \ l \in (0, 1), \tag{11.1.1}$$

the inequality holds, then, the function $f(x)$ is said to satisfy the Holder condition with l exponent in R^n, and the class of functions satisfying the condition (11.1.1) is denoted as $H^l(R^n)$.

Definition 11.1.2. If for any given pair of values $(x_1, y_1), (x_2, y_2) \in R_T^n$, $x_1 = x_1^{(1)}, x_1^{(2)}, ..., x_1^{(n)}, x_2 = x_2^{(1)}, x_2^{(2)}, ..., x_2^{(n)}$ holds,

$$|f(x_1, y_1) - f(x_2, y_2)| \leq \sum_{i=1}^{n} A_i |x_1^{(i)} - x_2^{(i)}|^l + A_{i+1}|y_1 - y_2|^{l/2}, \tag{11.1.2}$$
$$A_i = const > 0, \ l \in (0, 1),$$

then, the function $f(x, y)$ is said to satisfy the Holder condition with exponent $l, l/2$ on R_T^n, and the class of functions satisfying the condition with $l, l/2$, is denoted as $H^{l,l/2}(R_T^n)$.

If $\varphi(x) \in H^l(R^n)$, $f(x, y) \in H^{l,l/2}(R_T^n)$, then $H^{l+m}(R^n)$, $H^{l+m,(l+m)/2}(R_T^n)$ the norms in the spaces are defined as [11].

Cauchy problem. Find a solution $u(x, y)$ in the domain $(x, y) \in R_T^n$ of the following loaded heat equation:

$$u_y - k(y)\,\Delta u = \lambda D_{0y}^{-\alpha}(\alpha_0 u_x(0, y) + \beta_0 u(0, y)), \ \ (x, y) \in R_T^n \tag{11.1.3}$$

that satisfies the condition:

$$u\,(x,0)=\phi\,(x)\,,\; x\in R^n, \tag{11.1.4}$$

where $k(y)$, $\phi\,(x)$ are given real-valued sufficiently smooth functions, λ, α_0, $\beta_0\in R$, $D_{0y}^{-\alpha}$ is the Riemann–Liouville fractional integral operator of order α is defined by:

$$D_{0y}^{-\alpha}f(y)=\frac{1}{\Gamma(\alpha)}\int\limits_0^y\frac{f(z)dz}{(y-z)^{1-\alpha}},\; if\;\; \alpha>0,$$

$D_{0y}^{0}f(y)=f(y)$, if $\alpha=0$; $f(y)\in L_1(a,b)(a<b<+\infty)$.

The inhomogeneous problem (11.1.3) and (11.1.4) in the case of $\lambda\equiv 0$ is well-known, and has been studied in many works. This equation, considered as a heat equation, is investigated with various kinds of conditions such as initial Cauchy problems [12], mixed, nonlocal, and other types. At the beginning of the study of the Cauchy problem (11.1.3) and (11.1.4), we use the following formula:

$$W(x,y)=\int_{R^n}\chi(\xi)G(x,\sigma(y),\xi,0)d\xi+$$

$$+\int_0^{\sigma(y)}\frac{d\eta}{k(\sigma^{-1}(\eta))}\int_{R^n}F(\xi,\sigma^{-1}(\eta))G(x,\sigma(y),\xi,\eta)d\xi, \tag{11.1.5}$$

which is the solution of the following Cauchy problem for the heat equation with variable coefficients:

$$W_y-k\,(y)\,\Delta W=F\,(x,y)\,,\;(x,y)\in R_T^n,$$
$$W\,(x,0)=\chi\,(x)\,,\; x\in R^n.$$

In (11.1.5), the functions $\sigma\,(y)=\int_0^y k\,(\eta)\,d\eta$ and $\sigma^{-1}\,(y)$ are the inverse functions of $\sigma\,(y)$, $G(x,\sigma\,(y)\,,\xi,\eta)=G_1(x-\xi,\sigma\,(y)-\eta)$ is a fundamental solution of a differential operator with a variable coefficient $\partial/\partial y-k\,(y)\,\Delta$, where:

$$\xi=\left(\xi^{(1)},\xi^{(2)},...,\xi^{(n)}\right),\;\; \xi'=\left(\xi^{(1)},\xi^{(2)},...,\xi^{(n-1)}\right),\;\; d\xi=d\xi^{(1)}...d\xi^{(n)}.$$

We note that the Cauchy problem can also be studied if the α_0 and β_0 coefficients in Eq. (11.1.3) are functions of x and y.

It should be noted that in the work of Dikinov et al. [13], the problem related to a one-dimensional bounded medium was considered at one of the ends of which there is a heat source in this medium, the heat conduction equation. Later, in [14], studies were conducted on cases where the loaded term involved a combination (differential) of traces of the function $u(x,t)$. However, the loaded heat equation with a fractional operator was investigated in [10], [15], [16], [17], and [18].

We note that in this work, we investigated the Cauchy problem for the multidimensional fractionally loaded heat equation in Holder spaces. If $\alpha_0=0$ in Eq. (11.1.3), the problem is analyzed analogously to [19]. Hence, in this study, we focus on the case $\alpha_0\neq 0$ and $\beta_0=0$.

11.2 Investigation of the problem

Our main result is stated as follows:

Theorem 11.2.1. *If* $k(y) \in E := \left\{k(y) \in C^1[0,T], 0 < k_0 < k(y) \leq k_1 < \infty\right\}$, *and*

$$\phi(x) \in H^{l+3}\left(R^n\right), \tag{11.2.1}$$

$$\phi'(x) \leq \phi_0 = const > 0, \tag{11.2.2}$$

then the Cauchy problem admits a unique solution in the function space $u(x,y) \in H^{l+2,(l+2)/2}\left(R^n_T\right)$.

Before proceeding with the proof of Theorem 11.2.1, we first outline the method used to establish the unique solvability of the problem.

To prove Theorem 11.2.1, i.e., to solve the Cauchy problem, we transform it into an equivalent Volterra-type integral equation. By employing the theory of integral equations and imposing suitable conditions on the given functions, in accordance with the theorem's assumptions, we establish the existence of a unique solution to the derived equation. This, in turn, ensures the unique solvability of the original problem. Following this approach, we first reduce problem (11.1.3) and (11.1.4) to an integral equation of the Volterra type.

Using formula (11.1.5) and considering the properties of the fundamental solution, the Cauchy problem (11.1.3) and (11.1.4) is equivalently rewritten as follows:

$$\begin{aligned} u(x,y) = &\int_{R^n} G(x,\sigma(y);\xi,0)\phi(\xi)d\xi + \frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))}\times \\ &\times \int_{R^n}\int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta)-s\right)^{\alpha-1} u_x(0,s)\, G(x,\sigma(y);\xi,\eta)\, ds d\xi. \end{aligned} \tag{11.2.3}$$

Then, we compute the derivative $u(x,y)$ with respect to x for the case $n=1$:

$$\begin{aligned} u_x(x,y) = &\int_{R^1} G_x(x,\sigma(y);\xi,0)\,\phi(\xi)d\xi + \frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))}\times \\ &\times \int_{R^1}\int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta)-s\right)^{\alpha-1} u_x(0,s)\, G_x(x,\sigma(y);\xi,\eta)\, ds d\xi \end{aligned}$$

and we consider the point where $x=0$:

$$\begin{aligned} u_x(0,y) = &\int_{R^1} G_x(x,\sigma(y);\xi,0)|_{x=0}\,\phi(\xi)d\xi + \frac{\lambda}{\Gamma(\alpha)}\int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))}\times \\ &\times \int_{R^1}\int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta)-s\right)^{\alpha-1} u_x(0,s)\, G_x(x,\sigma(y);\xi,\eta)|_{x=0}\, ds d\xi. \end{aligned} \tag{11.2.4}$$

Thus by denoting $u_x(0,y) = v(0,y)$, we obtain the following Volterra-type integral equation of the second kind:

$$v(0,y) - \frac{\lambda}{\Gamma(\alpha)} \int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))} \int_{R^1} \int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} v(0,s) \frac{\xi}{2(\sigma(y)-\eta)}$$
$$\times G_x(x,\sigma(y);\xi,\eta))|_{x=0} ds d\xi = \int_{R^1} G_x(x,\sigma(y);\xi,0)|_{x=0}\phi(\xi) d\xi. \qquad (11.2.5)$$

Similarly, for $n = 2, 3, \ldots$, taking the derivative with respect to x_i $(i = 2, 3, \ldots)$, we obtain the corresponding two-dimensional, three-dimensional, and subsequently integral equations of the Volterra type (11.2.5), which are studied in a similar manner.

Lemma 11.2.2. *Let $k(t) \in E$, $\phi(x) \in H^{l+3}\left(R^1\right)$, and suppose that $|\phi'(x)| \le \phi_1$, with conditions (11.2.1) and (11.2.2) satisfied. Then, there exists a unique solution to the integral equation (11.2.5) in the function space $H^{l+3,(l+2)/2} \in \left(R_T^0\right)$.*

Proof. We prove this lemma using the method of successive approximations, constructing the following functional sequence:

$$v_0(0,y) = \int_{R^1} G_x(x,\sigma(y);\xi,0)|_{x=0}\phi(\xi) d\xi$$

$$v_1(0,y) = \frac{\lambda}{\Gamma(\alpha)} \int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))} \int_{R^1} \int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} v_0(0,s) \times$$
$$\times G_x(x,\sigma(y);\xi,\eta)|_{x=0} ds d\xi,$$

$$v_2(0,y) = \frac{\lambda}{\Gamma(\alpha)} \int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))} \int_{R^1} \int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} v_1(0,s) \times$$
$$\times G_x(x,\sigma(y);\xi,\eta)|_{x=0} ds d\xi,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (11.2.6)$$

$$v_n(0,y) = \frac{\lambda}{\Gamma(\alpha)} \int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))} \int_{R^1} \int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} \times$$
$$\times v_{n-1}(0,s) G_x(x,\sigma(y);\xi,\eta)|_{x=0} ds d\xi,$$

$$(x,y) \in R_T^1, \ \ n = 1, 2, \ldots\ldots$$

Therefore taking into account (11.2.1) and (11.2.2), along with the properties of the fundamental solution of the heat equation [12], the equality $\int_{R^n} G_1(x-\xi, \sigma(y) - \eta) d\xi = 1$, and introducing the notation

$$\phi_1 = max|\phi'(x)|, \qquad (11.2.7)$$

we estimate the functional sequence $\{v_n(0,y)\}$ defined with the integral Eq. (11.2.5), modulo in the class R_T^0, according to $G_{1x} = -G_{1\xi}$:

$$|v_0(0,y)|_T^{l+2,(l+2)/2} \le \int_{R^1} |\phi(\xi)|_T^{l+3,(l+2)/2} G_{1x}(x-\xi,\sigma(y))|_{x=0} d\xi$$

$$\begin{aligned}
&\le -\int_{R^1} |\phi(\xi)|_T^{l+3,(l+2)/2} G_{1\xi}(x-\xi,\sigma(y))|_{x=0} d\xi \\
&\le \int_{R^1} \left|\phi'(\xi)\right|_T^{l+3,(l+2)/2} G_1(x-\xi,\sigma(y))|_{x=0} d\xi \le \phi_1 .
\end{aligned}$$

Similarly, we evaluate

$$\begin{aligned}
|v_1(0,y)|_T^{l+2,(l+2)/2} &\le \frac{|\lambda|}{\Gamma(\alpha)} \int_0^{\sigma(y)} \frac{d\eta}{\left|k(\sigma^{-1}(\eta))\right|} \int_{R^1} \int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta)-s\right)^{\alpha-1} \times \\
&\quad \times |v_0(0,s)|_T^{l+2,(l+2)/2} G_1(-\xi,\sigma(y)-\eta) ds d\xi \\
&\le \frac{|\lambda|\phi_1 2\sqrt{k_1 T}}{k_0\sqrt{\pi}} \frac{y^\alpha}{\alpha!}, \\
|v_2(0,y)|_T^{l+2,(l+2)/2} &\le \frac{|\lambda|\,\phi_1 2\sqrt{k_1 T}}{k_0 \Gamma(\alpha)\,\Gamma(\alpha+1)} \int_0^{\sigma(y)} \frac{d\eta}{\left|k(\sigma^{-1}(\eta))\right|} \\
&\quad \times \int_{R^1} \int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta)-s\right)^{\alpha-1} \times \\
&\quad \times s^\alpha\, |v_1(0,s)|_T^{l+2,(l+2)/2} G_1(-\xi,\sigma(y)-\eta) ds d\xi \\
&\le \frac{\phi_1 \left(2|\lambda|\sqrt{k_1 T}\right)^2}{\left(k_0\sqrt{\pi}\right)^2} \frac{y^{2\alpha}}{(2\alpha)!} \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots \\
|v_n(0,y)|_T^{l+2,(l+2)/2} &\le \frac{\phi_1 \left(2\,|\lambda|\,\sqrt{k_1 T}\right)^{n-1}}{\sqrt{\pi}^{n-1} k_0^{n-1} \Gamma(\alpha)\,\Gamma((n-1)\alpha+1)} \int_0^{\sigma(y)} \frac{d\eta}{\left|k(\sigma^{-1}(\eta))\right|} \times \\
&\quad \times \int_{R^1} \int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta)-s\right)^{\alpha-1} s^\alpha\, |v_{n-1}(0,s)|_T^{l+2,(l+2)/2} \times \\
&\quad \times G_1(-\xi,\sigma(y)-\eta) ds d\xi \le \frac{\phi_1 \left(2|\lambda|\sqrt{k_1 T}\right)^n}{\left(k_0\sqrt{\pi}\right)^n} \frac{y^{n\alpha}}{(n\alpha)!} \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Thus we have constructed the following functional series:

$$\sum_{n=0}^{\infty} v_n(0,y). \tag{11.2.8}$$

Using the above estimates, we estimate (majorize) the obtained functional series with a numerical in the domain $(0,t) \in R_T^0$, i.e.,

$$\sum_{n=0}^{\infty} |v_n(0,y)| \le \sum_{n=0}^{\infty} \phi_1 A^n \frac{y^{n\alpha}}{(n\alpha)!} \le \phi_1 \sum_{n=0}^{\infty} \frac{\left(A^{\frac{1}{\alpha}}\right)^{n\alpha} T^{n\alpha}}{(n\alpha)!} \le$$

$$\leq \phi_1 \sum_{n=0}^{\infty} \frac{\left(A^{\frac{1}{\alpha}} T\right)^{n\alpha}}{(n\alpha)!} \leq \phi_1 exp\left(A^{\frac{1}{\alpha}} T\right).$$

Thus according to the latest estimates, according to the Weierstrass theorem, one can easily verify that the resulting functional series (11.2.8) converges absolutely and uniformly on the set H. Therefore the sequence of functions $v_k(0,t)$ defined by the integral Eq. (11.2.5), converges uniformly for some function $v(0,y)$ in the space $H^{l+2,(l+2)/2}\left(R_T^0\right)$.

Thus we have established the existence of a solution to (11.2.5). By substituting the obtained function into (11.2.3), we uniquely determine the solution to the Cauchy problem (11.1.3) and (11.1.4) in the class $H^{l+2,(l+2)/2}\left(R_T^1\right)$.

Now, we will prove that the integral Eq. (11.2.5) and thus the Cauchy problems (11.1.3) and (11.1.4) have a unique solution. To do this, first assume the opposite, i.e., let there be two different solutions $v_1(0,y)$ and $v_2(0,y)$ of the integral Eq. (11.2.5):

$$v_1(0,y) = \int_{R^1} G_1(-\xi, \sigma(y))\phi(\xi)d\xi + \frac{\lambda}{\Gamma(\alpha)} \int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))} \times$$
$$\times \int_{R^1} \int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} v_1(0,s) G_{1x}(-\xi, \sigma(y) - \eta) ds d\xi. \tag{11.2.9}$$

$$v_2(0,y) = \int_{R^1} G_1(-\xi, \sigma(y))\phi(\xi)d\xi + \frac{\lambda}{\Gamma(\alpha)} \int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))} \times$$
$$\times \int_{R^1} \int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} v_2(0,s) G_{1x}(-\xi, \sigma(y) - \eta) ds d\xi. \tag{11.2.10}$$

The difference of these functions v_1 and v_2, is denoted by $w(0,y)$:

$$w(0,y) = v_1(0,y) - v_2(0,y).$$

As a result, we obtain a homogeneous integral equation of the second kind:

$$w(0,y) = \int_{R^1} G_1(-\xi, \sigma(y))\, \phi(\xi)\, d\xi + \frac{\lambda}{\Gamma(\alpha)} \int_0^{\sigma(y)} \frac{d\eta}{k(\sigma^{-1}(\eta))} \times$$
$$\times \int_{R^1} \int_0^{\sigma^{-1}(\eta)} \left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} w(0,s)\, G_{1x}(-\xi, \sigma(y) - \eta)\, ds d\xi. \tag{11.2.11}$$

For each fixed $y \in [0,T]$, we define the supremum of the modulus of the function $w(0,y)$ as:

$$\overline{w} = \sup |w(0,y)|, \; y \in [0,T].$$

From the integral Eq. (11.2.11), we derive the following inequality:

$$|w(0,y)| \leq \frac{|\lambda|}{|\Gamma(\alpha)|} \int_0^{\sigma(y)} \frac{d\eta}{|k(\sigma^{-1}(\eta))|} \int_{R^1} \int_0^{\sigma^{-1}(\eta)} |\left(\sigma^{-1}(\eta) - s\right)^{\alpha-1} w(0,s)|\cdot$$

$$\cdot G_{1x}(-\xi, \sigma(y)-\eta)\,ds\,d\xi \le \overline{w}\frac{2}{k_0}\sqrt{\frac{k_1 T}{\pi}}\frac{y^{\alpha}}{\alpha!}.$$

By continuing this process, we obtain the following result for any arbitrary natural number n:

$$|w(0,y)| \le \overline{w}\left(\frac{2}{k_0}\sqrt{\frac{k_1 T}{\pi}}\right)^{n}\frac{y^{n\alpha}}{(n\alpha)!} \le \frac{c^n}{(n\alpha)!},$$

where $c = \frac{2}{k_0}\sqrt{\frac{k_1 T}{\pi}}T^{\alpha}$. This inequality implies that either $w(0,y)=0$ or $v_1(0,y) = v_2(0,y)$ as $n \to \infty$. Hence, the integral Eq. (11.2.5) has a unique solution.

Indeed, considering the theory of integral equations and expressing the solutions of (11.2.5) through the resolvent, it is straightforward to verify that for $\phi(x)=0$, we obtain $v(0,t) \equiv 0$. This completes the proof of the lemma.

As a consequence, under conditions (11.2.1) and (11.2.2), from (11.2.3) it follows that $u(x,y) \equiv 0$, for all $(x,y) \in R_T^n$. Thus we conclude that the Cauchy problem formulated at the beginning has a unique solution. *Theorem 11.2.1 is proved.* ☐

Declaration of competing interest

The authors declare that there is no conflict of interest regarding this work.

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