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The Soliton Solutions of the Stochastic Shallow Water Wave Equations in the Sense of Beta-Derivative

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Abstract: The stochastic shallow water wave equation (SSWWE) in the sense of the beta-derivative is considered in this study. The solutions of the SSWWE are obtained using the F-expansion technique with the Riccati equation and He's semi-inverse method. Since the shallow water equation has many uses in ocean engineering, including river irrigation flows, tidal waves, tsunami prediction, and weather simulations, the solutions discovered can be utilized to represent a wide variety of exciting physical events. We create many 2D and 3D graphs to demonstrate how the beta-derivative and Brownian motion affect the analytical solutions of the SSWWE.

Keywords: stochastic shallow water wave; beta-derivative; Brownian motion; *F*-expansion method

MSC: 35A20; 60H15; 83C15; 35Q51; 60H10

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1. Introduction

All physical phenomena are nonlinear in nature, and mathematical models are generally the most accurate way to represent them. To better explore and comprehend the nature of physical phenomena, partial differential equations (PDEs) have been modeled. Finding solutions for traveling waves is one of the most significant physical problems for these models. As a result, finding mathematical methods to produce exact solutions to PDEs has grown to be a large and essential task in nonlinear sciences.Recently, a number of approaches for dealing with PDEs have been proposed, such as spectral methods [1], Hirota's method [2], tanh-sech method [3,4], Jacobi elliptic function method [5], $exp(-\phi(\varsigma))$-expansion method [6], (G'/G)-expansion method [7,8], perturbation method [9,10], bifurcation analysis [11–13], etc.

Moreover, stochastic partial differential equations (SPDEs) are used to study physical, biological, and chemical systems that are subject to random influences. Over the past few decades, these models have been the subject of extensive investigation. The significance of taking stochastic effects into consideration when modeling complex systems has been emphasized. For example, the use of SPDEs to mathematically model complicated processes is becoming more popular in the domains of finance, mechanical and electrical engineering, biophysics, information systems, materials sciences, condensed matter physics, and climate systems [14,15]. Recently, some effort has been made to obtain the precise solutions for SPDEs; for example, [16–20] and the references therein.

Therefore, it is essential to take into account PDEs with some stochastic force. Here, we address the modified $(3+1)$-dimensional stochastic shallow water wave equation (SSWWE) with beta-derivative:

$$\mathbb{T}_{yz}^{2\beta}\mathcal{W}_t + \mathbb{T}_{xxxyz}^{5\beta}\mathcal{W} - 6\mathbb{T}_x^{\beta}\mathcal{W}\mathbb{T}_{xyz}^{3\beta}\mathcal{W} - 6\mathbb{T}_{xz}^{2\beta}\mathcal{W}\mathbb{T}_{xy}^{2\beta}\mathcal{W} + \gamma\mathbb{T}_{xyz}^{3\beta}\mathcal{W} = \delta(\mathbb{T}_{yz}^{2\beta}\mathcal{W})\mathcal{B}_t, \quad (1)$$

where $\mathbb{T}_y^{\beta} = \frac{\partial^{\beta}}{\partial y^{\beta}}$ is the beta-derivative, and $\mathbb{T}_{yz}^{2\beta} = \frac{\partial^{2\beta}}{\partial y^{\beta}\partial z^{\beta}}$, $\mathbb{T}_{xyz}^{3\beta} = \frac{\partial^{3\beta}}{\partial x^{\beta}\partial y^{\beta}\partial z^{\beta}}$, and it is defined later in next section, $\mathcal{W}(x,y,z,t)$ is the height of the Riemann wave, γ is a positive constant. $\mathcal{B}$ is the Brownian motion, δ represents the intensity of noise, and $(\mathbb{T}_{yz}^{2\beta}\mathcal{W})\mathcal{B}_t$ is multiplicative noise.

If we put $\delta = 0$ and $\beta = 1$, then we get the modified $(3+1)$-dimensional shallow water wave equation [21,22]:

$$\mathcal{W}_{yzt} + \mathcal{W}_{xxxyz} - 6\mathcal{W}_x\mathcal{W}_{xyz} - 6\mathcal{W}_{xz}\mathcal{W}_{xy} + \gamma\mathcal{W}_{xyz} = 0. \quad (2)$$

Equation (2) has various uses in ocean engineering, including river irrigation flows, tidal waves, tsunami prediction, weather simulations, and so on. Therefore, a number of authors have looked at the analytical solutions to Equation (2) by utilizing many different methods such as (G'/G)-expansion [23,24], exp-function [25], and Lie symmetry [26]. For the fractional-derivative of Equation (2), many authors obtained the exact solutions; for example, Phoosree et al. [27] used the simple equation method, and S. Duran [28] used the $(\frac{G'}{G}, \frac{1}{G})$-expansion method. Equation (2) with the beta-derivative and forced by a stochastic term is not considered.

Our contribution in this article is to deduce the analytical stochastic solutions of SSWWE (1) with the beta-derivative**.** To get these solutions, we use the F-expansion method with the Riccati equation and He's semi-inverse method. Physics researchers would find the solutions very helpful in defining several major physical processes because of the stochastic term and beta-derivative present in Equation (1). Additionally, by using the MATLAB software, we introduce numerous graphs to investigate the effects of noise and the beta-derivative on the exact solutions of the SSWWE (1).

An overview of this article is provided as follows: the beta-derivative and Brownian motion are defined, together with some of their properties, in Section 2. We obtain the wave equation of SSWWE (1) in Section 3. In Section 4, The exact solutions of the SSWWE are acquired utilizing F-expansion and He's semi-inverse methods. The impact of noise and the beta-derivative on the obtained solutions of SSWWE is analyzed in Section 5. Finally, the conclusions of the paper are introduced.

2. Beta-Derivative and Brownian Motion

The beta-derivative was recently suggested by Atangana et al. [29]. These derivatives may not be seen as fractional derivatives but can be considered to be a natural extension of the classical derivative [30]. From this point, let us define the beta-derivative for the function $f : (0,\infty) \to \mathbb{R}$ of order $\beta \in (0,1]$ as follows [29,31]:

Definition 1. *The beta-derivative for the function* $f : (0,\infty) \to \mathbb{R}$ *is defined as*

$$\mathbb{T}_y^{\beta} f(y) = \frac{d^{\beta} f}{dy^{\beta}} = \lim_{h\to 0} \frac{f(y + h(y + \frac{1}{\Gamma(\beta)})^{1-\beta}) - f(y)}{h}, \; 0 < \beta \le 1.$$

The beta-derivative meets the following characteristics [29]:

(1) $\mathbb{T}_y^{\beta} f(y) = (y + \frac{1}{\Gamma(\beta)})^{1-\beta}\frac{df}{dy}$,

(2) If $\mathbb{T}_y^{\beta}(f \circ g(y)) = (y + \frac{1}{\Gamma(\beta)})^{1-\beta} g'(y) f'(g(y))$,

(3) $\mathbb{T}_y^{\beta}(af + bg) = a\mathbb{T}_y^{\beta}(f) + b\mathbb{T}_y^{\beta}(g)$ for all a and b real number,

(4) $\mathbb{T}_y^{\beta}(a) = 0$.

Moreover, Brownian motion $\mathcal{B}$ is defined as follows [32]:

Definition 2. *The stochastic process $\{\mathcal{B}(\tau)\}_{\tau\geq 0}$ is referred to as Brownian motion if it meets the following criteria:*

1. $\mathcal{B}(0) = 0$,
2. $\mathcal{B}(\tau)$ *is continuous function of* $\tau \geq 0$,
3. $\mathcal{B}(\tau_2) - \mathcal{B}(\tau_1)$ *is independent for* $\tau_1 < \tau_2$,
4. $\mathcal{B}(\tau_2) - \mathcal{B}(\tau_1)$ *has a normal distribution* $N(0, \tau_2 - \tau_1)$.

We need the following lemma:

Lemma 1 ([32])**.** $\mathbb{E}(e^{\rho\mathcal{B}(\tau)}) = e^{\frac{1}{2}\rho^2\tau}$ *for* $\rho \geq 0$.

3. Wave Equation for SSWWE

The wave equation for the SSWWE (1) is created by using the next wave transformation:

$$\mathcal{W}(x,y,z,t) = \mathcal{Q}(\theta)e^{(\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t)}, \tag{3}$$

where $\mathcal{Q}$ is the deterministic function, and

$$\theta = \frac{\theta_1}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta + \frac{\theta_2}{\beta}(y + \frac{1}{\Gamma(\beta)})^\beta + \frac{\theta_3}{\beta}(z + \frac{1}{\Gamma(\beta)})^\beta + \theta_4 t, \tag{4}$$

where θ_1, θ_2, θ_3 and θ_4 are unknown constants. We observe that

$$\begin{aligned}
\mathcal{W}_t &= (\theta_4\mathcal{Q}' + \delta\mathcal{Q}\mathcal{B}_t)e^{(\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t)},\ \mathbb{T}^{2\beta}_{yz}\mathcal{W} = \theta_2\theta_3\mathcal{Q}''e^{(\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t)}, \\
\mathbb{T}^{2\beta}_{yz}\mathcal{W}_t &= (\theta_4\theta_2\theta_3\mathcal{Q}''' + \delta\theta_2\theta_3\mathcal{Q}''\mathcal{B}_t)e^{(\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t)}, \\
\mathbb{T}^{\beta}_{x}\mathcal{W} &= \theta_1\mathcal{Q}'e^{(\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t)},\ \mathbb{T}^{\beta}_{xz}\mathcal{W} = \theta_1\theta_3\mathcal{Q}''e^{(\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t)}, \\
\mathbb{T}^{2\beta}_{xy}\mathcal{W} &= \theta_1\theta_2\mathcal{Q}''e^{(\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t)},\ \mathbb{T}^{3\beta}_{xyz}\mathcal{W} = \theta_1\theta_2\theta_3\mathcal{Q}'''e^{(\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t)} \\
\mathbb{T}^{5\beta}_{xxxyz}\mathcal{W} &= \theta_2\theta_3\theta_1^3\mathcal{Q}'''''e^{(\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t)}.
\end{aligned} \tag{5}$$

Inserting Equation (5) into Equation (1), yields

$$\theta_2\theta_3\theta_1^3\mathcal{Q}''''' + \theta_4\theta_2\theta_3\mathcal{Q}''' + \gamma\theta_1\theta_2\theta_3\mathcal{Q}''' - 6\theta_2\theta_1^2\theta_3[(\mathcal{Q}'')^2 + \mathcal{Q}'\mathcal{Q}''']e^{(\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t)} = 0. \tag{6}$$

By considering the expectations on both sides, we obtain

$$\theta_2\theta_3\theta_1^3\mathcal{Q}''''' + \theta_4\theta_2\theta_3\mathcal{Q}''' + \gamma\theta_1\theta_2\theta_3\mathcal{Q}''' - 6\theta_2\theta_1^2\theta_3[(\mathcal{Q}'')^2 + \mathcal{Q}'\mathcal{Q}''']e^{(-\frac{1}{2}\delta^2 t)}\mathbb{E}e^{\delta\mathcal{B}(t)} = 0. \tag{7}$$

Thus, by using Lemma 2, Equation (7) turn into

$$\theta_1^3\mathcal{Q}''''' + (\theta_4 + \gamma\theta_1)\mathcal{Q}''' - 6\theta_1^2[\mathcal{Q}'\mathcal{Q}'']' = 0. \tag{8}$$

where $(\mathcal{Q}'')^2 + \mathcal{Q}'\mathcal{Q}''' = [\mathcal{Q}'\mathcal{Q}'']'$. Integrating once and ignoring the integral constant, we get

$$\mathcal{Q}'''' + \ell_1\mathcal{Q}'' - 2\ell_2\mathcal{Q}'\mathcal{Q}'' = 0, \tag{9}$$

where

$$\ell_1 = \frac{\gamma\theta_1 + \theta_4}{\theta_1^3} \text{ and } \ell_2 = \frac{3}{\theta_1}.$$

Integrating Equation (9) and ignoring the integral constant, we have

$$\mathcal{Q}''' + \ell_1\mathcal{Q}' + \ell_2(\mathcal{Q}')^2 = 0. \tag{10}$$

By using homogeneous balancing between $(\mathcal{Q}')^2$ with $\mathcal{Q}'''$ in Equation (10), we can conclude that
$$2P+2=P+3 \implies P=1.$$

4. Exact Solutions of SSWWE

The solutions of the wave Equation (10) are found using the F-expansion technique with the Riccati equation and He's semi-inverse method. The solutions of the SSWWE (1) are then obtained.

4.1. $F-$*Expansion Method with Riccati Equation*

Let us suppose the solution $\mathcal{Q}$ of Equation (10) has the form (with $P=1$)

$$\mathcal{Q}(\theta) = A_0 + A_1 F + \frac{B_1}{F}, \tag{11}$$

where F is the solutions of the Riccati equation

$$F' = F^2 + \lambda, \tag{12}$$

where λ is a real constant. Equation (12) has the following solutions:

$$F(\theta) = \sqrt{\lambda}\tan(\sqrt{\lambda}\theta) \text{ or } F(\theta) = -\sqrt{\lambda}\cot(\sqrt{\lambda}\theta), \tag{13}$$

if $\lambda > 0$, or

$$F(\theta) = -\sqrt{-\lambda}\tanh(\sqrt{-\lambda}\theta) \text{ or } F(\theta) = -\sqrt{-\lambda}\coth(\sqrt{-\lambda}\theta), \tag{14}$$

if $\lambda < 0$, or

$$F(\theta) = \frac{-1}{\theta}, \tag{15}$$

if $\lambda = 0$.

Plugging Equation (11) into Equation (10) yields

$$\begin{aligned}&(6A_1 - \ell_2 A_1^2)F^4 + (8\lambda A_1 + A_1\ell_1 + 2\lambda A_1^2 - B_1 A_1)F^2\\&+(2\lambda^2 A_1 - 2B_1\lambda - \ell_1 B_1 + \lambda\ell_1 A_1 + 2\ell_2 A_1 B_1\lambda - \lambda^2\ell_2 A_1^2 - \ell_2 B_1^2)\\&-\lambda B_1(8+\ell_1+2\lambda\ell_1 A_1 + 2\ell_2 B_1)F^{-2} - \lambda^2 B_1(\ell_2 B_1 + 6\lambda)F^{-4} = 0.\end{aligned}$$

Putting each coefficient F^k to zero

$$6A_1 - \ell_2 A_1^2 = 0,$$

$$8\lambda A_1 + A_1\ell_1 + 2\lambda A_1^2 - B_1 A_1 = 0,$$

$$2\lambda^2 A_1 - 2B_1\lambda - \ell_1 B_1 + \lambda\ell_1 A_1 + 2\ell_2 A_1 B_1\lambda - \lambda^2\ell_2 A_1^2 - \ell_2 B_1^2 = 0,$$

$$\lambda B_1(8+\ell_1+2\lambda\ell_1 A_1 + 2\ell_2 B_1) = 0,$$

and

$$\lambda^2 B_1(\ell_2 B_1 + 6\lambda) = 0.$$

When these equations are solved, we obtain three sets:

First set:

$$A_0 = \text{Free},\ A_1 = 2\theta_1,\ B_1 = 0, \text{ and } \theta_4 = 4\theta_1^3 - \gamma\theta_1. \tag{16}$$

Second set:

$$A_0 = \text{Free},\ A_1 = 0,\ B_1 = -2\lambda\theta_1, \text{ and } \theta_4 = 4\theta_1^3 - \gamma\theta_1. \tag{17}$$

Third set:

$$A_0 = \text{Free},\ A_1 = 2\theta_1,\ B_1 = -2\lambda\theta_1, \text{ and } \theta_4 = 16\theta_1^3 - \gamma\theta_1. \tag{18}$$

First set: Equation (10) has the solution

$$\mathcal{Q}(\theta) = A_0 + 2\theta_1 F(\theta).$$

For $F(\theta)$, three cases are present as follows:

Case 1: If $\lambda > 0$, then we obtain by using (13)

$$\mathcal{Q}(\theta) = A_0 + 2\theta_1\sqrt{\lambda}\tan(\sqrt{\lambda}\theta),$$

and

$$\mathcal{Q}(\theta) = A_0 - 2\theta_1\sqrt{\lambda}\cot(\sqrt{\lambda}\theta).$$

Hence, the solutions of SSWWE (1), by utilizing Equations (3) and (16), are

$$\mathcal{W}(x,y,z,t) = [A_0 + 2\theta_1\sqrt{\lambda}\tan(\sqrt{\lambda}\theta)]e^{(\delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t)}, \tag{19}$$

and

$$\mathcal{W}(x,y,z,t) = [A_0 - 2\theta_1\sqrt{\lambda}\cot(\sqrt{\lambda}\theta)]e^{(\delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t)}, \tag{20}$$

where θ is defined in Equation (4).

Case 2: If $\lambda < 0$, then we obtain by using (14)

$$\mathcal{Q}(\theta) = A_0 - 2\theta_1\sqrt{-\lambda}\tanh(\sqrt{-\lambda}\theta),$$

and

$$\mathcal{Q}(\theta) = A_0 - 2\theta_1\sqrt{-\lambda}\coth(\sqrt{-\lambda}\theta).$$

Hence, the solutions of SSWWE (1), by utilizing Equations (3) and (16), are

$$\mathcal{W}(x,y,z,t) = [A_0 - 2\theta_1\sqrt{-\lambda}\tanh(\sqrt{-\lambda}\theta)]e^{(\delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t)}, \tag{21}$$

and

$$\mathcal{W}(x,y,z,t) = [A_0 - 2\theta_1\sqrt{-\lambda}\coth(\sqrt{-\lambda}\theta)]e^{(\delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t)}, \tag{22}$$

where θ defined in Equation (4).

Case 3: If $\lambda = 0$, then we obtain by using (15)

$$\mathcal{Q}(\theta) = A_0 - \frac{2\theta_1}{\theta}.$$

Hence, the solutions of SSWWE (1), by utilizing Equations (3) and (16), are

$$\mathcal{W}(x,y,z,t) = [A_0 - \frac{2\theta_1}{\theta}]e^{(\delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t)}, \tag{23}$$

where θ is defined in Equation (4).

Second set: We have the same solutions as stated in First set when $\lambda > 0$ *and* $\lambda < 0$. While in the case of $\lambda = 0$, the solution of SSWWE (1) is

$$\mathcal{W}(x,y,z,t) = A_0 e^{(\delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t)}. \tag{24}$$

Third set: The solution of Equation (10) is

$$\mathcal{Q}(\theta) = A_0 + 2\theta_1 F(\theta) - \frac{2\theta_1\lambda}{F(\theta)}.$$

For $F(\theta)$, three cases are present as follows:

Case 1: If $\lambda > 0$, then by using (13), we get

$$\mathcal{Q}(\theta) = A_0 + 2\theta_1\sqrt{\lambda}[\tan(\sqrt{\lambda}\theta) - \cot(\sqrt{\lambda}\theta)]$$

Hence, the solutions of SSWWE (1), by utilizing Equations (3) and (18), are

$$\mathcal{W}(x,y,z,t) = [A_0 + 2\theta_1\sqrt{\lambda}(\tan(\sqrt{\lambda}\theta) - \cot(\sqrt{\lambda}\theta))]e^{(\delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t)}. \tag{25}$$

Case 2: If $\lambda < 0$, then by using (14), we have

$$\mathcal{Q}(\theta) = A_0 - 2\theta_1\sqrt{-\lambda}(\tanh(\sqrt{-\lambda}\theta) + \coth(\sqrt{-\lambda}\theta))$$

Hence, the solutions of SSWWE (1), by utilizing Equations (3) and (18), are

$$\mathcal{W}(x,y,z,t) = [A_0 - 2\theta_1\sqrt{-\lambda}(\tanh(\sqrt{-\lambda}\theta) + \coth(\sqrt{-\lambda}\theta))]e^{(\delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t)}. \tag{26}$$

Case 3: If $\lambda = 0$, then by using (15), we get

$$\mathcal{Q}(\theta) = A_0 + \frac{2\theta_1}{\theta}.$$

Hence, the solutions of SSWWE (1), by utilizing Equations (3) and (18), are

$$\mathcal{W}(x,y,z,t) = [A_0 + \frac{2\theta_1}{\theta}]e^{(\delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t)}, \tag{27}$$

where θ is defined in Equation (4).

4.2. He's Semi-Inverse Method

Using He's semi-inverse method mentioned in [33–35], we obtain the following variational formulations:

$$J(\mathcal{Q}) = \int_0^\infty \{\frac{1}{2}(\mathcal{Q}'')^2 - \frac{1}{2}\ell_1(\mathcal{Q}')^2 + \frac{1}{3}\ell_2(\mathcal{Q}')^3\}d\theta. \tag{28}$$

We assume the solution of Equation (9) according to [36] takes the form

$$\mathcal{Q}(\theta) = A\operatorname{sech}(\theta), \tag{29}$$

where A is an undefined constant. Substituting Equation (29) into Equation (28), we have

$$\begin{aligned} J &= \frac{1}{2}A^2\int_0^\infty [\operatorname{sech}^2(\theta)\tanh^4(\theta) + \operatorname{sech}^4(\theta)\tanh^2(\theta) + \operatorname{sech}^6(\theta) \\ &\quad -\ell_1\operatorname{sech}^2(\theta)\tanh^2(\theta) + \frac{2}{3}\ell_2 A\operatorname{sech}^3(\theta)\tanh^3(\theta)]d\theta \\ &= \frac{1}{2}A^2\int_0^\infty [(\operatorname{sech}^2(\theta) - \ell_1\operatorname{sech}^2(\theta)\tanh^2(\theta) + \frac{2}{3}\ell_2 A\operatorname{sech}^3(\theta)\tanh^3(\theta)]d\theta \\ &= \frac{A^2}{2} - \ell_1\frac{A^2}{6} - \frac{2}{45}\ell_2 A^3. \end{aligned}$$

Making J stationary with respect to A as follows

$$\frac{\partial J}{\partial A} = (1 - \frac{1}{3}\ell_1)A - \frac{2}{15}\ell_2 A^2 = 0. \tag{30}$$

By solving Equation (30), we obtain

$$A = \frac{15 - 5\ell_1}{2\ell_2}.$$

Therefore, the solution of Equation (9) is

$$\mathcal{Q}(\theta) = \frac{15 - 5\ell_1}{6\ell_2}\operatorname{sech}(\theta).$$

Now, the solution of SSWWE (1) is

$$\mathcal{W}(x,y,z,t) = \frac{15-5\ell_1}{2\ell_2}\text{sech}(\theta)e^{(\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t)}, \quad (31)$$

where θ is defined in Equation (4). Similarly, we can consider the solution of Equation (9) as

$$\mathcal{Q}(\theta) = B\text{sech}(\theta)\tanh^2(\theta).$$

With the same steps as before, we get

$$B = \frac{11(1199-213\ell_1)}{1456\ell_2}.$$

Hence, the solution of SSWWE (1) is

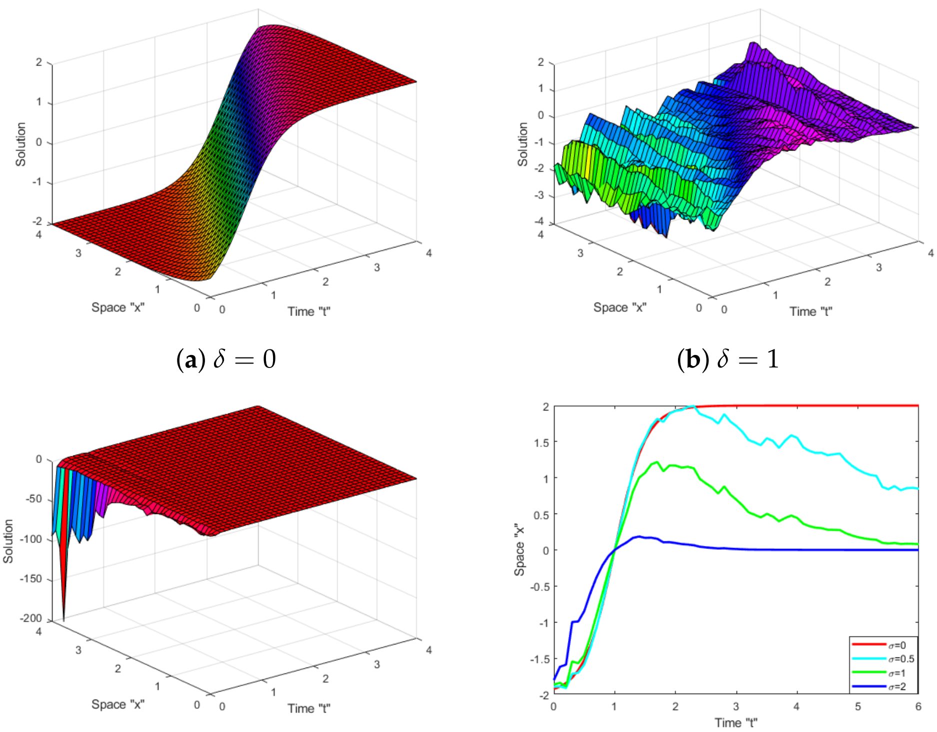
$$\mathcal{W}(x,y,z,t) = \frac{11(1199-213\ell_1)}{1456\ell_2}\text{sech}(\theta)\tanh^2(\theta)e^{(\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t)}, \quad (32)$$

where θ is defined in Equation (4).

5. Impacts of Noise and Beta-Derivative

We now investigate the effect of noise and the beta-derivative on the exact solution of the SSWWE (1). We provide a number of graphs to describe the behavior of these solutions. For specific solutions that have been obtained, such as (21) and (32), let us establish the parameters $\gamma = \theta_1 = 1$, $\theta_2 = -\theta_3 = 1$, $\theta_4 = -2$, $A_0 = a = 0$, $y = z = 1$, $x \in [0,4]$ and $t \in [0,4]$, so that we can model these graphs.

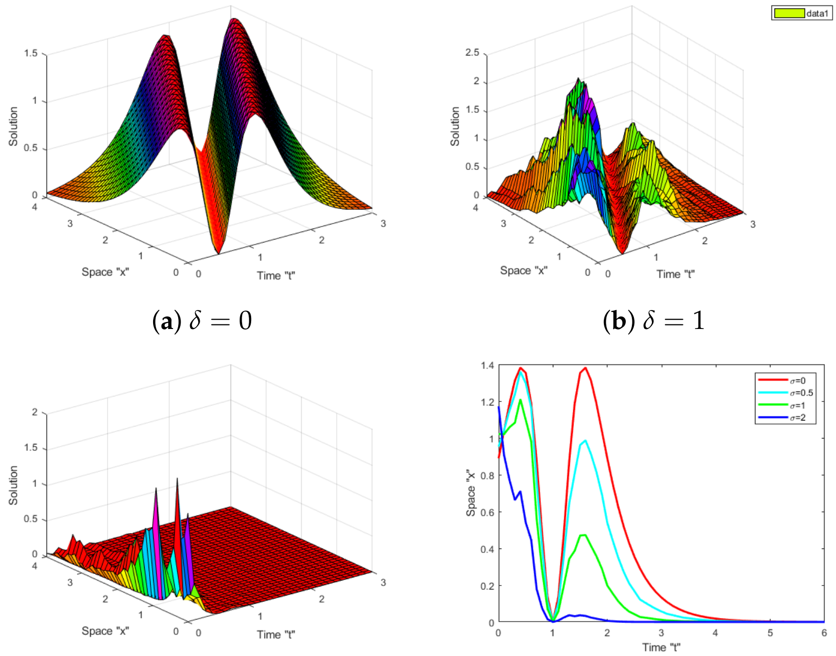
First the noise impacts: In Figures 1 and 2, we see the effect of noise on the solutions as follows:



(**a**) $\delta = 0$ (**b**) $\delta = 1$

(**c**) $\delta = 2$ (**d**) $\delta = 0,\ 0.5,\ 1,\ 2$

Figure 1. (**a**–**c**) Three-dimensional graph of solution $\mathcal{W}(x,y,z,t)$ in Equation (21) for various $\delta = 0,\ 1,\ 2$; (**d**) two-dimensional graph for various $\delta = 0,\ 1,\ 2$.



(**a**) $\delta = 0$ (**b**) $\delta = 1$

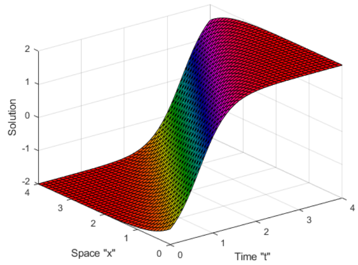
(**c**) $\delta = 2$ (**d**) $\delta = 0,\ 0.5,\ 1,\ 2$

Figure 2. (**a**–**c**) Three-dimensional graph of solution $\mathcal{W}(x, y, z, t)$ in Equation (32) for various $\delta = 0,\ 1,\ 2$; (**d**) two-dimensional graph for various $\delta = 0,\ 1,\ 2$.

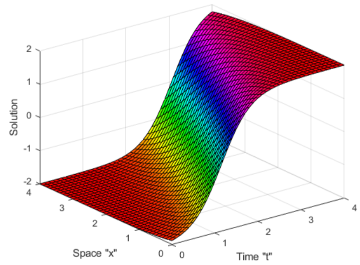
From Figures 1 and 2, we can deduce that when the noise is ignored (i.e., at $\delta = 0$), there are some different types of solutions, such as a periodic solution, kink solution, etc. When noise is added and its strength is increased by $\delta = 1, 2$, the surface becomes a great deal flatter after small transit patterns, and that was proved with the 2D graph. This indicates that the white noise impacts the SSWWE solutions and stabilizes them around zero.

Secondly, the beta derivative impacts: In Figures 3 and 4, if $\delta = 0$, we can see that the graph's shape is compressed as the value of β decreases:

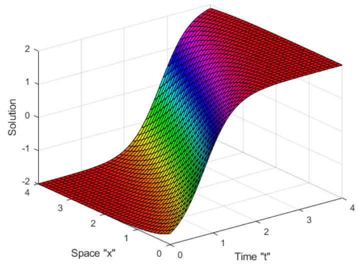
We deduced from Figures 3 and 4 that there is no overlap between the curves of the solutions. Furthermore, as the order of beta derivative decreases, the curves move to the right.



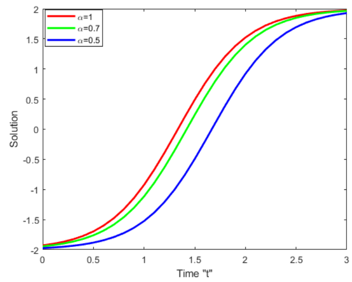
(**a**) $\delta = 0,\ \beta = 1$



(**b**) $\delta = 0,\ \beta = \frac{7}{10}$

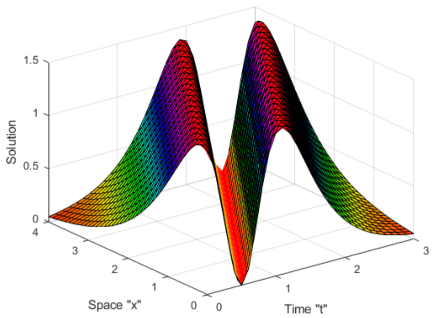


(**c**) $\delta = 0,\ \beta = \frac{5}{10}$

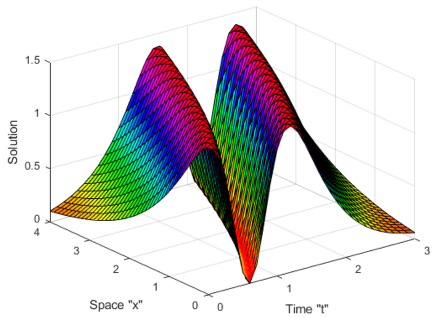


(**d**) $\delta = 0,\ \beta = 1,\ \frac{7}{10},\ \frac{5}{10}$

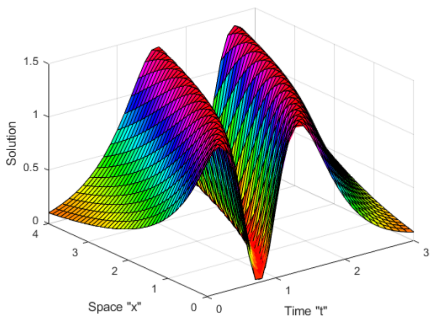
Figure 3. (**a**–**c**) Three-dimensional graph of Equation (21) with $\delta = 0$ and different values of $\beta = 1,\ \frac{7}{10},\ \frac{5}{10}$ (**d**) two-dimensional graph of Equation (21) with different values of $\beta = 1,\ \frac{7}{10},\ \frac{5}{10}$.



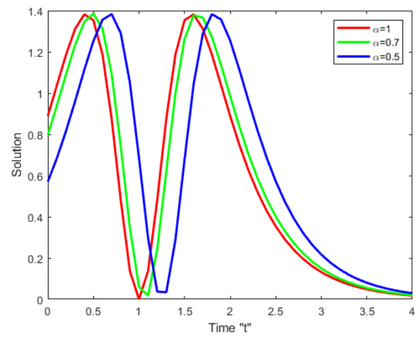
(**a**) $\delta = 0,\ \beta = 1$



(**b**) $\delta = 0,\ \beta = \frac{7}{10}$



(**c**) $\delta = 0,\ \beta = \frac{5}{10}$



(**d**) $\delta = 0,\ \beta = 1,\ \frac{7}{10},\ \frac{5}{10}$

Figure 4. (**a**–**c**) Three-dimensional graph of Equation (32) with $\delta = 0$ and different values of $\beta = 1,\ \frac{7}{10},\ \frac{5}{10}$; (**d**) two-dimensional graph of Equation (32) for different values of of $\beta = 1,\ \frac{7}{10}$, and $\frac{5}{10}$.

6. Conclusions

The stochastic shallow water equation (SSWWE) was investigated in the sense of the beta-derivative. By using the F-expansion method with the Riccati equation and He's semi-inverse method, we were able to find the exact stochastic solutions for SSWWE. These solutions are necessary to understand a wide range of interesting and difficult physical phenomena. In addition, the beta-derivative and multiplicative white noise effects on the analytical solution of SSWWE (1) were demonstrated using the MATLAB software. The beta-derivative moved the surface to the right as the order of derivative decreased, and we deduced that the multiplicative Brownian motion stabilized the solutions around zero.

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