Multiplicity results for a problem of $2-q$ Laplacian operator

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Abstract

This work investigates the existence and multiplicity of positive solutions for a class of Kirchhoff–type Laplacian problems. By applying the Nehari manifold approach together with the Lusternik–Schnirelmann category theory, we establish that the problem possesses at least $cat(\Omega)$ distinct positive solutions.

Keywords: positive solutions; Nehari Manifold; critical points; Ljusternik Schnirelmann category.
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1. Introduction

Here, we would like to consider the multiplicity of positive solutions for the Kirchhoff Laplacian type problem:

$$-\Delta u-\Delta_q u+\lambda V(x)u=\mu|u|^{r-2}u+|u|^{2^*-2}u,\qquad x\in\mathbb{R}^N,\tag{1}$$

where $N\geq 4$, $1<q<\frac{N}{N-1}$, $max\{2,2^*-q\}<r<2^*$, $V:\mathbb{R}^N\to\mathbb{R}$ is a nonnegative continuous function, there is some $M_0>0$ in which

$$meas\{x\in\mathbb{R}^N\quad V(x)\leq M_0\}<+\infty\tag{2}$$

and

$$\Omega:=int(V^{-1}(\{0\})\tag{3}$$

which is a non-empty bounded open set with smooth boundery.

For litrary, let us give a brief story: In special case, if $q=2$ the problem (1) is reduces to a simpler case that recently studied by Alves and Barros [4]

$$-\Delta u+\lambda V(x)u=\mu u^{r-1}+u^{2^*-1},\qquad x\in\mathbb{R}^n.\tag{4}$$

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In fact, using variational method and the Lusternik-Schnirelman category they showed existence of λ^* and μ^* positive such that for $\lambda \geq \lambda^*$ and $\mu \leq \mu^*$ problem (4) has at least $cat(\Omega)$ positive solutions for $4 < r < 6$ when $N = 3$ or $2 < r < 2^* = \frac{2N}{N-2}$ and $N \geq 4$: Rey in [10] proved that if $N \geq 5$; $\lambda = 0$ and $r = 2$; for μ small enough ,the number of solutions of problem (4) is at least $cat(\Omega)$: Here, cat stands for the Lusternik Schnirelmann category of Ω(see [8; 9]).

In [6], Yin and Yang have established the existence and multiplicity of positive solutions for the problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \mu|u|^{r-2}u + |u|^{2^*-2}u, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{5}$$

This problem plays an important role in the limit problem. Our main result is the following.

Theorem 1.1. *Assuming* (2) *and* (3). *There are* $\mu^*, \lambda^* > 0$ *such that problem* (1) *has at least* $cat(\Omega)$ *positive solutions for* $\mu \in (0, \mu^*)$ *and* $\lambda \in (\lambda^*, \infty)$.

In this paper, we fix used the following notations:

* The usual norms in $H^1(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ will be denoted by $\|\|$ and $|\ |_p$ respectively

** If f is a mensurable function, $\int f(x)dx$ will stand for $\int_{\mathbb{R}^N} f(x)dx$.

This paper is organized as follows: In Section 2, we will recall some required important points in the limit problem. In Section 3, we prove some technical results which are crucial in the proof of Theorem 1. In Section 4, we use the Lusternik-Schnirelmann category theory to prove of the main theorem. The letter C will be repeatedly used to show various positive constants whose exact values are not important.

2. The limit problem

Problem (5) can be reduced to a simple problem

$$\begin{cases} -\Delta u - \Delta_q u = \mu|u|^{r-2}u + |u|^{2^*-2}u, & in \ \ \Omega, \\ u = 0, & on \ \ \partial\Omega \end{cases} \tag{6}$$

Let $I_\mu : H_0^1(\Omega) \to \mathbb{R}$ be the energy functional for problem (6) which is

$$I_\mu(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{q}\int_\Omega |\nabla u|^q dx - \frac{\mu}{r}\int_\Omega |u|^r dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx.$$

Let S be the best Sobolev constant of the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ given by

$$S := \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |\nabla u|^{2^*} dx\right)^{\frac{2}{2^*}}} \tag{7}$$

Yin and Yang [6] proved that:

$$0 < c_\mu < \frac{1}{N} S^{\frac{N}{2}}, \qquad \forall \mu > 0 \tag{8}$$

where

$$c_\mu := \inf_{u \in N_\mu} I_\mu(u)$$

where

$$N_\mu := \{x \in H_0^1(\Omega) : u \neq 0 \ \ and \ \ I'_\mu(u)u = 0\},$$

is the Nehari manifold I_μ, Ω is a smooth bounded domain of $\mathbb{R}^N$. It is proved [6] that there is $r > 0$ small enough so that

$$\begin{aligned} \Omega_r^+ &:= \{x \in \mathbb{R}^N : \ \ dist(x, \Omega) < r\} \\ \Omega_r^- &:= \{x \in \Omega : \ \ dist(x, \partial\Omega) > r\} \end{aligned}$$

are homotopically equivalent to Ω. Without loss of generality. We can assume that $0 \in \Omega$ and $B_r(0) \subset \Omega$. Set

$$I_{\mu,r}(u) := \frac{1}{2} \int_{B_r(0)} |\nabla u|^2 dx + \frac{1}{q} \int_{B_r(0)} |\nabla u|^q dx - \frac{\mu}{p} \int_{B_r(0)} |u|^r dx - \frac{1}{2^*} \int_{B_r(0)} |u|^{2^*} dx.$$

then

$$0 < m(\mu) < \frac{1}{N} S^{\frac{N}{2}}.$$

Define

$$\begin{aligned} m(\mu) &:= \inf_{u \in N_{\mu,r}} I_{\mu,r}(u), \\ N_{\mu,r} &:= \{x \in H_0^1(\Omega) : u \neq 0 \ \ and \ \ I'_{\mu,r}(u)u = 0\}, \\ \beta_0 &: H_0^1(\Omega) \setminus \{0\} \to \mathbb{R}^N \\ \beta_0(u) &:= \frac{\int_\Omega |u|^{2^*} x dx}{\int_\Omega |u|^{2^*} dx}. \end{aligned} \tag{9}$$

We will recall and prove some lemmas which are crucial in the proof of the main theorem.

Lemma 2.1. $\lim_{\mu \to 0} c_\mu = \lim_{\mu \to 0} m(\mu) = \frac{1}{N} S^{\frac{N}{2}}$.

Lemma 2.2. *[6, lemma 3.3] There is $\mu^* > 0$ such that if $\mu \in (0, \mu^*)$ and $u \in N_\mu$ with $I_\mu(u) \leq m(\mu)$ then $\beta_0(u) \in \Omega_r^+$.*

Lemma 2.3. *[6, lemma 1.2] There exists a $\mu^* > 0$ such that for any $\mu \in (0, \mu^*)$ problem* (6) *possesses at least $cat(\Omega)$ positive solutions in $H_0^1(\Omega)$.*

3. Preliminary results

From now on, we fix the space $E \subset H^1(\mathbb{R}^N)$ given by

$$E = \{u \in H^1(\mathbb{R}^N) : \int V|u|^2 dx < +\infty\}$$

with inner product

$$\langle u, v\rangle_\lambda := \int (\nabla u \nabla v + \lambda V(x) uv) dx$$

and norm

$$\|u\|_\lambda := \left(\int (|\nabla u|^2 + \lambda V(x)|u|^2) dx\right)^{\frac{1}{2}}$$

Suppose that E is endowed with the norm $\|.\|_\lambda$ and we denote it by E_λ. It is well known that E_λ is a Hilbert space, moreover

$$\|u\|_\lambda \geq \Upsilon \|u\|, \qquad \forall u \in E_\lambda, \qquad \forall \lambda \geq 1 \tag{10}$$

which it shows that the embedding $E_\lambda \hookrightarrow H^1(\mathbb{R}^N)$ is continuous for $\lambda \geq 1$ and embedding

$$E_\lambda \hookrightarrow L^s(\mathbb{R}^N), \qquad \forall s \in [2, 2^*]$$

is also continuous for $\lambda \geq 1$.
Define $I_\eta : E_\lambda \to \mathbb{R}$ by

$$I_\eta(u) := \frac{1}{2}\|u\|_\lambda^2 + \frac{1}{q}\|u\|_q^q - \frac{\mu}{r}\int |u|^r dx - \frac{1}{2^*}\int |u|^{2^*} dx, \tag{11}$$

which is the energy functional respect to (1), is in $C^1(E_\lambda, \mathbb{R})$. Moreover

$$I'_\eta(u)v := \int (\nabla u \nabla v + \lambda V(x) uv) dx + \int (|\nabla u|^{q-2} \nabla u \nabla v dx - \mu \int |u|^{r-2} uv dx - \int |u|^{2^*-2} uv dx, \qquad u, v \in E \tag{12}$$

so

$$J_\eta(u) := I'_\eta(u)u = \|u\|_\lambda^2 + \|u\|_q^q - \mu \int |u|^r dx - \int |u|^{2^*} dx. \tag{13}$$

It is direct to see that critical points of I_η are weak solutions of (4). Set the Nehari manifold of $I_{\lambda,\mu}$ by

$$N_{\lambda,\mu} = \{u \in H \setminus \{0\} | I'_{\lambda,\mu}(u)u = 0\}.$$

For any $u \in N_{\lambda,\mu}$

$$I_\eta(u) = (\frac{1}{2} - \frac{1}{r})\|u\|_\lambda^2 + (\frac{1}{q} - \frac{1}{r})\|u\|_q^q + (\frac{1}{r} - \frac{1}{2^*})\int |u|^{2^*} dx > 0, \tag{14}$$

Thus, $I_{\lambda,\mu}$ is bounded from below on $N_{\lambda,\mu}$ so

$$c_{\lambda,\mu} := \inf_{u \in N_{\lambda,\mu}} I_{\lambda,\mu}(u)$$

.

exists. For any $u \in N_{\lambda,\mu}$

$$\begin{aligned} J'_{\eta}(u) &= 2\|u\|_{\lambda}^{2} + q\|u\|_{q}^{q} - r\mu\int |u|^{r}dx - 2^{*}\int |u|^{2^{*}}dx \\ &= (2-r)\|u\|_{\lambda}^{2} + (q-r)\|u\|_{q}^{q} + (r-2^{*})\int |u|^{2^{*}} < 0. \end{aligned} \tag{15}$$

We are going to show that the Mountain Pass Theorem is applicable.

Lemma 3.1. *Suppose that* $1 < q < 2 < r < 2^{*}$, $\lambda > 0$, $\mu > 0$. *Then*

- *i) there exist positive numbers* ρ *and* d *such that* $I_{\lambda,\mu}(u) \geq \rho$ *for* $\|u\|_{\lambda} = d$,
- *ii) there exists* $e \in C_{0}^{\infty}(\Omega)$ *such that* $\|e\|_{\lambda} > d$ *and* $I_{\lambda,\mu}(e) < 0$.

Proof. i) Using the Sobolev embedding

$$H^{1}(\mathbb{R}^{3}) \hookrightarrow L^{s}(\mathbb{R}^{3}), \qquad for \ \ 2 \leq s \leq 2^{*},$$

$$I_{\mu}(u) = \frac{1}{2}\|u\|_{\lambda}^{2} + \frac{1}{q}\|u\|_{q}^{q} - \frac{\mu}{r}\int_{\Omega} |u|^{r} - \frac{1}{2^{*}}\int_{\Omega} |u|^{2^{*}} \tag{16}$$

$$\geq \frac{1}{2}\|u\|_{\lambda}^{2} + \frac{1}{q}\|u\|_{q}^{q} - C\|u\|_{\lambda}^{r} - C\|u\|_{\lambda}^{2^{*}}. \tag{17}$$

Hence, there exist positive numbers $\rho > 0$ and $d > 0$ such that $I_{\lambda,\mu}(u) \geq \rho$ for $\|u\|_{\lambda} = d$.
ii) Fix $\phi \in C_{0}^{\infty}(\Omega)$ with $supp\phi \subset \Omega$. So

$$g(t) = I_{\eta}(t\phi) = \frac{t^{2}}{2}\|\phi\|_{\lambda}^{2} + \frac{t^{q}}{q}\|\phi\|_{q}^{q} - \frac{t^{r}\mu}{r}\int_{\Omega} |\phi|^{p} - \frac{t^{2^{*}}}{2^{*}}\int_{\Omega} |\phi|^{2^{*}}.$$

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Then $\lim_{t\to\infty} I_{\lambda,\mu}(t\phi) = -\infty$ Hance, there exists t_{u} positive such that $\|t_{u}\phi\| > d$ and $I_{\lambda,\mu}(t_{u}\phi) < 0$.

By Lemma 3.1, $I_{\lambda,\mu}$ possesses the mountain pass theorem in Willem [8]. We will denote by $m_{\lambda,\mu}$ the Mountain -Pass level, there is a $(PS)_{m_{\lambda,\mu}}$ sequence (u_{n}) for $I_{\lambda,\mu}$:

$$I_{\lambda,\mu}(u_{n}) \to m_{\lambda,\mu} := \inf_{\gamma\in\Gamma} max_{t\in[0,1]} I_{\lambda,\mu}(\gamma(t)) \qquad and \qquad I'_{\lambda,\mu}(u_{n}) \to 0,$$

where

$$\Gamma = \{\gamma \in C([0,1]), E_{\lambda}) : \gamma(0) = 0 \quad and \quad \gamma(1) = e\}.$$

Investigating of the following two assertions similar to [3; 4].

1) $c_{\eta} = m_{\eta}$.

2) There is $\sigma > 0$, which is independent of μ such that k$\|u\|_{\lambda} \geq \sigma$ for all $u \in N_{\eta}$.

Lemma 3.2. *There is* $\tau = \tau(\mu) > 0$ *such that the mountain pass level* c_η *verifies the following inequality*

$$0 < c_\eta < \frac{1}{N} S^{\frac{N}{2}} - \tau, \qquad \forall \lambda > 0.$$

PROOF. From (3)

$$I_{\lambda,\mu}(u) = I_\mu(u) \qquad \forall u \in H_0^1(\Omega),$$

so by definition of c_η and $c_\lambda, c_{\lambda,\mu} \le c_\mu$ for all $\eta > 0$. Then, it is enough to apply (2) to get the desired result.

Lemma 3.3. *Any* $(PS)_d$ *sequence* (w_n) *for* I_η *is bounded in* E_λ. *Moreover,*

$$\limsup_{n\to+\infty} \|w_n\|^2 \le \frac{2pd}{p-2}. \tag{18}$$

PROOF. Let (w_n) be $(PS)_d$ sequence for $I_{\lambda,\mu}$ in E_λ such that $I_{\lambda,\mu}(w_n) = d + o_n(1)$ and $I'_{\lambda,\mu}(w_n) = o_n(1)$.

$$\begin{aligned} I_{\lambda,\mu}(w_n) - \frac{1}{r} I'_{\lambda,\mu}(w_n)(w_n) &\le |I_{\lambda,\mu}(w_n)| + \frac{1}{p} \|I'_{\lambda,\mu}(w_n)(w_n)\|.\|w_n\|_\lambda \\ &\le d + o_n(1) + o_n(1)\|w_n\|_\lambda. \end{aligned} \tag{19}$$

On the other hand, for $n \in \mathbb{N}$

$$\begin{aligned} I_{\lambda,\mu}(w_n) - \frac{1}{r} I'_{\lambda,\mu}(w_n)(w_n) &= (\frac{1}{2} - \frac{1}{r})\|w_n\|_\lambda^2 + (\frac{1}{q} - \frac{1}{r})\|w_n\|^q + (\frac{1}{r} - \frac{1}{2^*}) \int |w_n|^{2^*} dx \\ &\ge (\frac{1}{2} - \frac{1}{r})\|w_n\|_\lambda^2 \end{aligned} \tag{20}$$

Combining the above inequalities, then for $n \in \mathbb{N}$ large enough

$$(\frac{1}{2} - \frac{1}{r})\|w_n\|_\lambda^2 \le d + o_n(1) + o_n(1)\|w_n\|_\lambda. \tag{21}$$

This proves boundedness. Doing lim sup of (21) then (18) follows.

Lemma 3.4. *Let* $\Theta > 0$. *If* $(w_n) \subset E_\lambda$ *is a* $(PS)_d$ *for* I_η *with* $0 \le d \le \Theta$, *then given* $\delta > 0$ *there are* $\lambda_* = \lambda_*(\delta, \Theta)$ *such that*

$$\limsup_{n\to+\infty} \int_{B_R^c} |w_n|^p dx < \delta, \qquad \forall \lambda > \lambda_*.$$

PROOF. See Lemma 3.6 of [4].

Corollary 3.1. *Let* $(v_n) \subset E_\lambda$ *be a sequence such that* $(\|v_n\|_{\lambda_n})$ *is bounded, where* $\lambda_n \to +\infty$ *If* $v_n \rightharpoonup 0$ *in* $H^1(\mathbb{R}^N)$, *then* $v_n \to 0$ *in* $L^p(\mathbb{R}^N)$.

PROOF. From Corollary 1 in [4].

Proposition 3.1. *There is* $\hat{\lambda} = \hat{\lambda}(\tau) > 0$ *such that* I_η *verifies the* $(PS)_{d_\lambda}$ *condition for any* $d_\lambda \in \left(0, \frac{1}{N}S^{\frac{N}{2}} - \tau\right)$ *for all* $\lambda \geq \hat{\lambda}$ *where* τ *is as in Lemma 3.2.*

PROOF. Let

$$I_\eta(w_n) \to d_\lambda \quad and \quad I'_\eta(w_n) \to 0.$$

By Lemma 3.3, there is $w \in E_\lambda$ such that

$$\begin{aligned} w_n \rightharpoonup w \qquad &\text{in} \qquad E_\lambda \\ w_n(x) \to w(x) \qquad a.e \ &\text{in} \qquad \mathbb{R}^N \\ w_n \to w \qquad \text{in} \qquad L^s_{Loc}(\mathbb{R}^N), \qquad &0 \leq s < 2^* \end{aligned}$$

Set $v_n := w_n - w$.

$$\int |w_n|^q dx = \int |v_n|^q dx + \int |w|^q dx + o_n(1)$$

and

$$\int |w_n|^{2^*} dx = \int |v_n|^{2^*} dx + \int |w|^{2^*} dx + o_n(1).$$

Then

$$\frac{1}{2}\|v_n\|^2_\lambda + \frac{1}{q}\|v_n\|^q_q - \frac{\mu}{r}\int |v_n|^r dx - \frac{1}{2^*}\int |v_n|^{2^*} dx = d_\lambda - I_{\lambda,\mu}(w) + o_n(1). \tag{22}$$

Since $I'_{\lambda,\mu}(w_n)(w_n) = 0_n(I)$ and $I'_{\lambda,\mu}(w)(w) = 0$, it follows that

$$\|v_n\|^2_\lambda + \|v_n\|^q_q - \mu\int |v_n|^r dx - \int |v_n|^{2^*} dx = o(1). \tag{23}$$

$$\frac{1}{2}\|v_n\|^2_\lambda + \frac{1}{q}\|v_n\|^q_q - \frac{1}{2^*}\int |v_n|^{2^*} dx = d_\lambda - I_{\lambda,\mu}(w) + o_n(1). \tag{24}$$

Since $I'_{\lambda,\mu}(w_n)w_n = o_n(1)$ and $I'_{\lambda,\mu}(w)w = 0$, it follows that

$$\|v_n\|^2_\lambda + \|v_n\|^q_q - \int |v_n|^{2^*} dx = o(1) \tag{25}$$

Assume that for a fix λ, $\|v_n\|^2_\lambda \to h_1 \geq 0$ and $\int |v_n|^{2^*} dx \to h_2 \geq 0$.

If $h_1 = 0$, we deduce that $v_n \to 0$ in E_λ, equivalently, $w_n \to w$ in E_λ, which it completes the proof.

Now assume that h_1 and h_2 are positive By Sobolev embedding

$$\|v_n\|^2_\lambda \leq C(\|v_n\|^r_\lambda + \|v_n\|^{2^*}_\lambda) + o_n(1). \tag{26}$$

Recalling that there is $C > 0$ verifying

$$|t|^r \leq \frac{1}{2C}|t|^2 + C|t|^{2^*}, \qquad \forall t \in \mathbb{R}$$

The last inequality ensures that

$$0 < C_1 := (\frac{1}{2C(C+1)})^{\frac{2}{2^*-2}} \leq \lim_{n\to+\infty} \|v_n\|^2_\lambda.$$

Then there is $C_2 > 0$ in which

$$h_1, h_2 \geq C_2 > 0. \tag{27}$$

On the other hand

$$S \leq \frac{\|v_n\|^2_\lambda}{\left(\int |v_n|^{2^*}dx\right)^{\frac{2}{2^*}}} \leq \frac{h_1}{(h_2 + o_\lambda(1))^{\frac{2}{2^*}}}.$$

Then

$$S^{\frac{N}{2}} \leq \liminf_{\lambda\to+\infty} h_1.$$

Using (24) and since $w \in N_{\lambda,\mu}$, $I_{\lambda,\mu} > 0$ (by (14)), so

$$d_\lambda \geq \frac{1}{q}(\|v_n\|^2_\lambda + \|v_n\|^q_q) - \frac{1}{2^*}\int_\Omega |v_n|^{2^*}dx$$
$$\liminf_{\lambda\to+\infty} d_\lambda \geq (\frac{1}{q} - \frac{1}{2^*}) \liminf_{\lambda\to+\infty} h_1 \geq \frac{1}{N}S^{\frac{N}{2}}.$$

But this is impossible, since

$$\limsup_{\lambda\to+\infty} d_\lambda \leq \frac{1}{N}S^{\frac{N}{2}} - \tau < \frac{1}{N}S^{\frac{N}{2}}.$$

There is $\hat{\lambda} > 0$ such that $h_1 = 0$ for all $\lambda > \hat{\lambda}$.

This reasult has a direct following corollary.

Corollary 3.2. *There is $\hat{\lambda} > 0$ such that $I_{\lambda,\mu}$ verifies the $(PS)_{d_\lambda}$ condition on $N_{\lambda,\mu}$ for any $d_\lambda \in \left(0, \frac{1}{N}S^{\frac{N}{2}} - \tau\right)$ and $\lambda > \hat{\lambda}$ where τ is as in Lemma 3.2.*

Theorem 3.5. *There is λ^* such that the mountain pass level $c_{\lambda,\mu}$ is a critical level of $I_{\lambda,\mu}$ for all $\lambda \geq \lambda^*$, that is $u_{\lambda,\mu} \in E_{\lambda,\mu}$ verifying*

$$I_{\lambda,\mu}(u_{\lambda,\mu}) = c_{\lambda,\mu} \qquad \textit{and} \qquad I'_{\lambda,\mu}(u_{\lambda,\mu}) = 0$$

PROOF. From Lemma 3.2, there is $\lambda^* = \lambda^*(\tau)$, $\forall\lambda \geq \lambda^*$, $c_{\lambda,\mu} < \frac{1}{N}S^{\frac{N}{2}} - \tau$. Proposition 3.1 implies that $I_{\lambda,\mu}$ satisfies in $(PS)_{c_{\lambda,\mu}}$ Thus, by mountain pass theorem due to Ambrosetti-Rabinowitz [1], $c_{\lambda,\mu}$ is a critical level of $I_{\lambda,\mu}$ for all $\lambda \geq \lambda^*$.

Definition 3.1. *$(u_n) \subset H^1(\mathbb{R}^N)$ is called a $(PS)_{c,\infty}$ if:*

$$\begin{aligned} &u_n \in E_{\lambda_n} \qquad \textit{and} \qquad \lambda_n \to +\infty, \\ &I_{\lambda_n,\mu}(u_n) \to c, \qquad \textit{for some} \qquad c \in R, \\ &\|I'_{\lambda_n,\mu}(u_n)\|_{E'_{\lambda_n}} \to 0. \end{aligned} \tag{28}$$

Theorem 3.6. *Let* (u_n) *be a* $(PS)_{c,\infty}$ *sequence for* $c \in \left(0, \frac{1}{N} S^{\frac{N}{2}}\right)$*. Then, there is a subsequence of* (u_n) *and* $u \in H^1(\mathbb{R}^N)$ *such that*

$$u_n \rightharpoonup u \qquad in \qquad H^1(\mathbb{R}^N).$$

- *i)* $u \equiv 0$ *in* $\mathbb{R}^N \setminus \Omega$*.*
- *ii)* $\|u_n - u\|^2_{\lambda_n} \to 0$*.*
- *iii) Moreover,*

$$\begin{aligned} u_n \to u \qquad in \qquad H^1(\mathbb{R}^N), \\ \lambda_n \int V(x)|u_n|^2 dx \to 0, \\ \int_{\mathbb{R}\setminus\Omega} (|\nabla u_n|^2 + \lambda_n V(x)|u_n|^2) dx \to 0, \\ \|u_n\|^2_{\lambda_n} \to \int_\Omega |\nabla u|^2 dx = \|u\|^2_{H^1_0(\Omega)}. \end{aligned} \tag{29}$$

- *iv)* *u is a weak solution of the problem* (6)*.*

Proof. As Lemma 3.3 implying that $(\|u_n\|_{\lambda_n})$ is bounded in R and so (10) implying that (u_n) is bounded in $H^1(\mathbb{R}^N)$ thus there exists a subsequence of (u_n) such that

$$u_n \rightharpoonup u \qquad in \qquad H^1(\mathbb{R}^N).$$

For i) set

$$C_m := \{x \in \mathbb{R}^3 : V(x) > \frac{1}{m}\}.$$

Hence,

$$\bigcup_{m=1}^{+\infty} C_m = \mathbb{R}^N \setminus \bar{\Omega}.$$

Note that,

$$\int_{C_m} |u_n|^2 dx \le \frac{m}{\lambda_n} \|u_n\|^2_{\lambda_n},$$

Fatou's Lemma implies that

$$\int_{C_m} |u|^2 dx \le \liminf_{n\to+\infty} \int_{C_m} |u_n|^2 dx \le \liminf_{n\to+\infty} \frac{m}{\lambda_n} \|u_n\|^2_{\lambda_n} = 0.$$

This implies that $u = 0$ almost everywhere in

$$\mathbb{R}^N \setminus \bar{\Omega}.$$

ii) From i) $\|u\|^2_{H^1_0(\Omega)} = \|u\|^2_{\lambda_n}$ and

$$\|u_n - u\|^2_{\lambda_n} = \|u_n\|^2_{\lambda_n} - \|u\|^2_{H^1_0(\Omega)} + o_n(1) \tag{30}$$

and since $(u_n - u)$ is bounded so $\|u_n\|_{\lambda_n}$ would be a bounded sequence.

$$\begin{aligned}\|u_n\|^2_{\lambda_n} - \|u_n\|^q_q &= I'_{\lambda,\mu}(u_n)u_n + \int (\mu|u_n|^r + |u_n|^{2^*})dx \\ &= \int (\mu|u_n|^r + |u_n|^{2^*})dx + o_n(1).\end{aligned} \tag{31}$$

On the other hand, since $I'_{\lambda,\mu}(u_n)u \to 0$ so

$$\int_\Omega \nabla u_n \nabla u dx + \int_\Omega |\nabla u_n|^{q-2}\nabla u_n \nabla u - \int_\Omega (\mu|u_n|^{r-2}u_n + |u_n|^{2^*}u_n)udx = o_n(1).$$

It follows that

$$\int |\nabla u|^2 dx + \int |\nabla u|^q dx - \int (\mu|u_n|^r + |u_n|^{2^*})dx = o_n(1), \tag{32}$$

Combining (30), (31) and (32)

$$\begin{aligned}\|u_n - u\|^2_{\lambda_n} &= \|u_n\|^2_{\lambda_n} + \|u_n\|^q_q - (\|u\|^2_{H^1_0(\Omega)} - \|u\|^q_q) - (\|u_n\|^q_q - \|u\|^q_q) \\ &= \int (\mu|u_n|^r + |u_n|^{2^*})dx - \int (\mu|u|^r + |u|^{2^*})dx + (\|u\|^q_q - \|u_n\|^q_q) + o_n(1)\end{aligned} \tag{33}$$

that is

$$\|v_n\|^2_{\lambda_n} = \mu|v_n|^r + |v_n|^{2^*} + (\|u\|^q_q - \|u_n\|^q_q) + o_n(1),$$

where $v_n = u_n - u$.
Corollary 3.1 implies that, $v_n \to 0$ in $L^r(\mathbb{R}^N)$ and from Brézis-Lieb lemma

$$\|v_n\|^2_{\lambda_n} + \|v_n\|^q_q = |v_n|^{2^*}_6 + o_n(1).$$

Now, the same arguments used in the proof of Proposition 3.1 shows that

$$\|v_n\|^2_{\lambda_n} \to 0.$$

iii) It comes from the following inequality and i) that $u \equiv 0$ on Ω^c:

$$0 \le \lambda_n \int V(x)|u_n|^2 dx = \lambda_n(\int_\Omega V(x)|u_n|^2 dx + \int_{\Omega^c} V(x)|u_n|^2 dx) = \lambda_n \int V(x)|u_n - u|^2 dx \le \|v_n\|^2_{\lambda_n}.$$

and

$$\|v_n\|^2_{\lambda_n} = \int (|\nabla v_n|^2 + \lambda_n V(x)|v_n|^2)dx \ge \int_{\Omega^c} (|\nabla v_n|^2 + \lambda_n V(x)|v_n|^2)dx \ge 0.$$

Finally

$$\|v_n\|^2_{\lambda_n} = \int (|\nabla v_n|^2 + \lambda_n V(x)|v_n|^2)dx = \int_\Omega (|\nabla v_n|^2 + \lambda_n V(x)|v_n|^2)dx + o_n(1) = \int_\Omega |\nabla v_n|^2 + o_n(1).$$

iv) For $\varphi \in C_0^\infty(\Omega)$ we have

$$I'_{\lambda,\mu}(u_n)\varphi := \int_\Omega \nabla u_n \nabla \varphi dx + \int_\Omega |\nabla u_n|^{q-2} \nabla u_n \nabla \varphi dx - \mu \int |u_n|^{r-2} u_n \varphi dx - \int |u_n|^4 u_n \varphi dx, \tag{34}$$

(u_n) is a $(PS)_{c,\infty}$ sequence, so

$$I'_{\lambda,\mu}(u_n)\varphi \to 0. \tag{35}$$

Since $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$

$$\int_\Omega \nabla u_n \nabla \varphi dx \to \int_\Omega \nabla u \nabla \varphi dx, \tag{36}$$

and

$$\int_\Omega (\mu |u_n|^{r-2} u_n + |u_n|^{2^*-2} u_n)\varphi dx \to \int_\Omega (\mu |u|^{r-2} u + |u|^{2^*-2} u)\varphi dx \tag{37}$$

and similar [2, Theorem 5.9]

$$\int_\Omega |\nabla u_n|^{q-2} \nabla u_n \nabla \varphi dx \to \int_\Omega |\nabla u|^{q-2} \nabla u \nabla \varphi dx. \tag{38}$$

Therefore,

$$\int_\Omega \nabla u \nabla \varphi dx + \int_\Omega |\nabla u|^{q-2} \nabla u \nabla \varphi dx = \mu \int |u|^{r-2} u \varphi dx + \int_\Omega |u|^{2^*-2} u \varphi dx, \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{39}$$

$C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ so

$$\int_\Omega \nabla u v dx + \int_\Omega |\nabla u|^{q-2} \nabla u v dx = \mu \int_\Omega |u|^{r-2} u v dx + \int_\Omega |u|^{2^*-2} u v dx, \qquad \forall v \in H_0^1(\Omega).$$

Lemma 3.7. *If $\lambda_n \to +\infty$, then $c_{\lambda_n,\mu} \to c_\mu$.*

PROOF. See Lemma 3.13 of [4].

The proof of corollaries 3, 4 and 4 are similar to those one in [4; 7].

Corollary 3.3. *Let $\lambda_n \in \mathbb{R}^+$ be a sequence verifying $\lambda_n \to +\infty$ and $u_{\lambda_n,\mu}$ the ground state solution obtained in Theorem 3.5. Then, there is a subsequence of $(u_{\lambda_n,\mu})$ still denoted by itself, and $u \in H_0^1(\Omega)$ such that $u_{\lambda_n,\mu} \to u$ in $H_0^1(\Omega)$ and u is a ground state solution of the limit problem* (6)*.*

Corollary 3.4. *There are $\lambda^* > 0$ large and $\mu^* > 0$ small such that*

$$m(\mu) < 2c_{\lambda,\mu}, \quad \lambda \geq \lambda^* \quad \text{and} \quad \forall \mu \in (0, \mu^*).$$

Corollary 3.5. *If $u \in E_\lambda$ is a nontrivial critical point of $I_{\lambda,\mu}$ such that $I_{\lambda,\mu} \leq m(\mu)$ then u is positive or u is negative.*

corollary 3.5 implies that the nontrivial critical points of $I_{\lambda,\mu}$ are positive solutions of problem (1).

4. Proof of main theorem

Choose $R > 0$ such that $\bar{\Omega} \subset B_R = \{x \in \mathbb{R}^N : |x| < R\}$ and set

$$\xi(t) = \begin{cases} 1, & 0 \le t \le R \\ \frac{R}{t}, & r \ge R. \end{cases} \tag{40}$$

Moreover, we define

$$\beta(u) := \frac{\int |u|^{2^*} \xi(|x|) x dx}{\int |u|^{2^*} dx} \qquad for \quad u \in N_{\lambda,\mu}.$$

Lemma 4.1. *There is $\lambda^* > 0$ such that if $u \in N_{\lambda,\mu}$ and $I_{\lambda,\mu}(u) \le m(\mu)$ then $\beta(u) \in \Omega_r^+$ for all $\lambda \ge \lambda^*$.*

Proof. If the conclusion is not true, then there would exist sequences $\lambda_n \to +\infty$ and $u_n \in N_{\lambda_n,\mu}$ in which $I_{\lambda_n,\mu}(u_n) \le m(\mu)$ and

$$\beta(u_n) \notin \Omega_r^+, \qquad \forall n \in \mathbb{N}.$$

Form (14), clearly the sequence $(\|u_n\|_{\lambda_n})$ is bounded in $\mathbb{R}$; (up to subsequence). There is, $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} & u_n \rightharpoonup u \qquad in \quad H^1(\mathbb{R}^N), \\ & u_n(x) \to u(x) \qquad a.e. \quad in \quad \mathbb{R}^N, \qquad (41) \\ & u_n \to u \qquad in \quad L^t_{Loc}(\mathbb{R}^N) \qquad for \qquad t \in [1, 2^*). \qquad (42) \end{aligned}$$

Moreover,

$$\|v_n\|^2_{\lambda_n} + \|v_n\|^q_q = \mu |v_n|^r_r + |v_n|^{2^*}_{2^*} + o_n(1),$$

where $v_n = u_n - u$. By Corollary 3.1 $|v_n|^r_r \to 0$, and so

$$\|v_n\|^2_{\lambda_n} + \|v_n\|^q_q = |v_n|^{2^*}_{2^*} + o_n(1).$$

Arguing as in the proof of Proposition 3.1,

$$\|v_n\|_{\lambda_n} \to 0$$

This limit combined with $\|v\|_\lambda \ge \sigma$ implies that

$$u_n \to u \qquad in \qquad H^1(\mathbb{R}^N). \ \ u \ne 0, I'_\mu(u)u = 0 \qquad \text{and} \qquad I_{\lambda,\mu}(u_n) \to I_\mu(u).$$

Thus $u \in N_\mu$ and $I_\mu(u) \le m(\mu)$ Applying the Lemma 2.2, so $\beta_0(u) \in \Omega^+_{r/2}$ then

$$\beta_0(u) = \lim_{n\to\infty} \beta(u_n) \in \Omega^+_{r/2}.$$

which is a contradiction.

Lemma 4.2. *If u is a critical point of $I_{\lambda,\mu}$ on $N_{\lambda,\mu}$ then it is a critical point of $I_{\lambda,\mu}$ in $H_0^1(\Omega)$.*

PROOF. Let $u \in N_{\lambda,\mu}$ then $I'_{\lambda,\mu}(u)u = 0$.
On the other hand, by the theory of Lagrange multipliers, there exists $\theta \in \mathbb{R}$ such that $I'_{\lambda,\mu}(u) = \theta J'_{\lambda,\mu}(u)$. Thus,

$$0 = I'_{\lambda,\mu}(u)u = \theta J'_{\lambda,\mu}(u)u.$$

Using (15), so $\theta = 0$, Thus u is a critical point of $I_{\lambda,\mu}$ in $H_0^1(\Omega)$.
Let $u_r \in H_0^1(B_r(0))$ is a positive radial ground state solution for $I_{\mu,r}$, that is

$$I_{\mu,r}(u_r) = m(\mu) = \inf_{u \in N_{\mu,r}} I_{\mu,r}(u) \qquad \text{and} \qquad I'_{\mu,r}(u_r) = 0.$$

Define $\phi : \Omega^- \to I_{\mu,r}^{m(\mu)}$, where $I_{\mu,r}^{m(\mu)} = \{u \in N_{\mu,r} : I_{\mu,r}(u) \leq m(\mu)\}$

$$\begin{aligned}
\phi_r(y)x &= u_r(|x-y|), \qquad x \in B_r(y) \\
\beta(\phi_r(y)) &= \frac{\int |\phi_r(y)(x)|^{2^*}\xi(|x|)x dx}{\int |\phi_r(y)(x)|^{2^*} dx} \\
&= \frac{\int_{B_r(y)} |u_r(|x-y|)|^{2^*}\xi(|x|)x dx}{\int_{B_r(y)} |u_r(|x-y|)|^{2^*} dx} \\
&= \frac{\int_{B_r(0)} |u_r(|z|)|^{2^*}(y+z) dz}{\int_{B_r(0)} |u_r(|z|)|^{2^*} d} \\
&= y, \qquad \forall y \in \Omega_r^-.
\end{aligned} \tag{43}$$

Lemma 4.3. *If $\mu \in (0, \mu^*)$, μ^* is given in Lemma 2.2, then*

$$cat(I_{\lambda,\mu}^{m(\mu)}) \geq cat(\Omega).$$

PROOF. Suppose that

$$I_{\lambda,\mu}^{m(\mu)} = A_1 \cup A_2 \cup ... \cup A_n.$$

where A_j, $j = 1, 2, ..., n$, is closed and contractible in $I_{\mu,r}^{m(\mu)}$, that is, there exists $h_j \in C([0,1] \times A_j, I_{\lambda,\mu}^{m(\mu)})$ such that

$$h_j(0,u) = u \qquad \text{and} \qquad h_j(1,u) = w_j \qquad \forall u \in A_j,$$

where $w_j \in A_j$ is fixed. Any $\phi_r^{-1}(A_j) = B_j$, is closed for $1 \leq j \leq n$,

$$\Omega_r^- = B_1 \cup B_2 \cup ... \cup B_n.$$

Consider the deformation map $g_j : [0,1] \times B_j \to \Omega_r^+$ given by

$$g_j(t,y) = \beta(h_j(t, \phi_r(y))).$$

Then

$$\begin{aligned}
g_j(0,y) &= \beta(h_j(0, \phi_r(y))) = \beta(\phi_r(y)) = y \\
g_j(1,y) &= \beta(h_j(1, \phi_r(y))) = \beta(w_j).
\end{aligned}$$

Thus any β_j is contractible in Ω_r^+. Lemma 4.1 implies that

$$cat(\Omega) = cat_{\Omega_r^+}(\Omega_r^-) \leq n.$$

To prove Theorem 1.1 we need the following results.

Proof. of Theorem 1.1. For $0 < \mu < \mu^*$ and $\lambda > \lambda^*$,

$$\Omega_r^- \xrightarrow{\phi_r} I_{\lambda,\mu}^{m(\mu)} \xrightarrow{\beta} \Omega_r^+$$

Let $u_r \in H_0^1(B_r) \subset E$ be a minimizer of $I_{\mu,r}$ on $N_{\mu,r}$ with $u_r > 0$ and $\varphi(x) =_r (|. - x|)$, so $\varphi(x) = 0$ in $\mathbb{R}^3 \setminus \Omega$ for every $x \in \Omega_r^-$.

$$\varphi(x) \in N_{\lambda,\mu} \qquad and \qquad I_{\lambda,\mu}(\varphi(x)) = I_{\lambda,r}(\varphi(x)) = m(\mu).$$

Clearly, I and β are even and $\beta \circ \varphi$ is a homotopy equivalence.
Thus Lemma 4.3 implies that $cat(\Omega) \leq cat(I_{\lambda,\mu}^{m(\mu)})$. Since $I_{\lambda,\mu}$ satisfies the $(PS)_c$ condition on $N_{\lambda,\mu}$ for $c \leq m(\mu)$ An standard Lusternik-Schnirelmann theory and Lemma 4.3 yields at least $cat(\Omega)$ of critical points in $N_{\lambda,\mu}$ and consequently, critical points in E_λ. Corollary 3.2 conclude that $I_{\lambda,\mu}$ has at least $cat(\Omega)$ positive solutions.

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