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NUMERICAL STABILITY OF THE BRANCHED CONTINUED FRACTION EXPANSION OF THE HORN'S HYPERGEOMETRIC FUNCTION H_4

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In this paper, we consider some numerical aspects of branched continued fractions as special families of functions to represent and expand analytical functions of several complex variables, including generalizations of hypergeometric functions. The backward recurrence algorithm is one of the basic tools of computation approximants of branched continued fractions. Like most recursive processes, it is susceptible to error growth. Each cycle of the recursive process not only generates its own rounding errors but also inherits the rounding errors committed in all the previous cycles. On the other hand, in general, branched continued fractions are a non-linear object of study (the sum of two fractional-linear mappings is not always a fractional-linear mapping). In this work, we are dealing with a confluent branched continued fraction, which is a continued fraction in its form. The essential difference here is that the approximants of the continued fraction are the so-called figure approximants of the branched continued fraction. An estimate of the relative rounding error, produced by the backward recurrence algorithm in calculating an n th approximant of the branched continued fraction expansion of Horn's hypergeometric function H_4 , is established. The derivation uses the methods of the theory of branched continued fractions, which are essential in developing convergence criteria. The numerical examples illustrate the numerical stability of the backward recurrence algorithm.

1. Introduction. Numerous studies show that branched continued fraction expansions provide a useful means for representing and extending of special functions, including generalized hypergeometric functions [3, 33], Appell's hypergeometric functions [11, 20, 25], Horn's hypergeometric functions [2, 4, 5, 6, 15], Lauricella–Saran's hypergeometric functions [1, 12, 24], and also some other functions [10, 17, 18, 29]. To render branched continued fractions more useful in computational, one needs to know more about their numerical stability, which is the main concern of this paper.

The backward recurrence algorithm for computing the n th approximant

$$f_n = 1 + a_{1,0} + \frac{a_{0,1}}{1 + a_{1,1} + \frac{a_{0,2}}{1 + a_{1,2} + \frac{a_{0,3}}{1 + \dots + a_{1,n-2} + \frac{a_{0,n-1}}{1 + a_{1,n-1} + a_{0,n}}}}$$

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of a branched continued fraction

$$1 + a_{1,0} + \frac{a_{0,1}}{1 + a_{1,1} + \frac{a_{0,2}}{1 + a_{1,2} + \frac{a_{0,3}}{1 + \dots}}} \quad (1)$$

consists of setting $G_n^{(n)} = 1$ and computing successively, from tail to head,

$$G_k^{(n)} = 1 + a_{1,k} + \frac{a_{0,k+1}}{G_{k+1}^{(n)}}, \quad n-1 \geq k \geq 0.$$

Thus,

$$f_n = G_0^{(n)}.$$

It should be noted that the branched continued fraction (1) is a confluent branched continued fraction, which is a continued fraction in its form. The essential difference here is that the approximants of the continued fraction are the so-called figure approximants of the branched continued fraction (see, [8, p. 18]).

Branched continued fractions of the structure (1) appeared thanks to works [4, 16] and are related to Horn's hypergeometric function H_4 (see [23])

$$H_4(a, b; c, d; \mathbf{z}) = \sum_{r,s=0}^{\infty} \frac{(a)_{2r+s} (b)_s z_1^r z_2^s}{(c)_r (d)_s r! s!}, \quad |z_1| < p, |z_2| < q,$$

where $a, b, c, d \in \mathbb{C}$; $c, d \notin \{0, -1, -2, \dots\}$; p and q are positive numbers such that $4p = (q-1)^2$ and $q \neq 1$; $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)\dots(\alpha+n-1)$, $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$. The paper [4] provides the formal expansion

$$\frac{H_4(a, b; c, b; \mathbf{z})}{H_4(a+1, b; c+1, b; \mathbf{z})} = 1 - z_2 - \frac{h_1 z_1}{1 - z_2 - \frac{h_2 z_1}{1 - z_2 - \frac{h_3 z_1}{1 - \dots}}}, \quad (2)$$

where

$$h_k = \frac{(2c - a + k - 1)(a + k)}{(c + k - 1)(c + k)}, \quad k \geq 1, \quad (3)$$

as well as,

$$\frac{H_4(a, d+1; c, d; \mathbf{z})}{H_4(a+1, d+1; c, d+1; \mathbf{z})} = 1 - \frac{d-a}{d} z_2 - \frac{m_1 z_1}{1 - z_2 - \frac{m_2 z_1}{1 - z_2 - \frac{m_3 z_1}{1 - \dots}}}, \quad (4)$$

where $m_1 = \frac{2(a+1)}{c}$, $m_k = \frac{(2c - a + k - 3)(a + k)}{(c + k - 2)(c + k - 1)}$ ($k \geq 2$), and

$$\frac{H_4(a, d+1; c, d; \mathbf{z})}{H_4(a, d+2; c, d+1; \mathbf{z})} = 1 + \frac{v_0 z_2}{1 + v_1 z_2 + \frac{u_1 z_1}{1 + v_2 z_2 + \frac{u_2 z_1}{1 + \dots}}}, \quad (5)$$

$$\text{where } v_0 = \frac{a}{d(d+1)}, \quad v_1 = \frac{a}{d+1} - 1, \quad u_1 = -\frac{2(a+1)}{c},$$

$$v_k = -1, \quad u_k = -\frac{(2c-a+k-3)(a+k)}{(c+k-2)(c+k-1)} \quad (k \geq 2).$$

Some questions of convergence of the expansions (2), (4), and (5) were discussed in [4, 13, 14, 15]. Numerical aspects related to the backward recurrence algorithm for computing the approximants of continued fractions were considered in [7, 9, 27, 28, 30]. Some analogous results concerning branched continued fractions can be found in [19, 21, 22, 25, 31, 32].

2. Estimates of relative rounding error. In this section, we will establish an estimate of the relative rounding error produced by the backward recurrence algorithm in calculating the n th approximant of (2).

Let us recall the necessary concepts. Let n be an arbitrary fixed natural number. For each $1 \leq k \leq n$, let $\widehat{a}_{1,k-1}$ and $\widehat{a}_{0,k}$ denote rounded values of the elements $a_{1,k-1}$ and $a_{0,k}$, respectively, of a given branched continued fraction (1). The number

$$\widehat{f}_n = 1 + \widehat{a}_{1,0} + \frac{\widehat{a}_{0,1}}{1 + \widehat{a}_{1,1} + \frac{\widehat{a}_{0,2}}{1 + \widehat{a}_{1,2} + \frac{\widehat{a}_{0,3}}{1 + \dots + \widehat{a}_{1,n-2} + \frac{\widehat{a}_{0,n-1}}{1 + \widehat{a}_{1,n-1} + \widehat{a}_{0,n}}}}$$

is the computed (approximate) value of $f_n = G_0^{(n)}$.

Definition 1. A *numerical stability set* Ω is a set to which for any $\varepsilon > 0$ one can find $\delta > 0$ depending only on ε and Ω such that, for all $n \geq 1$

$$\left| \widehat{f}_n - f_n \right| < \varepsilon \cdot |f_n|$$

for every branched continued fraction (1) with all $a_{1,k-1}, a_{0,k} \in \Omega$ and $\widehat{a}_{1,k-1}, \widehat{a}_{0,k} \in \Omega$ such that, for all $k \geq 1$

$$\left| \frac{\widehat{a}_{1,k-1} - a_{1,k-1}}{a_{1,k-1}} \right| < \delta \quad \text{and} \quad \left| \frac{\widehat{a}_{0,k} - a_{0,k}}{a_{0,k}} \right| < \delta.$$

We set

$$G_n^{(n)}(\mathbf{z}) = 1, \quad n \geq 1, \quad (6)$$

and

$$G_k^{(n)}(\mathbf{z}) = 1 - z_2 - \frac{h_{k+1}z_1}{1 - z_2 - \frac{h_{k+2}z_1}{1 - \dots - z_2 - \frac{h_{n-1}z_1}{1 - z_2 - h_n z_1}}}$$

for $1 \leq k \leq n-1$, $n \geq 2$, where h_k , $1 \leq k \leq n-1$, $n \geq 2$, are defined by (3). It follows that

$$G_k^{(n)}(\mathbf{z}) = 1 - z_2 - \frac{h_{k+1}z_1}{G_{k+1}^{(n)}(\mathbf{z})}, \quad 1 \leq k \leq n-1, \quad n \geq 2, \quad (7)$$

and n th approximant of (2) we write as

$$f_n(\mathbf{z}) = 1 - z_2 - \frac{h_1 z_1}{G_1^{(k)}(\mathbf{z})}.$$

Let α_1 , α_2 , and β_k , $1 \leq k \leq n$, denote the relative errors in the rounded values \widehat{z}_1 , \widehat{z}_2 , and \widehat{h}_k , $1 \leq k \leq n$, of z_1 , z_2 , and h_k , $1 \leq k \leq n$, respectively, so that

$$\widehat{z}_1 = z_1(1 + \alpha_1), \quad \widehat{z}_2 = z_2(1 + \alpha_2), \quad \widehat{h}_k = h_k(1 + \beta_k), \quad 1 \leq k \leq n. \quad (8)$$

Similarly, let $\varepsilon_k^{(n)}$, $0 \leq k \leq n$, denote the relative errors in $\widehat{G}_k^{(n)}(\widehat{\mathbf{z}})$, the approximation to $G_k^{(n)}(\mathbf{z})$ from (6)–(7) and

$$G_0^{(n)}(\mathbf{z}) = 1 - z_2 - \frac{h_1 z_1}{G_1^{(n)}(\mathbf{z})}$$

using \widehat{z}_1 , \widehat{z}_2 , and \widehat{h}_k , $1 \leq k \leq n$. Thus,

$$\widehat{G}_k^{(n)}(\widehat{\mathbf{z}}) = G_k^{(n)}(\mathbf{z})(1 + \varepsilon_k^{(n)}), \quad 0 \leq k \leq n, \quad (9)$$

and

$$\widehat{G}_n^{(n)}(\widehat{\mathbf{z}}) = G_n^{(n)}(\mathbf{z}) = 1, \quad \text{and} \quad \varepsilon_n^{(n)} = 0. \quad (10)$$

Also, for convenience, let $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\beta}_k$, $1 \leq k \leq n$, and $\widehat{\varepsilon}_k^{(n)}$, $0 \leq k \leq n$, denote the relative errors defined by $z_1 = \widehat{z}_1(1 + \widehat{\alpha}_1)$, $z_2 = \widehat{z}_2(1 + \widehat{\alpha}_2)$, $h_k = \widehat{h}_k(1 + \widehat{\beta}_k)$, $1 \leq k \leq n$, and

$$G_k^{(n)}(\mathbf{z}) = \widehat{G}_k^{(n)}(\widehat{\mathbf{z}})(1 + \widehat{\varepsilon}_k^{(n)}), \quad 0 \leq k \leq n,$$

respectively.

Next, we establish recurrence relations for relative errors $\varepsilon_k^{(n)}$, $0 \leq k \leq n - 1$. For arbitrary k , $0 \leq k \leq n - 1$, one obtains

$$\begin{aligned} \varepsilon_k^{(n)} &= \frac{\widehat{G}_k^{(n)}(\widehat{\mathbf{z}}) - G_k^{(n)}(\mathbf{z})}{G_k^{(n)}(\mathbf{z})} = \frac{1}{G_k^{(n)}(\mathbf{z})} \left(1 - \widehat{z}_2 - \frac{\widehat{h}_{k+1} \widehat{z}_1}{\widehat{G}_{k+1}^{(n)}(\widehat{\mathbf{z}})} \right) - 1 = \\ &= \frac{1}{G_k^{(n)}(\mathbf{z})} \left(1 - z_2(1 + \alpha_2) - \frac{h_{k+1}(1 + \beta_{k+1})z_1(1 + \alpha_1)}{G_{k+1}^{(n)}(\mathbf{z})(1 + \varepsilon_{k+1}^{(n)})} \right) - 1 = \\ &= \frac{1}{G_k^{(n)}(\mathbf{z})} - \frac{z_2(1 + \alpha_2)}{G_k^{(n)}(\mathbf{z})} - \frac{h_{k+1}(1 + \beta_{k+1})z_1(1 + \alpha_1)(1 + \widehat{\varepsilon}_{k+1}^{(n)})}{G_k^{(n)}(\mathbf{z})G_{k+1}^{(n)}(\mathbf{z})} - 1. \end{aligned}$$

It follows from (7) that

$$\frac{1}{G_k^{(n)}(\mathbf{z})} = 1 + \frac{z_2}{G_k^{(n)}(\mathbf{z})} + \frac{h_{k+1} z_1}{G_k^{(n)}(\mathbf{z})G_{k+1}^{(n)}(\mathbf{z})}.$$

Then,

$$\begin{aligned} \varepsilon_k^{(n)} &= \frac{z_2}{G_k^{(n)}(\mathbf{z})} - \frac{z_2(1 + \alpha_2)}{G_k^{(n)}(\mathbf{z})} - \frac{h_{k+1} z_1}{G_k^{(n)}(\mathbf{z})G_{k+1}^{(n)}(\mathbf{z})} ((1 + \beta_{k+1})(1 + \alpha_1)(1 + \widehat{\varepsilon}_{k+1}^{(n)}) - 1) = \\ &= -\frac{z_2 \alpha_2}{G_k^{(n)}(\mathbf{z})} - \frac{h_{k+1} z_1}{G_k^{(n)}(\mathbf{z})\widehat{G}_{k+1}^{(n)}(\widehat{\mathbf{z}})} (\beta_{k+1} + \alpha_1 + \beta_{k+1} \alpha_1) - \frac{h_{k+1} z_1}{G_k^{(n)}(\mathbf{z})G_{k+1}^{(n)}(\mathbf{z})} \widehat{\varepsilon}_{k+1}^{(n)}. \end{aligned}$$

Thus, for each $0 \leq k \leq n - 1$,

$$\varepsilon_k^{(n)} = -\frac{z_2 \alpha_2}{G_k^{(n)}(\mathbf{z})} - \frac{h_{k+1} z_1}{G_k^{(n)}(\mathbf{z})\widehat{G}_{k+1}^{(n)}(\widehat{\mathbf{z}})} (\beta_{k+1} + \alpha_1 + \beta_{k+1} \alpha_1) - \frac{h_{k+1} z_1}{G_k^{(n)}(\mathbf{z})G_{k+1}^{(n)}(\mathbf{z})} \widehat{\varepsilon}_{k+1}^{(n)}. \quad (11)$$

Similarly, for relative errors $\widehat{\varepsilon}_k^{(n)}$, $0 \leq k \leq n-1$, one obtains

$$\widehat{\varepsilon}_k^{(n)} = -\frac{\widehat{z}_2 \widehat{\alpha}_2}{\widehat{G}_k^{(n)}(\mathbf{z})} - \frac{\widehat{h}_{k+1} \widehat{z}_1}{\widehat{G}_k^{(n)}(\widehat{\mathbf{z}}) \widehat{G}_{k+1}^{(n)}(\mathbf{z})} (\widehat{\beta}_{k+1} + \widehat{\alpha}_1 + \widehat{\beta}_{k+1} \widehat{\alpha}_1) - \frac{\widehat{h}_{k+1} \widehat{z}_1}{\widehat{G}_k^{(n)}(\widehat{\mathbf{z}}) \widehat{G}_{k+1}^{(n)}(\widehat{\mathbf{z}})} \varepsilon_{k+1}^{(n)}. \quad (12)$$

Combining (11) and (12) with

$$g_k^{(n)}(\mathbf{z}) = -\frac{h_k z_1}{G_{k-1}^{(n)}(\mathbf{z}) G_k^{(n)}(\mathbf{z})}, \quad \widehat{g}_k^{(n)}(\widehat{\mathbf{z}}) = -\frac{\widehat{h}_k \widehat{z}_1}{\widehat{G}_{k-1}^{(n)}(\widehat{\mathbf{z}}) \widehat{G}_k^{(n)}(\widehat{\mathbf{z}})}, \quad 1 \leq k \leq n-1, \quad (13)$$

one easily obtains the relation

$$\varepsilon_0^{(n)} = \sum_{k=1}^n \frac{(-1)^k}{\widetilde{G}_{k-1}^{(n)}} \left(z_{2,k} \alpha_{2,k} + \frac{\widetilde{h}_k z_{1,k} (\widetilde{\beta}_k + \alpha_{1,k} + \widetilde{\beta}_k \alpha_{1,k})}{\widetilde{G}_k^{(n)}} \right) \prod_{r=1}^{k-1} \widetilde{g}_r^{(n)}, \quad (14)$$

where, for $p = 1, 2$, $z_{p,k} = \begin{cases} \widehat{z}_p, & \text{if } k \text{ even,} \\ z_p, & \text{if } k \text{ odd,} \end{cases}$ $\alpha_{p,k} = \begin{cases} \widehat{\alpha}_p, & \text{if } k \text{ even,} \\ \alpha_p, & \text{if } k \text{ odd,} \end{cases}$ and

$$\widetilde{h}_k = \begin{cases} \widehat{h}_k, & \text{if } k \text{ even,} \\ h_k, & \text{if } k \text{ odd,} \end{cases} \quad \widetilde{g}_k^{(n)} = \begin{cases} g_k^{(n)}(\mathbf{z}), & \text{if } k \text{ even,} \\ \widehat{g}_k^{(n)}(\widehat{\mathbf{z}}), & \text{if } k \text{ odd,} \end{cases} \quad \widetilde{G}_k^{(n)} = \begin{cases} G_k^{(n)}(\mathbf{z}), & \text{if } k \text{ even,} \\ \widehat{G}_k^{(n)}(\widehat{\mathbf{z}}), & \text{if } k \text{ odd.} \end{cases}$$

For convenience, we set $\varepsilon_n = \varepsilon_0^{(n)}$. Now we are ready to prove our main result.

Theorem 1. *Let there exist a constant α , $0 < \alpha < 1$, such that*

$$|\alpha_1| \leq \alpha, \quad |\alpha_2| \leq \alpha, \quad \text{and} \quad |\beta_k| \leq \alpha \quad \text{for all } k \geq 1, \quad (15)$$

where α_1 , α_2 , and β_k , $k \geq 1$, are relative errors of z_1 , z_2 , and h_k , $k \geq 1$, respectively, which are defined in (8) for all $n \geq 1$. Then:

(A) *The set*

$$\mathbb{H}_{h,l} = \{ \mathbf{z} \in \mathbb{C}^2 : |z_1| < l(1-l)/(2h), |z_2| < (1-l)/2 \}, \quad (16)$$

where

$$h = \max_{k \in \mathbb{N}} \{ |h_k|, |\widehat{h}_k| \}, \quad l \in (0, 1/3) \cup (1/3, 1), \quad (17)$$

forms the numerical stability set of the branched continued fraction (2).

(B) *If ε_n denotes the relative errors of n th approximant of (2), then, for $n \geq 1$,*

$$|\varepsilon_n| \leq \frac{4\alpha}{(1+l+|3l-1|)(1-\alpha)} \left(\frac{1-l}{2} + \frac{2l(1-l)}{1+l+|3l-1|} \left(2 + \frac{\alpha}{1-\alpha} \right) \right) \frac{1-\eta^n}{1-\eta}, \quad (18)$$

where

$$\eta = \begin{cases} 2l/(1-l), & \text{if } 1 < l < 1/3, \\ (1-l)/(2l), & \text{if } 1/3 < l < 1. \end{cases}$$

Remark 1. It follows from [15, Theorem 2] that if a and c are complex constants such that

$$|h_k| + \operatorname{Re}(h_k) \leq pq(1 - q) \quad \text{for all } k \geq 1,$$

where h_k , $k \geq 1$, are defined by (3), $c \notin \{0, -1, -2, \dots\}$, p is a positive number, $0 < q < 1$, then the branched continued fraction (2) converges uniformly on every compact subset of the domain (16).

Proof of Theorem 1. We consider the periodic continued fraction

$$(1+l)/2 - \frac{l(1-l)/2}{(1+l)/2 - \frac{l(1-l)/2}{(1+l)/2 - \frac{l(1-l)/2}{(1+l)/2 - \dots}}}, \quad (19)$$

which is equivalent to

$$(1+l)/2 - \frac{l(1-l)/(1+l)}{1 - \frac{2l(1-l)/(1+l)^2}{1 - \frac{2l(1-l)/(1+l)^2}{1 - \dots}}}, \quad (20)$$

since $(1+l)/2 \neq 0$ (see, [26, Section 2.3]).

It is easy to show that $-\frac{1}{4} < -\frac{2l(1-l)}{(1+l)^2}$, i.e. the elements of (20) satisfy Theorem 3.2 in [26]. According to this theorem, the continued fraction (20) converges, and its value is

$$\begin{aligned} f^* &= \frac{1+l}{2} \left(1 - \frac{1 - 2\sqrt{1/4 - 2l(1-l)/(1+l)^2}}{2} \right) = \\ &= \frac{1+l}{2} \left(1 - \frac{1+l - |1-3l|}{2(1+l)} \right) = \frac{1+l + |1-3l|}{4}. \end{aligned}$$

The continued fraction (19), as equivalent to (20), also converges to the value f^* . In addition, it is easy to show that the approximants

$$f_n^* = (1+l)/2 - \frac{l(1-l)/2}{(1+l)/2 - \frac{l(1-l)/2}{(1+l)/2 - \dots - \frac{l(1-l)/2}{(1+l)/2}}$$

forms a monotonically descending sequence.

Let n be an arbitrary fixed natural number. Let us prove that

$$|G_k^{(n)}(\mathbf{z})| > f_{n-k}^*, \quad 0 \leq k \leq n-1,$$

where $G_k^{(n)}(\mathbf{z})$, $0 \leq k \leq n-1$, are defined by (6)–(7).

If $k = n-1$, we have

$$|G_{n-1}^{(n)}(\mathbf{z})| \geq 1 - |z_2| - |h_n||z_1| > \frac{1+l}{2} - \frac{l(1-l)}{2} > (1+l)/2 - \frac{l(1-l)/2}{(1+l)/2} = f_1^*.$$

Assuming that the inequality (20) is true if $k = s+1 \leq n-1$. Then, for $k = s$ from (7) we obtain

$$|G_s^{(n)}(\mathbf{z})| \geq 1 - |z_2| - \frac{|h_{s+1}||z_1|}{|G_{s+1}^{(n)}(\mathbf{z})|} > (1+l)/2 - \frac{l(1-l)/2}{f_{n-s}^*} = f_{n-s}^*.$$

Since $f_n^* > f^*$ for all $n \geq 1$, then $|G_k^{(n)}(\mathbf{z})| > f^*$ for each $0 \leq k \leq n-1$. Considering $z_1 \neq 0$, let us estimate the values $g_k^{(n)}(\mathbf{z})$, $1 \leq k \leq n-1$, which are defined in (13).

For any k , $1 \leq k \leq n-1$, one obtains

$$\begin{aligned} |g_k^{(n)}(\mathbf{z})| &= \left| \frac{h_k z_1}{G_{k+1}^{(n)}(\mathbf{z}) G_k^{(n)}(\mathbf{z})} \right| = \left| \frac{h_k z_1 / G_k^{(n)}(\mathbf{z})}{1 - z_2 - h_k z_1 / G_k^{(n)}(\mathbf{z})} \right| = \frac{1}{\left| \frac{1 - z_2}{h_k z_1} G_k^{(n)}(\mathbf{z}) - 1 \right|} \leq \\ &\leq \frac{1}{\frac{1 - |z_2|}{|h_k||z_1|} |G_k^{(n)}(\mathbf{z})| - 1} < \frac{1}{\frac{(1+l)(1+l+|1-3l|)}{4l(l-1)} - 1} = \begin{cases} \frac{2l}{1-l}, & \text{if } 1 < l < \frac{1}{3}, \\ \frac{1-l}{2l}, & \text{if } \frac{1}{3} < l < 1. \end{cases} \end{aligned}$$

Now since $\widehat{\mathbf{z}} = (\widehat{z}_1, \widehat{z}_2) \in \Omega^{h,l}$, we have

$$|\widehat{G}_k^{(n)}(\widehat{\mathbf{z}})| > f_{n-k}^* \quad \text{for each } 0 \leq k \leq n-1, \quad (21)$$

and

$$\widehat{g}_k^{(n)}(\widehat{\mathbf{z}}) < \eta \quad \text{for each } 1 \leq k \leq n-1, \quad (22)$$

where $\widehat{G}_k^{(n)}(\widehat{\mathbf{z}})$, $0 \leq k \leq n-1$, and $\widehat{g}_k^{(n)}(\widehat{\mathbf{z}})$, $1 \leq k \leq n-1$, are defined in (9)–(10) and (13), respectively. Further, from the conditions of this theorem it follows

$$|z_{1,k}| < \frac{l(1-l)}{2h}, \quad |z_{2,k}| < \frac{1-l}{2}, \quad |\widetilde{h}_k| \leq h \quad \text{for each } 1 \leq k \leq n.$$

Thus, from (14) we have

$$|\varepsilon_n| \leq \sum_{k=1}^n \frac{1}{|\widetilde{G}_{k-1}^{(n)}|} \left(|z_{2,k}| |\alpha_{2,k}| + \frac{|\widetilde{h}_k| |z_{1,k}| (|\widetilde{\beta}_k| + |\alpha_{1,k}| + |\widetilde{\beta}_k| |\alpha_{1,k}|)}{|\widetilde{G}_k^{(n)}|} \right) \prod_{r=1}^{k-1} |\widetilde{g}_r^{(n)}|.$$

Using (21)–(22), we get

$$|\varepsilon_n| \leq \frac{4\alpha}{(1+l+|3l-1|)(1-\alpha)} \left(\frac{1-l}{2} + \frac{2l(1-l)}{1+l+|3l-1|} \left(2 + \frac{\alpha}{1-\alpha} \right) \right) \sum_{k=1}^n \eta^{k-1},$$

which is equal to (18), since $\sum_{k=1}^n \eta^{k-1} = \frac{1-\eta^n}{1-\eta}$.

Finally, it follows from (18) that there exists a constant C such that $|\varepsilon_n| \leq \alpha C / (1-\alpha)$ for all $n \geq 1$. It is easy to show that, if

$$|\alpha_1| \leq \alpha < \frac{\varepsilon}{\varepsilon + C}, \quad |\alpha_2| \leq \alpha < \frac{\varepsilon}{\varepsilon + C},$$

and $|\beta_k| \leq \alpha < \varepsilon / (\varepsilon + C)$ for all $k \geq 1$, where ε is an arbitrary positive constant, then $|\varepsilon_n| < \varepsilon$ for all $n \geq 1$. This fact proves that the conditions from Definition 1 are fulfilled. \square

From Theorem 1 we have the following.

Corollary 1. *Let there exist a constant α , $0 < \alpha < 1$, satisfying (15), where α_1 , α_2 , and β_k , $k \geq 1$, are relative errors of z_1 , z_2 , and h_k , $k \geq 1$, respectively, of the branched continued fraction*

$$\frac{1}{1 - z_2 - \frac{h_1 z_1}{1 - z_2 - \frac{h_2 z_1}{1 - \dots}}}, \quad (23)$$

where $h_1 = 2/c$, $h_k = \frac{k(2c + k - 3)}{(c + k - 2)(c + k - 1)}$ for all $k \geq 2$. Then:

- (A) The set (16), where h and l are defined in (17), forms the numerical stability set of (23).
 (B) If ε_n denotes the relative errors of the n th approximant of the branched continued fraction (23), then estimate (18) holds for all $n \geq 1$.

Remark 2. Results similar to Theorem 1 can also be obtained for the other two expansions in (4) and (5). In the general case, the problem of studying the numerical stability set of all three expansions remains open.

3. Numerical experiments. To illustrate the numerical stability of the backward recurrence algorithm, we describe the numerical examples taken from the branched continued fraction representation of the function (see [4])

$$H_4(1, b; 1, b; \mathbf{z}) = \sqrt{(1 - z_2)^2 - 4z_1} = \frac{1}{1 - z_2 - \frac{2z_1}{1 - z_2 - \frac{z_1}{1 - z_2 - \frac{z_1}{1 - z_2 - \dots}}}}. \quad (24)$$

We considered the values of the n th approximants $f_n(\mathbf{z})$, $1 \leq n \leq 100$, correctly rounded to 28 decimal digits for the points $(0.125, 0.25)$ and $(0.0625, -0.25)$, respectively. We also considered an approximation to each $f_n(\mathbf{z})$ obtained by the backward recurrence algorithm, correctly rounded to 14 decimal digits. We take the values $\widehat{f}_n(0.125, 0.25)$ correctly rounded to 14 decimal places for all $1 \leq n \leq 100$.

Further, calculations of $f_n(0.125, 0.25)$ and $\widehat{f}_n(0.125, 0.25)$, for $55 \leq n \leq 96$, showed that $|\widehat{f}_n(0.125, 0.25) - f_n(0.125, 0.25)|/|f_n(0.125, 0.25)|$ decreases and for $97 \leq n \leq 100$ are equal zero, as well as for $n = 1, 2$. The relative rounding error at the point $(0.0625, -0.25)$ does not decrease or increase for $1 \leq n \leq 100$. In the case of the backward recurrence algorithm, $\widehat{f}_n(0.0625, -0.25)$ remains correctly rounded to 14 decimal places for all $1 \leq n \leq 100$. Finally, from Corollary 1, one can obtain rigorous bounds for the relative rounding error $|\varepsilon_n|$ for each $n \geq 1$, which are entirely consistent with those found numerically in these examples.

Our calculations were performed using Maple software 2022.2 for Windows.

4. Conclusions. This paper concerns the establishment of the numerical stability sets of the branched continued fraction in the domains of its convergence. An estimate of the relative rounding error, produced by the backward recurrence algorithm in calculating an n th approximant of the branched continued fraction expansion of Horn's hypergeometric function H_4 , is established. It has provided to investigate the numerical stability of the bi-disc.

The considered numerical experiments are entirely consistent with theoretical calculations. Both of them, in particular, show that the stability of the backward recurrence algorithm depends not only on the calculation value of the elements of the branched continued fraction but also on the domain of convergence. Further research is to study wider sets of numerical stability.

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