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A Finite-Interval Schrödinger Potential Exactly Solvable in Terms of Angular Spheroidal Functions

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Abstract—We consider the one-dimensional stationary Schrödinger equation on a finite interval for a symmetric rational potential with repulsive endpoint divergences and a minimum at the midpoint. An affine coordinate transformation followed by a single gauge transformation reduces the equation exactly to the canonical angular spheroidal equation with spheroidal order $m = 1$. The bound-state energies are determined by an inverse characteristic-value condition, while the corresponding wavefunctions are expressed through regular angular spheroidal functions of the first kind. The resulting spectrum is exact but implicit, in the modern spectral sense of exact solvability. Dimensionless energy levels are calculated for representative coupling strengths and verified independently by finite-difference diagonalization. The model provides a compact example of a finite-interval quantum system whose complete bound-state problem is governed by classical spheroidal spectral theory rather than by the more general confluent Heun representation.

Keywords: Schrödinger equation, exactly solvable potential, confluent Heun equation, spheroidal wavefunctions, characteristic values, bound states

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1. INTRODUCTION

The relation between exactly solvable Schrödinger equations and special functions is one of the most productive themes in mathematical physics. Classical examples include the harmonic oscillator [1], the Coulomb problem [2], and the Morse potential [3], whose Schrödinger equations reduce, after suitable transformations, to the Kummer confluent hypergeometric equation, together with the Eckart [4] and Pöschl–Teller [5] potentials, which reduce to the Gauss hypergeometric equation. The factorization method and later shape-invariant and supersymmetric approaches unified and enlarged this class [6, 7].

Several natural extensions beyond the hypergeometric class—especially those involving an additional singular point, an accessory parameter, or the further coalescence of singularities—lead to the general Heun equation or one of its confluent forms [8–12]. Such equations occur in quantum mechanics and general relativity and also in atomic, molecular, and optical physics, including laser-driven two-state systems [10, 13, 14]. In this broader setting, exact solvability need not mean that every energy level is an elementary algebraic expression. A problem may still be regarded as exactly solvable when its admissible eigenfunctions and exact quantization condition are analytically determined in terms of established special functions, even if the energy spectrum is specified only implicitly (see, e.g., [15]).

The potential studied here belongs to this broader class. It diverges at both endpoints of a finite interval, is symmetric about the midpoint, and admits a direct confluent-Heun representation [12]. Mathematically, these endpoint divergences make the associated boundary-value problem singular, although the potential is smooth throughout the open interval. Its algebraic structure is nevertheless more restrictive than the general Heun form suggests. The denominator of the potential is converted by an affine coordinate transformation into the characteristic factor of the Legendre and spheroidal equations. A gauge transformation then generates the first-derivative and centrifugal terms of the canonical angular spheroidal equation.

Spheroidal wavefunctions originated in the separation of the Helmholtz equation in spheroidal coordinates and now occur in electromagnetic scattering, acoustics, molecular physics, and relativistic wave equations [16–19]. Unlike many special functions used in quantum mechanics, angular spheroidal func-

tions are intrinsically spectral objects: regularity at the endpoints selects a discrete sequence of characteristic values. This feature is central to the present problem. The strength of the potential fixes the spheroidal characteristic value, whereas the quantum-mechanical energy determines the spheroidal deformation parameter. The spectrum is therefore obtained from an inverse characteristic-value problem.

To our knowledge, although this potential was previously identified within the confluent-Heun class [12], its more restrictive reduction to the angular spheroidal equation and the resulting inverse characteristic-value formulation of the complete bound-state problem have not been reported. The present work identifies this hidden specialization and derives the corresponding spectrum and wavefunctions.

The purpose of this paper is to present this exact reduction in a compact physics-oriented form. Only the indispensable algebra is retained. The operator-theoretic details can be treated by the standard theory of lower-bounded singular Sturm-Liouville operators on finite intervals [20–25]; here they enter only through the physical requirement that the wavefunction vanish at the repulsive endpoint divergences. The main emphasis is placed on the transformation, the spectral condition, the explicit wavefunctions, and representative numerical results.

2. FINITE-INTERVAL POTENTIAL AND TRANSFORMED SCHRÖDINGER EQUATION

We consider

$$H\psi(x) = E\psi(x), \quad H = -\frac{\hbar^2}{2M}\frac{d^2}{dx^2} + V(x), \qquad 0 < x < 1 \tag{1}$$

with the potential

$$V(x) = V_0 + \frac{V_0}{4x(x-1)}, \quad V_0 < 0. \tag{2}$$

Here x is the coordinate on the unit interval, $\psi(x)$ is the stationary wavefunction, E is the energy eigenvalue, H is the Hamiltonian, M is the particle mass, $\hbar$ is the reduced Planck constant, and $V(x)$ is the potential energy; $V_0 < 0$ is an energy-scale parameter. Because $x(x-1) < 0$ on the open interval, the endpoint-divergent contribution is positive. The potential is symmetric, $V(1-x) = V(x)$, has $V(1/2) = 0$, and diverges to $+\infty$ at both endpoints.

Thus the particle is confined by two repulsive endpoint barriers generated by the potential itself. The symmetry about $x = 1/2$ implies that the eigenfunctions may be chosen with definite parity about the midpoint. The natural lower-bounded self-adjoint realization is the Friedrichs realization of the corresponding singular Sturm-Liouville operator; its spectrum on the finite interval is discrete, and the physical wavefunctions vanish at both endpoints [20–25].

In dimensionless form, define

$$g = -\frac{MV_0}{2\hbar^2} > 0 \Rightarrow u_g(x) = \frac{MV(x)}{2\hbar^2} = g\left(\frac{1}{4x(1-x)} - 1\right). \tag{3}$$

Here $g > 0$ is the dimensionless coupling strength and $u_g(x)$ the dimensionless potential. Figure 1 shows $u_g(x)$ for $g = 1, 5, 10$; increasing g raises the endpoint walls while leaving the minimum fixed at zero.

The central algebraic simplification follows from the affine coordinate transformation

$$z = 2x - 1, \quad x = \frac{1+z}{2} \Leftrightarrow x(1-x) = \frac{1-z^2}{4}, \quad -1 < z < 1. \tag{4}$$

Using $d^2/dx^2 = 4d^2/dz^2$, the Schrödinger equation becomes

$$\frac{d^2\psi}{dz^2} + \left(\frac{M(E-V_0)}{2\hbar^2} + \frac{MV_0}{2\hbar^2(1-z^2)}\right)\psi = 0. \tag{5}$$

Introducing the nonnegative energy-dependent spheroidal parameter c through

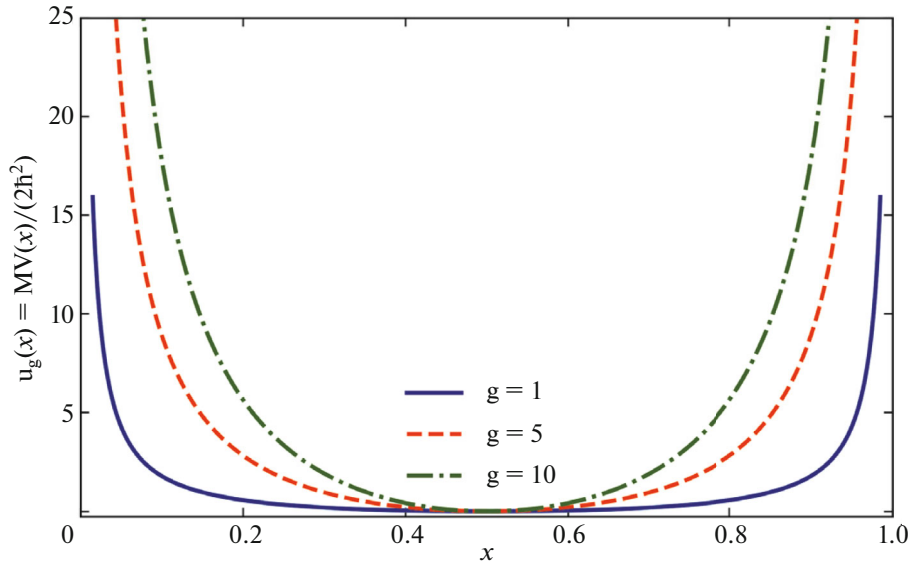


Fig. 1. Dimensionless potential $u_g(x) = MV(x)/(2\hbar^2)$ for $g = 1, 5,$ and 10. The potential is symmetric about $x = 1/2$, vanishes at the midpoint, and diverges to $+\infty$ at both endpoints.

$$c^2 = \frac{M(E - V_0)}{2\hbar^2}, \tag{6}$$

the transformed equation assumes the compact form

$$\psi''(z) + \left(c^2 - \frac{g}{1 - z^2}\right)\psi(z) = 0. \tag{7}$$

Here and below, primes denote differentiation with respect to z. This equation already displays the characteristic factor $1 - z^2$, but it is not yet in canonical spheroidal form because it contains no first-derivative term.

3. REDUCTION TO THE ANGULAR SPHEROIDAL EQUATION

The canonical angular spheroidal equation is written here as

$$(1 - z^2)Y'' - 2zY' + \left(\lambda + c^2(1 - z^2) - \frac{m^2}{1 - z^2}\right)Y = 0. \tag{8}$$

Here $Y(z)$ is the angular spheroidal function, m is the spheroidal order, λ is the separation constant (characteristic value), and c is the spheroidal parameter defined in Eq. (6).

To generate the canonical first-derivative and centrifugal terms, we apply the gauge transformation

$$\psi(z) = \sqrt{1 - z^2}\,Y(z). \tag{9}$$

Substitution into Eq. (7) gives

$$(1 - z^2)Y'' - 2zY' + \left(-g + c^2(1 - z^2) - \frac{1}{1 - z^2}\right)Y = 0. \tag{10}$$

Comparison with the canonical form gives the complete parameter identification

$$m = 1, \quad \lambda = -g = \frac{MV_0}{2\hbar^2}, \quad c^2 = \frac{M(E - V_0)}{2\hbar^2}. \tag{11}$$

No approximation has been introduced. The transformed Schrödinger equation is therefore exactly equivalent, on the open interval, to the angular spheroidal equation with spheroidal order one. The square-root gauge factor simultaneously generates the canonical first-derivative and centrifugal terms.

This reduction is more specialized than the direct confluent-Heun representation. The latter remains mathematically valid and places the potential within a broad family of singular equations [8–12]. The spheroidal representation, however, exploits additional algebraic structure and brings with it a mature spectral theory, including characteristic values, recurrence relations, asymptotic expansions, and stable numerical implementations [16–19].

4. BOUND-STATE SPECTRUM AND EXACT WAVEFUNCTIONS

For spheroidal order $m = 1$ and fixed $c \geq 0$, regularity at $z = \pm 1$ selects a discrete sequence of characteristic values, denoted by

$$\lambda_n^{(1)}(c^2), \quad n = 1,2,3,\ldots. \tag{12}$$

Here $n = 1,2,3,\ldots$ is the spheroidal degree, and the superscript (1) denotes the order $m = 1$. A regular angular spheroidal function of degree n and order one has $n-1$ zeros in $(-1,1)$; hence the spheroidal degree coincides with the one-based ordering of the quantum bound states.

At zero spheroidal parameter, $c = 0$, the angular spheroidal equation reduces to the associated-Legendre equation, and the characteristic values become

$$\lambda_n^{(1)}(0) = n(n+1). \tag{13}$$

The characteristic-value branch satisfies the DLMF derivative bound [19]:

$$-1 < \frac{d\lambda_n^{(1)}(c^2)}{d(c^2)} < 0. \tag{14}$$

The potential fixes the separation constant at $\lambda = -g$. Consequently, the admissible values of the spheroidal parameter, denoted by c_n, are determined by the inverse characteristic-value equation

$$\lambda_n^{(1)}\left(c_n^2\right) = -g = \frac{MV_0}{2\hbar^2}. \tag{15}$$

Equation (14) shows that each characteristic-value branch is strictly decreasing as a function of c^2. Since Eq. (13) places the branch above $-g$ at $c = 0$, while the standard large-c behavior sends it to $-\infty$, Eq. (15) has a unique positive solution c_n for every n.

The corresponding energy levels follow directly from the definition of c:

$$E_n = V_0 + \frac{2\hbar^2}{M}c_n^2. \tag{16}$$

The spectrum is therefore exact but implicit. Instead of terminating a hypergeometric series, one solves a one-dimensional characteristic-value equation. This is the distinctive quantization mechanism of the model.

Let $S_n^{(1)}$ denote the regular angular spheroidal function of order one and degree n, with its dependence on c^2 and z displayed explicitly in Eq. (17). Transforming back to x gives

$$\psi_n(x) = N_n\sqrt{x(1-x)}S_n^{(1)}\left(c_n^2, 2x-1\right), \tag{17}$$

where the factor of two produced by $\sqrt{1-z^2} = 2\sqrt{x(1-x)}$ has been absorbed into the normalization constant N_n.

The regular spheroidal solution behaves as $(1-z^2)^{1/2}$ near the endpoints. Consequently, the physical wavefunction is asymptotic to x at the left endpoint and to $1-x$ at the right endpoint. This behavior is compatible with the endpoint-divergent confining barriers. The second independent spheroidal solution is singular at $z = \pm 1$ and does not produce an admissible bound-state wavefunction.

Parity follows directly from spheroidal symmetry:

Table 1. Dimensionless bound-state energies $\varepsilon_n = ME_n/(2\hbar^2)$ obtained from the spheroidal characteristic-value equation

n	$g = 1$	$g = 5$	$g = 10$
1	2.679728	3.422421	4.183469
2	10.414981	12.390749	14.503959
3	22.955468	25.770905	28.929572
4	40.372622	43.803270	47.773130
5	62.692030	66.601355	71.216427

$$\psi_n(1-x) = (-1)^{n-1}\psi_n(x). \tag{18}$$

Thus $n = 1,3,5,\ldots$ give states even about $x = 1/2$, whereas $n = 2,4,6,\ldots$ give odd states. The state with $n = 1$ is the nodeless even ground state.

The exact solution is readily implemented in systems that provide spheroidal characteristic values and regular spheroidal functions. In the Wolfram Language convention, SpheroidalEigenvalue [n,1,c] corresponds to the DLMF characteristic value $\lambda_n^{(1)}(c^2)$ associated with Eq. (8), so Eq. (15) is implemented directly:

$$\text{SpheroidalEigenvalue}\,[n,1,c] = -g$$

followed by substitution into SpheroidalPS[n, 1, c, 2 x − 1] and normalization. In an alternative commonly used convention, the shifted characteristic value is defined by

$$A_n^{(1)}(c^2) = \lambda_n^{(1)}(c^2) + c^2, \tag{19}$$

in which case the equivalent inverse root equation is

$$A_n^{(1)}\left(c_n^2\right) = c_n^2 - g. \tag{20}$$

The defining differential equation should therefore always be checked before numerical comparison.

5. NUMERICAL ILLUSTRATION

For numerical presentation, we use the dimensionless energy

$$\varepsilon_n = \frac{ME_n}{2\hbar^2}. \tag{21}$$

With $V_0 = -2\hbar^2 g/M$, Eq. (16) gives $\varepsilon_n = c_n^2 - g$, where c_n is determined by the inverse characteristic-value condition in Eq. (15). Table 1 gives the first five levels for three coupling strengths. The eigenvalues increase with g because the repulsive endpoint walls become stronger. The ground-state shift is moderate, whereas the higher levels retain the approximately quadratic growth expected for confinement on a finite interval. For fixed g, the high-energy spectrum has the leading asymptotic behavior of the unit-width infinite square well, with a subleading logarithmic correction generated by the endpoint divergences:

$$\varepsilon_n = \frac{\pi^2 n^2}{4} + \frac{g}{2}\ln n + O(1), \quad n \to \infty, \tag{22}$$

$$\frac{c_n}{n} \to \frac{\pi}{2}, \quad n \to \infty. \tag{23}$$

Since $c_n^2 = \varepsilon_n + g$, Eq. (22) implies the limit in Eq. (23). The relevant high-energy regime is therefore a simultaneous large-n, large-c_n limit, rather than the fixed-c large-degree limit of the standard DLMF expansion. The Legendre limit, the fixed-parameter large-degree limit, and the inverse-characteristic-value limit considered here are distinct and noncommuting limiting procedures. The logarithmic term reflects the inverse-distance behavior of the potential near both endpoints and is consistent with singular Sturm–Liouville eigenvalue asymptotics [26]. A detailed derivation, including the constant term and numerical residual analysis, will be reported separately.

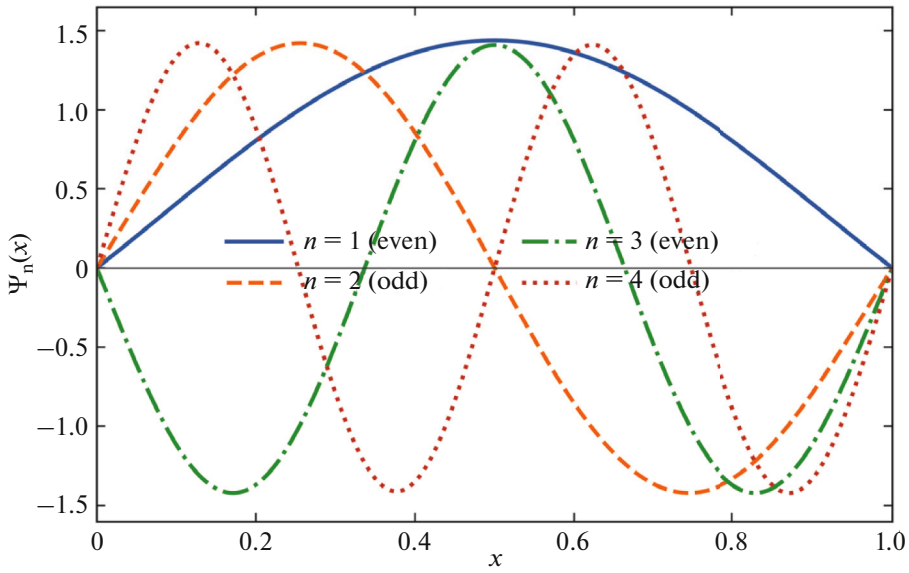


Fig. 2. First four normalized eigenfunctions for $g = 5$. The states alternate between even and odd parity about $x = 1/2$ and vanish linearly at both endpoints.

As an independent check, the original Schrödinger operator was discretized directly on a uniform grid of 4000 interior points using a centered second-order finite difference. For the first five levels, the maximum absolute discrepancy between direct diagonalization and the spheroidal result was 8.1×10^{-5} for $g = 1$, 8.5×10^{-5} for $g = 5$, and 9.1×10^{-5} for $g = 10$. The errors decrease under grid refinement and confirm the characteristic-value quantization independently of the analytical transformation.

Figure 2 shows the first four normalized wavefunctions for $g = 5$. The expected parity alternation is explicit. The wavefunctions vanish linearly at both endpoints, while the number of interior nodes increases by one from level to level.

The numerical calculation involves only a scalar root search for each degree n. Once c_n is determined, the energy follows algebraically and the wavefunction is evaluated from the regular spheroidal function. No shooting method or direct discretization is required for the exact construction itself; the finite-difference calculation above is used only as an independent verification.

6. DISCUSSION AND CONCLUSIONS

The potential considered here provides a simple but nontrivial example of exact solvability outside the classical hypergeometric family. Its endpoint divergences are strong enough to create intrinsic confinement, yet its denominator has precisely the algebraic form needed for a reduction to the angular spheroidal equation. The exact reduction is given by the gauge transformation and parameter identification in Eqs. (9) and (11), while the spectrum and wavefunctions are determined by Eqs. (12) and (15)–(17).

The solution is exact in a spectral rather than elementary-algebraic sense: the energies are defined by roots of a classical characteristic-value function. This is no less complete than exact quantization through zeros, special-function parameters, or transcendental equations encountered in many standard quantum systems.

The most distinctive feature is the inversion of the usual spheroidal spectral problem. In conventional applications the deformation parameter is fixed, and regularity determines the characteristic value. Here the potential fixes the characteristic value, while the energy must adjust through the deformation parameter. This inverse viewpoint is both analytically transparent and computationally efficient.

The spheroidal formulation also clarifies the relation to the confluent Heun equation. The Heun representation classifies the differential equation correctly but does not by itself expose the full spectral specialization. The angular spheroidal equation is a much more constrained Sturm–Liouville problem with established characteristic values, orthogonality properties, asymptotic theory, and mature numerical algorithms. Recognizing this narrower structure is therefore the essential analytical step.

The principal contribution is therefore not the introduction of a new special function or a wholly new potential, but the recognition that a previously known confluent-Heun potential possesses a substantially more specialized angular-spheroidal structure. This recognition converts the complete bound-state problem into a standard but inverted spheroidal characteristic-value problem.

Several extensions are natural. One may study asymptotic approximations to the inverse characteristic-value equation, classify other singular potentials whose Heun representations collapse to spheroidal or related classical equations, and examine asymmetric or relativistic generalizations. The present result suggests that additional exactly solvable potentials may be found not by introducing new special functions, but by identifying hidden reductions to already developed spectral families.

In summary, the finite-interval potential defined in Eq. (2) admits an exact angular-spheroidal solution. The energy levels are obtained from spheroidal characteristic values, the physical wavefunctions are regular spheroidal functions multiplied by the endpoint factor $\sqrt{x(1-x)}$, and the numerical implementation reduces to a stable one-dimensional root problem. The compact formulation places the model naturally alongside other exactly solvable Schrödinger potentials while retaining the distinctive spectral structure generated by the endpoint divergences.

AUTHOR CONTRIBUTION

All authors contributed to the conception and design of the study, mathematical development, special-function analysis, and preparation of the manuscript. All authors reviewed and approved the final version.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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