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Effects of the Wiener Process and Beta Derivative on the Exact Solutions of the Kadomtsev–Petviashvili Equation

Farah M. Al-Askar [1], Clemente Cesarano [2] and Wael W. Mohammed [3,4,*]

[1] Department of Mathematical Science, College of Science, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; famalaskar@pnu.edu.sa
[2] Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II 39, 00186 Rome, Italy; c.cesarano@uninettuno.it
[3] Department of Mathematics, College of Science, University of Ha'il, Ha'il 2440, Saudi Arabia
[4] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
* Correspondence: wael.mohammed@mans.edu.eg

Abstract: We take into account the (2 + 1)-dimensional stochastic Kadomtsev–Petviashvili equation with beta-derivative (SKPE-BD) in this paper. To develop new hyperbolic, trigonometric, elliptic, and rational solutions, the Riccati equation and Jacobi elliptic function methods are employed. Because the KP equation is required for explaining the development of quasi-one-dimensional shallow-water waves, the solutions obtained can be used to interpret various attractive physical phenomena. To display how the multiplicative white noise and beta-derivative impact the exact solutions of the SKPE-BD, we plot a few graphs in MATLAB and display different 3D and 2D figures. We deduce how multiplicative noise stabilizes the solutions of SKPE-BD at zero.

Keywords: stochastic KP; fractional KP; stability by noise; exact solution; beta derivative

MSC: 60H15; 83C15; 60H10; 35Q51; 35A20

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1. Introduction

Fractional differential equations (FDEs) are often used in relation to optical fibers, chemical kinematics, solid-state physics, electrical circuits, nuclear-physics, fluid mechanics, elastic media, quantum field theory, plasma physics, neural physics, mathematical biology, and other domains [1–7]. Also, many physical phenomena, such as fluid dynamics, elasticity, heat, electrodynamics, gravity, sound electrostatics, quantum mechanics, and diffusion, are described by fractional-order derivatives. Consequently, it is essential in mathematical physics to seek exact solutions for FDEs. In recent years, multiple approaches for dealing with FDEs have been devised, such as the (G'/G)-expansion method [8,9], Kudryashov method [10], first-integral method [11], sine–cosine method [12,13], $exp(-\phi(\varsigma))$-expansion [14], direct algebraic method [15], perturbation method [16,17], tanh-sech [18,19], sine-Gordon expansion [20], Jacobi elliptic function [21], etc.

Recently, beta-derivative (BD), a new conformable fractional derivative, was proposed by Atangana et al. in [22]. From here, the BD for $\mathcal{Y} : (0, \infty) \to \mathbb{R}$ of order $\beta \in (0, 1]$ is defined as follows:

$$\mathbb{D}_x^{\beta}\mathcal{Y}(x) = \lim_{\epsilon \to 0} \frac{\mathcal{Y}(x + \epsilon(x + \frac{1}{\Gamma(\beta)})^{1-\beta}) - \mathcal{Y}(x)}{\epsilon}.$$

The beta-derivative satisfies the next features for any constant a and b:

(1) $\mathbb{D}_x^{\beta}[a] = 0,$
(2) $\mathbb{D}_x^{\beta}[a\mathcal{R}(x) + b\mathcal{Y}(x)] = a\, \mathbb{D}_x^{\beta}\mathcal{R}(x) + b\, \mathbb{D}_x^{\beta}\mathcal{Y}(x),$
(3) $\mathbb{D}_x^{\beta}\mathcal{Y}(\theta) = (x + \frac{1}{\Gamma(\beta)})^{1-\beta}\frac{d\mathcal{Y}}{dx}$, (4) If $\theta = \frac{a}{\beta}(x + \frac{1}{\Gamma(\beta)})^{\beta}$, then $\mathbb{D}_x^{\beta}\mathcal{Y}(\theta) = a\frac{d\mathcal{Y}}{d\theta}$.

On the contrary, it is now well known that randomness or fluctuations play an essential role in a wide range of phenomena. Consequently, random impacts have assumed a greater role in demonstrating numerous physical processes that take place in disciplines such as telecommunications, cryptography, computer science, ecology, biology, information theory, signal processing, neuroscience, chemistry, image processing, physics, and finance, among others [23–25]. Partial differential equations are appropriate mathematical equations for modeling complex systems in the presence of noise or random effects.

It is essential to consider FDEs with a stochastic term. Therefore, we look at the following (2 + 1)-dimensional stochastic Kadomtsev–Petviashvili equation with beta-derivative (SKPE-BD):

$$\mathbb{D}_x^{\beta}[\mathcal{R}_t + 6\mathcal{R}\mathbb{D}_x^{\beta}\mathcal{R} + \mathbb{D}_{xxx}^{\beta}\mathcal{R} + \gamma\mathcal{R}\mathcal{W}_t] + \rho\mathbb{D}_{yy}^{\beta}\mathcal{R} = 0, \tag{1}$$

where $\mathcal{R}$ denotes the rescaled velocities and the rescaled wave amplitude in surface shallow-water waves, $\rho = \pm 1$, γ is the noise strength and it is a real number, $\mathcal{W}_t(t) = \frac{\partial \mathcal{W}(t)}{\partial t}$ is the derivative of the Wiener process $\mathcal{W}(t)$, and $\mathcal{R}\mathcal{W}_t$ is an Itô multiplicative noise.

When $\gamma = 0$ and $\beta = 1$, we attain the Kadomtsev–Petviashvili (KP) equation [26,27] that can be used to characterize the development of quasi-one-dimensional shallow-water waves whenever the impacts of viscosity and surface tension are negligible:

$$\frac{\partial}{\partial x}[\frac{\partial \mathcal{R}}{\partial t} + 6\mathcal{R}\frac{\partial \mathcal{R}}{\partial x} + \frac{\partial^3 \mathcal{R}}{\partial x^3}] + \rho\frac{\partial^2 \mathcal{R}}{\partial y^2} = 0. \tag{2}$$

The KP equation (2) has numerous applications in fluid dynamics and plasma physics. The equation is widely used to study various physical phenomena, such as the propagation of waves and solitons in different media. As such, the KP equation has been crucial in advancing our understanding of complex nonlinear systems. Its importance lies in its ability to accurately model and predict the behavior of waves and solitons, which has applications in a wide range of fields including oceanography, optics, and plasma physics. As a result, several approaches to acquiring the exact solutions of KP Equation (2) have been suggested, such as sine–cosine [28], Hirota's bilinear method [29], Hirota's method [30], trial equation method [31], novel generalized (G'/G)-expansion [32], extended mapping method [33], F-expansion method [34], etc.

Our contribution here is to find the exact solutions for SKPE-BD (1). To obtain these solutions, we utilize the Riccati equation method (RE-Method) and Jacobi elliptic function method (JEF-Method). Because Equation (1) is used in describing the propagation of waves on the surface of shallow water, the acquired solutions of the SKPE-BD (1) will help researchers to gain a deeper understanding of these phenomena and make predictions about their behavior. Additionally, the obtained solutions can also be used in practical applications, such as designing improved tsunami warning systems or optimizing wave energy converters. Furthermore, we investigate the effect of BD and noise on the analytical solutions of the SKPE-BD (1) by providing some graphs via the MATLAB program 2022b .

Following is the structure of the paper: In Section 2, the wave equation of SKPE-BD (1) is derived. In Section 3, the RE-Method and JEF-Method are utilized to obtain the exact solution of the SKPE-BD (1). In Section 4, we can examine the effect of the Wiener process and the beta-derivative on the achieved solutions of the SKPE-BD. In Section 5, we discuss the physical meaning of the obtained results. Finally, the conclusions of the paper are offered in Section 6.

2. Traveling Wave Equation for SKPE-BD

The wave equation for SKPE-BD (1) is found by using

$$\mathcal{R}(x,y,t) = \mathcal{Y}(\xi)e^{[-\gamma\mathcal{W}(t) - \frac{1}{2}\gamma^2 t]}, \quad \xi = [\frac{1}{\beta}(x + \frac{1}{\Gamma(\beta)})^{\beta} + \frac{1}{\beta}(y + \frac{1}{\Gamma(\beta)})^{\beta} - \lambda t], \tag{3}$$

where $\mathcal{Y}$ is a deterministic and real function. It is worth noting that

$$\frac{\partial \mathcal{R}}{\partial t} = [-\lambda\mathcal{Y}' - \gamma\mathcal{Y}\frac{\partial \mathcal{W}}{\partial t}]e^{[-\gamma\mathcal{W}(t)-\frac{1}{2}\gamma^2 t]}, \tag{4}$$

and

$$\mathbb{D}_x^\beta \mathcal{R} = \mathcal{Y}' e^{[-\gamma\mathcal{W}(t)-\frac{1}{2}\gamma^2 t]}, \mathbb{D}_{xxx}^\beta \mathcal{R} = \mathcal{Y}''' e^{[-\gamma\mathcal{W}(t)-\frac{1}{2}\gamma^2 t]}, \mathbb{D}_{yy}^\beta \mathcal{R} = \mathcal{Y}'' e^{[-\gamma\mathcal{W}(t)-\frac{1}{2}\gamma^2 t]}. \tag{5}$$

Inserting Equation (3) into Equation (1) and using (4) and (5), we obtain

$$\mathcal{Y}'''' + (\rho-\lambda)\mathcal{Y}'' + 6[\mathcal{Y}\mathcal{Y}'' + (\mathcal{Y}')^2]e^{[-\gamma\mathcal{W}(t)-\frac{1}{2}\gamma^2 t]} = 0.$$

Taking into account the expectations of both sides, we achieve

$$\mathcal{Y}'''' + (\rho-\lambda)\mathcal{Y}'' + 6[\mathcal{Y}\mathcal{Y}'' + (\mathcal{Y}')^2]e^{-\frac{1}{2}\gamma^2 t}\mathbb{E}e^{[-\gamma\mathcal{W}(t)]} = 0. \tag{6}$$

Since $\mathcal{W}(t)$ is normal process, hence $\mathbb{E}(e^{-\gamma\mathcal{W}(t)}) = e^{\frac{1}{2}\gamma^2 t}$ for any real number γ. Therefore, Equation (6) becomes

$$\mathcal{Y}'''' + (\rho-\lambda)\mathcal{Y}'' + 6(\mathcal{Y}\mathcal{Y}')' = 0, \tag{7}$$

where we replaced $\mathcal{Y}\mathcal{Y}'' + (\mathcal{Y}')^2$ by $(\mathcal{Y}\mathcal{Y}')'$. Integrating Equation (7) twice and ignoring the integration constant, we have

$$\mathcal{Y}'' - (\lambda-\rho)\mathcal{Y} + 3\mathcal{Y}^2 = 0. \tag{8}$$

3. Exact Solutions of SKPE-BD

To obtain exact solutions for SKPE-BD (1), we employ two alternative methods: the RE-Method [35] and JEF-Method [36].

3.1. RE-Method

Let us assume the solution $\mathcal{Y}$ of Equation (8) is

$$\mathcal{Y}(\xi) = \sum_{j=0}^{K} a_j Z^j, \tag{9}$$

where $\mathcal{R}$ solves the Riccati equation

$$Z' = Z^2 + b, \tag{10}$$

with b is a unknown constant. Equation (10) has the following solutions:

$$Z = \frac{-1}{\xi}, \tag{11}$$

if $b = 0$, or

$$Z = \sqrt{b}\tan(\sqrt{b}\xi) \ \text{ or } Z = -\sqrt{b}\cot(\sqrt{b}\xi), \tag{12}$$

if $b > 0$, or

$$Z = -\sqrt{-b}\tanh(\sqrt{-b}\xi) \ \text{ or } Z = -\sqrt{-b}\coth(\sqrt{-b}\xi), \tag{13}$$

if $b < 0$.

In order to compute the parameter K in Equation (9), we balance $\mathcal{Y}^2$ with $\mathcal{Y}''$ in Equation (8) to obtain

$$2K = K + 2,$$

then

$$K = 2. \tag{14}$$

Rewriting Equation (9), with $K = 2$, as

$$\mathcal{Y}(\xi) = a_0 + a_1 Z + a_2 Z^2. \tag{15}$$

Substituting Equation (15) into Equation (8) we obtain

$$\begin{aligned}
&(6a_2 + 3a_2^2)Z^4 + (2a_1 + 3a_1 a_2)Z^3 \\
&+(8ba_2 - (\lambda - \rho)a_2 + 3a_1^2 + 6a_0 a_2)Z^2 \\
&+(2a_1 b - (\lambda - \rho)a_1 + 6a_0 a_1)Z \\
&+(2b^2 a_2 - (\lambda - \rho)a_0 + 3a_0^2) = 0.
\end{aligned}$$

We derive by setting each coefficient of Z^j to zero

$$6a_2 + 3a_2^2 = 0,$$

$$2a_1 + 3a_1 a_2 = 0,$$

$$8ba_2 - (\lambda - \rho)a_2 + 3a_1^2 + 6a_0 a_2 = 0,$$

$$2a_1 b - (\lambda - \rho)a_1 + 6a_0 a_1 = 0,$$

and

$$2b^2 a_2 - (\lambda - \rho)a_0 + 3a_0^2 = 0.$$

The next two families are obtained by solving these equations:
First family:

$$a_0 = \frac{-2}{3}b,\ a_1 = 0,\ a_2 = -2,\ \lambda = \rho + 4b. \tag{16}$$

Second family:

$$a_0 = -2b,\ a_1 = 0,\ a_2 = -2,\ \lambda = \rho - 4b. \tag{17}$$

First family: There are three cases relying on b.
Case 1: If $b = 0$, then the solution of (8), by using (11) and (15), is

$$\mathcal{Y}(\xi) = \frac{-2}{\xi^2}.$$

Consequently, the solution of SKPE-BD (1) is

$$\mathcal{R}(x,y,t) = -2\Big[\frac{1}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta + \frac{1}{\beta}(y + \frac{1}{\Gamma(\beta)})^\beta - \rho t\Big]^{-2} e^{[-\gamma \mathcal{W}(t) - \frac{1}{2}\gamma^2 t]}. \tag{18}$$

Case 2: If $b > 0$, then the solutions of (8), using (12) and (15), are

$$\mathcal{Y}(\xi) = \frac{-2}{3}b - 2b\tan^2(\sqrt{b}\xi),$$

or

$$\mathcal{Y}(\xi) = \frac{-2}{3}b - 2b\cot^2(\sqrt{b}\xi).$$

As a result, the solutions of SKPE-BD (1) are

$$\mathcal{R}(x,y,t) = [\frac{-2}{3}b - 2b\tan^2(\sqrt{b}\xi)]e^{[-\gamma \mathcal{W}(t) - \frac{1}{2}\gamma^2 t]}, \tag{19}$$

or

$$\mathcal{R}(x,y,t) = [\frac{-2}{3}b - 2b\cot^2(\sqrt{b}\xi)]e^{[-\gamma \mathcal{W}(t) - \frac{1}{2}\gamma^2 t]}. \tag{20}$$

Case 3: If $b < 0$, then the solutions of (8), by using (13) and (15), are

$$\mathcal{Y}(\xi) = \frac{-2}{3}b + 2b\tanh^2(\sqrt{-b}\xi),$$

or

$$\mathcal{Y}(\xi) = \frac{-2}{3}b + 2b\coth^2(\sqrt{-b}\xi).$$

Thence, the solutions of SKPE-BD (1) are

$$\mathcal{R}(x,y,t) = [\frac{-2}{3}b + 2b\tanh^2(\sqrt{-b}\xi)]e^{[-\gamma\mathcal{W}(t)-\frac{1}{2}\gamma^2 t]}, \tag{21}$$

or

$$\mathcal{R}(x,y,t) = [\frac{-2}{3}b + 2b\coth^2(\sqrt{-b}\xi)]e^{[-\gamma\mathcal{W}(t)-\frac{1}{2}\gamma^2 t]}. \tag{22}$$

where $\xi = \frac{1}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta + \frac{1}{\beta}(y + \frac{1}{\Gamma(\beta)})^\beta - (\rho + 4b)t$.

Second family: There are three cases also relying on b.

Case 1: If $b = 0$, then we have the same solution as announced before in the first set.

Case 2: If $b > 0$, then the solutions of (8), using (12) and (15), are

$$\mathcal{Y}(\xi) = -2b - 2b\tan^2(\sqrt{b}\xi),$$

or

$$\mathcal{Y}(\xi) = -2b - 2b\cot^2(\sqrt{b}\xi).$$

Thence, the solutions of SKPE-BD (1) are

$$\mathcal{R}(x,y,t) = [-2b - 2b\tan^2(\sqrt{b}\xi)]e^{[-\gamma\mathcal{W}(t)-\frac{1}{2}\gamma^2 t]}, \tag{23}$$

or

$$\mathcal{R}(x,y,t) = [-2b - 2b\cot^2(\sqrt{b}\xi)]e^{[-\gamma\mathcal{W}(t)-\frac{1}{2}\gamma^2 t]}. \tag{24}$$

Case 3: If $b < 0$, then the solution of (8), using (13) and (15), are

$$\mathcal{Y}(\xi) = -2b + 2b\tanh^2(\sqrt{-b}\xi) = -2b\text{sech}^2(\sqrt{-b}\xi),$$

or

$$\mathcal{Y}(\xi) = -2b + 2b\coth^2(\sqrt{-b}\xi) = 2b\text{csch}^2(\sqrt{-b}\xi).$$

As a result, the solutions of SKPE-BD (1) are

$$\mathcal{R}(x,y,t) = -2b\text{sech}^2(\sqrt{-b}\xi)e^{[-\gamma\mathcal{W}(t)-\frac{1}{2}\gamma^2 t]}, \tag{25}$$

or

$$\mathcal{R}(x,y,t) = 2b\text{csch}^2(\sqrt{-b}\xi)e^{[-\gamma\mathcal{W}(t)-\frac{1}{2}\gamma^2 t]}. \tag{26}$$

where $\xi = \frac{1}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta + \frac{1}{\beta}(y + \frac{1}{\Gamma(\beta)})^\beta - (\rho - 4b)t$.

Remark 1. *If we put $\beta = 1$ and $\gamma = 0$ in Equation (25), then we obtain the solution (2), reported in [37].*

3.2. JEF-Method

In this subsection, we use the JEF-method [36]. Assuming the solutions to Equation (8) has the form (with $K = 2$):

$$\mathcal{Y}(\xi) = \hbar_0 + \hbar_1 Z(\xi) + \hbar_2 Z^2(\xi), \tag{27}$$

where $\hbar_0$, $\hbar_1$, and $\hbar_2$ are undefined constants and $Z(\xi) = sn(\xi, \kappa)$ is the Jacobi elliptic sine function for $0 < \kappa < 1$. Differentiating Equation (27) twice,

$$\mathcal{Y}''(\xi) = 2\hbar_2 - \hbar_1(\kappa^2+1)Z - 4\hbar_2(\kappa^2+1)Z^2 + 2\hbar_1\kappa^2 Z^3 + 6\hbar_2\kappa^2 Z^4. \tag{28}$$

Plugging Equations (27) and (28) into Equation (8), we have

$$\begin{gathered}(2\kappa^2\hbar_2 + 3\hbar_2^2)Z^4 + (2\kappa^2\hbar_1 + 6\hbar_1\hbar_2)Z^3 \\ +[6\hbar_0\hbar_2 - 4\hbar_2(\kappa^2+1) - \hbar_2(\lambda-\rho) + 3\hbar_1^2]Z^2 \\ -[(\kappa^2+1)\hbar_1 + \hbar_1(\lambda-\rho) - 6\hbar_0\hbar_1]Z + (2\hbar_2 - \hbar_0(\lambda-\rho) + 3\hbar_0^2) = 0.\end{gathered}$$

Equating coefficient of Z^n to zero for $n = 4,3,2,1,0$:

$$2\kappa^2\hbar_2 + 3\hbar_2^2 = 0,$$

$$2\kappa^2\hbar_1 + 6\hbar_1\hbar_2 = 0,$$

$$6\hbar_0\hbar_2 - 4\hbar_2(\kappa^2+1) - \hbar_2(\lambda-\rho) + 3\hbar_1^2 = 0,$$

$$(\kappa^2+1)\hbar_1 + \hbar_1(\lambda-\rho) - 6\hbar_0\hbar_1 = 0,$$

and

$$2\hbar_2 - \hbar_0(\lambda-\rho) + 3\hbar_0^2 = 0.$$

When these equations are solved, we derive

$$\hbar_0 = \frac{2(\kappa^2+1) + 2\sqrt{\kappa^4-\kappa^2+1}}{3},\ \hbar_1 = 0,\ \hbar_2 = -2\kappa^2,\ \lambda = \rho + 4\sqrt{\kappa^4-\kappa^2+1},$$

or

$$\hbar_0 = \frac{2(\kappa^2+1) - 2\sqrt{\kappa^4-\kappa^2+1}}{3},\ \hbar_1 = 0,\ \hbar_2 = -2\kappa^2,\ \lambda = \rho - 4\sqrt{\kappa^4-\kappa^2+1},$$

Thus, Equation (8), by using (27), has the solution

$$\mathcal{Y}(\xi) = \frac{2(\kappa^2+1) + 2\sqrt{\kappa^4-\kappa^2+1}}{3} - 2\kappa^2 sn^2(\xi,\kappa),$$

or

$$\mathcal{Y}(\xi) = \frac{2(\kappa^2+1) - 2\sqrt{\kappa^4-\kappa^2+1}}{3} - 2\kappa^2 sn^2(\xi,\kappa).$$

Hence, the solutions of SKPE-BD (1) is

$$\mathcal{R}(x,y,t) = [\frac{2(\kappa^2+1) + 2\sqrt{\kappa^4-\kappa^2+1}}{3} - 2\kappa^2 sn^2(\xi,\kappa)]e^{[-\gamma\mathcal{W}(t) - \frac{1}{2}\gamma^2 t]}, \tag{29}$$

where $\xi = \frac{1}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta + \frac{1}{\beta}(y + \frac{1}{\Gamma(\beta)})^\beta - (\rho + 4\sqrt{\kappa^4-\kappa^2+1})t$, or

$$\mathcal{R}(x,y,t) = [\frac{2(\kappa^2+1) - 2\sqrt{\kappa^4-\kappa^2+1}}{3} - 2\kappa^2 sn^2(\xi,\kappa)]e^{[-\gamma\mathcal{W}(t) - \frac{1}{2}\gamma^2 t]}, \tag{30}$$

where $\xi = \frac{1}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta + \frac{1}{\beta}(y + \frac{1}{\Gamma(\beta)})^\beta - (\rho - 4\sqrt{\kappa^4-\kappa^2+1})t$. If $\kappa \to 1$, then Equation (29) changes to

$$\mathcal{R}(x,y,t) = 2\text{sech}^2(\xi)e^{[-\gamma\mathcal{W}(t) - \frac{1}{2}\gamma^2 t]}, \tag{31}$$

or

$$\mathcal{R}(x,y,t) = [\frac{2}{3} - 2\tanh^2(\xi)]e^{[-\gamma\mathcal{W}(t) - \frac{1}{2}\gamma^2 t]}. \tag{32}$$

In a similar way, we can replace sn in (27) by cn to obtain the solutions of Equation (8) as follows:

$$\mathcal{Y}(\xi) = [\frac{2\sqrt{\kappa^4 - \kappa^2 + 1} - 2(2\kappa^2 - 1)}{3} + 2\kappa^2 cn^2(\xi, \kappa)].$$

Therefore, the solutions of the SKPE-BD (1) is

$$\mathcal{R}(x, y, t) = [\frac{2\sqrt{\kappa^4 - \kappa^2 + 1} - 2(2\kappa^2 - 1)}{3} + 2\kappa^2 cn^2(\xi, \kappa)]e^{[-\gamma \mathcal{W}(t) - \frac{1}{2}\gamma^2 t]}. \quad (33)$$

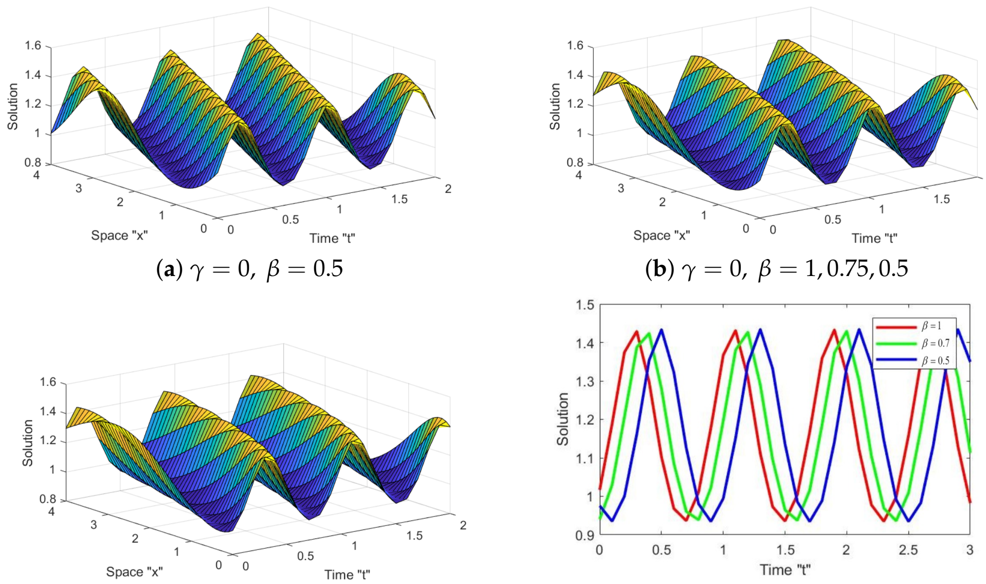
where $\xi = \frac{1}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta + \frac{1}{\beta}(y + \frac{1}{\Gamma(\beta)})^\beta - (\rho + 4\sqrt{\kappa^4 - \kappa^2 + 1})t$. If $\kappa \to 1$, then the solutions (33) takes the form

$$\mathcal{R}(x, y, t) = [2\text{sech}^2(\xi)]e^{[-\gamma \mathcal{W}(t) - \frac{1}{2}\gamma^2 t]}. \quad (34)$$

4. The Effect of the Wiener Process and Beta Derivative

Here, the impact of SWP and BD on the analytical solutions of the SKPE-BD (1) is discussed. We illustrate the behavior of these solutions through a number of graphs. For different γ (noise strength), we generate certain figures for some found solutions including Equations (29) and (31). First, let us define the parameters $\rho = 1$ and $\kappa = 0.5$. Also, let $t \in [0, 2]$ and $x \in [0, 4]$.

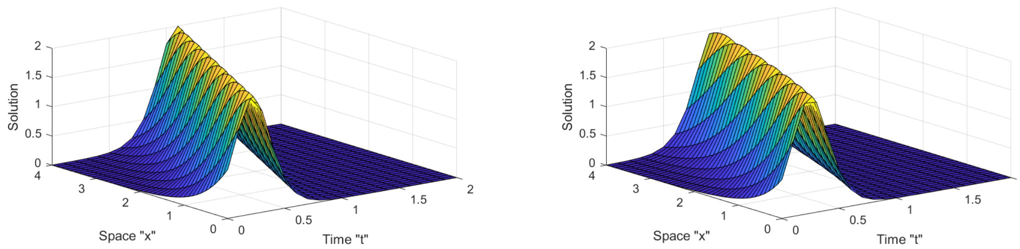
First the beta derivative effects: In Figures 1 and 2, if $\gamma = 0$, we notice that the profile of the graphs is pressed as the value of β decreases:



(**a**) $\gamma = 0,\ \beta = 0.5$ (**b**) $\gamma = 0,\ \beta = 1, 0.75, 0.5$

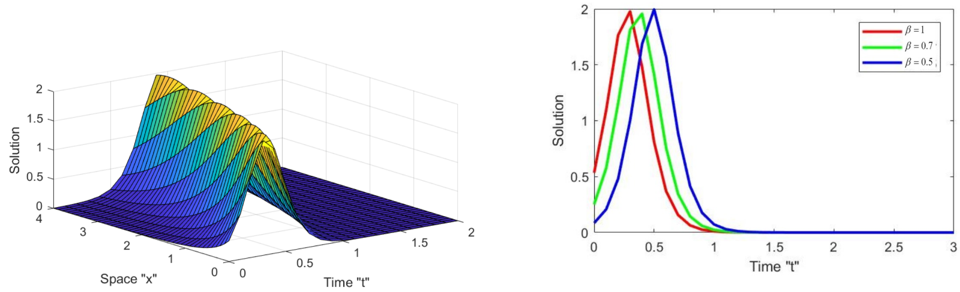
(**c**) $\beta = 0.5$ (**d**) $\beta = 1,\ 0.7,\ 0.5$

Figure 1. (**a**–**c**) Display of 3D-graph of Equation (29) with $\gamma = 0$ and several values of $\beta = 1, 0.75, 0.5$, and (**d**) shows 2D-graph of Equation (29) with several values of $\beta = 1, 0.75, 0.5$.



(**a**) $\gamma = 0,\ \beta = 1$ (**b**) $\gamma = 0,\ \beta = 0.75$

Figure 2. *Cont.*



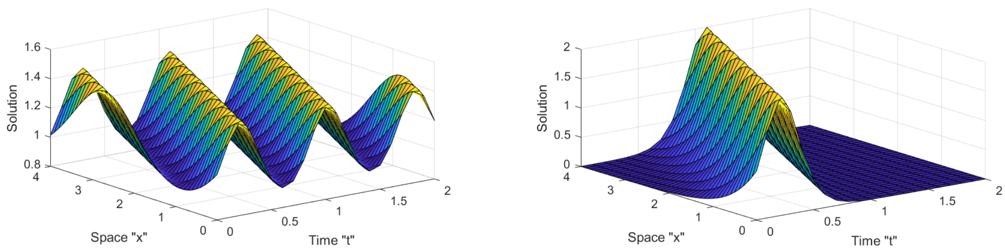
(**c**) $\gamma = 0,\ \beta = 0.5$ (**d**) $\gamma = 0,\ \beta = 0.25$

Figure 2. (**a**–**c**) Display of 3D-graph of Equation (31) with $\gamma = 0$ and several values of $\beta = 1, 0.75, 0.5$, and (**d**) shows 2D-graph of Equation (31) for several values of $\beta = 1, 0.75, 0.5$.

We deduced from Figures 1 and 2 that no overlap exists between the contours of the solutions. Additionally, the surface moves to the right as the order of the beta derivative decreases.

Second the noise effects:

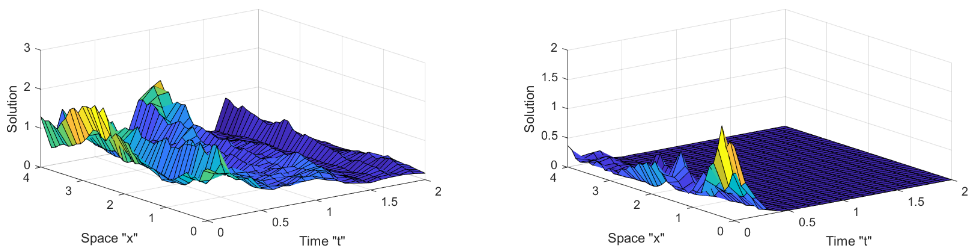
In Figure 3, the surface is not flat and contains various imperfections when $\gamma = 0$ (i.e., there is no noise).



$\gamma = 0$ $\gamma = 0$

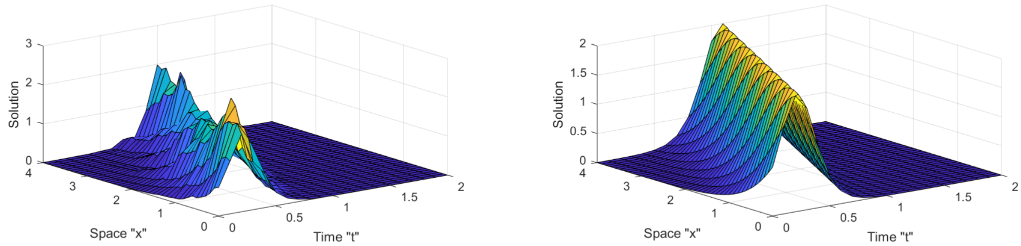
Figure 3. Diplay of 3D-profile of solution $\mathcal{R}(x, y, t)$ in Equations (29) and (31).

Meanwhile, we can see in Figures 4 and 5, after small movement patterns, the surface becomes more flat:



(**a**) $\gamma = 1$ (**b**) $\gamma = 2$

Figure 4. Display of 3D-profile of solution $\mathcal{R}(x, y, t)$ in Equation (29) for various $\gamma = 1, 2$.



(**a**) $\gamma = 1$ (**b**) $\gamma = 2$

Figure 5. Diplay of 3D-profile of solution $\mathcal{R}(x, y, t)$ in Equation (31) for various $\gamma = 1, 2$.

In the end, we can deduce from Figures 3–5 that several solutions exist when noise is disregarded (i.e., at $\gamma = 0$), such as periodic solutions, kink solutions, and others. After minor transit patterns, the surface becomes significantly flattened when noise occurs and its intensity is increased by $\gamma = 1, 2$. This demonstrates that the multiplicative white noise has an impact on the SKPE-BD solutions and stabilizes them at zero.

5. Discussion and Physical Meaning

In this paper, we take into consideration stochastic Kadomtsev–Petviashvili equation with beta-derivative (SKPE-BD). Finding an exact stochastic solution to the KP equation is a challenging task due to its nonlinearity and complexity. Here, we applied two methods, the RE-Method and JEF-Method, to attain the exact solutions for this equation. The first method provided solutions in the form of trigonometric, hyperbolic, and rational functions, while the second method gave elliptic solutions. Additionally, the specific characteristics of the stochastic term play a crucial role in the effect it has on the solution. Overall, understanding the stochastic effects is essential for accurately modeling and analyzing systems in the presence of uncertainty. The obtained solutions provide insights into the behavior of waves in different physical systems and can aid in the development of innovative technologies. They serve as foundational tools for advancing our understanding of nonlinear wave phenomena and can lead to significant advancements in fields such as plasma physics, fluid dynamics, and engineering.

6. Conclusions

In the current study, the stochastic (2 + 1)-dimensional Kadomtsev–Petviashvili equation with beta derivative (SKPE-BD) was derived. By employing two distinct methods such as the Riccati equation and Jacobi elliptic function, we obtained the exact solutions of SKPE-BD (1). Due to the importance of KP in the field of fluid dynamics and plasma physics, the acquired solutions are important for illustrating a wide range of intriguing and complex physical phenomena. Finally, the MATLAB tool was applied to illustrate the effect of SWP and BD on the obtained solutions of the SKPE-BD (1). We deduced that the beta-derivative shifted the surface to the left when the fractional-order derivative increased and the Wiener process stabilized the solutions at zero.

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