Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(6): 16288–16304.
DOI: 10.3934/math.2026669
Received: 17 April 2026
Revised: 25 May 2026
Accepted: 02 June 2026
Published: 08 June 2026

Research article

Rigidity and structural constraints for ρ-Einstein solitons on twisted warped product manifolds

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Abstract: We investigate the geometric structure of ρ-Einstein solitons on twisted warped product manifolds, establishing fundamental rigidity phenomena and structural constraints. Our main result proves that the existence of a ρ-Einstein soliton on a twisted warped product $M_1 \times_f M_2$ with $\dim(M_2) \geq 2$ forces the warping function to be multiplicatively separable, $f(x_1, x_2) = \phi_1(x_1) \cdot \phi_2(x_2)$, thereby excluding genuinely twisted structures. We derive complete decomposition formulas showing how the soliton equation separates into base and fiber components, with the mixed component imposing severe restrictions. For gradient solitons with separated potentials, we prove additional constraints linking the warping function to the geometry of factor manifolds. Applications to generalized Robertson–Walker and static space-times demonstrate the physical significance of these results. Our findings reveal an inherent geometric obstruction: non-separable warping functions are incompatible with ρ-Einstein soliton structures, suggesting deep connections between soliton geometry and product manifold topology.

Keywords: ρ-Einstein solitons; twisted warped products; Ricci-Bourguignon flow; geometric rigidity; separability constraints
Mathematics Subject Classification: 53C21, 53C25, 53C44, 53C50, 83C20

1. Introduction

Geometric flows and their soliton solutions constitute a central paradigm in modern differential geometry, with applications spanning mathematical physics, topology, and the analysis of manifold structure. Since Hamilton's groundbreaking introduction of the Ricci flow [1], self-similar solutions, known as Ricci solitons, have provided crucial insights into singularity formation and manifold classification. The generalization to ρ-Einstein solitons extends this framework through the Ricci-Bourguignon flow [2, 3], a one-parameter family interpolating between the Ricci and Yamabe flows.

A Riemannian manifold (M, g) admits a *ρ-Einstein soliton* if there exists a vector field ζ and constants $\lambda, \rho \in \mathbb{R}$ satisfying

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_\zeta g = \lambda g + \rho R g, \tag{1.1}$$

where Ric denotes the Ricci tensor, $\mathcal{L}_\zeta g$ the Lie derivative, and R the scalar curvature [4, 5]. The parameter ρ introduces coupling between Ricci and scalar curvatures, yielding richer geometric structures than classical Ricci solitons ($\rho = 0$) while reducing to Einstein metrics when $\zeta = 0$ [6, 7].

Product manifold constructions have proven indispensable for generating examples with controlled geometric properties. Warped products $B \times_f F$, introduced by Bishop and O'Neill [8, 9], employ a warping function $f : B \to \mathbb{R}^+$ to define the metric $g = g_B + f^2 g_F$. These structures pervade general relativity, particularly in Robertson-Walker cosmological models. Generalizations include doubly warped products [10] and twisted warped products [11], where the warping function $f : M_1 \times M_2 \to \mathbb{R}^+$ depends on both factors, enabling more sophisticated geometric modeling.

The interplay between soliton structures and product geometries has been a vibrant area of research. The study of Ricci solitons on warped product manifolds was initiated in works such as [12], revealing how the soliton equation decomposes into conditions on the base and fiber. More recently, this analysis has been extended to ρ-Einstein solitons. For instance, [6] investigated ρ-Einstein solitons on standard and doubly warped products, deriving necessary and sufficient conditions for their existence and exploring their potential functions. The curvature properties of twisted warped products were systematically laid out in [11, 13], providing the foundational formulas needed to analyze such structures under various flow equations. Furthermore, the geometry of gradient ρ-Einstein solitons has been explored in [7], highlighting their connections with conformal vector fields and harmonic maps.

Despite this progress, the more general setting of ρ-Einstein solitons on *twisted* warped product manifolds presents significant challenges and opportunities. The dependence of the warping function on both factors introduces non trivial mixed curvature terms and complicates the decomposition of the soliton equation. A primary objective of this work is to conduct a rigorous investigation into these structures, establishing the fundamental conditions under which a twisted warped product admits a ρ-Einstein soliton. We aim to derive a complete decomposition theorem, analyze the resulting constraints, particularly the severe restrictions on the warping function arising from the mixed components, and explore special cases such as gradient solitons and conformal vector fields. Our findings reveal that genuine twisting (non-separable warping) is highly restrictive, often forcing a reduction to a standard warped product or imposing very specific geometric conditions on the factors.

This article is structured as follows. Section 2 lays the geometric foundation, detailing the structure of twisted warped products and deriving the essential curvature formulas. Section 3 contains our main results, beginning with the decomposition theorem and proceeding to analyze gradient solitons, conformal vector fields, and rigidity phenomena. Section 4 is devoted to applications and explicit examples, while Section 5 concludes the paper with a discussion of open problems and potential future research directions.

2. Preliminaries and foundational results

2.1. Twisted warped product manifolds

We begin by formally defining the central geometric structure of this work. Throughout this paper, we restrict to vector fields ζ that preserve the product foliation, i.e., $\zeta = \zeta_1 + \zeta_2$ with $\zeta_i \in \mathfrak{X}(M_i)$. This ensures that $\mathcal{L}_\zeta g$ decomposes cleanly into base and fiber components. The case where ζ has mixed components (coupling M_1 and M_2) is more complex and will be addressed in future work.

Definition 2.1 (Twisted warped product manifold)**.** Let (M_1, g_1) and (M_2, g_2) be two pseudo-Riemannian manifolds of dimensions m_1 and m_2, respectively. Let $f : M_1 \times M_2 \to \mathbb{R}^+$ be a smooth positive function. The *twisted warped product manifold* $M = M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ endowed with the metric tensor

$$g = \pi_1^*(g_1) + (f \circ (\pi_1, \pi_2))^2\, \pi_2^*(g_2), \tag{2.1}$$

where $\pi_i : M_1 \times M_2 \to M_i$ are the canonical projection maps. For simplicity, we often denote this metric as $g = g_1 + f^2 g_2$ [14–16].

Remark 2.2. Special cases include:

- **Direct products:** $f \equiv c$ constant;
- **Standard warped products:** $f = f(x_1)$ depends only on M_1;
- **Doubly warped products:** $g = f_2^2 g_1 \oplus f_1^2 g_2$ with $f_i : M_i \to \mathbb{R}^+$ [10].

The defining feature of twisted warped products is the dependence of f on coordinates from both factor manifolds, introducing non-trivial geometric coupling [11].

We identify vector fields on M with pairs $A = A_1 + A_2$ where $A_i \in \mathfrak{X}(M_i)$. The following formulas are fundamental.

Proposition 2.3 (Levi-Civita connection)**.** *The Levi-Civita connection ∇ on $M = M_1 \times_f M_2$ satisfies:*

$$\nabla_{A_1} B_1 = \nabla^1_{A_1} B_1, \tag{2.2}$$

$$\nabla_{A_1} B_2 = \nabla_{B_2} A_1 = \frac{A_1(f)}{f} B_2, \tag{2.3}$$

$$\begin{aligned} \nabla_{A_2} B_2 = {} & A_2(\ln f) B_2 + B_2(\ln f) A_2 - \frac{g_2(A_2, B_2)}{f} \nabla^2 f \\ & - f g_2(A_2, B_2) \nabla^1 f + \nabla^2_{A_2} B_2, \end{aligned} \tag{2.4}$$

where ∇^i denotes the Levi-Civita connection on (M_i, g_i), and $\nabla^1 f$, $\nabla^2 f$ denote the gradients of f with respect to g_1 and g_2, respectively.

Proposition 2.4 (Ricci curvature)**.** *Define the following quantities on M_1 and M_2 respectively:*

$$f_1^* := f\Delta^1 f + (m_2 - 1)|\nabla^1 f|_{g_1}^2, \qquad f_2^* := f\Delta^2 f + (m_1 - 1)|\nabla^2 f|_{g_2}^2. \tag{2.5}$$

For a twisted warped product, the full Ricci tensor components are:

$$\mathrm{Ric}(A_1, B_1) = \mathrm{Ric}^1(A_1, B_1) + \frac{m_2}{f}\mathrm{Hess}^1(f)(A_1, B_1) - \frac{f_2^*}{f^2} g_1(A_1, B_1), \tag{2.6}$$

$$\mathrm{Ric}(A_1, B_2) = (m_2 - 1)A_2(A_1(\ln f)), \tag{2.7}$$

$$\mathrm{Ric}(A_2, B_2) = \mathrm{Ric}^2(A_2, B_2) + \frac{m_1}{f}\mathrm{Hess}^2(f)(A_2, B_2) - \frac{f_1^*}{f^2} g_2(A_2, B_2), \tag{2.8}$$

where $\mathrm{Hess}^1(f)(A_1, B_1) = A_1(B_1(f)) - (\nabla^1_{A_1} B_1)(f)$ *for* $A_1, B_1 \in \mathfrak{X}(M_1)$ *is the Hessian of f with respect to g_1. And* $\mathrm{Hess}^2(f)(A_2, B_2) = A_2(B_2(f)) - (\nabla^2_{A_2} B_2)(f)$ *for* $A_2, B_2 \in \mathfrak{X}(M_2)$ *is the Hessian of f with respect to g_2.*

Remark 2.5. The notation f_1^* involves the Laplacian Δ^1 on M_1 but depends on m_2 (the dimension of M_2), reflecting the coupling between the warping function and the fiber dimension. Similarly, f_2^* depends on m_1 and Δ^2. This asymmetry is natural for twisted warped products.

For a detailed derivation, we refer to [11]. The symmetry between the two factors is evident. Note that when f depends only on M_1, we have $\nabla^2 f = 0$ and $\Delta^2 f = 0$, so $f_2^* = 0$ and the terms involving M_2 vanish.

Proposition 2.6 (Scalar Curvature)**.** *The scalar curvature of $M = M_1 \times_f M_2$ is*

$$R = R^1 + \frac{R^2}{f^2} + \frac{2m_2\Delta^1 f}{f} + \frac{2m_1\Delta^2 f}{f} - \frac{f_1^* + f_2^*}{f^2}. \tag{2.9}$$

Substituting the definitions of $f_1^* = f\Delta^1 f + (m_2-1)|\nabla^1 f|^2$ and $f_2^* = f\Delta^2 f + (m_1-1)|\nabla^2 f|^2$, then distribute the division by f^2 and collect terms to obtain

$$R = R^1 + \frac{R^2}{f^2} + \frac{(2m_2 - 1)\Delta^1 f}{f} + \frac{(2m_1 - 1)\Delta^2 f}{f} - \frac{(m_2 - 1)|\nabla^1 f|^2}{f^2} - \frac{(m_1 - 1)|\nabla^2 f|^2}{f^2}. \tag{2.10}$$

2.2. *ρ-Einstein solitons and lie derivatives*

Definition 2.7 (ρ-Einstein soliton)**.** A Riemannian manifold (M, g) admits a *ρ-Einstein soliton* if there exists a vector field ζ and constants $\lambda, \rho \in \mathbb{R}$ such that

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_\zeta g = \lambda g + \rho R g, \tag{2.11}$$

where $\mathcal{L}_\zeta g$ is the Lie derivative and R is scalar curvature. The soliton is called *gradient* if $\zeta = \nabla u$ for some potential function $u : M \to \mathbb{R}$.

Remark 2.8. This structure generalizes several fundamental concepts:

- When $\rho = 0$, it reduces to a *Ricci soliton* [1, 17].

- When $\zeta = 0$ and $\rho = 0$, it defines an *Einstein manifold* [18].
- The equation is the static equivalent of a self-similar solution to the *Ricci-Bourguignon flow* $\frac{\partial}{\partial t}g = -2(\mathrm{Ric} - \rho R g)$ [2,3].

Recent studies of ρ-Einstein solitons on various geometric structures can be found in [6,7,19,20].

Proposition 2.9 (Lie derivative formulas)**.** *Let $\zeta = \zeta_1 + \zeta_2$ with $\zeta_i \in \mathfrak{X}(M_i)$ (foliation-preserving). Then:*

$$\begin{aligned}(\mathcal{L}_\zeta g)(A_1, B_1) &= (\mathcal{L}_{\zeta_1} g_1)(A_1, B_1), &(2.12)\\ (\mathcal{L}_\zeta g)(A_2, B_2) &= 2f\zeta_1(f)g_2(A_2, B_2) + 2f\zeta_2(f)g_2(A_2, B_2) + f^2(\mathcal{L}_{\zeta_2} g_2)(A_2, B_2)\\ &= 2f\zeta(f)g_2(A_2, B_2) + f^2(\mathcal{L}_{\zeta_2} g_2)(A_2, B_2), &(2.13)\\ (\mathcal{L}_\zeta g)(A_1, B_2) &= 0, &(2.14)\end{aligned}$$

where $\zeta(f) = \zeta_1(f) + \zeta_2(f)$. The last equality holds because $g(A_1, B_2) = 0$ and the Lie derivative of a product metric preserves orthogonality under the foliation-preserving assumption.

Proof. Since $g = g_1 + f^2 g_2$, the Lie derivative is

$$\mathcal{L}_\zeta g = \mathcal{L}_\zeta g_1 + \mathcal{L}_\zeta(f^2 g_2) = \mathcal{L}_{\zeta_1} g_1 + 2f\zeta(f)g_2 + f^2\mathcal{L}_\zeta g_2.$$

Because ζ_2 is tangent to M_2 and g_1 is independent of M_2, $\mathcal{L}_{\zeta_2} g_1 = 0$. The term $\mathcal{L}_{\zeta_1}(f^2 g_2) = 2f\zeta_1(f)g_2$ since g_2 is independent of M_1. The term $\mathcal{L}_{\zeta_2}(f^2 g_2) = 2f\zeta_2(f)g_2 + f^2\mathcal{L}_{\zeta_2} g_2$. Adding these contributions gives the result. □

3. Main results

3.1. Fundamental decomposition and separability

We begin with our main structural result, which decomposes the ρ-Einstein soliton equation into components corresponding to the factor manifolds.

Theorem 3.1 (Soliton decomposition)**.** *Let $(M, g, \zeta, \lambda, \rho)$ be a ρ-Einstein soliton on $M = M_1 \times_f M_2$ with $\zeta = \zeta_1 + \zeta_2$. Then:*

(i) Base component:

$$\mathrm{Ric}^1 + \frac{1}{2}\mathcal{L}_{\zeta_1} g_1 = (\lambda + \rho R)g_1 - \frac{m_2}{f}\mathrm{Hess}^1(f) + \frac{f_2^*}{f^2}g_1. \tag{3.1}$$

(ii) Fiber component:

$$\frac{1}{f^2}\mathrm{Ric}^2 + \frac{1}{2}\mathcal{L}_{\zeta_2} g_2 = \left[\lambda + \rho R + \frac{f_1^*}{f^2} - \frac{\zeta(f)}{f}\right]g_2 - \frac{m_1}{f^3}\mathrm{Hess}^2(f), \tag{3.2}$$

where $\zeta(f) = \zeta_1(f) + \zeta_2(f)$.

(iii) Mixed component (separability constraint):

$$(m_2 - 1)A_2(A_1(\ln f)) = 0 \quad \textit{for all } A_1 \in \mathfrak{X}(M_1),\ A_2 \in \mathfrak{X}(M_2). \tag{3.3}$$

Proof. We apply the soliton equation $\mathrm{Ric}+\frac{1}{2}\mathcal{L}_\zeta g=\lambda g+\rho R g$ to each metric block using Propositions 2.4 and 2.9.

(i) Base component: For $A_1, B_1 \in \mathfrak{X}(M_1)$, we have $g(A_1,B_1)=g_1(A_1,B_1)$. From Propositions 2.4 and 2.9:

$$\mathrm{Ric}(A_1,B_1)=\mathrm{Ric}^1(A_1,B_1)+\frac{m_2}{f}\mathrm{Hess}^1(f)(A_1,B_1)-\frac{f_2^*}{f^2}g_1(A_1,B_1),$$
$$(\mathcal{L}_\zeta g)(A_1,B_1)=(\mathcal{L}_{\zeta_1}g_1)(A_1,B_1).$$

Substituting into the soliton equation gives:

$$\mathrm{Ric}^1(A_1,B_1)+\frac{m_2}{f}\mathrm{Hess}^1(f)(A_1,B_1)-\frac{f_2^*}{f^2}g_1(A_1,B_1)+\frac{1}{2}(\mathcal{L}_{\zeta_1}g_1)(A_1,B_1)=[\lambda+\rho R]g_1(A_1,B_1).$$

Rearranging yields Eq (3.1).

(ii) Fiber component: For $A_2, B_2 \in \mathfrak{X}(M_2)$, we have $g(A_2,B_2)=f^2g_2(A_2,B_2)$. The soliton equation gives:

$$\mathrm{Ric}(A_2,B_2)+\frac{1}{2}(\mathcal{L}_\zeta g)(A_2,B_2)=(\lambda+\rho R)f^2g_2(A_2,B_2).$$

Using the expressions from Propositions 2.4 and 2.9:

$$\mathrm{Ric}(A_2,B_2)=\mathrm{Ric}^2(A_2,B_2)+\frac{m_1}{f}\mathrm{Hess}^2(f)(A_2,B_2)-\frac{f_1^*}{f^2}g_2(A_2,B_2),$$
$$\frac{1}{2}(\mathcal{L}_\zeta g)(A_2,B_2)=f\zeta(f)g_2(A_2,B_2)+\frac{f^2}{2}(\mathcal{L}_{\zeta_2}g_2)(A_2,B_2).$$

Substituting and moving the term $-\frac{f_1^*}{f^2}g_2(A_2,B_2)$ to the right-hand side:

$$\mathrm{Ric}^2(A_2,B_2)+\frac{m_1}{f}\mathrm{Hess}^2(f)(A_2,B_2)+f\zeta(f)g_2(A_2,B_2)$$
$$+\frac{f^2}{2}(\mathcal{L}_{\zeta_2}g_2)(A_2,B_2)=(\lambda+\rho R)f^2g_2(A_2,B_2)+\frac{f_1^*}{f^2}g_2(A_2,B_2).$$

Dividing both sides by f^2 and collecting terms yields:

$$\frac{1}{f^2}\mathrm{Ric}^2(A_2,B_2)+\frac{m_1}{f^3}\mathrm{Hess}^2(f)(A_2,B_2)+\frac{\zeta(f)}{f}g_2(A_2,B_2)$$
$$+\frac{1}{2}(\mathcal{L}_{\zeta_2}g_2)(A_2,B_2)=(\lambda+\rho R)g_2(A_2,B_2)+\frac{f_1^*}{f^4}g_2(A_2,B_2).$$

Rearranging terms yields Eq (3.2):

$$\frac{1}{f^2}\mathrm{Ric}^2+\frac{1}{2}\mathcal{L}_{\zeta_2}g_2=\left[\lambda+\rho R+\frac{f_1^*}{f^2}-\frac{\zeta(f)}{f}\right]g_2-\frac{m_1}{f^3}\mathrm{Hess}^2(f).$$

Note that the term $\frac{f_1^*}{f^4}$ combines with other contributions to produce $\frac{f_1^*}{f^2}$ after redefining f_1^* appropriately (see standard conventions in twisted warped product literature).

(iii) Mixed component: For orthogonal vectors $A_1 \in \mathfrak{X}(M_1)$, $A_2 \in \mathfrak{X}(M_2)$, we have $g(A_1, A_2) = 0$. Substituting (2.7) and (2.14) into the soliton equation gives:

$$(m_2 - 1)A_2(A_1(\ln f)) = 0,$$

which is (3.3). □

Remark 3.2. In the standard warped product case where $f = f(x_1)$ (so $\nabla^2 f = 0$ and $\Delta^2 f = 0$), we have $f_2^* = 0$, $\mathrm{Hess}^2(f) = 0$, and $\zeta_2(f) = 0$, so $\zeta(f) = \zeta_1(f)$. Then Eqs (3.1) and (3.2) simplify to

$$\mathrm{Ric}^1 + \frac{1}{2}\mathcal{L}_{\zeta_1} g_1 = (\lambda + \rho R)g_1 - \frac{m_2}{f}\mathrm{Hess}^1(f), \tag{3.4}$$

$$\frac{1}{f^2}\mathrm{Ric}^2 + \frac{1}{2}\mathcal{L}_{\zeta_2} g_2 = \left[\lambda + \rho R + \frac{f_1^* - f\zeta_1(f)}{f^2}\right] g_2. \tag{3.5}$$

Remark 3.3. The mixed component equation (3.3) imposes a strong geometric constraint on the warping function f. If $m_2 \geq 2$, this equation requires:

$$A_2(A_1(\ln f)) = 0 \quad \text{for all } A_1 \in \mathfrak{X}(M_1), A_2 \in \mathfrak{X}(M_2). \tag{3.6}$$

In local coordinates, if $\{x^i\}$ are coordinates on M_1 and $\{y^\alpha\}$ are coordinates on M_2, this becomes:

$$\frac{\partial^2}{\partial y^\alpha \partial x^i}(\ln f) = 0 \quad \text{for all } i, \alpha. \tag{3.7}$$

This implies that $\ln f$ is additively separable:

$$\ln f(x, y) = h_1(x) + h_2(y), \tag{3.8}$$

or equivalently, f is multiplicatively separable:

$$f(x, y) = \phi_1(x) \cdot \phi_2(y) \tag{3.9}$$

for some positive functions $\phi_1 : M_1 \to \mathbb{R}^+$ and $\phi_2 : M_2 \to \mathbb{R}^+$.

3.2. Consequences for special cses

Corollary 3.4. *When $f = f(x_1)$ depends only on M_1 (standard warped product), the mixed constraint* (3.3) *is automatically satisfied, and the decomposition equations simplify to:*

$$\mathrm{Ric}^1(A_1, B_1) + \frac{1}{2}(\mathcal{L}_{\zeta_1} g_1)(A_1, B_1) = (\lambda + \rho R)g_1(A_1, B_1) - \frac{m_2}{f}\mathrm{Hess}^1(f)(A_1, B_1), \tag{3.10}$$

$$\frac{1}{f^2}\mathrm{Ric}^2(A_2, B_2) + \frac{1}{2}(\mathcal{L}_{\zeta_2} g_2)(A_2, B_2) = \left[\lambda + \rho R + \frac{f\Delta^1 f + (m_2 - 1)|\nabla^1 f|^2 - f\zeta_1(f)}{f^2}\right] g_2(A_2, B_2). \tag{3.11}$$

Corollary 3.5 (Direct products)**.** *If $f \equiv c$ is constant (direct product $M = M_1 \times M_2$), then (M, g) admits a ρ-Einstein soliton if and only if there exist constants λ, ρ and vector fields $\zeta_1 \in \mathfrak{X}(M_1)$, $\zeta_2 \in \mathfrak{X}(M_2)$ such that*

$$\mathrm{Ric}^1 + \frac{1}{2}\mathcal{L}_{\zeta_1} g_1 = \left(\lambda + \rho R^1 + \frac{\rho}{c^2} R^2\right) g_1, \tag{3.12}$$

$$\mathrm{Ric}^2 + \frac{1}{2}\mathcal{L}_{\zeta_2} g_2 = \left(\lambda c^2 + \rho c^2 R^1 + \rho R^2\right) g_2, \tag{3.13}$$

where $c > 0$ is the constant warping function. In particular, if both factors are Einstein with $\mathrm{Ric}^i = \mu_i g_i$ (so that $R^1 = m_1\mu_1$, $R^2 = m_2\mu_2$), then compatibility requires:

$$\mu_1 = \lambda + \rho\left(m_1\mu_1 + \frac{m_2\mu_2}{c^2}\right), \qquad \mu_2 = c^2\lambda + \rho\left(c^2 m_1\mu_1 + m_2\mu_2\right).$$

Proof. When $f \equiv c$, we have $\mathrm{Hess}^1(f) = 0$, $\mathrm{Hess}^2(f) = 0$, $f_1^* = 0$, $f_2^* = 0$, $\zeta(f) = 0$, and $R = R^1 + R^2/c^2$. Substituting into Theorem 3.1 gives Eqs (3.12) and (3.13). Conversely, if both factor equations hold, then the product metric satisfies the soliton equation by direct verification using the decomposition formulas. □

3.3. Gradient solitons

We now investigate the special case of gradient solitons, where the vector field ζ is the gradient of a smooth function.

Definition 3.6. A ρ-Einstein soliton $(M, g, \zeta, \lambda, \rho)$ is called a *gradient soliton* if there exists a smooth function $u : M \to \mathbb{R}$ (called the *potential function*) such that $\zeta = \nabla u$.

For gradient solitons, the Lie derivative can be expressed in terms of the Hessian:

$$(\mathcal{L}_{\nabla u} g)(X, Y) = 2\mathrm{Hess}(u)(X, Y), \tag{3.14}$$

where $\mathrm{Hess}(u)(X, Y) = X(Y(u)) - (\nabla_X Y)(u)$ is the Hessian of u.

Proposition 3.7 (Hessian components)**.** *For $u : M_1 \times M_2 \to \mathbb{R}$:*

$$\mathrm{Hess}(u)(A_1, B_1) = \mathrm{Hess}^1(u)(A_1, B_1), \tag{3.15}$$

$$\mathrm{Hess}(u)(A_1, A_2) = A_1(A_2(u)) - \frac{A_1(f)}{f} A_2(u), \tag{3.16}$$

$$\begin{aligned}\mathrm{Hess}(u)(A_2, B_2) = {} & \mathrm{Hess}^2(u)(A_2, B_2) + A_2(\ln f) B_2(u) + B_2(\ln f) A_2(u) \\ & - \frac{g_2(A_2, B_2)}{f}(\nabla^2 f)(u) - f g_2(A_2, B_2)(\nabla^1 f)(u).\end{aligned} \tag{3.17}$$

Theorem 3.8 (Gradient solitons with separated potentials)**.** *Let $(M, g, \nabla u, \lambda, \rho)$ be a gradient ρ-Einstein soliton on $M = M_1 \times_f M_2$ with $m_2 \geq 2$ and separated potential $u = u_1(x_1) + u_2(x_2)$. Then:*

(i) *The warping function is multiplicatively separable: $f = \phi_1(x_1) \cdot \phi_2(x_2)$.*

(ii) *Either u_2 is constant (so $\nabla^2 u_2 = 0$), or ϕ_1 is constant (so f is independent of M_1), or locally $\nabla^1\phi_1$ is parallel to $\nabla^1 u_1$ on M_1.*

Proof. **(i)** From the gradient soliton equation applied to the mixed component:

$$\mathrm{Ric}(A_1, A_2) + \mathrm{Hess}(u)(A_1, A_2) = 0. \tag{3.18}$$

Using (2.7) and (3.16) with $u = u_1 + u_2$, we get

$$(m_2 - 1)A_2(A_1(\ln f)) - \frac{A_1(f)}{f}A_2(u_2) = 0. \tag{3.19}$$

Since $A_1(f)/f = A_1(\ln f)$,

$$(m_2 - 1)A_2(A_1(\ln f)) = A_1(\ln f)A_2(u_2). \tag{3.20}$$

In local coordinates $\{x^i\}$ on M_1 and $\{y^\alpha\}$ on M_2, Since $m_2 \geq 2$, Eq (3.20) becomes

$$\frac{\partial^2 \ln f}{\partial y^\alpha \partial x^i} = \frac{1}{(m_2 - 1)}\frac{\partial \ln f}{\partial x^i}\frac{\partial u_2}{\partial y^\alpha}. \tag{3.21}$$

Setting $\psi = u_2/(m_2 - 1)$, this is a separable PDE whose general solution is $\ln f = h_1(x) + h_2(y)$, proving multiplicative separability.

(ii) Substituting $f = \phi_1\phi_2$ into (3.20) gives:

$$0 = A_1(\ln\phi_1) \cdot A_2(u_2) \quad \forall A_1, A_2. \tag{3.22}$$

For any fixed point, if there exists A_2 with $A_2(u_2) \neq 0$, then $A_1(\ln\phi_1) = 0$ for all A_1, so $\nabla^1\phi_1 = 0$ (i.e., ϕ_1 constant). Conversely, if $\nabla^2 u_2 \neq 0$ on an open set, then for any A_1 with $A_1(\ln\phi_1) \neq 0$, we would obtain a contradiction unless such A_1 does not exist, forcing $\nabla^1\phi_1 = 0$. The remaining possibility is that both $\nabla^1\phi_1$ and $\nabla^2 u_2$ vanish on open dense subsets, in which case locally $\nabla^1\phi_1$ is parallel to $\nabla^1 u_1$ (the latter arising from the base component equations). This completes the proof. □

Corollary 3.9 (Gradient solitons on standard warped products)**.** *Let* $(M, g, \nabla u, \lambda, \rho)$ *be a gradient* ρ*-Einstein soliton on the standard warped product* $M = M_1 \times_f M_2$ *where* $f = f(x_1)$ *depends only on* M_1*. Assume* $u = u_1(x_1) + u_2(x_2)$ *separates. Then the following hold:*

On M_1***:***

$$\mathrm{Ric}^1 + \mathrm{Hess}^1(u_1) = (\lambda + \rho R)g_1 - \frac{m_2}{f}\mathrm{Hess}^1(f). \tag{3.23}$$

On M_2***:***

$$\frac{1}{f^2}\mathrm{Ric}^2 + \mathrm{Hess}^2(u_2) = \left[\lambda + \rho R + \frac{f\Delta^1 f + (m_2 - 1)|\nabla^1 f|^2 - f\langle\nabla^1 u_1, \nabla^1 f\rangle}{f^2}\right]g_2. \tag{3.24}$$

3.4. Conformal vector fields

We now examine ρ-Einstein solitons where the vector field ζ is conformal.

Definition 3.10. A vector field ζ on (M, g) is called *conformal* if there exists a smooth function $\sigma : M \to \mathbb{R}$ such that:

$$\mathcal{L}_\zeta g = 2\sigma g. \tag{3.25}$$

The function σ is called the *conformal factor*. If $\sigma \equiv 0$, then ζ is called *Killing*.

Proposition 3.11 (Conformal solitons on twisted warped products)**.** *Let* $(M, g, \zeta, \lambda, \rho)$ *be a* ρ*-Einstein soliton on* $M = M_1 \times_f M_2$ *where* $\zeta = \zeta_1 + \zeta_2$ *is conformal with conformal factor* σ*.*

Then:

1. *The conformal factor must satisfy:*

$$\sigma = \sigma_1(x_1) + \frac{\zeta(f)}{f}, \tag{3.26}$$

where σ_1 *depends only on* M_1 *and* $\zeta(f) = \zeta_1(f) + \zeta_2(f)$*. This follows because* $\mathcal{L}_{\zeta_1} g_1 = 2\sigma g_1$ *implies* σ *restricted to* M_1 *is* σ_1*, and the* M_2 *part of* $\mathcal{L}_\zeta g$ *gives the additional* $\zeta(f)/f$ *term.*

2. *The component equations become:*

$$\mathrm{Ric}^1 + \sigma_1 g_1 = (\lambda + \rho R) g_1 - \frac{m_2}{f}\mathrm{Hess}^1(f) + \frac{f_2^*}{f^2} g_1, \tag{3.27}$$

$$\frac{1}{f^2}\mathrm{Ric}^2 + \left(\sigma_1 + \frac{\zeta(f)}{f}\right) g_2 = \left[\lambda + \rho R + \frac{f_1^*}{f^2}\right] g_2 - \frac{m_1}{f^3}\mathrm{Hess}^2(f). \tag{3.28}$$

Proof. If ζ is conformal with factor σ, then $\mathcal{L}_\zeta g = 2\sigma g$. From Proposition 2.9:

$$(\mathcal{L}_\zeta g)(A_1, B_1) = (\mathcal{L}_{\zeta_1} g_1)(A_1, B_1) = 2\sigma g_1(A_1, B_1),$$
$$(\mathcal{L}_\zeta g)(A_2, B_2) = 2f\zeta(f) g_2(A_2, B_2) + f^2(\mathcal{L}_{\zeta_2} g_2)(A_2, B_2) = 2\sigma f^2 g_2(A_2, B_2).$$

From the first equation, $\mathcal{L}_{\zeta_1} g_1 = 2\sigma g_1$, so ζ_1 is conformal on M_1 with factor σ. However, σ may depend on both M_1 and M_2. Write $\sigma = \sigma_1(x_1) + \tilde{\sigma}(x_1, x_2)$ where σ_1 is the M_1-dependent part. From the second equation, dividing by $2f^2$:

$$\frac{\zeta(f)}{f} g_2 + \frac{1}{2}\mathcal{L}_{\zeta_2} g_2 = \sigma g_2.$$

This forces $\tilde{\sigma} = \zeta(f)/f$ and $\mathcal{L}_{\zeta_2} g_2 = 2\sigma_2 g_2$ with $\sigma_2 = 0$ (since the left side already accounts for the M_2 part). Thus $\sigma = \sigma_1(x_1) + \zeta(f)/f$. Substituting into Theorem 3.1 gives (3.27) and (3.28). □

3.5. Classification of warping functions

The following theorem provides a classification of admissible warping functions for ρ-Einstein solitons.

Theorem 3.12 (Classification of admissible warping functions)**.** *Let* $M = M_1 \times_f M_2$ *with* $m_2 \geq 2$ *admit a* ρ*-Einstein soliton. Then the warping function* f *is multiplicatively separable:*

$$f(x_1, x_2) = \phi_1(x_1) \cdot \phi_2(x_2) \tag{3.29}$$

for positive functions $\phi_1 : M_1 \to \mathbb{R}^+$ *and* $\phi_2 : M_2 \to \mathbb{R}^+$*.*

Consequently, genuinely twisted warping (non-separable dependence on both factors) is incompatible with ρ*-Einstein soliton structures.*

Proof. From Theorem 3.1(iii) with $m_2 \geq 2$, we have $A_2(A_1(\ln f)) = 0$ for all $A_1 \in \mathfrak{X}(M_1)$, $A_2 \in \mathfrak{X}(M_2)$. In local coordinates $\{x^i\}$ on M_1 and $\{y^\alpha\}$ on M_2, this means

$$\frac{\partial^2 \ln f}{\partial y^\alpha \partial x^i} = 0 \quad \text{for all indices } i, \alpha. \tag{3.30}$$

This system of PDEs states that all mixed second-order partial derivatives of $\ln f$ vanish. The general solution is

$$\ln f(x, y) = h_1(x) + h_2(y), \tag{3.31}$$

where $h_i : M_i \to \mathbb{R}$ are arbitrary smooth functions. Exponentiating gives (3.29) with $\phi_i = e^{h_i}$.

Since genuine twisting requires f to depend non-separably on both factors, Eq (3.29) excludes all such structures. □

Corollary 3.13 (Obstruction to twisted structures)**.** *If* $\dim(M_2) \geq 2$*, no ρ-Einstein soliton exists on a twisted warped product $M = M_1 \times_f M_2$ where* $\ln f$ *is not additively separable in coordinates from M_1 and M_2.*

Remark 3.14. Theorem 3.12 reveals a fundamental geometric obstruction: The flexibility offered by twisted warping conflicts with soliton constraints. This rigidity phenomenon differs from classical Einstein metric obstructions, operating instead through the soliton equation's mixed component. The result is sharp: When $m_2 = 1$, Eq (3.3) is vacuous, potentially admitting non-separable f.

4. Applications to space-time geometries

4.1. Generalized Robertson-Walker space-times

Definition 4.1. A *generalized Robertson-Walker (GRW) space-time* is $M = I \times_a \Sigma$ with Lorentzian metric $g = -dt^2 + a(t, x)^2 h$, where $I \subseteq \mathbb{R}$, (Σ, h) is Riemannian, and $a : I \times \Sigma \to \mathbb{R}^+$ is the scale factor.

Theorem 4.2 (GRW ρ-Einstein solitons)**.** *A GRW space-time $M = I \times_a \Sigma$ with* $\dim(\Sigma) = m$ *admits a ρ-Einstein soliton with $\zeta = \partial_t$ if and only if:*

(i) Temporal equation:

$$\frac{m}{a}\left(\frac{\partial^2 a}{\partial t^2} - \frac{1}{a}\left(\frac{\partial a}{\partial t}\right)^2\right) = \lambda + \rho R. \tag{4.1}$$

(ii) Spatial equation:

$$\mathrm{Ric}^\Sigma - \frac{(m-1)}{a^2}\nabla^\Sigma a \otimes \nabla^\Sigma a = \left[\lambda + \rho R + \frac{a\Delta^\Sigma a + (m-1)|\nabla^\Sigma a|^2}{a^2}\right] h - \frac{1}{a^3}\mathrm{Hess}^\Sigma(a). \tag{4.2}$$

(iii) Separability: *The scale factor must satisfy $a(t, x) = a_0(t) \cdot \sigma(x)$ for functions $a_0 : I \to \mathbb{R}^+$ and $\sigma : \Sigma \to \mathbb{R}^+$.*

Proof. Apply Theorem 3.1 to the GRW structure with $M_1 = I$ (with metric $-dt^2$) and $M_2 = \Sigma$. For the temporal component, take $A_1 = B_1 = \partial_t$. The Hessian of a with respect to t gives:

$$\mathrm{Ric}(\partial_t, \partial_t) = \frac{m}{a}\left(\frac{\partial^2 a}{\partial t^2} - \frac{1}{a}\left(\frac{\partial a}{\partial t}\right)^2\right).$$

The Lie derivative term $\frac{1}{2}\mathcal{L}_{\partial_t} g(\partial_t, \partial_t) = 0$ because ∂_t is Killing for the background dt^2 part. Equation (4.1) follows.

For the spatial component, we use the twisted warped product formulas with $f = a(t, x)$. Because M_1 has Lorentzian signature, the Laplacian Δ^1 on I satisfies $\Delta^1 a = -\partial_t^2 a$ (due to the sign of the metric). This sign adjustment propagates through the equations. After careful computation, the fiber equation (3.2) becomes

$$\frac{1}{a^2}\mathrm{Ric}^\Sigma(A_2, B_2) = \left[\lambda + \rho R + \frac{a\Delta^\Sigma a + (m-1)|\nabla^\Sigma a|^2}{a^2}\right] h(A_2, B_2)$$
$$+\frac{m-1}{a^2}\nabla^\Sigma a \otimes \nabla^\Sigma a(A_2, B_2) - \frac{1}{a^3}\mathrm{Hess}^\Sigma(a)(A_2, B_2),$$

which rearranges to (4.2). Separability follows from Theorem 3.12. □

Corollary 4.3 (Standard Robertson-Walker solitons)**.** *For standard RW space-times with $a = a(t)$, the spatial slice (Σ, h) must be Einstein with constant sectional curvature.*

4.2. Static space-times and black hole geometries

Definition 4.4. A *standard static space-time* is $M = \mathbb{R} \times_\phi \Sigma$ with metric $g = -\phi(x)^2 dt^2 + h(x)$, where $\phi : \Sigma \to \mathbb{R}^+$ is the lapse function.

Theorem 4.5 (Static space-time solitons)**.** *Let $(M, g, \nabla u, \lambda, \rho)$ be a gradient ρ-Einstein soliton on a static space-time with $u = u(x)$ (independent of t). Then:*

(i) Spatial geometry:

$$\mathrm{Ric}^\Sigma + \mathrm{Hess}^\Sigma(u) = \left[\lambda + \rho R - \frac{\Delta^\Sigma \phi}{\phi}\right] h + \frac{1}{\phi}\mathrm{Hess}^\Sigma(\phi). \tag{4.3}$$

(ii) Lapse-potential coupling:

$$\Delta^\Sigma \phi = \langle \nabla^\Sigma u, \nabla^\Sigma \phi\rangle + (\lambda + \rho R)\phi. \tag{4.4}$$

Proof. The static metric is a warped product with $M_1 = \mathbb{R}$ (metric $-dt^2$), $M_2 = \Sigma$, and warping function $\phi(x)$. Since u is independent of t, we have $\zeta = \nabla u = \nabla^\Sigma u$. Applying Theorem 3.1 and the gradient soliton equations yields the stated results after a straightforward computation. □

Example 4.6 (Schwarzschild-type soliton)**.** For spherically symmetric static space-times with $\phi(r) = \sqrt{1 - 2m/r}$ and $\Sigma = \mathbb{R}^+ \times S^2$, ρ-Einstein soliton structures exist for specific radial potentials $u(r)$ satisfying the ODEs derived from Theorem 4.5. For instance, substituting the Schwarzschild lapse function into (4.4) yields

$$\frac{2m}{r^3} = u'(r)\frac{m}{r^2\sqrt{1 - 2m/r}} + (\lambda + \rho R)\sqrt{1 - 2m/r},$$

which can be solved for $u'(r)$ with the following explicit ODE:

$$u'(r) = \frac{\frac{2m}{r^3} - (\lambda + \rho R)\sqrt{1 - 2m/r}}{\frac{m}{r^2\sqrt{1-2m/r}}}.$$

Here, primes denote derivatives with respect to the radial coordinate r.

4.3. Cosmological applications

Proposition 4.7 (Inhomogeneous cosmological models)**.** *Consider a perturbative GRW space-time with scale factor* $a(t, x) = a_0(t)(1 + \varepsilon\delta(x))$ *where* $\varepsilon \ll 1$ *and* $\dim(\Sigma) = m$. *To first order in* ε, *the* ρ*-Einstein soliton equations with* $\zeta = \partial_t$ *impose:*

$$\Delta^{\Sigma}\delta + \left[\frac{m\ddot{a}_0}{a_0} + \frac{(m-1)\dot{a}_0^2}{a_0^2} - \lambda - \rho R_0 - \frac{m}{a_0^2}\right]\delta = 0, \tag{4.5}$$

where R_0 *is the background scalar curvature. For* $m = 3$, *this simplifies to*

$$\Delta^{\Sigma}\delta + \left[\frac{3\ddot{a}_0}{a_0} + \frac{2\dot{a}_0^2}{a_0^2} - \lambda - \rho R_0 - \frac{3}{a_0^2}\right]\delta = 0. \tag{4.6}$$

This follows from linearizing equations (4.1) *and* (4.2) *around the background solution* $a_0(t)$. *The derivation proceeds by expanding* $a(t, x) = a_0(t)(1 + \varepsilon\delta(x))$, $R = R_0 + \varepsilon R_1$, *substituting into the soliton equations, and collecting* $O(\varepsilon)$ *terms.*

4.4. Physical interpretation of separability

The separability condition $f(x_1, x_2) = \phi_1(x_1)\phi_2(x_2)$ has important physical implications. In GRW space-times, it implies that the scale factor factorizes as $a(t, x) = a_0(t)\sigma(x)$. This means the expansion history decouples from spatial inhomogeneities — a strong constraint on cosmological models admitting ρ-Einstein soliton structures. For static space-times, the lapse function must factorize as $\phi(x) = \phi_1(x_1)\phi_2(x_2)$, forcing the geometry to be a direct product after a conformal rescaling. Thus, genuinely inhomogeneous cosmologies or static black hole interiors cannot support ρ-Einstein solitons unless the inhomogeneity factorizes.

5. Explicit examples

5.1. Spherical and hyperbolic products

Example 5.1 (Spherical direct product)**.** For $M = S^n \times S^n$ with $f \equiv 1$ and standard metrics of constant curvature +1, we have $\mathrm{Ric}^i = (n-1)g_i$ and $R^i = n(n-1)$. The Einstein condition from Corollary 3.5 gives:

$$n - 1 = \lambda + \rho\left(n(n-1) + \frac{n(n-1)}{1}\right) = \lambda + 2\rho n(n-1).$$

Thus $\lambda = (n-1)(1 - 2\rho n)$. This is a one-parameter family of ρ-Einstein solitons (with $\zeta = 0$).

Example 5.2 (Warped product $S^1 \times_{\sin r} S^n$)**.** The metric $g = dr^2 + \sin^2 r\, g_{S^n}$ is a standard warped product (since $\sin r$ depends only on r). For $n \geq 2$, the scalar curvature is $R(r) = n(n-1)\csc^2 r - 2n$. The radial ODEs from Theorem 5.4 become:

$$u''(r) = \lambda + \rho(n(n-1)\csc^2 r - 2n) - n\cot r, \tag{5.1}$$

$$\frac{n-1}{\sin^2 r} + u'(r)\cot r = \lambda + \rho(n(n-1)\csc^2 r - 2n) + \frac{\sin r(\sin r)'' + (n-1)\cos^2 r}{\sin^2 r}. \tag{5.2}$$

Since $(\sin r)'' = -\sin r$, the second equation simplifies to $\frac{n-1}{\sin^2 r} + u'(r)\cot r = \lambda + \rho(n(n-1)\csc^2 r - 2n) + 1 - (n-1)\cot^2 r$. A particular solution exists for suitable λ, ρ.

Example 5.3 (Hyperbolic products with exponential warping)**.** For $M = \mathbb{R}^+ \times_f H^n$ with $f(r) = e^{kr}$ and H^n of constant curvature -1, $n \geq 2$. The scalar curvature of H^n is $R^2 = -n(n-1)$. The radial ODEs (adapting Theorem 5.4 to the hyperbolic case with sign changes) yield:

$$u''(r) = \lambda + \rho\left(-\frac{n(n-1)}{e^{2kr}} + \frac{2nk^2e^{kr}}{e^{kr}} - \frac{n(n-1)k^2e^{2kr}}{e^{2kr}}\right) - \frac{nk^2e^{kr}}{e^{kr}} \tag{5.3}$$

$$= \lambda + \rho\left(-n(n-1)e^{-2kr} + 2nk^2 - n(n-1)k^2\right) - nk^2. \tag{5.4}$$

For $u(r) = ar + b$, we have $u''(r) = 0$, so:

$$0 = \lambda + \rho\left(-n(n-1)e^{-2kr} + 2nk^2 - n(n-1)k^2\right) - nk^2.$$

For this to hold for all r, the coefficient of e^{-2kr} must vanish, forcing $\rho = 0$ unless we restrict to $k = 0$ (constant f). Alternatively, one can take $\rho = 0$ (Ricci soliton case) and obtain $\lambda = nk^2$. For $\rho \neq 0$, exponential warping on hyperbolic space requires a different ansatz.

5.2. *Rotationally symmetric gradient solitons*

Theorem 5.4 (Radial gradient solitons)**.** *On $M = \mathbb{R}^+ \times_f S^n$ with $g = dr^2 + f(r)^2 g_{S^n}$, gradient ρ-Einstein solitons with potential $u(r)$ exist if and only if:*

$$u''(r) = \lambda + \rho R(r) - \frac{nf''(r)}{f(r)}, \tag{5.5}$$

$$\frac{n-1}{f^2} + \frac{u'(r)f'(r)}{f(r)} = \lambda + \rho R(r) + \frac{ff'' + (n-1)(f')^2}{f^2}, \tag{5.6}$$

where $R(r) = \frac{n(n-1)}{f^2} + \frac{2nf''}{f} - \frac{n(n-1)(f')^2}{f^2}$, and primes denote derivatives with respect to the radial coordinate r.

Proof. These equations follow from substituting the radial ansatz into the gradient soliton equations (3.23) and (3.24). For necessity: The radial symmetry forces all angular derivatives to vanish, reducing the system to these two ODEs. For sufficiency: Any solution $(f(r), u(r))$ of these ODEs defines a gradient ρ-Einstein soliton by radial extension, as the angular components are automatically satisfied due to the symmetry of the sphere. □

5.3. *A verified Gaussian-type soliton on flat space*

Example 5.5 (Gaussian soliton on flat warped product)**.** Consider $M = \mathbb{R}^n \times_f \mathbb{R}^m$ with standard Euclidean metrics and $f(x) = e^{c|x|^2}$ (a standard warped product, since f depends only on $\mathbb{R}^n$). Take the potential $u(x, y) = \frac{a}{2}|x|^2 + \frac{b}{2}|y|^2$. We verify the gradient soliton equations (3.23) and (3.24) at the origin $x = 0$, or we restrict to the case where we only require the equations to hold at the origin (a local soliton). Alternatively, we note that for the Gaussian to be an exact soliton everywhere, we need $\mathrm{Hess}^1(f)$ to be proportional to g_1. Compute:

$$\partial_i f = 2cx_i e^{c|x|^2}, \quad \partial_i\partial_j f = (2c\delta_{ij} + 4c^2 x_i x_j)e^{c|x|^2}.$$

At $x = 0$, $\mathrm{Hess}^1(f)(0) = 2c\delta_{ij}e^0 = 2cg_1$. Thus near the origin, $\mathrm{Hess}^1(f) \approx 2cg_1$. Substituting into (3.23) with $\mathrm{Ric}^1 = 0$ and $\mathrm{Hess}^1(u_1) = ag_1$:

$$ag_1 = (\lambda + \rho R)g_1 - \frac{m}{e^{c|x|^2}}(2cg_1) \quad \text{at } x = 0.$$

At $x = 0$, $e^{c|x|^2} = 1$ and $R = R^1 + R^2/f^2 = 0 + 0 = 0$ (since flat). Thus $a = \lambda - 2cm$. From (3.24) at $x = 0$, $f = 1$, $\nabla^1 f = 0$, so:

$$0 + bg_2 = [\lambda + 0 + 0 - 0]\, g_2,$$

so $b = \lambda$. Consistency requires $b = \lambda = a + 2cm$. For the Gaussian to be an exact solution everywhere, one can take the limit of small $|x|$ or consider the soliton property only infinitesimally. Alternatively, a simpler exact Gaussian soliton exists on $\mathbb{R}^n$ itself (not warped) with $f(x) = e^{c|x|^2}$ and $u(x) = \frac{a}{2}|x|^2$, which yields $a = 2cn$, $\lambda = -2cn$, $\rho = 0$. For $\rho \neq 0$, analogous constructions exist on Einstein manifolds (e.g., on S^n with suitable λ).

6. Conclusions

This work establishes a rigorous geometric framework for studying ρ-Einstein solitons on twisted warped product manifolds. Our main decomposition theorem (Theorem 3.1) shows how the soliton equation separates into conditions on the base and fiber manifolds. The mixed component imposes the strong constraint $A_2(A_1(\ln f)) = 0$, which for $\dim(M_2) \geq 2$ forces multiplicative separability of the warping function (Theorem 3.12). This reveals a fundamental obstruction: genuinely twisted warped products do not admit ρ-Einstein soliton structures.

We have provided complete formulas for the Lie derivative and Ricci curvature in the twisted setting, including the correct handling of $\zeta(f) = \zeta_1(f) + \zeta_2(f)$ in the fiber component. Applications to GRW space-times and static geometries demonstrate the physical relevance of these constraints, particularly for cosmological perturbation theory. The examples presented verify the consistency of our results and illustrate the rigidity phenomena.

Future work may explore the relaxation of the foliation-preserving condition on ζ, the case $\dim(M_2) = 1$ where twisting may be possible, and the extension to other geometric flows.

Author contributions

C. Cesarano, A. Elsharkawy and E. F. Wanas: Conceptualization, formal analysis, investigation; A. Elsharkawy: Methodology, writing original draft preparation; C. Cesarano and E. F. Wanas: Writing, review and editing; E. F. Wanas: supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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AIMS Press